Pion transition form factor in the Regge approach and incomplete vector-meson dominance

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The concept of incomplete vector-meson dominance and Regge models is applied to the transition form factor of the pion. First, we argue that variants of the chiral quark model fulfilling the chiral anomaly may violate the Terazawa-West unitarity bounds, as these bounds are based on unverified assumptions for the real parts of the amplitudes, precluding a possible presence of polynomial terms. A direct consequence is that the transition form factor need not necessarily vanish at large values of the photon virtuality. Moreover, in the range of the *BABAR* experiment, the Terazawa-West bound is an order of magnitude above the data, thus is of formal rather than practical interest. Then we demonstrate how the experimental data may be properly explained with incomplete vector-meson dominance in a simple model with one state, as well as in more sophisticated Regge models. Generalizations of the simple Regge model along the lines of Dominguez result in a proper description of the data, where one may adjust the parameters in such a way that the Terazawa-West bound is satisfied or violated. We also impose the experimental constraint from the $Z \to \pi_0 \gamma$ decay. Finally, we point out that the photon momentum asymmetry parameter may noticeably influence the precision analysis.

I. INTRODUCTION

The pion transition form factor, $F_{\pi^0\gamma\gamma^*}$, measured in the $e^+e^- \to \pi^0\gamma$ annihilation, has been a particularly interesting object of study since its value at the origin is fixed by the chiral anomaly [1,2] while the behavior at very high Euclidean momenta has been predicted via perturbative quantum chromodynamics (pQCD), apparently holding at sufficiently high but unspecified scales [3,4]. However, objections have been raised [5] and reiterated [6] against the applicability of pQCD to exclusive processes. The recent measurement of $F_{\pi^0\gamma\gamma^*}$ by the *BABAR* Collaboration [7], where the pion transition form factor goes visibly above the conventional pQCD prediction at scales $Q^2 > 15$ GeV^2, raised serious doubts concerning the theoretical understanding of the exclusive processes, also in the asymptotic region. In the pQCD approach based on factorization, one uses the light-cone Feynman rules and the Efremov-Radyushkin-Brodsky-Lepage evolution of the pion distribution amplitude (PDA) [3,4,8–16], which results asymptotically in the leading-twist Brodsky-Lepage term $F_{\pi^0\gamma\gamma^*} \sim 2f_\pi/Q^2$, in vivid contradiction to the recent *BABAR* [7] data.

Several ideas have been proposed to circumvent the problem. Radyushkin [17] pointed out that the presence of the possible end-point singularities in the PDA (as found by the authors in the chiral quark models at the low-energy quark-model scale [18]), together with essentially switched-off evolution and regulated quark propagators,

is capable of reproducing the data in the available Q^2 domain. Similar conclusions were reached by Polyakov [19]. In this approach asymptotically $F_{\pi^0\gamma\gamma^*} \sim \log(Q^2/\mu^2)/Q^2$, with the log in the numerator indicating the breaking of factorization. We note that the same asymptotics follows in the spectral quark model; see Eq. (14.1) of Ref. [20]. In fact, serious concerns have been spoken out on the validity of factorization [21] in the considered process. Dorokhov [22–24] proposed the use of the fixed-mass constituent quark model to evaluate the triangle diagram of Fig. 1, which is capable of reproducing the *BABAR* data, however the needed value of the constituent quark mass is uncomfortably low, $M \sim 135$ MeV. In this model the asymptotics has the form $\sim[\log(Q^2/\mu^2)]^2/Q^2$. Possible need of the higher-twist terms has been invoked in Ref. [25]. The calculation of Kotko and Praszalowicz [26] in the nonlocal chiral quark model inspired by the instanton-liquid model of QCD produced the result in agreement with the data at lower values of Q^2. The influence of nonperturbative gluonic components of the pion stemming from instantons has been considered in Ref. [27]. Mikhailov and Stefanis [16] showed that the significant experimental growth of the transition form fac-
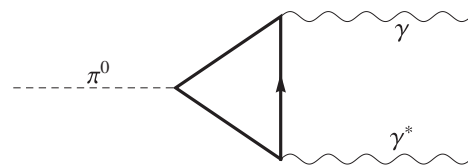

FIG. 1. The triangle diagram used to evaluate the pion transition form factor in quark models, with one real and one virtual photon (the crossed diagram not shown).

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tor between 10 and 40 GeV^2 cannot be explained in terms of higher-order pQCD corrections at the next-to-next-to-leading order level. Finally, in a very recent paper [28] Dorokhov reiterates the findings of Ref. [22] in the non-local model, where the asymptotic behavior is $\sim \log(Q^2/\mu^2)/Q^2$. The nonlocal model of Ref. [28] also requires a very low constituent quark mass in order to reproduce the data in the whole Q^2 range.

Given the fact that the recent *BABAR* data [7] predate the conventional pQCD expectations at such high virtualities as 15–30 GeV^2, it seems adequate to question any implicit assumptions in theoretical analyses. In this paper we analyze critically some of the assumptions underlying certain high-energy theorems which, if holding, would forbid the simple pre-*BABAR* fits based on the incomplete vector-meson dominance (IVMD) [29] (see also Ref. [30] where this result is upgraded with tiny consequences for the muon $g - 2$). The theorems in question, the Terazawa-West (TW) bounds [31,32], were recalled by Dorokhov in Ref. [24].

The TW bounds [31,32] (for a review see, e.g., [33]) are central to our discussion; they are reviewed in Sec. III. Their derivation uses the Schwarz inequality to predict that for physical momenta $\mathrm{Im}F_{\pi^0\gamma\gamma^*} = \mathcal{O}(1/\sqrt{q^2})$ (the first bound). If one further *assumes* that there are no polynomial contributions to the real part of the form factor, one may conclude with the help of a dispersion relation that $F_{\pi^0\gamma\gamma^*} = \mathcal{O}(1/Q)$ for spacelike momenta Q (the second bound). The assumption concerning the real part is clearly spelled out in the original works and, to our knowledge, has never been credibly questioned. We should remind the reader that this was a pre-QCD bound, so we may test the validity of the assumption by analyzing a particular model where *both* the anomaly is satisfied and the second Terazawa-West bound is violated. Actually, a reanalysis of the issue using the quark model and corresponding triangle graphs was presented in Ref. [34] as an example case where the expected bound behavior holds. Here we will consider another variant of the quark model which in fact does not fulfill the assumptions, hence leads to a violation of the second bound. The key point is to realize that the anomaly fixes the value of the form factor at the origin, while the finite Q behavior can be determined regardless of the value at $Q = 0$. This construction requires a subtraction constant which actually implies that at large Q the pion transition form factor does not go to zero. These arguments suggest that pQCD might in fact be computing just the subtracted form factor. Clearly, the issue cannot be settled unless one could go smoothly over all the available energy range. Of course, quark models are not QCD itself, and it is quite possible that our considerations do not apply directly to the QCD analyses. However, the present paper unveils a warning that from purely field-theoretic reasons an asymptotically nonvanishing pion transition form factor cannot be rejected.

With this finding in mind we analyze in detail the predictions of IVMD and the Regge models for $F_{\pi^0\gamma\gamma^*}$. Regge models with (infinitely many) tree-level meson and glueball exchanges are a realization of the large-N_c limit [35] which, unlike the quark models, incorporates confinement. In particular, these models were insightful in analyzing condensate issues [36–44], form factors [45,46], or the nature of scalar states [47].

On purely phenomenological grounds, we will show that the data for $F_{\pi^0\gamma\gamma^*}$ can be fitted very well in a simple vector-meson exchange model with IVMD. In more elaborate variants of the Regge approach the data may be reproduced (with different choices of the model parameters) almost equally well with or without the fulfillment of the TW bounds.

There has also been an important interest in the pion transition form factor after the proposal of determining the rare $Z \to \pi^0\gamma$ decay [48], which probes the timelike value $q^2 = M_Z^2$. The experimental bound given by the Particle Data Group [49] implies a large suppression as compared to the $q^2 = 0$ anomaly point, but not necessarily as much as predicted by many authors, where $\Gamma(Z \to \pi^0\gamma) \sim 10^{-11}$ GeV (see, e.g., [50]).

The paper is organized as follows. As a preparatory material, in Sec. II we provide some basic kinematics, in Sec. III we review the TW bounds, and in Sec. IV recall the bound from the $Z \to \pi^0\gamma$ decay. In Sec. V we use several variants of the quark model to inquire on the validity of the TW bounds. As we will show, these models are not capable of describing the data with realistic model parameters, but pose a warning on the second TW bound. We use this insight in Sec. VI to propose a more realistic Vector Meson Dominance description of the data. The analysis is complemented in Sec. VII with the inclusion of the large N_c motivated infinite tower of radial Regge-like excitations of the ρ meson. The important role of photon asymmetry is discussed briefly in Sec. VIII. Finally, in Sec. IX we list our main conclusions.

II. THE KINEMATICS

With the outgoing momenta and polarizations of the photons denoted as q_1, e_1 and q_2, e_2 (see Fig. 1), one finds the amplitude

$$\Gamma^{\mu\nu}_{\pi^0\gamma^*\gamma^*}(q_1, q_2) = \epsilon_{\mu\nu\alpha\beta}e_1^\mu e_2^\nu q_1^\alpha q_2^\beta F_{\pi\gamma^*\gamma^*}(Q^2, A), \quad (1)$$

where the pion transition form factor $F_{\pi\gamma^*\gamma^*}$ depends on the total virtuality, Q^2, and the photon momentum asymmetry parameter, A,

$$Q^2 = -(q_1^2 + q_2^2), \qquad A = \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2}, \qquad -1 \leq A \leq 1. \tag{2}$$

Equivalently, $q_1^2 = -\frac{(1+A)}{2}Q^2$, $q_2^2 = -\frac{(1-A)}{2}Q^2$. For large

virtualities, assuming factorization, one finds the standard twist decomposition of the pion transition form factor [8],

$$F_{\pi^0 \gamma^* \gamma^*}(Q^2, A) = J^{(2)}(A) \frac{1}{Q^2} + \dots, \qquad (3)$$

with

$$J^{(2)}(A) = \frac{4 f_\pi}{N_c} \int_0^1 dx \frac{\varphi_\pi^{(2)}(x)}{1 - (2x - 1)^2 A^2}, \qquad (4)$$

involving the pion parton distribution amplitude PDA, $\varphi(x)$.

Taking the asymptotic form $\varphi(x) = 6x(1 - x)$ in Eq. (4) gives the Brodsky-Lepage asymptotic result [8]

$$J^{(2)}(A = 1) = \frac{6}{N_c f_\pi}. \qquad (5)$$

An analysis of the lowest Gegenbauer moments a_2 and a_4 of $\varphi(x)$ has been carried out in Refs. [13,15,51,52]. Higher twists have been analyzed in the framework of the light-cone sum rules [53]. An attempt of a direct measurement of the DA has been presented by the E791 Collaboration [54]. For a concise review on all these developments see, e.g., Ref. [55] and references therein.

Although theoretically one may take the maximally asymmetric case of $A = 1$, in the experiment this situation is not possible to achieve. For instance, in the *BABAR* kinematic setup $-q_1^2 < 0.6$ GeV^2 and $-q_2^2 > 3$ GeV^2, suggesting $A \sim 0.9$–0.97. As advanced in Ref. [56], we note that the model results presented in the following sections become quite sensitive to the precise value of A at large Q^2, hence in precision analyses the effects of kinematic cuts should be incorporated.

At this point, before undertaking further elaborations, an important qualification is in order. Equation (4) provides a gauge-invariant high-energy-motivated definition of the pion PDA, which formally determines a nonperturbative matrix element. A purely low-energy definition of the PDA involves link operators between the quark fields at different space-time points and depends on the renormalization scale. A nontrivial issue has to do with the equivalence of both definitions and the scale at which this identification makes sense. We refer to previous works [18,20,45,57] for a detailed discussion on this delicate and nontrivial issue. An outstanding outcome was that in several models

$$\varphi(x) = 1 \qquad (6)$$

at the model low-energy scale; the end-point behavior would be bended after the QCD evolution to higher scales [see Eq. (5.26) of Ref. [58] for the detailed asymptotic form of the PDA near the end points after evolution]. While the identification of the PDA through Eq. (4) indeed requires taking asymptotically large momenta, it does not necessarily follow that the asymptotic expression itself holds at the so far measured momenta.

III. TERAZAWA-WEST BOUNDS

Terazawa [31] and West [32] (for a review see, e.g., [33]) derived unitarity bounds for the pion transition form factor. The derivation considers the photon propagator, the charge form factor of the pion, and the transition form factor. The Schwarz inequality is used to show that

$$\text{Im} F_{\pi^0 \gamma \gamma^*}(q^2) = \mathcal{O}(1/\sqrt{q^2}) \qquad (7)$$

for timelike momenta, $q^2 > 4m_\pi^2$, which we term *the first TW bound*. Under a further important assumption that there are no polynomial contributions to the real part of the form factor, one may also conclude that $|F_{\pi^0 \gamma \gamma^*}(q^2)| = \mathcal{O}(1/\sqrt{q^2})$. Then the dispersion relation (see Sec. 14 of Ref. [33]) yields that

$$|F_{\pi^0 \gamma \gamma^*}(Q^2)| = \mathcal{O}(1/Q) \qquad (8)$$

for large spacelike momenta Q, which we call *the second TW bound*.

The assumption of the absence of the polynomial terms (in particular, a constant) from the real part of the pion transition form factor is equivalent to validity of the unsubtracted dispersion relation. As mentioned in the Introduction, this crucial assumption is clearly spelled out as such in the original papers; in the following Sections of the paper we will investigate situations where the subtraction constant is present in the pion transition form factor.

The constant in the bound (8) may be given [33] in terms of the photon spectral density and the pion quark structure function, namely,

$$|F_{\pi^0 \gamma \gamma^*}(Q^2)| < \frac{2\sqrt{\Pi(\infty)}}{Q} \int_0^1 dx \sqrt{\frac{F_1(x, Q^2)}{x(1 - x)}}, \qquad (9)$$

where

$$\Pi(s) = \frac{s}{16\pi^3 \alpha_{\text{QED}}^2} \sigma_{e^+ e^- \to \text{hadrons}}(s), \qquad (10)$$

with the asymptotic value

$$\Pi(\infty) = \frac{1}{12\pi^2} \sum_i e_i^2, \qquad (11)$$

e_i denoting the charge of the quark of flavor i in the natural units. The pion structure function is

$$F_1(x, Q^2) = \frac{1}{2} \sum_i e_i^2 [q_i(x, Q^2) + \bar{q}_i(x, Q^2)], \qquad (12)$$

with $q_i(x, Q^2)$ and $\bar{q}_i(x, Q^2)$ denoting the quark and antiquark distribution functions at momentum Q.

With the help of the Sutton-Martin-Roberts-Stirling [59] and the Gluck-Reya-Vogt [60] parametrizations of the pion parton distribution functions we obtain, via integrating from $x = 10^{-5}$ to 1, a numerical estimate for the coeffi-

cient in the bound (9) which actually exhibits a rather mild dependence on Q^2. With Q^2 in the range $10-40$ GeV^2 we find for the LO and next-to-leading order (NLO) evolved parton distribution functions the bounds

$$|F_{\pi^0\gamma\gamma^*}(Q^2)| < \frac{0.85(1)}{Q} \quad \text{(LO)},$$
$$< \frac{0.75(1)}{Q} \quad \text{(NLO)}, \tag{13}$$

with the small uncertainty reflecting the considered Q interval. Therefore the bound is completely "inefficient" in the considered range of momenta, as it is an order of magnitude above the *BABAR* data. For that reason it is of formal rather than practical interest.[1]

IV. BOUND FROM THE $Z \to \pi^0\gamma$ DECAY

Another bound for the pion transition form factor comes from the rare $Z \to \pi^0\gamma$ decays. In that process, proceeding via a quark loop, only the vector coupling of the Z boson to the quarks contributes [48], hence

$$\frac{F_{Z\to\pi^0\gamma}(q^2)}{F_{Z\to\pi^0\gamma}(0)} = \frac{F_{\pi^0\gamma^*\gamma}(q^2)}{F_{\pi_0\gamma^*\gamma}(0)}. \tag{14}$$

The experimental limit $\Gamma(Z \to \pi^0\gamma) < 5 \times 10^{-5}\Gamma_{\text{tot}}(Z) = 10.25 \times 10^{-5}$ GeV, provided by the Particle Data Group [49], implies

$$|F_{Z\to\pi^0\gamma}(M_Z^2)/F_{Z\to\pi^0\gamma}(0)| < 0.17. \tag{15}$$

This experimental bound is not as stringent as predicted by many authors, where $\Gamma(Z \to \pi^0\gamma) \sim 10^{-11}$ GeV (see, e.g., [50]). The bound can be used in models which predict the form factor in the timelike region, e.g. the Regge models considered in the following sections.

V. QUARK MODELS

The role of the constituent vs. current quarks in the π^0 and ω transition form factors in triangle diagrams was recognized in Ref. [61]. The diagram of Fig. 1 is superficially linearly divergent, but the requirement of the gauge invariance guarantees convergence. In the constituent quark model (CQM) a direct computation of the triangle yields, for the general kinematic case,

$$F_{\pi^0\gamma^*\gamma^*}(Q^2, A) = \frac{1}{4\pi^2 f_\pi} G(Q^2, A), \tag{16}$$

where the loop function (we work in the strict chiral limit of $m_\pi = 0$),

$$G(Q^2, A) = \frac{2M^2}{Q^2} \int_0^1 \frac{dx}{x}$$
$$\times \log\left[\frac{2M^2 + (1 + A)x(1 - x)Q^2}{2M^2 + (1 - A)x(1 - x)Q^2}\right], \tag{17}$$

is normalized to unity at the origin, $G(0, A) = 1$. For the special case $A = 1$

$$G(Q^2, A = 1) \equiv G(Q^2)$$
$$= \frac{2M^2}{Q^2} \int_0^1 \frac{dx}{x} \log\left[1 + x(1 - x)\frac{Q^2}{M^2}\right]. \tag{18}$$

We note that the evaluation is covariant, thus different in philosophy from the conventional light-cone analysis. In the latter case the soft part of the diagram containing the PDA is factorized, while the hard part of the diagram is evaluated according to the light-cone Feynman rules. On the other hand, in CQM one evaluates the diagram of Fig. 1 with the instant-form Feynman rules, corresponding to local or nonlocal variants of the model. That way one is insensitive to the issues of factorization. However, it is not clear how credible this approach is at high virtualities, in particular, in local models, where the virtual quark propagator carries, even asymptotically, a large constituent mass. In the study of this section we are, however, primarily interested in formal aspects, namely, the TW bounds in a field-theoretic model, thus we shall not be concerned with the issue whether or not a low-energy quark model can be realistically used to obtain the asymptotic transition form factor according to Fig. 1.

Chiral quark models are particular realizations of the large-N_c limit. A variant of the CQM, the well-known Georgi-Manohar model [62], allows the quarks to carry an axial charge different from unity, $g_A^Q \neq 1$. The relevant part of the Lagrangian of the model is

$$L = \bar{q}(i\slashed{\partial} + g_A^Q \slashed{A}\gamma_5 - M)q + \frac{f_\pi^2}{4}\text{Tr}(\partial_\mu U^\dagger \partial^\mu U) + \text{WZW},$$
$$A_\mu = \frac{i}{2}(u^\dagger \partial_\mu u - u\partial_\mu u^\dagger), \quad u = e^{i\vec{\pi}\cdot\vec{\tau}/(2f)}, \quad U = u^2,$$
$$\tag{19}$$

where $f = 93$ MeV is the pion decay constant and WZW denotes the Wess-Zumino-Witten term [63,64]. There has been some discussion on whether or not $1 - g_A^Q = \mathcal{O}(N_c^0)$. This issue was answered in the affirmative way in Ref. [65], where the $\mathcal{O}(N_c^0)$ departure from unity is basically due to the t-channel exchanges in the Adler-Weisberger sum rule for the pion-quark scattering. In the Nambu–Jona-Lasinio model with vector mesons, the $A_1 - \pi$ mixing is an explicit leading-N_c source [66–68] of the effect. Because of the chiral anomaly, the field representation used to describe the $g_A^Q \neq 1$ situation is relevant. In Ref. [69] it was shown that the apparent anomalous inequivalence of effective theories may be compensated by

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[1]We are taking Q^2 large enough as to neglect the higher twist corrections. Actually the bound never crosses the pQCD limit $2f_\pi/Q^2$ for the lowest momenta considered in Ref. [60], namely, $Q_0^2 \sim 0.26$ GeV^2 at LO and $Q_0^2 \sim 0.40$ GeV^2 at NLO.

including extra terms in the effective action. More specifically, the original Georgi-Manohar model *does not* contain the anomalous piece, such that the Wess-Zumino-Witten (WZW) action should be added *a posteriori*, as in Eq. (19). An example on how various vertex functions are modified by naively including $g_A^Q \neq 1$ and violating the anomaly in chiral quark models is presented in Ref. [70]. The subtraction procedure described above to restore the anomaly and based on [69] was addressed in Ref. [71] (see also [72]).

A direct calculation of the pion transition form factor in the Georgi-Manohar model supplied with the WZW action yields the result

$$F_{\pi^0 \gamma \gamma^*}(Q^2) = \frac{1}{4\pi^2 f_\pi} + \frac{g_A^Q}{4\pi^2 f_\pi}[G(Q^2) - 1], \quad (20)$$

with $G(Q^2)$ given by Eq. (18). This expression satisfies the anomaly and the dispersion relation but *does not* vanish at infinity, since

$$F_{\pi^0 \gamma \gamma^*}(Q^2) = \frac{1 - g_A^Q}{4\pi^2 f_\pi} + \frac{g_A^Q M^2}{4\pi^2 f_\pi}\frac{[\log(Q^2/M^2)]^2}{Q^2} + \dots \quad (21)$$

fulfills the first TW bound (for the imaginary part) but *does not* fulfill the second bound (for the real part).

Similarly, within the spectral quark model [20] a direct implementation of g_A^Q is equally possible [73] and one gets the result of Eq. (20) with

$$G(Q^2) = \frac{1}{3}\left[\frac{2m_\rho^2}{m_\rho^2 + Q^2} + \frac{m_\rho^2}{Q^2}\log\left(\frac{m_\rho^2 + Q^2}{m_\rho}\right)\right]. \quad (22)$$

With $g_A^Q = 1$ this model fulfills qualitatively the result of Radyushkin with a similar mass scale [17], since

$$F_{\pi^0 \gamma \gamma^*}(Q^2) = \frac{1 - g_A^Q}{4\pi^2 f_\pi} + \frac{g_A^Q m_\rho^2}{12\pi^2 f_\pi}\frac{[\log(Q^2/m_\rho^2)]}{Q^2} + \dots \quad (23)$$

In the *local* quark models we have the general relation

$$F_{\pi^0 \gamma \gamma^*}(t) = \frac{1 - g_A^Q}{4\pi^2 f_\pi} + \frac{g_A^Q}{12\pi^2 f_\pi}\left[2F_{\pi^+}^{\text{em}}(t) + \int_0^t ds F_{\pi^+}^{\text{em}}(s)\right], \quad (24)$$

which correlates the charge and transition form factors. This relation shows that in these models even in the case $g_A^Q - 1$ one cannot have complete VMD simultaneously for the two form factors.

Unfortunately, attempts to fit the experimental data with the quark-model formulas, with g_A equal or different from unity, are not numerically successful in the whole momentum range, unless one uses unrealistic model parameters. The purpose of the above calculations was different. Our

examples show in an explicit manner that the second TW may be violated in models consistent with all field-theoretic constraints, such as covariance, gauge invariance, chiral symmetry, and anomaly matching. Importantly, they provide instances where the form factor does not vanish asymptotically. The calculation also illustrates that the $Q^2 = 0$ value and the finite Q^2 values are independent. The possibility of disobeying the second TW bound will be explored in the following sections.

VI. VECTOR MESON DOMINANCE MODELS

We now come to the core of our paper and consider several implementations of IVMD, both in a simple approach with a single vector meson, as well as in more sophisticated Regge models with infinitely-many radially-excited states. It will turn out that the pion transition form factor can be accurately described with this phenomenological method. The coupling of photons to vector mesons has a long history and Ref. [74] comprehensively reviews the interplay between universality, the vector-meson dominance, and the low-energy theorems. We will highlight first the issue of IVMD for the charge form factor, such that our points are later on more easily made for the transition form factor.

A. Charge form factor

The charge form factor is the famous case where VMD can be implemented:

$$F_{\pi^+}^{\text{em}}(t) = \frac{M_V^2}{M_V^2 - t} \sim -\frac{M_V^2}{t} + \dots \quad (25)$$

with $t = -Q^2$. Here and in the following the asymptotic behavior at $t \to \infty$ is indicated by the $\sim$ sign. When the unregularized quark-loop mean squared radius [75] is matched to (25), one gets the relation

$$M_V^2 = 24\pi^2 f_\pi^2/N_c. \quad (26)$$

There is no way to match the successful form factor of Eq. (25) to the pQCD result.[2]

If one uses a once-subtracted dispersion relation and imposes the current conservation, $F_{\pi^+}^{\text{em}}(0) = 1$, one gets

$$F_{\pi^+}^{\text{em}}(t) - 1 = \frac{1}{\pi}\int_{t_0}^\infty \frac{t}{t'}\frac{\text{Im}F_{\pi^+}^{\text{em}}(t')}{t' - t - i\epsilon}dt', \quad (27)$$

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[2]If we nevertheless match Eq. (25) to pQCD, the only possible solution is $\alpha_s(Q) = \pi/2$, which yields a too-small scale $Q \sim 300$ MeV. This matching to pQCD may seem weird but need not necessarily be conceptually wrong. Note that in the string model calculation of the $q\bar{q}$ potential one gets $V_{\bar{q}q}(r) = -\pi/12r$, whereas from pQCD $V_{\bar{q}q}(r) = -4\alpha_S/3r$. This yields $\alpha = \pi/16$, which means a safely high scale of $\mu = 2$ GeV for $\Lambda_{\text{QCD}} = 240$ MeV. The string model describes accurately the lattice data in the short distance region. In the case of the charge form factor with VMD, the required scale is much smaller.

which yields

$$F_{\pi^+}^{\text{em}}(-\infty) - 1 = -\frac{1}{\pi} \int_{t_0}^{\infty} \frac{\text{Im}F_{\pi^+}^{\text{em}}(t')}{t'} dt'. \quad (28)$$

Saturation with a single resonance gives the result

$$F_{\pi^+}^{\text{em}}(t) = 1 + \frac{a}{2} \frac{t}{M_V^2 - t} \sim 1 - \frac{a}{2} + \frac{a}{2} \frac{M_V^2}{-t} + \dots \quad (29)$$

When the coupling $a/2 = f_{\rho\gamma}g_{\rho\pi\pi}/m_\rho^2 \equiv g_{\rho\pi\pi}/g_{\rho e^+e^-}$ becomes unity, the form factor vanishes at infinity. Sakurai's universality indeed requires $g_{\rho\pi\pi} = g_{\rho e^+e^-}$. In general, however, we may have $g_{\rho\pi\pi} \neq g_{\rho e^+e^-}$. Note that positivity requires $a < 2$, such that $F_{\pi^+}^{\text{em}}(-Q^2) < 1$. A fit to the experimental data [76–79] yields $a/2 = 0.99(2)$ and $M_V = 679(36)$ MeV. Thus the data for the pion charge form factor are consistent with the complete VMD, but certainly do not preclude IVMD, with $a/2$ departing slightly from unity.

Caldi and Pagels [80] obtained a similar expression as Eq. (29) for the pion form factor from a direct photon contribution and a momentum-dependent $\rho - \gamma$ vertex (see also Ref. [81]). The discussed properties are nicely displayed by the hidden-symmetry approach by Bando *et al.* [82], where there is a contact piece and the vector meson term, which dominates completely when the Kawarabayashi-Suzuki-Fayyazuddin-Riazuddin relation is fulfilled. Actually, in Ref. [83] the equivalence of this approach to more conventional ones is established. The interplay between VMD and universality was analyzed by Schechter [84] (see also Ref. [85]). Symmetry breaking effects have been analyzed in Ref. [86]. More attempts including predictions for meson decays can be found in [87].

Of course, the nonvanishing of the low-energy representation (the one-resonance saturation) need not be taken as a fundamental problem. The only feature we see is that this nonvanishing represents more accurately the unknown high-energy data. If we had infinitely many states, we could fit that data and by separating explicitly the contribution from the lowest ρ state. We see that the effect of all other states does behave as the a constant, which is slowly dependent on Q in a *wide* energy range.

Ideally, one should take the $e^+e^- \to \pi^+\pi^-$ data over all possible momenta. However, obviously the experimental data are available only up to a certain maximum value, $s_{\text{max}} = 4\Lambda^2$. Thus, even if the form factor vanishes at infinity, we have

$$F_{\pi^+}^{\text{em}}(t) - 1 = \frac{1}{\pi} \int_{4m_\pi^2}^{4\Lambda^2} \frac{t}{s} \frac{\text{Im}F_{\pi^+}^{\text{em}}(s)}{s - t} ds$$
$$+ \frac{1}{\pi} \int_{4\Lambda^2}^{\infty} \frac{t}{s} \frac{\text{Im}F_{\pi^+}^{\text{em}}(s)}{s - t} ds, \quad (30)$$

where the last term is weakly momentum-dependent and

hence resembles a constant behavior assumed in IVMD, as discussed above.

B. Transition form factor

For the $\pi^0\gamma\gamma^*$ form factor the complete VMD with just one state implies

$$F_{\pi^0\gamma\gamma^*}(t) = \frac{1}{4\pi^2 f_\pi} \frac{M_V^2}{M_V^2 - t} \sim -\frac{M_V^2}{4\pi^2 f_\pi t} + \dots. \quad (31)$$

Note that while the anomaly value does not depend on N_c explicitly, the high momentum behavior does. When matching Eq. (31) to the pQCD result of Eq. (5) is done, one gets independently the relation (26),

$$M_V^2 = 24\pi^2 f_\pi^2/N_c. \quad (32)$$

Despite this appealing property, the parametrization (31) fails to describe the experimental data in the high momentum region.

If we incorporate the possibility that the form factor need not vanish at infinity, we may write a once-subtracted dispersion relation [88],

$$F_{\pi^0\gamma\gamma^*}(t) - F_{\pi^0\gamma\gamma^*}(0) = \frac{1}{\pi} \int_{t_0}^{\infty} \frac{t}{t'} \frac{\text{Im}F_{\pi^0\gamma\gamma^*}(t')}{t' - t - i\epsilon} dt'. \quad (33)$$

The influence of the well-known timelike region does not determine unambiguously when the onset of the pQCD takes place. Actually, the single VMD model shows that even in the spacelike region with momenta as low as $Q^2 \sim m_\rho^2$ the effects of the chiral logs and final state interactions are meager. Taking the limit $t \to -\infty$ we get

$$F_{\pi^0\gamma\gamma^*}(-\infty) - F_{\pi^0\gamma\gamma^*}(0) = -\frac{1}{\pi} \int_{t_0}^{\infty} \frac{\text{Im}F_{\pi^0\gamma\gamma^*}(t')}{t'} dt'. \quad (34)$$

This shows that $F_{\pi^0\gamma\gamma^*}(-\infty) < F_{\pi^0\gamma\gamma^*}(0)$ if $\text{Im}F_{\pi^0\gamma\gamma^*}(t) > 0$. When we saturate the absorptive part with just one resonance, we get

$$F_{\pi^0\gamma\gamma^*}(-Q^2) = \frac{1}{4\pi^2 f_\pi}\left[1 - c\frac{Q^2}{M_V^2 + Q^2}\right], \quad (35)$$

with M_V denoting the vector-meson mass. The coefficient c is related to the $\rho \to \pi\gamma$ decay,

$$c = c_{\rho\pi\gamma} = \frac{-2ef_\rho}{m_\rho} \frac{\mathcal{A}(\rho^0 \to \pi^0\gamma)}{\mathcal{A}(\pi^0 \to \gamma\gamma)} = 1.022 \pm 0.051, \quad (36)$$

where the value of the estimate is obtained from the decay width $\Gamma(\rho^+ \to \pi^+\gamma) = 68 \pm 7$ KeV.

If we fit the parameters in formula (35) with the CLEO data only, we get

$$c = 0.998(18), \qquad M_V = 777(44) \text{ MeV}, \quad (37)$$

with $\chi^2/\text{DOF} = 0.54$, hence c is consistent with unity, in

agreement with previous determinations assuming complete VMD, however not excluding IVMD. On the other hand, the fit to the combined CELLO, CLEO, and *BABAR* data yields the result

$$c = 0.986(2), \qquad M_V = 748(14) \text{ MeV,} \qquad (38)$$

hence c is different from unity at the level of 7 standard deviations, while it remains consistent with estimate (36). As seen from Fig. 2, the agreement with the experimental data over all the momentum range is remarkable, with $\chi^2/\text{DOF} = 0.7$. The contours of the confidence levels are displayed in Fig. 3.

The logarithmic slope at the origin reads

$$b_\pi = -\left[\frac{1}{F_{\pi^0\gamma\gamma^*}(Q)}\frac{d}{dQ^2}F_{\pi^0\gamma\gamma^*}(Q)\right]\bigg|_{Q^2=0}. \qquad (39)$$

Numerically, taking the optimum parameters (38), we get the value

$$b_\pi = \frac{c}{M_V^2} = 1.76(7) \text{ GeV}^{-2}. \qquad (40)$$

Our IVMD model estimate is in a very good agreement with the average experimental values quoted in the PDG [49]: $b_\pi = (1.76 \pm 0.22)$ GeV^{-2}. This value is very close to that originally reported by the CELLO Collaboration [89], obtained from an extrapolation from high-Q^2 data to low Q^2 by means of generalized vector meson dominance, $b_\pi = (1.4 \pm 1.3 \pm 2.6)$ GeV^{-2} given in [90], and $b_\pi = (1.4 \pm 0.8 \pm 1.4)$ GeV^{-2} given in [91].

To end this section, we turn to the rare decay $Z \to \pi^0\gamma$, which probes the transition form factor in the physical region. From Eq. (35) with parameters (38) we get $|F_{Z\to\pi^0\gamma}(M_Z^2)/F_{Z\to\pi^0\gamma}(0)| = 0.014(2)$, a comfortable order of magnitude smaller than the upper experimental bound

FIG. 3. The $\Delta\chi^2 = 2.3$ and 4.6 contours, corresponding to the 68% and 90% confidence levels, in the $M_V - c$ plane for the fit with the IVMD ansatz (35). The central values yield $\chi^2/\text{DOF} = 0.6$.

(15), but not as small as predicted by several models ($\sim 10^{-4}$, see, e.g., [50] and references therein).

VII. REGGE MODELS

In the previous section we have considered the simplest possible implementation of IVMD, with just one vector-meson state. However, the large-N_c limit of QCD involves tree-level diagrams with infinitely many states, including the radial excitations. That way the matching to the QCD correlators can be accomplished [42,43,47], as well as the correct asymptotic behavior of the pion charge form factor [46] may be obtained. In this section we analyze the pion transition form factors in the framework of Regge models with infinitely many radially excited vector-meson states.

Based on the success of the Veneziano-Lovelace-Shapiro dual resonance model (see, e.g., [92,93] and references therein) Suura [94] and Frampton [95] proposed analytic models. The general form reads

$$F_{\pi^0\gamma^*\gamma^*}(Q^2, A) = \sum_{V_\rho, V_\omega}\frac{F_{V_\rho}(q_1^2)F_{V_\omega}(q_2^2)G_{\pi V_\rho V_\omega}(q_1^2, q_2^2)}{(q_1^2 - M_{V_\rho}^2)(q_2^2 - M_{V_\omega}^2)}$$
$$+ (q_1 \leftrightarrow q_2), \qquad (41)$$

where F_{V_ρ} and F_{V_ω} are the current-vector meson couplings, while $G_{\pi V_\rho V_\omega}$ is the coupling of two vector mesons to the pion. The situation is depicted in Fig. 4. At the soft photon point, corresponding to the neutral pion decay $\pi^0 \to 2\gamma$, the chiral anomaly matching condition imposes the normalization
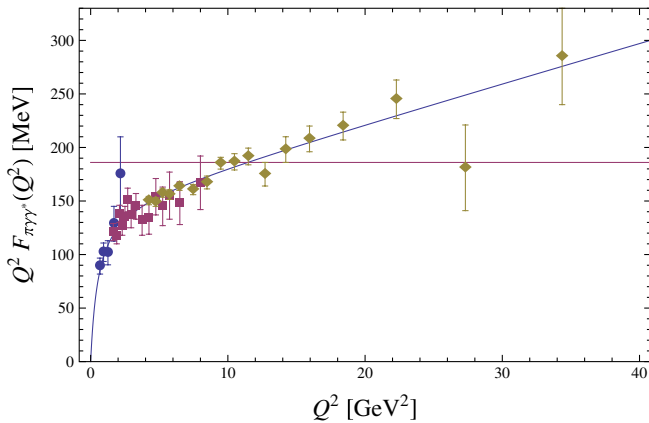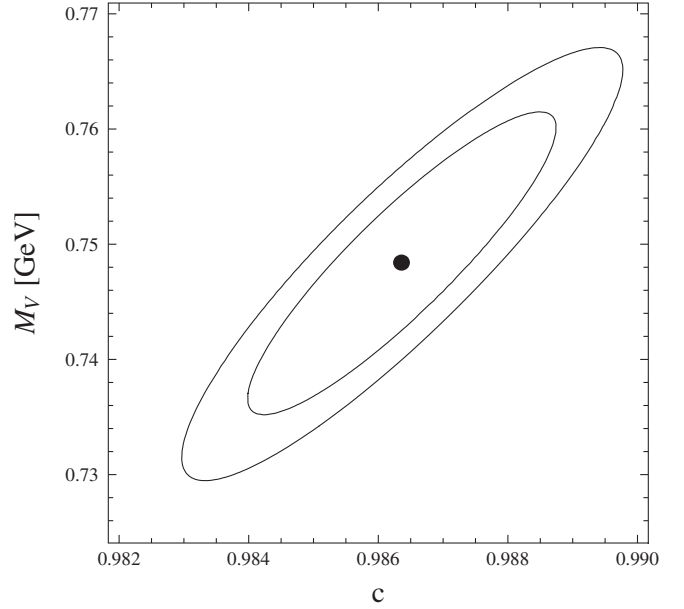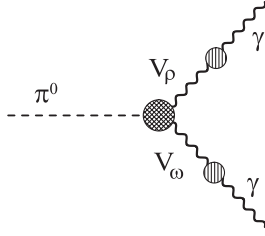

FIG. 2 (color online). The pion transition form factor in the spacelike region $Q^2 = -t > 0$. The dots, squares, and diamonds correspond to the CELLO [89], CLEO [97], and *BABAR* [7] data, correspondingly. The line is the incomplete-vector-meson-dominance fit with formula (35) and parameters (38).

FIG. 4.   The pion transition form factor in Regge models. The labels V_ρ and V_ω denote the infinite vector meson towers with the ρ and ω quantum numbers.

$$F_{\pi^0\gamma^*\gamma^*}(0,0) = \sum_{V_\rho V_\omega} \frac{2F_{V_\rho}(0)F_{V_\omega}(0)G_{\pi V_\rho V_\omega}(0,0)}{M_{V_\rho}^2 M_{V_\omega}^2}$$
$$= \frac{1}{4\pi^2 f_\pi}. \qquad (42)$$

This consistency constraint, realized in nature, can be always satisfied in models by an appropriate choice of the couplings. In Ref. [45] we analyzed the transition form factor with the help of formula (41) with the coupling constants F_V taken as constants (as requested asymptotically by the matching of spectral densities to QCD) and allowing for constant diagonal couplings in $G_{\pi V_\rho V_\omega}$. As a result, a constant PDA of Eq. (6) was extracted. The corresponding pion transition form factor has the asymptotic behavior $\sim \log(Q^2/\mu^2)/Q^2$, however with physical values of the model parameters it overshoots the data in the region above $Q^2 \sim 1$ GeV^2, thus does not properly reproduce the data.[3]

In the analysis below we use a more sophisticated approach, namely, the factorizable product suggested by Dominguez [96] in the dual description of radiative decays,

$$F_{\pi^0\gamma\gamma^*}(t) = \frac{1}{4\pi^2 f_\pi} f_b(t), \qquad (43)$$

where

$$f_b(t) = \frac{1}{B(b-1, \frac{M_V^2}{a})} \sum_{n=0}^{\infty} \frac{\Gamma(2-b+n)}{\Gamma(n+1)\Gamma(2-b)}$$
$$\times \frac{1}{an + M_V^2 - t}. \qquad (44)$$

The function $f_b(t)$ depends on three parameters: the lowest-lying vector-meson mass, M_V, the string tension, $\sigma = a/(2\pi)$, and the asymptotic fall-off parameter, b. The mass formula is then $M_V(n)^2 = an + M_V^2$. The function (45) fulfills the normalization condition

$$f_b(0) = 1. \qquad (45)$$

---

[3]As already mentioned, we stress that taking the $Q^2 \to \infty$ limit is an operational way of extracting the leading twist PDA; this is different than describing the current data within this limit.

For $x \gg y$ one has $B(x,y) \sim \Gamma(y)x^{-y}$, hence in the asymptotic region of $M_V^2 - t \gg (b-1)a$ we find

$$f_b(t) \sim \frac{\Gamma(\frac{M_V^2}{a} + b - 1)}{\Gamma(\frac{M_V^2}{a})} \left(\frac{a}{M_V^2 - t}\right)^{b-1}. \qquad (46)$$

The TW bounds are satisfied if $b > 1.5$.

We remark that in Ref. [46] this version of the Regge approach was used to describe the charge form factor. We have shown that it can be accurately reproduced up to $Q^2 \sim 6$ GeV^2, while the pQCD result greatly undershoots the experiment. The onset of pQCD occurs at extremely high ("cosmological") values of Q^2.

We now proceed with the application of the presented Regge model to the pion transition form factor. Taking $a = 1.3$ GeV^2 (which correspond to the string tension $\sigma = (455$ MeV$)^2$), a χ^2 fit of formula (43) to the joint CELLO, CLEO, and *BABAR* data yields

$$M_V = 0.672(25) \text{ GeV}, \qquad b = 1.81(3), \qquad (47)$$

with $\chi^2/\text{DOF} = 1$. The fit is shown in Fig. 5 with the dashed line. Since $b > 1.5$, the fit satisfies the TW bounds. For the logarithmic slope at the origin we find $b_\pi = 2.1(2)$ GeV^{-2}. We also get, with smoothing the spectral density in the physical region, the ratio $|F_{Z\to\pi^0\gamma}(M_Z^2)/F_{Z\to\pi^0\gamma}(0)| = 0.0014(4)$, which is comfortably below the experimental bound and an order of magnitude smaller than the IVMD fit of Sec. VI B.

We can further try to improve the agreement with the *BABAR* data by explicitly separating the first pole as follows:
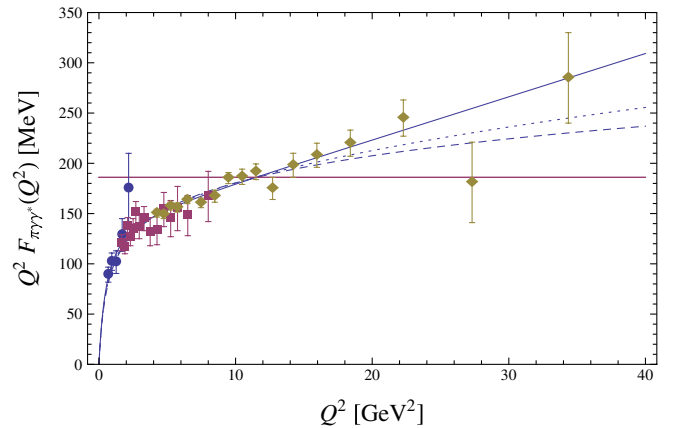

FIG. 5 (color online).   The pion transition form factor in the spacelike region $Q^2 = -t > 0$. The dots, squares, and diamonds correspond to the CELLO [89], CLEO [97], and *BABAR* [7] data, correspondingly. The dashed line is the results of the Regge fit with formula (43) and parameters (47). The dotted line shows the Regge fit with Eq. (48) and parameters (49). Finally, the solid line corresponds to the subtracted Regge model of Eq. (50) and parameters (51).

$$F_{\pi^0\gamma\gamma^*}(t) = \frac{1}{4\pi^2 f_\pi}\left[c\frac{M_V^2}{M_V^2 - t} + (1-c)\frac{\bar{f}_b(t)}{\bar{f}_b(0)}\right], \quad (48)$$

where $\bar{f}_b(t)$ is obtained from Eq. (45) with the $n = 0$ term omitted from the sum. We impose the TW bound by setting $b = 1.5$. Then (with $a = 1.3$ GeV^2) we find

$$c = 0.95(2), \qquad M_V = 709(9) \text{ MeV}, \qquad (49)$$

with $\chi^2/\text{DOF} = 0.85$, $b_\pi = 1.9(1)$ GeV^{-2}, and $|F_{Z\to\pi^0\gamma}(M_Z^2)/F_{Z\to\pi^0\gamma}(0)| = 0.01(1)$. The corresponding curve is presented in Fig. 5 with the dotted line.

Finally, we take a subtracted Regge model, in analogy to the model of Sec. VI B of the form

$$F_{\pi^0\gamma\gamma^*}(t) = \frac{1}{4\pi^2 f_\pi}[1 - c + cf_b(t)]. \qquad (50)$$

Setting $M_V = 770$ MeV and $a = 1.3$ GeV^2 yields the optimum values for the remaining parameters

$$c = 0.984(4), \qquad b = 2.05(3), \qquad (51)$$

with $\chi^2/\text{DOF} = 0.7$. Since the value of b is consistent with 2, the model gives very similar results to the model with the single vector-meson state considered in Sec. VI B. Hence we find $b_\pi = 1.69(1)$ GeV^{-2}, and $|F_{Z\to\pi^0\gamma}(M_Z^2)/F_{Z\to\pi^0\gamma}(0)| = 0.021(4)$. The corresponding curve is presented in Fig. 5 with the solid line. Within the present model we see that the higher radially excited states of the vector mesons, $\rho', \rho'', \dots$ and $\omega', \omega'', \dots$, are weakly coupled.

We also zoom the low-Q^2 range in Fig. 6. All the considered fits practically overlap in the displayed range, in agreement with the observation that IVMD could not be rejected even at $Q^2 \sim 8$ GeV^2. Our analysis shows that higher energy data do in fact favor IVMD at the level of 7 standard deviations.
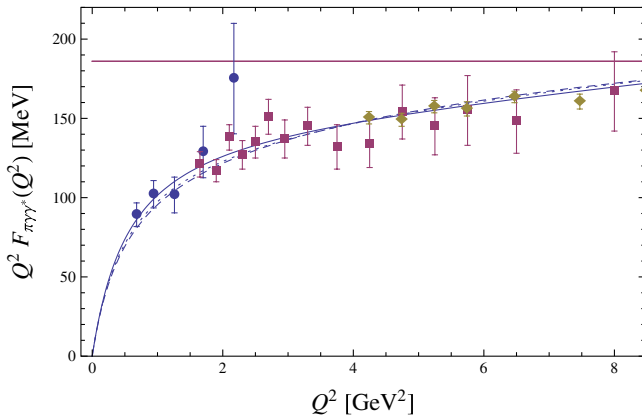

FIG. 6 (color online). Same as Fig. 5 for a smaller range of Q^2.

VIII. ASYMMETRY PARAMETER

As mention in the Introduction, in the *BABAR* kinematic setup $-q_1^2 < 0.6$ GeV^2 and $-q_2^2 > 3$ GeV^2, suggesting $A \sim 0.9$–0.97, hence A is not strictly 1. This departure has significance for the fits and the obtained parameters, which should not be forgotten in precision analyses. We take as an example the IMVD model, which now becomes

$$F_{\pi^0\gamma\gamma^*}(-Q^2)$$

$$= \frac{1}{4\pi^2 f_\pi} \times \left[1 - c\left(1 - \frac{4M_V^4}{4M_V^4 + 4M_V^2 Q^2 + (1-A^2)Q^4}\right)\right].$$
$$(52)$$

Fitting with $A = 1$, 0.975, and 0.95 yields, respectively, $c = 0.986$, 0.978, 0.974, and $M_V = 748$, 754, 768 MeV. We note significantly different values of the optimum parameters, with M_V closest to the physical value for $A = 0.95$. The corresponding curves are displayed in Fig. 7.

IX. CONCLUSIONS

These are our main findings:
(i) The Terazawa-West bound, asserted for the real part of the pion transition form factor, need not be fulfilled in field-theoretic approaches. We provide an explicit counter-example with the low-energy Georgi-Manohar model, where the chiral anomaly is fulfilled but the transition form factor does not vanish at $t \to -\infty$. Provided this feature holds in QCD, it opens a possibility of explaining the *BABAR* experimental data with models incorporating the incomplete vector-meson dominance.
(ii) Moreover, the coefficient in the Terazawa-West bound, estimated with the help of a phenomenological parametrization of the pion parton distribution functions, is large, extending an order of magnitude above the *BABAR* data. Thus, even if it holds in the absence of polynomial contributions to the real part,
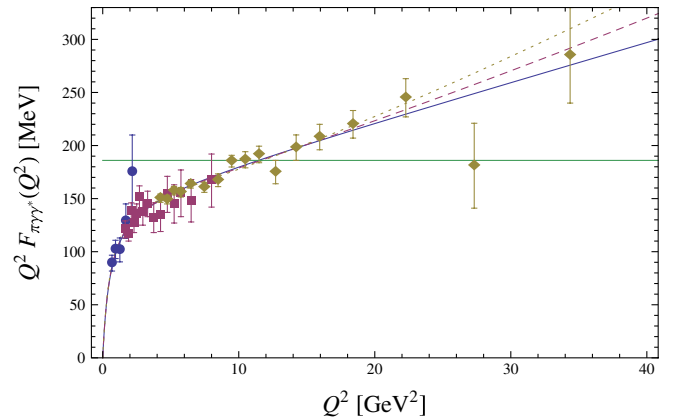

FIG. 7 (color online). The IMVD fit of Eq. (53) with $A = 1$, 0.975, and 0.95, denoted with solid, dashed, and dotted lines, respectively.

it is completely ineffective for the momentum range of interest.

(iii) Already the simplest model with the incomplete vector-meson dominance, incorporating a single vector-meson state, is capable of reproducing the data in the whole available experimental range, $0 < Q^2 < 35$ GeV^2.

(iv) Within the Regge approach, where infinitely many radially excited states are included, the data can be fitted both ways: satisfying or violating the TW bound. The agreement with the experiment is satisfactory, both near $Q^2 \simeq 0$, where the anomaly value and the slope of the form factor are reproduced, as well as in the intermediate CLEO range and the high-Q^2 *BABAR* range.

(v) An additional constraint on the models in the time-like region of momenta follows from the rare $Z \to \pi^0 \gamma$ decay. We use this bound in our considerations. For the considered models it is comfortably satisfied.

(vi) Finally, we note that the numerical fits are quite sensitive to the photon momentum asymmetry parameter, A, which leads to sensitivity in the physical parameters, such as the vector meson mass, and sensitivity of the transition form factor in the asymptotic range. Since A is not strictly 1, the effect of the kinematic cuts should be considered in precision analyses.

ACKNOWLEDGMENTS

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