PHYSICAL REVIEW D **81,** 087901 (2010)

Cold quark matter, quadratic corrections, and gauge/string duality

Oleg Andreev*

Arnold Sommerfeld Center for Theoretical Physics, LMU-München, Theresienstrasse 37, 80333 München, Germany

(Received 1 February 2010; published 5 April 2010)

We make an estimate of the quadratic correction in the pressure of cold quark matter using gauge/string duality.

DOI: 10.1103/PhysRevD.81.087901 PACS numbers: 11.25.Tq, 11.25.Wx, 12.38.Lg

I. INTRODUCTION

The further progress in understanding the phase structure of quantum chromodynamics (QCD) at nonzero temperature and density is very important in view of the recent experiments at RHIC and also planned at LHC and FAIR. From the theoretical point of view, it can be divided into some regions of "phase space": perturbative QCD, lattice QCD, phenomenological models, etc.

As known, perturbative QCD works well at asymptotically high densities [1] or temperatures [2], where the QCD coupling α_s is small. For lower densities and temperatures the MIT bag model can be of use [3]. It is a phenomenological model that effectively includes strong interactions via a bag constant.

The bag model has taken an interesting turn with the detailed analysis [4] of the lattice data [5]. The surprise of this analysis is that at zero chemical potential it reveals a term quadratic in temperature as the leading correction to the ideal gas term in the pressure. The most recent data [6] indicate that this is generic: that the equation of state of the MIT bag model gets modified as[1]

$$p(T) = aT^4 - T_*^2 T^2 - B, \qquad \text{with } T_{\min} < T < T_{\max}. \tag{1}$$

As noted, the novelty is the T^2 term, while the remaining two are the standard bag terms, with B a bag constant and a a parameter. A common choice is to take a from perturbation theory up to one loop order. $T_{\min}$ is close to a critical temperature T_c (or some approximate "T_c" for a crossover). A small difference between T_c and $T_{\min}$ may vary with the model. $T_{\max}$ is set by perturbation theory such that to leading orders it is applicable only for temperatures higher than $T_{\max}$.

While numerical simulations on the lattice can be of use at nonzero temperature when the quark density is quite small, standard Monte Carlo techniques are not of use in cold dense matter because of poor convergence called the sign problem. On the other hand, there are arguments in the literature [7] that in QCD long perturbative series (or the UV renormalons) result in the so-called quadratic corrections. From this point of view, the T^2 term in (1) is nothing else but an example of the quadratic correction. If so, then it is natural to expect that at zero temperature the equation of state of the MIT bag model can also get modified by a quadratic correction as[2]

$$p(\mu) = b\mu^4 - \mu_*^2 \mu^2 - B, \qquad \text{with } \mu_{\min} < \mu < \mu_{\max}, \tag{2}$$

where again b is a parameter to be fixed from perturbation theory up to one loop order. The μ^2 term is a quadratic correction. $\mu_{\min}$ is expected to be close to a critical value of μ, while $\mu_{\max}$ is set by leading orders of perturbation theory. Note that μ stands for the baryon chemical potential, here and below.

Until recently, the lattice formulation and effective field theories were the main computational tools to deal with nonweakly coupled gauge theories. The situation changed drastically with Maldacena duality (AdS/CFT) [10] that resumed interest in another tool, string theory. The original duality was for conformal theories, but various perturbations (deformations) produce gauge/string duals with a mass gap, confinement, and chiral symmetry breaking [11].

In this Letter we continue a series of recent studies [12–14] devoted to a search for an effective string theory description of strong interactions. Since precise recipes for finding the string theory dual to QCD are still unknown, our strategy is based on deformations of AdS/CFT. The deformation we are pursuing turned out to be successful in providing a systematic approach to the quadratic corrections. Indeed, in [12], the quadratic correction was found in the two-current correlator. Later, the model was extended to Euclidean signature for computing the heavy quark potential, where the quadratic correction occurs as a linear term in the potential at short distances [13]. Subsequent comparison [15] with the meson spectrum made it clear that the model should be taken seriously. Moreover, it was also extended to finite temperature. As a result, the T^2 term in the pressure (1) was found [14]. In addition, this model results in the spatial string tension [16] and the expectation value of the Polyakov loop [17], which are remarkably

*Also at Landau Institute for Theoretical Physics, 119334 Moscow, Russia.

[1]Rob Pisarski called it a "fuzzy" bag model for the pressure [4].

[2]Such a parametrization of the quark matter equation of state was also considered in the context of phenomenology of hybrid stars [8] and Quarkyonic phase [9].

consistent with the lattice. Thus, there are reasons to believe that this deformation also provides a good approximation for a string dual to cold quark matter.

II. THE MODEL

Let us first explain the model to be considered. We take the following ansatz for the ten-dimensional background geometry which is a deformed product of the extremal Reissner-Nordström black hole in Euclidean AdS_5 and a five-dimensional sphere (compact space X)[3]

$$ds^2 = \frac{l^2}{z^2} H(f dt^2 + d\vec{x}^2 + f^{-1} dz^2) + H^{-1} d\Omega_X,$$
$$H = e^{(1/2)cz^2}, \qquad f = (1 - (z/z_+)^2)^2 (1 + 2(z/z_+)^2), \tag{3}$$

where $z_+ = (2/q^2)^{(1/6)}$. The deformation is due to the same z-dependent factor H as those of [12–14], with c being a parameter whose value will be fixed shortly. Note that (3) is smooth and complete at $z = z_+$ such that the inverse period of t is equal to $\beta^{-1} = T = 0$, with T the temperature. We also take a constant dilaton and discard other (if any) background fields.

Given the background metric (3), we can now find the corresponding gauge potential from the condition of Weyl invariance on a string world sheet. To leading order in α' it is given by

$$\beta^A_\mu = \nabla^\nu F_{\mu\nu} + O(\alpha') = 0. \tag{4}$$

Here β^A_μ is, in fact, a renormalization group beta function on the world sheet.

For a pure electric potential $A_0(z)$, (4) becomes

$$\partial_z(\sqrt{g} g^{00} g^{zz} \partial_z A_0) = 0. \tag{5}$$

The solution is given by $A_0(z) = C_1 e^{cz^2} + C_2$, with C_i constants. If we choose the constants so that at $c = 0$ the solution is reduced to that of Reissner-Nordström (A6), we find

$$A_0(z) = i\left(-\frac{\sqrt{3}}{2}\frac{q}{c}(e^{cz^2} - 1) + \mu\right). \tag{6}$$

Finally, we impose the condition $A_0(z_+) = 0$ and as a result get μ as a function of q

$$\mu(q) = \frac{\sqrt{3}}{2}\frac{q}{c}(e^{cz_+^2} - 1). \tag{7}$$

Following the AdS/CFT dictionary [11], we identify the parameters μ and ρ, as defined in (A7), with the baryon chemical potential and the baryon number density, respectively.

[3]For completeness, we include a brief summary of the relevant results concerning the Reissner-Nordström black holes in the Appendix.

III. ESTIMATE OF THE QUADRATIC CORRECTION

Our first goal will be to analyze the nondeformed model. That is, we take $c = 0$ for (3) and (6). Using the formulas (3) and (A7), we can show that Eq. (7) yields, up to a constant multiple, a unique solution $\rho(\mu)$. Explicitly, it is given by

$$\rho(\mu) = 4b\mu^3. \tag{8}$$

This is the desired result, corresponding in QCD to the fact that for low temperatures and large chemical baryon potentials the baryon number density is proportional to the cube of the potential.

To fix the constant of proportionality, we need some knowledge of the exact string theory dual to QCD or some additional insight. Since the former is beyond our grasp at present, we match the parameter b with that of perturbative QCD neglecting perturbative interactions among the quarks. In doing so, we first find the pressure by integrating $\frac{dp}{d\mu} = \rho$. In terms of the quark chemical potential $\mu_q = \mu/N_c$, it is $p(\mu_q) = bN_c^4\mu_q^4$. Finally, we have

$$b = \frac{1}{12\pi^2}\frac{N_f}{N_c^3}. \tag{9}$$

Here N_f is the number of quark flavors and N_c is the number of colors.

Now let us return and discuss the deformed model. At large baryon density (or equivalently at large q) it is reasonable to represent (7) as a series $\frac{\sqrt{3}}{2}\frac{q}{c}\sum_{n=1}\frac{c^n}{n!} \times (\frac{2}{q^2})^{(n/3)}$. If we take the two leading terms of the series, then we can easily invert the function $\mu(q)$. Finally, using Eq. (A7), we find the leading correction to (8)

$$\rho(\mu) = 4b\left(\mu^3 - \frac{1}{2}\mu_\star^2\mu + O(1)\right), \qquad \text{with } \mu_\star = 3\sqrt{\frac{c}{2}}. \tag{10}$$

In the homogeneous case the pressure is obtained by integrating the above expression over μ

$$p(\mu) = b(\mu^4 - \mu_\star^2\mu^2 + O(1)). \tag{11}$$

This is our main result. It includes the μ^2 term, as expected.

Making an estimate requires some numerics. First, let us consider the light (u, d) quarks. In this case, the value of c is fixed from the slope of the Regge trajectory of $\rho(n)$ mesons [18]. This gives $c \approx 0.9\ \mathrm{GeV}^2$ [12]. So, we find for the value of the quadratic correction

$$\mu_\star^2 \approx 4.1\ \mathrm{GeV}^2. \tag{12}$$

In contrast, a simple estimate of the corresponding coefficient of perturbative QCD with $N_f = 2$ results in [1]

$$\frac{27}{2}(m_u^2 + m_d^2) \approx 6 \times 10^{-4}\ \text{GeV}^2. \tag{13}$$

Here we have used that $m_u = 3$ MeV and $m_d = 6$ MeV.

Thus, our model predicts that the μ^2 term being negligible in a pure perturbative region $\mu_{\max} < \mu$ gets strongly enhanced in the intermediate region $\mu_{\min} < \mu < \mu_{\max}$.

Next, let us discuss the effect of the strange quark. For $N_f = 3$, (13) becomes $9(m_u^2 + m_d^2 + m_s^2)$, which is certainly valid near the upper limit $\mu_{\max}$, where $\mu \gg 3m_s$. A simple algebra shows that its value is of order 0.1 GeV^2, with $m_s \approx 0.1$ GeV. It is still smaller than (12), so the effect of the strange quark is not dominant.

Finally, let us estimate the range of μ for the model of interest. A crude estimate of the lower limit can be made by using the positivity of the baryon density and the pressure. It gives that $\mu_{\min}$ is of order $\mu_\star$. If we assume that as at finite T on the lattice [6], where $T_{\min} \sim 1.5T_c$, in the model of interest $\mu_{\min} \sim 1.5\mu_c$, then using (12) we arrive at a critical chemical potential of 1.3 GeV, which is reasonable phenomenologically. A crude estimate of the upper limit can be made by assuming that at $\mu = \mu_{\max}$ the contribution of the μ^2 term in the pressure is 1 order of magnitude smaller than that of the leading μ^4 term. This gives $\mu_{\max} \sim 3.3\mu_\star$ or, in terms of μ_c, $\mu_{\max} \sim 5\mu_c$.

IV. CONCLUDING COMMENTS

(i) Having derived the equation of state, we can easily develop finite μ thermodynamics. In particular, for the energy density, we have $\epsilon = b(3\mu^4 - \mu_\star^2\mu^2 + O(1))$. Combining with (11), we find the expression for the trace anomaly

$$\frac{\epsilon - 3p}{\mu^4} = 2b\frac{\mu_\star^2}{\mu^2} + O(1). \tag{14}$$

In addition, for the speed of sound $C_s^2 = \frac{dp}{d\epsilon}$ we get

$$C_s^2(\mu) = \frac{1}{3}\left(1 - \frac{1}{3}\frac{\mu_\star^2}{\mu^2} + O(1)\right). \tag{15}$$

All the above formulas are similar to those of [14] at finite T.

(ii) Here we used the model based on the deformation of the Reissner-Nordström solution. Certainly, such a phenomenologically motivated way is out of the mainstream of (academic) AdS/CFT, where the background geometry follows from the equations of supergravity and fundamental matter is introduced via D-brane embeddings in the probe approximation with $N_c \gg N_f$. One of the advantages of our approach is that it allows us to incorporate the backreaction due to the gauge potential on the background geometry. What really fits better to QCD remains to be seen.

(iii) In the phenomenological parametrization of [8] the coefficient in front of the μ^2 term arises from the strange quark mass as well as color superconductivity. As a result, it is proportional to $m_s^2 - 4\Delta^2$. Its value is 1 order of magnitude smaller than ours (12).

The formula (2) was also suggested, by analogy with the deformed bag model (1), in the context of Quarkyonic matter [9]. Our interpretation of the μ^2 term as a power correction differs from that of [9], where it is interpreted as due to nonperturbative corrections. However, in the intermediate region of interest some matching conditions between the two regimes may be possible.

ACKNOWLEDGMENTS

This work was supported in part by DFG within the Emmy-Noether Program under Grant No. HA 3448/3-1 and the Alexander von Humboldt Foundation under Grant No. PHYS0167. We would like to thank M. Haack, S. Hofmann, and especially V. I. Zakharov for discussions and comments.

APPENDIX

In this appendix we review the relevant results concerning the Reissner-Nordström solutions in five dimensions. Most of this material can be found in [19].

For the Einstein-Maxwell action with cosmological constant, we take

$$I = -\frac{1}{16\pi G_N}\int d^5x\sqrt{g}\left(R - l^2F^2 + \frac{12}{l^2}\right). \tag{A1}$$

Here G_N is the five-dimensional Newton's constant.

With a pure electric gauge potential[4]

$$A_0(r) = i\left(-\frac{\sqrt{3}}{2l}\frac{q}{r^2} + \mu\right), \tag{A2}$$

a solution of the equations of motion for the metric (with Euclidean signature) takes the spherically symmetric form

$$ds^2 = fdt^2 + f^{-1}dr^2 + r^2d\Omega_3^2, \qquad f = 1 - \frac{m}{r^2} + \frac{q^2}{r^4} + \frac{r^2}{l^2}. \tag{A3}$$

The parameters m and q are, respectively, related to the mass and charge of the black hole as

$$\mathcal{M} = \frac{3\text{Vol}(\mathbf{S}^3)}{16\pi G_N}m, \qquad Q = \frac{\sqrt{3}\text{Vol}(\mathbf{S}^3)}{4\pi G_N}q. \tag{A4}$$

Here Vol($\mathbf{S}^3$) is the volume of a unit 3-sphere.

The solution (A3) is asymptotic at $r = \infty$ to $\mathbf{S}^3 \times \mathbf{S}^1$. A scaling that reduces it to a solution with $\mathbf{R}^3 \times \mathbf{S}^1$ may be made as follows. If we introduce a dimensionless parameter λ and make the transformation $r \to \lambda^{(1/4)}r$, $t \to \lambda^{-(1/4)}t$, $m \to \lambda l^6 m$, $q \to \lambda^{(3/4)}l^5 q$, then in the large λ limit we obtain

[4]The parameter μ is reserved for future use.

$$ds^2 = \frac{l^2}{z^2}(f dt^2 + f^{-1} dz^2 + d\vec{x}^2),$$
$$f = 1 - mz^4 + q^2 z^6, \tag{A5}$$

where $z = l^2/r$. In the process, we have also introduced local coordinates y_i near a point $P \in \mathbf{S}^3$ such that $d\Omega_3^2 = \sum dy_i^2$, and then set $x_i = \lambda^{(1/4)} l y_i$.

Having derived the desired solution for the metric, we can easily obtain that for the gauge potential. From (A2), we have

$$A_0(z) = i\left(-\frac{\sqrt{3}}{2} q z^2 + \mu\right). \tag{A6}$$

When we go to $\mathbf{R}^3 \times \mathbf{S}^1$, we get that the radius of $\mathbf{S}^3$ is proportional to $\lambda^{(1/4)}$ and so diverges for $\lambda \to \infty$. Hence, the corresponding volume is also becoming infinite and looks like $V_3 = \lambda^{(3/4)} l^3 \mathrm{Vol}(\mathbf{S}^3)$. If we introduce the charge density $\rho = Q/V_3$, then the second equation of (A4) becomes

$$\rho = 3\sqrt{3} b q, \tag{A7}$$

where $b = l^2/(12\pi G_N)$. The difference between $\mathbf{S}^3 \times \mathbf{S}^1$ and $\mathbf{R}^3 \times \mathbf{S}^1$ is obvious: in the first case q is related to the charge of the black hole, while in the second case it is related to its charge density.

The metric (A5) is smooth and complete if the period of t is $\beta = \frac{4\pi}{|f'(z_+)|}$, where z_+ is the smallest real positive root of $f(z) = 0$.

For $T = 0$, the black hole becomes extremal so that $4m^3 = 27q^4$. In this case the function $f(z)$ takes the form

$$f = (1 - (z/z_+)^2)^2 (1 + 2(z/z_+)^2),$$
$$\text{with } z_+ = (2/q^2)^{(1/6)}. \tag{A8}$$

[1] See, for example, B. A. Freedman and L. D. McLerran, Phys. Rev. D **16**, 1169 (1977); V. Baluni, Phys. Rev. D **17**, 2092 (1978); J. Kapusta, Nucl. Phys. **B148**, 461 (1979); E. S. Fraga, R. D. Pisarski, and J. Schaffner-Bielich, Phys. Rev. D **63**, 121702 (2001); A. Kurkela, P. Romatschke, and A. Vuorinen, arXiv:0912.1856 [Phys. Rev. D (to be published)].
[2] See, for example, E. Braaten and A. Nieto, Phys. Rev. D **51**, 6990 (1995); J. O. Andersen and M. Strickland, Ann. Phys. (N.Y.) **317**, 281 (2005); A. Hietanen, K. Kajantie, M. Laine, K. Rummukainen, and Y. Schroder, Phys. Rev. D **79**, 045018 (2009).
[3] A. Chodos, R. L. Jaffe, K. Johnston, and C. B. Thorn, Phys. Rev. D **10**, 2599 (1974).
[4] R. D. Pisarski, Prog. Theor. Phys. Suppl. **168**, 276 (2007).
[5] G. Boyd *et al.*, Phys. Rev. Lett. **75**, 4169 (1995).
[6] M. Cheng *et al.*, Phys. Rev. D **77**, 014511 (2008); M. Panero, Phys. Rev. Lett. **103**, 232001 (2009).
[7] For a review and references, see V. I. Zakharov, Nucl. Phys. B, Proc. Suppl. **74**, 392 (1999); see also S. Narison and V. I. Zakharov, Phys. Lett. B **679**, 355 (2009); E. Megias, E. Ruiz Arriola, and L. L. Salcedo, arXiv:0912.0499.
[8] M. Alford, M. Braby, M. Paris, and S. Reddy, Astrophys. J. **629**, 969 (2005).
[9] L. McLerran and R. D. Pisarski, Nucl. Phys. **A796**, 83 (2007).
[10] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[11] For a review, see O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000); see also for an updated reference list R. C. Myers and S. E. Vazquez, Classical Quantum Gravity **25**, 114008 (2008); S. S. Gubser and A. Karch, Annu. Rev. Nucl. Part. Sci. **59**, 145 (2009); S. J. Brodsky and G. de Teramond, arXiv:1001.1978.
[12] O. Andreev, Phys. Rev. D **73**, 107901 (2006).
[13] O. Andreev and V. I. Zakharov, Phys. Rev. D **74**, 025023 (2006); **76**, 047705 (2007).
[14] O. Andreev, Phys. Rev. D **76**, 087702 (2007).
[15] F. Giannuzzi, Phys. Rev. D **78**, 117501 (2008).
[16] O. Andreev and V. I. Zakharov, Phys. Lett. B **645**, 437 (2007); O. Andreev, Phys. Lett. B **659**, 416 (2008).
[17] O. Andreev and V. I. Zakharov, J. High Energy Phys. 04 (2007) 100; O. Andreev, Phys. Rev. Lett. **102**, 212001 (2009).
[18] R. R. Metsaev, arXiv:hep-th/0002008; A. Karch, E. Katz, D. T. Son, and M. A. Stephanov, Phys. Rev. D **74**, 015005 (2006).
[19] L. J. Romans, Nucl. Phys. **B383**, 395 (1992); E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998); A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, Phys. Rev. D **60**, 104026 (1999).