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Rigorous limits on the interaction strength in quantum field theory

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We derive model-independent, universal upper bounds on the operator product expansion coefficients in unitary 4-dimensional conformal field theories. The method uses the conformal block decomposition and the crossing symmetry constraint of the 4-point function. In particular, the operator product expansion coefficient of three identical dimension d scalar primaries is found to be bounded by $\simeq 10(d-1)$ for $1 < d < 1.7$. This puts strong limits on unparticle self-interaction cross sections at the LHC.

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In this paper we will answer, in a particular well-defined context, the question: Is there an upper bound to the interaction strength in relativistic quantum field theory (rQFT)?

Intuitive reasons suggest that such a bound exists. Take QCD as a representative real-world example. At energies E above the scale $\Lambda_{\rm QCD} \sim 1$ GeV, this is a perturbative theory of interacting quarks and gluons, and the interaction strength is measured by the dimensionless running coupling $g_s(E)$. The coupling starts small at very high energies $E \gg \Lambda_{\rm QCD}$ and grows at low energies, formally becoming infinite at $E \sim \Lambda_{\rm QCD}$. However, perturbative expansion breaks down before this happens. L-loop diagrams are suppressed by factors $\sim(g_s^2/16\pi^2)^L$. As soon as $g_s \sim 4\pi$, all loop orders contribute equally. Thus in perturbation theory it is impossible to get couplings stronger than about 4π.

To recall what happens beyond perturbation theory, let us look at the same theory at energies below $\Lambda_{\rm QCD}$. In this regime the appropriate degrees of freedom are hadrons, and their interactions can be described by an effective Lagrangian. For instance, pion-pion scattering at low energies is described by the chiral Lagrangian

$$\mathcal{L} = \frac{f_\pi^2}{4}\,{\rm Tr}|\partial_\mu U|^2 + \cdots, \qquad U = \exp(i\pi^a\sigma^a/f_\pi),$$

where $f_\pi \simeq 93$ MeV is the pion decay constant, and $\cdots$ stands for the chiral symmetry breaking terms. The dimensionless quartic pion coupling defined from the $2 \to 2$ scattering amplitude grows with energy as $\lambda \sim (E/f_\pi)^2$. If the chiral Lagrangian is valid up to $E \sim \Lambda_{\rm QCD}$ and is stable under radiative corrections, we should have $\lambda(\Lambda_{\rm QCD})/16\pi^2 \lesssim 1$, or $\Lambda_{\rm QCD} \lesssim 4\pi f_\pi$. Experimentally this bound is satisfied and near saturated, which forms the basis of the naive dimensional analysis [1] method of estimating couplings in strongly coupled theories.

While the above arguments are appealing, at present it is unknown if they can be turned into a theorem, or even how to formulate such a general theorem. In order to make progress, in what follows we will assume that we have a conformal field theory (CFT), i.e. an rQFT invariant under the action of the conformal group [2].

CFTs form an important subclass of rQFTs. Presumably, any unitary, scale invariant rQFT is conformally invariant. This is proved in $D = 2$ spacetime dimensions under very mild technical assumptions [3], and no counterexamples are known in $D \geq 3$. Unitarity is however crucial here: without unitarity simple physical counterexamples exist. We are interested in applications to particle physics; thus we will assume unitarity, and will work in $D = 4$.

There are many known or conjectured classes of four-dimensional CFTs. For example, $\mathcal{N} = 1$ supersymmetric QCD with N_c colors and N_f flavors flows to a CFT in the infrared as long as $3/2 < N_f/N_c < 3$ [4]. Large N_c analysis [5] and lattice simulations [6] suggest that a similar ‘‘conformal window’’ exists also without supersymmetry. Another famous example is the $\mathcal{N} = 4$ super Yang-Mills (SYM), conformal for any coupling and any N_c. At large ’t Hooft coupling and large N_c it can be described via the AdS/CFT correspondence [7]. Many deformations preserving conformal symmetry are known on both field theory and gravity sides of the correspondence [7]. Our discussion will be general and will in principle apply to all the above examples.

The $D = 4$ conformal group is finite dimensional; it is obtained from the Poincaré group by adding the generators of dilatation $\mathcal{D}$ and of special conformal transformations $\mathcal{K}_\mu$. The local quantum fields $O(x)$ are eigenstates of $\mathcal{D}$, $[\mathcal{D}, O(0)] = i\Delta O(0)$, where Δ is the scaling dimension. The $\mathcal{K}_\mu$ acts as a lowering operator for the scaling dimension, and the corresponding ‘‘lowest-weight states’’, i.e. fields satisfying $[\mathcal{K}_\mu, O(0)] = 0$, play a special role. They are called *primaries*. All other fields can be obtained from primaries by taking derivatives and are called *descendants*.

Conformal symmetry constrains the 2- and 3-point functions of primary fields to have particularly simple form. For scalar primaries, we have

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$$\langle O_i(x_1)O_j(x_2)\rangle = \delta_{ij}(x_{12}^2)^{-\Delta},$$
$$\langle O_i(x_1)O_j(x_2)O_k(x_3)\rangle = c_{ijk}(x_{12}^2)^{\rho_{kij}}(x_{13}^2)^{\rho_{jik}}(x_{23}^2)^{\rho_{ijk}},$$
$$x_{ij}^2 \equiv (x_i - x_j)^2,$$
$$\rho_{ijk} \equiv (\Delta_i - \Delta_j - \Delta_k)/2. \tag{1}$$

The first equation says that a diagonal basis can be chosen in the space of primary fields, and sets the normalization. The second equation then defines coefficients c_{ijk}. These same coefficients appear in the operator product expansion (OPE)

$$O_i(x)O_j(0) \sim (x^2)^{-(\Delta_i+\Delta_j)/2}\{1 + c_{ijk}(x^2)^{\Delta_k/2}O_k(0) + \cdots\},$$

where $\cdots$ stands for the contributions of higher spin primaries and of descendants.

In CFT, any n-point function can be, in principle, reduced to a sum of products of 2-point functions by repeated application of the OPE, with coefficients given by products of c_{ijk}'s. In this sense, c_{ijk}'s play in CFT a role similar to that of the coupling constants in perturbation theory, measuring interaction strength. We thus have the following CFT version of our initial question: Is there an upper bound to the OPE coefficients, valid in an arbitrary unitary CFT in $D = 4$? We will now proceed to show that such a universal bound indeed exists.

Let us pick a Hermitian scalar primary ϕ of scaling dimension d and consider its OPE with itself:

$$\phi(x)\phi(0) \sim (x^2)^{-d}\Big\{\mathbb{1} + \sum_{l=0,2,4\cdots}\sum_{\Delta\geq\Delta\min(l)} c_{\Delta,l}\frac{x^{\mu_1}\cdots x^{\mu_l}}{(x^2)^{(l-\Delta)/2}} \times O_{\mu_1\cdots\mu_l}(0) + \cdots\Big\}.$$

This time we show explicitly contributions of both scalars ($l = 0$) and of higher spin primaries $O_{\mu_1\cdots\mu_l}$ which are symmetric traceless tensors. Spin l has to be even by the Bose symmetry. Unitarity implies lower bounds on the dimension Δ of a spin l primary [8]:

$$\Delta_{\min}(l = 0) = 1, \qquad \Delta_{\min}(l \geq 1) = l + 2.$$

Only special fields may saturate these bounds: a free scalar ($l = 0$), conserved currents ($l = 1$), and the stress tensor ($l = 2$). Higher l conserved currents, present in free theories, also saturate the bounds.

An interesting object to study is the 4-point function of ϕ, constrained by conformal symmetry to have the form

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = g(u, v)/(x_{12}^{2d}x_{34}^{2d}), \tag{2}$$

where $u = x_{12}^2x_{34}^2/(x_{13}^2x_{24}^2)$, $v = x_{14}^2x_{23}^2/(x_{13}^2x_{24}^2)$ are the conformal cross ratios. The same 4-point function can be reduced to a sum of 2-point functions by applying the OPE in the 12 and 34 channels. Cross terms of different primary families drop out because of Eq. (1) and its higher spin analog. Resumming the terms involving the same primary and its descendants, we get the *conformal block decomposition*

$$g(u, v) = 1 + \sum p_{\Delta,l}g_{\Delta,l}(u, v), \qquad p_{\Delta,l} \equiv c_{\Delta,l}^2, \tag{3}$$

where the functions $g_{\Delta,l}$ are known explicitly [9]

$$g_{\Delta,l}(u, v) = \frac{(-)^l}{2^l}\frac{z\bar{z}}{z - \bar{z}}[k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})],$$
$$k_\beta(x) \equiv x^{\beta/2}{}_2F_1(\beta/2, \beta/2, \beta; x),$$
$$u = z\bar{z},$$
$$v = (1 - z)(1 - \bar{z}).$$

This decomposition is expected to converge at least in the circle $|z| < 1$, $|\bar{z}| < 1$ [10].

The 4-point function (2) must be crossing symmetric under the $x_1 \leftrightarrow x_2$ and $x_1 \leftrightarrow x_3$ exchanges. The first crossing is manifest since only even spins contribute to the OPE. The second one gives a nontrivial constraint

$$v^d g(u, v) = u^d g(v, u). \tag{4}$$

Decomposition (3) must be consistent with this constraint. Separating the contribution of the unit operator, we obtain the *sum rule*

$$1 = \sum p_{\Delta,l}F_{d,\Delta,l}(u, v),$$
$$F_{d,\Delta,l}(u, v) \equiv [v^d g_{\Delta,l}(u, v) - u^d g_{\Delta,l}(v, u)]/(u^d - v^d). \tag{5}$$

This equation can be used to get an upper bound on $c_{\Delta,l}$.

Crucially, coefficients $c_{\Delta,l}$ are real, and thus $p_{\Delta,l} \geq 0$. This is related to the absence of parity violation in the conformal 3-point function of two scalars and a symmetric tensor [11]. Equation (5) then allows a geometric interpretation: when $p_{\Delta,l} \geq 0$ are allowed to vary, the right-hand side fills a convex cone C_d in the vector space $\mathcal{V}$ whose elements are two-variable functions. We say that this cone is *generated* by functions $F_{d,\Delta,l}(u, v)$. Equation (5) expresses the fact that the function $f(u, v) \equiv 1$ belongs to this cone. It will follow from Eq. (7) below that there is no vanishing linear combination of the F's with positive coefficients, so that C_d is really a cone and, in particular, does not fill the whole space.

Let us pick a particular field $O_{\bar{\Delta},\bar{l}}$ and rewrite (5) as

$$1 - p_{\bar{\Delta},\bar{l}}F_{d,\bar{\Delta},\bar{l}}(u, v) = \sum p_{\Delta,l}F_{d,\Delta,l}(u, v). \tag{6}$$

As $p_{\bar{\Delta},\bar{l}}$ is increased, the vector corresponding to the left-hand side of this equation moves in the vector space. Suppose that for all $p_{\bar{\Delta},\bar{l}}$ above some critical value $p_{\rm cr}$ this vector stays out of the cone C_d. Then $p_{\rm cr}$ provides a bound on the squared OPE coefficient $|c_{\bar{\Delta},\bar{l}}|^2$. This bound will depend on d, $\bar{\Delta}$, $\bar{l}$, but will be valid in any unitary CFT.

To find $p_{\rm cr}$, we employ the method of linear functionals developed in [11]. Recall that a linear functional is a linear map Λ from $\mathcal{V}$ to real numbers, Λ: $\mathcal{V} \to \mathbb{R}$, $\Lambda[\alpha_i F_i] = \alpha_i \Lambda[F_i]$. Suppose that we found a functional which is positive on all functions generating the cone C_d:

$$\Lambda[F_{d,\Delta,l}] \geq 0, \qquad \Lambda[1] = 1. \tag{7}$$

The second condition is imposed for normalization. Since for such Λ Eq. (5) implies $\Lambda[1 - p_{\bar{\Delta},\bar{l}} F_{d,\bar{\Delta},\bar{l}}] \geq 0$, we would get an upper bound:

$$p_{\bar{\Delta},\bar{l}} \leq p_{\rm cr}(\Lambda) \equiv 1/\Lambda[F_{d,\bar{\Delta},\bar{l}}]. \tag{8}$$

To make this bound as strong as possible, we will impose, in addition to (7), an extremality condition

$$\Lambda[F_{d,\bar{\Delta},\bar{l}}] \to \max. \tag{9}$$

We will use linear functionals given by a finite linear combination of derivatives evaluated at a given point:

$$\Lambda[F] \equiv \sum_{n,m\geq 0, n+m\leq N} \lambda_{n,m} F^{(2n,2m)}, \qquad N = 3,$$
$$F^{(2n,2m)} \equiv \partial_a^{2n} \partial_b^{2m} F|_{a=b=0}, \qquad z = 1/2 + a + b,$$
$$\bar{z} = 1/2 + a - b. \tag{10}$$

Here $\lambda_{n,m}$ are fixed real numbers defining the functional. The symmetric point $a = b = 0$ is chosen as in [11] since the sum rule is expected to converge fastest here, and because the functions $F_{d,\Delta,l}$ are even in both variables with respect to this point. This is why only even-order derivatives are included in (10).

Equations (7) and (9), define a *linear programming* optimization problem for the coefficients $\lambda_{n,m}$. (The constraints are given by linear equations and inequalities, and the cost function is also linear.) Although the number of constraints in (7) is formally infinite, they can be reduced to a finite number by discretizing Δ and truncating at large Δ and l, where the constraints approach a calculable asymptotic form. The reduced problem can be efficiently solved by well-known numerical methods, such as the simplex method. A found solution can be then checked to see if it also solves the full problem. This procedure was developed and successfully used for a related but different problem in [11].

Using this procedure, we computed bounds on the OPE coefficients $c_{\phi\phi O}$ when O is a scalar field ($\bar{l} = 0$). We will now present our numerical results [12]. Figure 1 concerns the case when the dimension of ϕ is close to that of a free field, $1 < d \leq 1.1$. Notice the bell-shaped form of the bound, peaked at $\bar{\Delta} \simeq 2$. This shape makes it tempting to draw an analogy with the Breit-Wigner formula, especially since the dilatation operator $\mathcal{D}$ plays the role of energy in radial quantization. For $d \to 1$ the bound evidently tends to zero everywhere except near $\bar{\Delta} = 2$. This means that the free field theory limit is approached continuously: for $d = 1$ the only scalar operator in the $\phi \times \phi$ OPE is the $:\phi^2:$ of dimension 2. In Fig. 2 we present a similar plot for $1.2 \leq d \leq 1.7$. Notice that the bounds in Figs. 1 and 2 go to zero as $\bar{\Delta} \to 1$. This is expected in view of the general theorem that a dimension 1 scalar must be free, hence decoupled from everything else in the CFT.

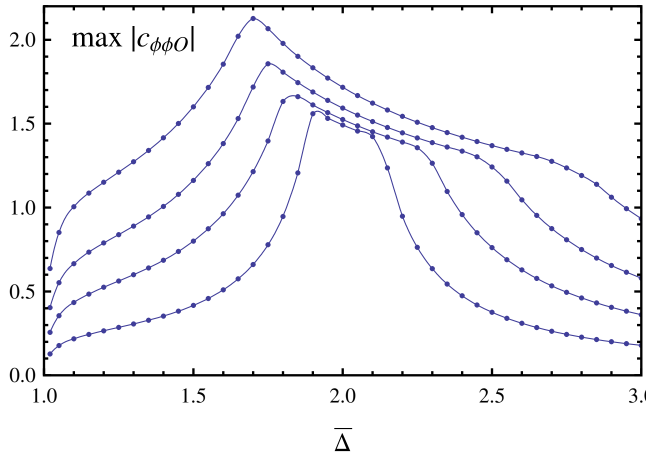

FIG. 1 (color online). Theoretical upper bound for $c_{\phi\phi O}$ as a function of the dimension $\bar{\Delta}$ of the scalar field O. The curves correspond to the ϕ's dimension fixed at $d = 1.005$, 1.02, 1.05, 1.1 (from the bottom up). The bound was computed for each of the shown points, with interpolation in between.

Starting from $d \simeq 1.75$, we found that there is no functional of the form (10) satisfying the constraints (7); that is why we only give bounds for $d \leq 1.7$. We expect that a bound exists also for larger d, but to find it one needs to use more general functionals, e.g. with higher N in Eq. (10). This will also give improved bounds in the range of d that we considered.

On the other hand, the restriction to $\bar{\Delta} \leq 3$ in Figs. 1 and 2 is not essential: our method would also give bounds beyond this range. In fact, any of the functionals derived for $\bar{\Delta} \leq 3$ could be used to compute a suboptimal but valid bound for larger $\bar{\Delta}$ (as well as for $\bar{l} > 0$) via Eq. (8).

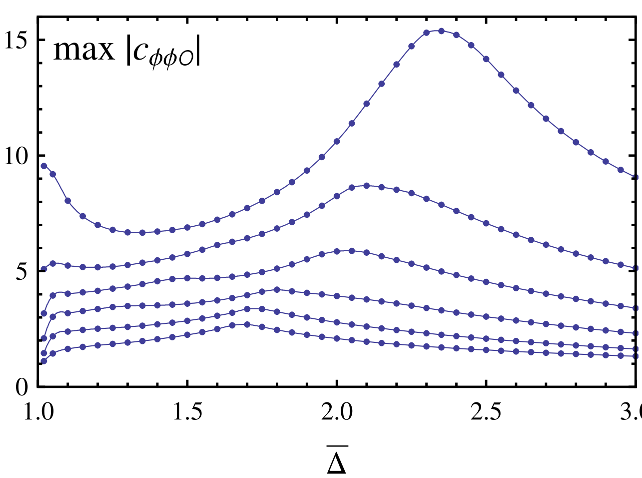

FIG. 2 (color online). Same as Fig. 1 for the ϕ's dimension fixed at $d = 1.2$, 1.3, 1.4, 1.5, 1.6, 1.7 (from the bottom up).

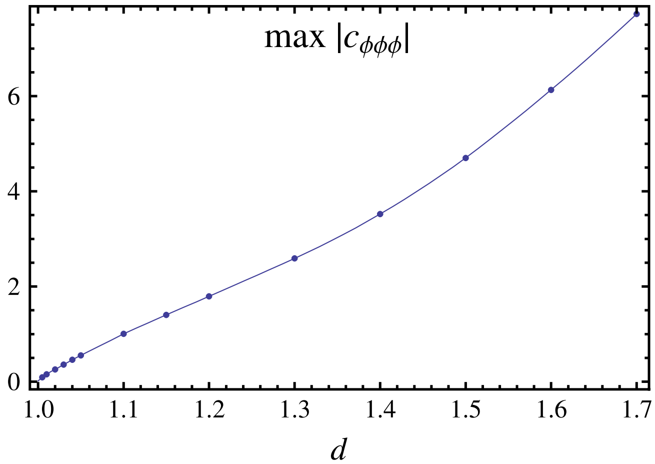

FIG. 3 (color online). Theoretical upper bound for the OPE coefficient $c_{\phi\phi\phi}$ as a function of ϕ's dimension d.

It is interesting to study the asymptotic behavior of the bound at large $\bar{\Delta}$. A simple upper estimate can be obtained from the known asymptotics of $F_{d,\bar{\Delta},\bar{l}}$ and its derivatives [11], assuming that the functional Λ in (8) is $\bar{\Delta}$ independent. So one concludes that the bound cannot grow faster than $|c_{\phi\phi O}| = \mathcal{O}(q^{\bar{\Delta}})$, $q = (\sqrt{2} + 1)/2$.

It would be also interesting to derive analogous bounds in two spacetime dimensions, where explicit expressions for conformal blocks are also known [9].

As a phenomenoligical application of our results, consider the unparticle physics scenario [13]. Unparticle self-interactions were considered in [14,15] a prominent feature of such scenarios, giving rise to processes like $gg \to \phi \to \phi\phi \to 4\gamma$. The cross section is proportional to the square of the self-coupling OPE coefficient $c_{\phi\phi\phi}$, where ϕ is a scalar from a hidden-sector CFT (*unparticle*) with non-renormalizible couplings to gluons and photons. In [14], the values of these coefficients were assumed unconstrained by prime principles, and only experimental constraints from the Tevatron were imposed, which led to a possibility of spectacularly large cross sections at the LHC. In Fig. 3 we plot our theoretical upper bound on $c_{\phi\phi\phi}$ (extracted from Figs. 1 and 2 by setting $\bar{\Delta} = d$). The values of $c_{\phi\phi\phi}$ used in [14] exceed our bound by 2–4 orders of magnitude.[1] A revision of the studies in [14,16], taking into account our bounds, is necessary.

As a purely field-theoretical application, consider the $\mathcal{N} = 4$ SYM theory, conformal for any value of the 't Hooft coupling $\lambda = g^2_{YM} N_c$. The region of small λ is accessible via perturbation theory, while large λ (and large N_c) are accessible via the AdS/CFT correspondence. Moreover, the large N_c theory is integrable, which allows one to interpolate between the two regimes and perform various nontrivial checks [17]. As λ is increased from 0 to ∞, the spectrum of the theory changes, with some anomalous dimensions becoming large. For example, at large N_c the fields which do not map onto supergravity modes on $AdS_5 \times S^5$ have anomalous dimensions growing for large λ as $\lambda^{1/4}$ [18]. Can the OPE coefficients have similar growth? From our results, assuming that they can be extended to $d > 1.7$ as discussed above, it follows that no matter how large λ is, the OPE coefficients *of fields with low dimensions* will stay bounded. It should be noted that this conclusion is nontrivial only for small N_c, since otherwise the OPE $O_1 \times O_2$ is known to factorize, with the composite "multitrace" fields $:O_1 O_2:$ appearing with the coefficient $1 + \mathcal{O}(1/N_c^2)$ while all other fields $1/N_c$ suppressed [19].

In summary, we have presented theoretical upper bounds on the OPE coefficients of two identical scalars and a third scalar, valid in an arbitrary unitary CFT. Our results are based on imposing crossing symmetry on the conformal block decomposition of a scalar 4-point function. They imply that interaction strength remains limited even in theories like $\mathcal{N} = 4$ SYM where a coupling λ can be taken to infinity. They also lead to strong bounds on the cross sections of unparticle self-interaction–type processes at future colliders.

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[1]The unparticle OPE coefficient C_d used in [14] is related to our normalization via $C_d = g^3_{d/2}(|B_d|/g_d)^{3/2} c_{\phi\phi\phi}$ where B_d is given in [14] and $g_d = 4^{2-d}\pi^2\Gamma(2-d)/\Gamma(d)$.

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