Schwinger-DeWitt technique for quantum effective action in brane induced gravity models

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We develop the Schwinger-DeWitt technique for the covariant curvature expansion of the quantum effective action for brane induced gravity models in curved spacetime. This expansion has a part nonanalytic in Dvali-Gabadadze-Porrati type scale parameter m, leading to the cutoff scale which is given by the geometric average of the mass of the quantum field in the bulk M and m. This cutoff $M_{\mathrm{cutoff}} = \sqrt{Mm}$ is much higher than the analogous strong coupling scale of the Dvali-Gabadadze-Porrati model treated by weak field expansion in the tree-level approximation. The lowest orders of this curvature expansion are calculated for the case of the scalar field in the $(d+1)$-dimensional bulk with the brane carrying the d-dimensional kinetic term of this field. The ultraviolet divergences in this model are obtained for a particular case of $d = 4$.

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I. INTRODUCTION

Modified theories of gravity in the form of braneworld models can in principle account for the phenomenon of dark energy [1,2] as well as for nontrivial compactifications of multidimensional string models. It becomes increasingly more obvious that one should include in such models the analysis of quantum effects beyond the tree-level approximation [3]. This is the only way to reach an ultimate conclusion on the resolution of such problems as the presence of ghosts [4] and low strong-coupling scale [5–7]. Quantum effects in brane models are also important for the stabilization of extra dimensions [8], fixing the crossover scale in the Brans-Dicke modification of the Dvali-Gabadadze-Porrati (DGP) model [9] and in the recently suggested mechanism of the cosmological acceleration generated by the four-dimensional conformal anomaly [10].

A general framework for treating quantum effective actions in brane models (or, more generally, models with timelike and spacelike boundaries) was recently suggested in [11–14]. The main peculiarity of these models is that due to quantum field fluctuations on the branes the field propagator is subject to generalized Neumann boundary conditions involving normal and tangential derivatives on the brane/boundary surfaces. This presents both technical and conceptual difficulties, because such boundary conditions are much harder to handle than the simple Dirichlet ones. The method of [13] provides a systematic reduction of the generalized Neumann boundary conditions to Dirichlet conditions. As a by-product it disentangles from the quantum effective action the contribution of the surface modes mediating the brane-to-brane propagation, which play a very important role in the zero-mode localization mechanism of the Randall-Sundrum type [15]. The purpose of this work is to make the next step—to extend a well-known Schwinger-DeWitt technique [16–19] to the calculation of this contribution in the DGP model in a weakly curved spacetime in the form of the *covariant* curvature expansion.

Briefly the method of [13] looks as follows. The action of a (free field) brane model generally contains the bulk and the brane parts,

$$S[\Phi] = \frac{1}{2}\int_{\mathbf{B}} d^{d+1}X\sqrt{G}\Phi(X)F(\nabla_X)\Phi(X) + \frac{1}{2}\int_{\mathbf{b}} d^d x\sqrt{g}\varphi(x)\varkappa(\nabla_x)\varphi(x), \tag{1}$$

where the $(d+1)$-dimensional bulk and the d-dimensional brane coordinates are labeled, respectively, by $X = X^A$ and $x = x^\mu$, and the boundary values of bulk fields $\Phi(X)$ on the brane/boundary $\mathbf{b} = \partial\mathbf{B}$ are denoted by $\varphi(x)$,

$$\Phi(X)|_{\mathbf{b}} = \varphi(x), \tag{2}$$

G and g are the determinants of the bulk and brane metrics, respectively. Brane metric $g_{\mu\nu}$ is considered as induced from the bulk metric G_{AB} via embedding.

The kernel of the bulk Lagrangian is given by the second-order differential operator $F(\nabla_X)$, whose covariant derivatives ∇_X in (1) are integrated by parts in such a way that they form bilinear combinations of first-order derivatives acting on two different fields. Integration by parts in the bulk gives nontrivial surface terms on the brane/boundary. In particular, this operation results in the Wronskian relation for generic test functions $\phi_{1,2}(X)$,

$$\int_{\mathbf{B}} d^{d+1}X\sqrt{G}(\Phi_1\vec{F}(\nabla_X)\Phi_2 - \Phi_1\overleftarrow{F}(\nabla_X)\Phi_2) = -\int_{\mathbf{b}} d^d x\sqrt{g}(\Phi_1\vec{W}\Phi_2 - \Phi_1\overleftarrow{W}\Phi_2). \tag{3}$$

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Arrows everywhere here indicate the direction of action of derivatives either on Φ_1 or Φ_2.

The brane part of the action contains as a kernel some local operator $\hat{\varkappa}(\nabla)$, $\nabla \equiv \nabla_x$. Its order in derivatives depends on the model in question. In the Randall-Sundrum model [15], for example, it is for certain gauges just an ultralocal multiplication operator generated by the tension term on the brane. In the Dvali-Gabadadze-Porrati (DGP) model [1] this is a second-order operator induced by the brane Einstein term on the brane, $\hat{\varkappa}(\nabla) \sim \nabla\nabla/m$, where m is the DGP scale which is of the order of magnitude of the horizon scale, being responsible for the cosmological acceleration [2]. In the context of the Born-Infeld action in D-brane string theory with vector gauge fields, $\varkappa(\nabla)$ is a first-order operator [20].

In all these cases the variational procedure for the action (1) with dynamical (not fixed) fields on the boundary $\varphi(x)$ naturally leads to generalized Neumann boundary conditions of the form

$$(\vec{W}(\nabla_X) + \varkappa(\nabla))\Phi|_{\mathbf{b}} = 0, \tag{4}$$

which uniquely specify the propagator of quantum fields and, therefore, a complete Feynman diagrammatic technique for the system in question. The method of [13] allows one to systematically reduce this diagrammatic technique to the one subject to the Dirichlet boundary conditions $\Phi|_{\mathbf{b}} = 0$. The main additional ingredient of this reduction procedure is the brane operator $\boldsymbol{F}^{\rm brane}(x, x')$ which is constructed from the Dirichlet Green's function $G_D(X, X')$ of the operator $F(\nabla)$ in the bulk,

$$\boldsymbol{F}^{\rm brane}(x, x') = -\vec{W}(\nabla_X) G_D(X, X') \overleftarrow{W}(\nabla_{X'})|_{X=e(x), X'=e(x')} + \varkappa(\nabla)\delta(x, x'). \tag{5}$$

This expression expresses the fact that the kernel of the Dirichlet Green's function is being acted upon both arguments by the Wronskian operators with a subsequent restriction to the brane, with $X = e(x)$ denoting the brane embedding function.

As shown in [13], this operator determines the brane-to-brane propagation of the physical modes in the system with the classical action (1) (its inverse is the brane-to-brane propagator) and additively contributes to its full one-loop effective action according to

$$\varGamma_{\text{1-loop}} \equiv \tfrac{1}{2}{\rm Tr}_N^{(d+1)} \ln F = \tfrac{1}{2}{\rm Tr}_D^{(d+1)} \ln F + \tfrac{1}{2}{\rm Tr}^{(d)} \ln \boldsymbol{F}^{\rm brane}, \tag{6}$$

where ${\rm Tr}_{D,N}^{(d+1)}$ denotes functional traces of the bulk theory subject to Dirichlet and Neumann boundary conditions, respectively, while ${\rm Tr}^{(d)}$ is a functional trace in the boundary d-dimensional theory. The full quantum effective action of this model is obviously given by the functional determinant of the operator $F(\nabla_X)$ subject to the generalized Neumann boundary conditions (4), and the above equation reduces its calculation to that of the Dirichlet boundary conditions plus the contribution of the brane-to-brane propagation.

Here we apply (6) to a simple model of a scalar field which mimics, in particular, the properties of the brane induced gravity models and the DGP model [1]. This is the $(d + 1)$-dimensional massive scalar field $\Phi(X) = \Phi(x, y)$ with mass M living in the *curved* half-space $y \geq 0$ with the additional d-dimensional kinetic term for $\varphi(x) \equiv \Phi(x, 0)$ localized at the brane (boundary) at $y = 0$,

$$S[\phi] = \frac{1}{2}\int_{y\geq 0} d^{d+1}X\sqrt{G}((\nabla_X\Phi(X))^2 + M^2\Phi^2(X) + P(X)\Phi^2(X)) + \frac{1}{4m}\int d^dx\sqrt{g}((\nabla_x\varphi(x))^2 + \mu^2\varphi^2(x) + p(x)\varphi^2(x)). \tag{7}$$

Here and in what follows we work in a Euclidean (positive-signature) spacetime. Therefore, this action corresponds to the following choice of $F(\nabla_X)$ in terms of $(d+1)$-dimensional covariant d'Alembertian (Laplacians)

$$F(\nabla_X) = -\Box^{(d+1)} + M^2 + P = -G^{AB}\nabla_A\nabla_B + M^2 + P. \tag{8}$$

In the normal Gaussian coordinates its Wronskian operator is given by $W = -\partial_y$—the normal derivative with respect to outward-pointing normal to the brane, and the boundary operator $\varkappa(\nabla)$ equals in terms of the d-dimensional d'Alembertian

$$\varkappa(\nabla) = \frac{1}{2m}(-\Box + \mu^2 + p), \qquad \Box = \Box^{(d)} \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu, \tag{9}$$

where the dimensional parameter m mimics the role of the DGP scale [1]. Thus, the generalized Neumann boundary conditions in this model involve second-order derivatives tangential to the brane,

$$\left(\partial_y - \frac{-\Box + \mu^2 + p}{2m}\right)\Phi(X)\bigg|_{\mathbf{b}} = 0, \tag{10}$$

cf. (4) with $W = -\partial_y$ and $\varkappa$ given by (9).

As was shown [14], the flat space brane-to-brane operator for such a model without potential terms has the form of the pseudodifferential operator with the flat-space $\Box$,

$$\boldsymbol{F}^{\rm brane}(\nabla) = \frac{1}{2m}(-\Box + \mu^2 + 2m\sqrt{M^2 - \Box}). \tag{11}$$

In the massless case of the DGP model [1], $M = 0$, this operator is known to mediate the gravitational interaction on the brane, interpolating between the four-dimensional Newtonian law at intermediate distances and the five-dimensional law at the horizon scale $\sim 1/m$ [6].

Here we generalize this construction to a curved spacetime and expand the brane-to-brane operator and its effective action in covariant curvature series. This is the expansion in powers of the bulk curvature ${}^{\mathbf{B}}R$, extrinsic curvature of the brane $k_{\alpha\beta}$, the potential terms of the bulk P and brane p operators and their covariant derivatives—all taken at the location of the brane. The expansion starts with the approximation (11) based on the *full covariant* d'Alembertian on the brane. We present a systematic technique of calculating curvature corrections in (6) and rewrite their nonlocal operator coefficients—functions of the covariant $\Box$-in the form of the generalized (weighted) proper time representation.

The success of the conventional Schwinger-DeWitt method is based on the fact that the one-loop effective action of the operator, say $-\Box + M^2$, has a proper time representation

$$\frac{1}{2}\,\mathrm{Tr}\ln(M^2-\Box) = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\, e^{-sM^2}\,\mathrm{Tr}e^{s\Box}. \tag{12}$$

In view of the well-known small time expansion for the heat kernel [16,17],

$$e^{s\Box}\delta(x,x') = \frac{1}{(4\pi s)^{d/2}} D^{1/2}(x,x') e^{-\sigma(x,x')/2s} \sum_{n=0}^{\infty} s^n a_n(x,x'), \tag{13}$$

($\sigma(x,x')$ is the geodesic world function, $D(x,x')$ is the associated Van Vleck determinant and $a_n(x,x')$ are the Schwinger-DeWitt or Gilkey-Seely coefficients) the curvature expansion eventually reduces to the calculation of the coincidence limits of $a_n(x,x')$ and a trivial proper time integration resulting in the inverse mass expansion

$$\frac{1}{2}\,\mathrm{Tr}\ln(M^2-\Box) = -\frac{1}{2}\frac{M^d}{(4\pi)^{d/2}}\sum_{n=0}^{\infty}\frac{\Gamma(n-d/2)}{M^{2n}} \times \int dx\sqrt{g}\,a_n(x,x). \tag{14}$$

As we will show below, the calculation of the brane effective action differs from the conventional Schwinger-DeWitt case in that the proper time integral (12) contains in the integrand a certain extra weight function $w(s)$, and instead of just $\mathrm{Tr}e^{s\Box}$ one has to calculate the trace of the heat kernel acted upon by a certain local differential operator $\mathrm{Tr}(W(\nabla)e^{s\Box})$. This again reduces to the calculation of the coincidence limits—this time of the multiple covariant derivatives of $a_n(x,x')$, $\sigma(x,x')$ and $D(x,x')$—the task easily doable within a conventional DeWitt recurrence procedure.

The result of this calculation is peculiar. Unlike the usual Schwinger-DeWitt expansion (14) the brane effective action takes the form

$$\frac{1}{2}\,\mathrm{Tr}\ln\boldsymbol{F}^{\mathrm{brane}} = \left(\frac{Mm}{4\pi}\right)^{d/2}\sum_{N=0}^{\infty}\frac{1}{M^N}\sum_{i\le N}\frac{O(\mathfrak{M}^{2N-i})}{m^{N-i}} + \frac{M^d}{(4\pi)^{d/2}}\sum_{N=0}^{\infty}\frac{1}{M^N}\sum_{i\le N} m^i O(\mathfrak{M}^{N-i}), \tag{15}$$

where $O(\mathfrak{M}^k)$ represent the integrals over the brane/boundary space of local invariants of dimensionality k in units of mass or inverse length. With this notation, in particular, $a_n(x,x) = O(\mathfrak{M}^{2n})$. More generally, these invariants (or spacetime covariant higher-dimensional operators) are composed of the powers of the bulk and brane curvature, extrinsic curvature of the brane/boundary, the potential terms of the bulk and brane operators and their covariant derivatives.

The main difference of (15) from (14) is that in addition to a usual part analytic in m with a typical M-dependence [second series in (15)] we also have the part singular in $m\to 0$ with a qualitatively different analytic dependence on the bulk mass ($M^{d/2-N}$ instead of M^{d-N}). This property was recently discovered for the effective potential in the toy model of the DGP type [14]. Physically this leads to an essential modification of the perturbation theory cutoff—the domain of validity of the local expansion $\mathfrak{M} \ll M_{\mathrm{cutoff}}$. It reduces this cutoff from $M_{\mathrm{cutoff}} = M$ to

$$M_{\mathrm{cutoff}} = \sqrt{Mm}. \tag{16}$$

In physically interesting brane models with $m \ll M$ this implies essential reduction of M_{cutoff} and signifies the problem of a low strong coupling scale [6]. While in [6] this phenomenon was observed in the tree-level theory, here we extend it to the quantum one-loop approximation.

As an application of this generalized Schwinger-DeWitt expansion we calculate the one-loop brane effective action of the quantum scalar field with the accuracy $O(\mathfrak{M}^2)$. In this approximation the basis of local curvature invariants includes one structure as a cosmological term, two structures linear in the extrinsic curvature and the potential term of the brane operator (9) and seven structures of dimensionality ($\mathfrak{M}^2$),

$$O(\mathfrak{M}^0) = \int_{\mathbf{b}} d^dx\sqrt{g}, \tag{17}$$

$$O(\mathfrak{M}^1) = \int_{\mathbf{b}} d^dx\sqrt{g}k, \qquad \int_{\mathbf{b}} d^dx\sqrt{g}\frac{p}{2m}, \tag{18}$$

$$\begin{aligned} O(\mathfrak{M}^2) = &\int_{\mathbf{b}} d^dx\sqrt{g}\,{}^{\mathbf{B}}R, \qquad \int_{\mathbf{b}} d^dx\sqrt{g}\,{}^{\mathbf{B}}R_{nn},\\ &\int_{\mathbf{b}} d^dx\sqrt{g}k^2_{\alpha\beta}, \qquad \int_{\mathbf{b}} d^dx\sqrt{g}k^2, \qquad \int_{\mathbf{b}} d^dx\sqrt{g}P,\\ &\int_{\mathbf{b}} d^dx\sqrt{g}\left(\frac{p}{2m}\right)^2, \qquad \int_{\mathbf{b}} d^dx\sqrt{g}k\frac{p}{2m}. \end{aligned} \tag{19}$$

Here ${}^{\mathbf{B}}R$ is a bulk scalar curvature, ${}^{\mathbf{B}}R_{nn} = {}^{\mathbf{B}}R_{AB}n^A n^B$ is the projection of the bulk Ricci tensor on the normal vector n^A to the brane, $k_{\alpha\beta}$ is the extrinsic curvature of the brane and $k = g^{\alpha\beta}k_{\alpha\beta}$ is its trace. In this article we use the following sign convention for extrinsic curvature: $k_{\alpha\beta} = e^A_{(\alpha}e^B_{\beta)}(\nabla_X)_A n_B$, where $\{e^A_\alpha\}$ is a holonomic basis tangent to brane, n_A-unit inward normal vector to the brane. P and p represent the bulk and brane potential terms of relevant operators $F(\nabla_X)$ and $\varkappa(\nabla_x)$ introduced above. Below we find explicit coefficients of these structures in (15) as nontrivial functions of mass parameters M, m, and μ and find UV divergences in this model.

II. PERTURBATION THEORY FOR THE BULK GREEN'S FUNCTION AND BRANE-TO-BRANE INVERSE PROPAGATOR

In normal Gaussian coordinates the covariant bulk d'Alembertian decomposes as $\Box^{(d+1)}_X = \partial_y^2 + \Box_x(y) + \ldots$, where ellipses denote (depending on spin) terms at most linear in derivatives and $\Box_x(y)$ is a covariant d'Alembertian on the slice of constant coordinate y. Therefore, the full bulk operator takes the form

$$\begin{aligned} F(\nabla) &= -\Box^{(d+1)}_X + M^2 + P(X) \\ &= -\partial_y^2 - \Box + M^2 - V(X|\partial_y, \nabla) \equiv F_0 - V, \\ \Box &\equiv \Box_x(0), \end{aligned} \tag{20}$$

in which all nontrivial y-dependence is isolated as a perturbation term $V(X|\partial_y, \nabla) \equiv V(y, \partial_y)$—a first-order differential operator in y, proportional to the extrinsic and bulk curvatures, and of second order in brane derivatives ∇ which we do not explicitly indicate here by assuming that they are encoded in the operator structure of $V(y, \partial_y)$. In particular, it includes the difference $\Box_x(0) - \Box(y) \equiv \Box - \Box(y)$ expandable in Taylor series in y.

The kernel of the bulk Green's function can formally be written as a y-dependent nonlocal operator acting on the d-dimensional brane—some nonpolynomial function of the brane covariant derivative

$$G_D(X, X') = G_D(y, y'|\nabla)\delta(x, x'). \tag{21}$$

The perturbation expansion for $G_D(y, y'|\nabla)$ is usual

$$G_D = G_D^0 + G_D^0 V G_D^0 + \ldots = G_D^0 \sum_{n=0}^{\infty} (V G_D^0)^n, \tag{22}$$

where G_D^0 is the propagator for operator F_0 obeying Dirichlet boundary conditions and the composition law includes the integration over the bulk coordinates, like, for example, in the first subleading term

$$G_D^0 V G_D^0(y, y') = \int_0^\infty dy'' G_D^0(y, y'') V(y'', \partial_{y''}) G_D^0(y'', y'). \tag{23}$$

The lowest order Green's function in the half-space of the DGP model setting—the Green's function of $F_0 = -\partial_y^2 - \Box + M^2$ subject to Dirichlet conditions on the brane $y = 0$ and at infinity—reads as follows

$$G_D^0(y, y') = \frac{e^{-|y-y'|\sqrt{M^2-\Box}} - e^{-(y+y')\sqrt{M^2-\Box}}}{2\sqrt{M^2 - \Box}}. \tag{24}$$

We want to stress that here we assume the exact (curved) d-dimensional d'Alembertian $\Box$ depending on the induced metric of the brane $g_{\mu\nu}(x)$. This means that in the lowest order approximation the underlying spacetime is not flat, but rather has a nontrivial but constant in y metric of constant y slices. Correspondingly in the zeroth order we have

$$\begin{aligned} -[\vec{W} G_D(y, y') \overleftarrow{W}]^0_{y=y'=0} &= -\vec{\partial}_y G_D^0(y, y') \overleftarrow{\partial}_y|_{y=y'=0} \\ &= \sqrt{M^2 - \Box}. \end{aligned} \tag{25}$$

The perturbation of the bulk operator can be expanded in Taylor series in y, so that it reads

$$\begin{aligned} V &= k(x, y)\partial_y + \Box(y) - \Box - P(x, y) \\ &\equiv \sum_{k=0}^{\infty} \frac{1}{k!} u_k y^k \partial_y - \sum_{k=0}^{\infty} \frac{1}{k!} v_k y^k, \end{aligned} \tag{26}$$

where $u_k(\nabla)$ and $v_k(\nabla)$ form a set of y-independent *local d-dimensional covariant* operators of maximum second order in ∇_x. The coefficients of these operators are given by the powers of the bulk and brane curvature, the extrinsic curvature of the brane, the potential term P and the covariant derivatives of all these quantities—all of them taken at the brane.

Below we present them up to the first order in the bulk and brane curvature and to the second order in the extrinsic curvature of the brane. Working in a Gauss normal coordinate system for the case of a single quantum scalar field they read as

$$\begin{aligned} u_0 &= k(x, y)|_{y=0} \equiv k, \\ u_1 &= \partial_y k(x, y)|_{y=0} = -{}^{\mathbf{B}}R_{nn} - k^2_{\mu\nu} \end{aligned} \tag{27}$$

and

$$\begin{aligned} v_0 &= P(x, y)|_{y=0} \equiv P, \\ v_1 &= \partial_y(-\Box_x(y) + P(x, y))|_{y=0} \\ &= 2k^{\alpha\beta}\nabla_\alpha\nabla_\beta + 2(\nabla_\alpha k^{\alpha\beta})\nabla_\beta - (\nabla^\beta k)\nabla_\beta + O(\mathfrak{R}^3), \\ v_2 &= \partial_y^2(-\Box_x(y) + P(x, y))|_{y=0} \\ &= (-2{}^{\mathbf{B}}R^\alpha{}_n{}^\beta{}_n - 6k^{\alpha\mu}k_\mu{}^\beta)\nabla_\alpha\nabla_\beta + O(\mathfrak{R}^3). \end{aligned} \tag{28}$$

Here we everywhere omit the argument x of all quantities located on the brane, ∇_α is the d-dimensional covariant derivative on the brane, $k_{\alpha\beta}$ denotes the extrinsic curvature of the brane, $k = g^{\alpha\beta}k_{\alpha\beta}$ is its trace. Subscript n denotes the projection of the relevant bulk index to the normal vector n_A, in particular ${}^{\mathbf{B}}R_{nn} = {}^{\mathbf{B}}R^C{}_{ACB}n^A n^B$.

Another source of differential operators with coefficients of growing power in the curvatures and their derivatives is the commutation of $\Box$ with all x-dependent quantities involved, like

$$\begin{aligned} [\Box, u_0] &= [\Box, k(x)] = 2(\nabla^\alpha k)\nabla_\alpha + O(\mathfrak{M}^3) \\ [\Box, v_1] &= 4(\nabla^\mu k^{\alpha\beta})\nabla_\mu\nabla_\alpha\nabla_\beta + O(\mathfrak{M}^3). \end{aligned} \tag{29}$$

In all these equations $O(\mathfrak{M}^l)$ denotes the accuracy in powers of the dimensionful quantities—curvatures, operator potential terms and their derivatives—with which the relevant quantity is calculated within the local Schwinger-DeWitt technique. In fact l associated with $O(\mathfrak{M}^l)$ indicates the dimensionality (in mass units) of the coefficient of the higher-derivative term of the relevant operator. In [17] it was called a *background dimensionality*, in contrast to a total dimensionality of the quantity. In what follows we will denote the background dimensionality of the operator $A = O(\mathfrak{M}^l)$ by

$$\mathrm{Dim}\, A = l. \tag{30}$$

Thus for an operator of the form $(\nabla^a R^m P^l k^n)\nabla^b$, where the coefficient of the bth order derivative is given by a monomial of curvatures and potential terms and their covariant derivatives of the total order a, the background dimensionality is given by

$$\begin{aligned} \mathrm{Dim}\,(\nabla^a R^m P^l k^n)\nabla^b &= \mathrm{Dim} O(\mathfrak{M}^{a+2m+2l+n}) \\ &= a + 2m + 2l + n, \end{aligned} \tag{31}$$

whereas its total dimensionality is, of course, $[(\nabla^a R^m P^l k^n)\nabla^b] = a + b + 2m + 2l + n$.

The background dimensionality in fact counts the order of the perturbation theory in powers of the local quantities $R = O(\mathfrak{M}^2)$, $P = O(\mathfrak{M}^2)$, $k = O(\mathfrak{M}^1)$ and their derivatives. The orders of this perturbation theory have the form $\mathfrak{M}^N/M^M$, where M is the bulk mass playing the role of the cutoff, $\mathfrak{M} \ll M$, beyond which the local expansion does not apply. The background dimensionalities of relevant local operators, like (28) and (29), always contribute to N. On the contrary, free (acting to the right) derivatives of these operators may effect the overall power of the cutoff M in the denominator (by reducing it) so that they are not indicative of the accuracy of the Schwinger-DeWitt expansion. In other words, any quantity of the background dimensionality l contributes to the lth order of the local expansion $\mathfrak{M}^l$ and higher, whereas the total dimensionality of this quantity does not determine the order of this expansion. This explains a distinguished role of the *background dimensionality* vs the total one.

The next calculational step consists in the substitution of (25) and (26) into (22), and it leads to exactly calculable integrals over y. The integration over y results in a nonlocal series in inverse powers of $\sqrt{M^2 - \Box}$—this is obvious from the y-expansion of (26), because every extra power of y brings one extra inverse power of $\sqrt{M^2 - \Box}$. Each kth order of this series arises in the form of the following nonlocal chain of square root "propagators,"

$$\frac{1}{(\sqrt{M^2 - \Box})^{l_1}} V_1 \frac{1}{(\sqrt{M^2 - \Box})^{l_2}} V_2 \ldots V_{p-1} \frac{1}{(\sqrt{M^2 - \Box})^{l_p}}, \quad l_1 + l_2 + \ldots + l_p = k,$$

with some differential operators V_i as its vertices. With the aid of the commutation relations like (29) all these propagators can be systematically commuted to the right of the expression by the price of extra commutator terms of the same structure, and the perturbation expansion finally takes the form

$$-[\vec{W} G_D(y, y') \overleftarrow{W}]_{y=y'=0} = \sqrt{M^2 - \Box} - \sum_{k=1}^{\infty} U_k(\nabla) \frac{1}{(\sqrt{M^2 - \Box})^{k-1}}, \tag{32}$$

where $U_k(\nabla)$ is a set of certain local covariant differential operators acting on the brane. The dimensionality of each $U_k(\nabla)$ is the inverse length to the power k, which is composed of the dimensionalities of bulk and extrinsic curvatures and covariant derivatives all taken on the brane at $y = 0$.

With $\varkappa(\nabla)$ given by (9) the brane-to-brane operator (5) reads

$$\boldsymbol{F}^{\,\mathrm{brane}}(\nabla) = \boldsymbol{F}_0 - \sum_{k=1}^{\infty} U_k(\nabla) \frac{1}{(\sqrt{M^2 - \Box})^{k-1}}, \tag{33}$$

$$\boldsymbol{F}_0 = \frac{1}{2m}(-\Box + \mu^2 + 2m\sqrt{M^2 - \Box}), \tag{34}$$

Here we absorbed the potential term p of the operator $\varkappa(\nabla)$ into the first term, $k = 1$, of the perturbation part of $\boldsymbol{F}^{\mathrm{brane}}(\nabla)$ by redefining the U_1 term, $U_1 \to U_1 - p/2m$, because p should of course be treated on equal footing with other perturbations. We do not introduce a new notation for U_1, and this should not lead to a confusion because Eq. (32) will not be used in what follows. The rest of the perturbation part is induced from the bulk and does not depend on the boundary conditions on the brane encoded in the operator $\varkappa(\nabla)$. Note that the zeroth-order term here is a nontrivial nonlocal operator because $\Box$ is a curved space d-dimensional D'Alembertian acting on the brane.

In the $\mathfrak{M}^2$ approximation (involving the terms linear in the bulk curvature and potential P and the terms quadratic in the extrinsic curvature k and the brane potential p) the operator coefficients $U_k(\nabla)$ extend to $k = 6$. Higher order coefficients go beyond the $\mathfrak{M}^2$-approximation. The calculations show that for a single scalar field they read as

$$U_1(\nabla) = -\frac{1}{2}u_0 - \frac{p}{2m} = -\frac{1}{2}k - \frac{p}{2m} \qquad U_2(\nabla) = -\frac{1}{4}u_1 - \frac{1}{2}v_0 - \frac{1}{8}u_0u_0 = \frac{1}{4}{}^{\mathbf{B}}R_{nn} + \frac{1}{4}k^2_{\mu\nu} - \frac{1}{2}P - \frac{1}{8}k^2,$$
$$U_3(\nabla) = -\frac{1}{8}u_2 - \frac{1}{8}[u_0, v_0] - \frac{1}{8}[\Box, u_0] - \frac{1}{8}u_1u_0 - \frac{1}{4}v_1 = -\frac{1}{2}k^{\alpha\beta}\nabla_\alpha\nabla_\beta - \frac{1}{2}k^{\alpha\beta}{}_{;\alpha}\nabla_\beta + O(\mathfrak{M}^3),$$
$$U_4(\nabla) = -\frac{1}{8}v_2 + O(\mathfrak{M}^3) = \Big(\frac{1}{4}{}^{\mathbf{B}}R^\alpha{}_n{}^\beta{}_n + \frac{3}{4}k^{\alpha\mu}k_\mu{}^\beta\Big)\nabla_\alpha\nabla_\beta + O(\mathfrak{M}^3), \tag{35}$$
$$U_5(\nabla) = -\frac{1}{8}[\Box, v_1] + O(\mathfrak{M}^3) = -\frac{1}{2}(\nabla^\mu k^{\alpha\beta})\nabla_\mu\nabla_\alpha\nabla_\beta + O(\mathfrak{M}^3),$$
$$U_6(\nabla) = \frac{5}{32}v_1v_1 + O(\mathfrak{M}^3) = \frac{5}{8}k^{\alpha\beta}k^{\mu\nu}\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu + O(\mathfrak{M}^3).$$

As mentioned above, each U_k has a total dimensionality k in units of mass. Except the case of $k = 2$, for which $U_2 = O(\mathfrak{M}^2)$, their background dimensionality is $U_k = O[\mathfrak{M}^{[(k+2)/3]}]$, where the square brackets denote the integer part of a fractional number.

III. PERTURBATION THEORY FOR THE BRANE EFFECTIVE ACTION

Perturbation theory for the effective action immediately follows from the perturbation series (33) for the operator $\boldsymbol{F}^{\rm brane}$. The brane effective action can be rewritten as

$$\frac{1}{2}\,{\rm Tr}\ln\boldsymbol{F}^{\rm brane} = \frac{1}{2}\,{\rm Tr}\ln\boldsymbol{F}_0 + \frac{1}{2}\,{\rm Tr}\ln\Big(1 - \sum_{k\geq1}U_k(\nabla) \times \frac{1}{(\sqrt{M^2-\Box})^{k-1}}\,\frac{1}{\boldsymbol{F}_0}\Big), \tag{36}$$

and reexpanded in powers of the perturbation series term under the logarithm sign. After commuting all the square root propagators $1/(\sqrt{M^2-\Box})^k$ and the propagators $1/\boldsymbol{F}_0$ to the right this expansion takes the form

$$\frac{1}{2}\,{\rm Tr}\ln\boldsymbol{F}^{\rm brane} = \frac{1}{2}\,{\rm Tr}\ln\boldsymbol{F}_0 - \frac{1}{2}\sum_{k\geq0,l\geq1}{\rm Tr}W_{kl}(\nabla) \times \frac{1}{(\sqrt{M^2-\Box})^k}\,\frac{1}{\boldsymbol{F}_0{}^l}, \tag{37}$$

with a new set of local covariant differential operators $W_{kl}(\nabla)$ acting on the brane. For dimensional reasons the total dimensionality of $W_{kl}(\nabla)$ is $k + l$ in units of mass, because $\boldsymbol{F}_0$ like $\sqrt{M^2-\Box}$ has a unit dimensionality.

These operators are composed of the products of the operators $U_k(\nabla)$ introduced above and their multiple commutators with the "propagators" $1/(\sqrt{M^2-\Box})^k$ and $1/\boldsymbol{F}_0$. These commutators are based on the multiple use of the formula

$$f(\Box)B - Bf(\Box) = \sum_{k=1}^{\infty}\frac{1}{k!}{\rm ad}^k_\Box B\cdot\partial^k_\Box f(\Box), \quad {\rm ad}_\Box B \equiv [\Box, B], \tag{38}$$

which leads to the following structure of $W_{kl}(\nabla)$

$$W_{kl}(\nabla) = \sum_{p=0}^{\max\{0,l-2\}}\frac{1}{m^p}W_{kl,p}(\nabla). \tag{39}$$

Here, modulo the powers of $p/2m$ originating from $U_1(\nabla)$, the coefficients $W_{kl,p}(\nabla)$ are m-independent, and the negative powers of m follow from the differentiation of the propagator $1/\boldsymbol{F}_0$ with respect to $\Box$ participating in the commutators $[U_k, 1/\boldsymbol{F}_0]$,

$$\partial_\Box\frac{1}{\boldsymbol{F}_0} = \frac{1}{2}\,\frac{1}{\boldsymbol{F}_0{}^2}\Big(\frac{1}{m} + \frac{1}{\sqrt{M^2-\Box}}\Big).$$

As we see, each such differentiation results in the extra power of the propagator $1/\boldsymbol{F}_0$ and an extra term proportional to $1/m$. For a $W_{kl}(\nabla)$ with a given l the highest order of $1/m$ is limited by $\max\{0, l-2\}$. This is because p differentiations increase the power of $1/\boldsymbol{F}_0$ from some initial l' to at least $l = l' + p$, and the initial $l' \geq 2$, because the commutation of $1/\boldsymbol{F}_0$ with other quantities begins only when the number of these propagators exceeds two [one should remember that one propagator $1/\boldsymbol{F}_0$ always stands to the right of everything else, see Eq. (36)]. This explains the upper limit of summation over p in (39) and, as we will later see, underlies the regularity of the Neumann limit $m \to \infty$.

The background dimensionality of $W_{kl,p}(\nabla)$ is a monotonically growing function of all its indices and can be shown to satisfy the bound

$${\rm Dim}\,W_{kl,p}(\nabla) \geq \Big[\frac{k+l+2p+2}{3}\Big], \tag{40}$$

where square brackets denote an integer part of the fractional number. This bound follows from the observation that for any $W_{kl,p}$, composed of the chain of U_{k_i} and n commutators with $\Box$, the total dimensionality $[W_{kl,p}] = k + l + p$ equals $[W_{kl,p}] = \sum_i k_i + 2n$. On the other hand, the background dimensionality in addition to the sum of ${\rm Dim}U_{k_i} = [(k_i + 2)/3]$ contains at least one extra unit of mass per each commutation with $\Box$. Therefore

$$\text{Dim}\,W_{kl,p}(\nabla) \geq \sum_i \left[\frac{k_i+2}{3}\right] + n \geq \left[\frac{\sum_i k_i + 2}{3}\right] + n = \left[\frac{(k+l+p)+n+2}{3}\right], \tag{41}$$

where we used the above counting of the total dimensionality $\sum_i k_i + 2n = k + l + p$. The bound (40) then follows from the fact that the overall negative power of m does not exceed the number of commutations n, $0 \leq p \leq n$.

Relevant W_{kl} which can contribute up to $O(\mathfrak{M}^2)$ order inclusive are

$$W_{01} = U_1 = -\frac{1}{2}k - \frac{p}{2m}, \qquad W_{11} = U_2 = \frac{1}{4}{}^{\mathbf{B}}R_{nn} + \frac{1}{4}k^2_{\mu\nu} - \frac{1}{2}P - \frac{1}{8}k^2, \qquad W_{02} = \frac{1}{2}U_1^2 = \frac{1}{2}\left(\frac{1}{2}k + \frac{p}{2m}\right)^2,$$
$$W_{21} = U_3 = -\frac{1}{2}k^{\alpha\beta}\nabla_\alpha\nabla_\beta - \frac{1}{2}(\nabla_\alpha k^{\alpha\beta})\nabla_\beta + O(\mathfrak{M}^3), \qquad W_{31} = U_4 = \left(\frac{1}{4}{}^{\mathbf{B}}R^\alpha{}_n{}^\beta{}_n + \frac{3}{4}k^{\alpha\mu}k_\mu{}^\beta\right)\nabla_\alpha\nabla_\beta + O(\mathfrak{M}^3),$$
$$W_{22} = U_3U_1 + \frac{1}{2}U_2U_2 = \frac{1}{2}\left(\frac{1}{2}k + \frac{p}{2m}\right)k^{\alpha\beta}\nabla_\alpha\nabla_\beta + O(\mathfrak{M}^3), \qquad W_{41} = U_5 = -\frac{1}{2}(\nabla^\mu k^{\alpha\beta})\nabla_\mu\nabla_\alpha\nabla_\beta + O(\mathfrak{M}^3),$$
$$W_{51} = U_6 = \frac{5}{8}k^{\alpha\beta}k^{\mu\nu}\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu + O(\mathfrak{M}^3),$$
$$W_{42} = U_5U_1 + U_4U_2 + \frac{1}{2}U_3U_3 + U_3[\Box, U_1] = \frac{1}{8}k^{\alpha\beta}k^{\mu\nu}\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu + O(\mathfrak{M}^3). \tag{42}$$

IV. GENERALIZED PROPER TIME METHOD

The further calculation is based on the possibility to express the nonlocal structures in (37) in terms of the heat kernel of the box operator $\Box$, which admits a well-known curvature expansion. This is the set of proper time representations which differ from a usual Schwinger integral by nontrivial weight functions $w_{kl}(s)$

$$\ln \boldsymbol{F}_0 = -\ln(2m) - \int_0^\infty \frac{ds}{s} w_{00}(s) e^{-s(M^2-\Box)}, \tag{43}$$

$$\frac{1}{(\sqrt{M^2-\Box})^k}\frac{1}{\boldsymbol{F}_0{}^l} = \int_0^\infty \frac{ds}{s} w_{kl}(s) e^{-s(M^2-\Box)}, \qquad (l \geq 1). \tag{44}$$

These weight functions can be found as follows. First, decompose the operator $\boldsymbol{F}_0$ defined by (34) into the product of two factors linear in $\sqrt{M^2-\Box}$

$$\boldsymbol{F}_0 = \frac{1}{2m}(\sqrt{M^2-\Box} - m_+)(\sqrt{M^2-\Box} - m_-), \tag{45}$$

where $m_\pm$ denote the roots of the quadratic equation $x^2 + 2mx - M^2 + \mu^2 = 0$,

$$m_+ = -m + \sqrt{m^2 + M^2 - \mu^2}, \quad m_- = -m - \sqrt{m^2 + M^2 - \mu^2}. \tag{46}$$

Therefore

$$\ln \boldsymbol{F}_0 = -\ln 2m + \ln(\sqrt{M^2-\Box} - m_+) + \ln(\sqrt{M^2-\Box} - m_-),$$
$$\frac{1}{(\sqrt{M^2-\Box})^k \boldsymbol{F}_0{}^l} = \sum_{a=1}^k \frac{B^a_{kl}(m_+,m_-)}{(\sqrt{M^2-\Box})^a} + \sum_{b=1}^l\left(\frac{D^b_{kl}(m_+,m_-)}{(\sqrt{M^2-\Box}-m_+)^b} + (m_+ \leftrightarrow m_-)\right). \tag{47}$$

Here the second equation is the result of the decomposition of its left-hand side into partial fractions with the coefficients B^a_{kl} and D^b_{kl}

$$B^a_{kl}(m_+,m_-) = \frac{(-m/2)^l}{(\triangle)^{2l}}\frac{1}{(l-1)!(k-a)!}\times\sum_{b=1}^l\frac{\Gamma(2l-b)\Gamma(k-a+b)}{\Gamma(l-b+1)\Gamma(b)}\times\left(\frac{1}{m_+{}^{k-a}}\left(\frac{m_+-m_-}{m_+}\right)^b + \frac{1}{m_-{}^{k-a}}\times\left(\frac{m_- - m_+}{m_-}\right)^b\right), \tag{48}$$

and

$$D^b_{kl}(m_+,m_-) = \left(\frac{m}{\triangle}\right)^l\frac{1}{\Gamma(k)\Gamma(l)}\frac{(-1)^{l-b}}{m_+{}^{k+l-b}}\times\sum_{p=0}^{l-b}\frac{\Gamma(k+l-b-p)}{\Gamma(l-b+1-p)}\frac{\Gamma(l+p)}{p!}\times\left(\frac{m_+}{m_+-m_-}\right)^p. \tag{49}$$

Now, in addition to a simple proper time representation of the square root propagators

$$\frac{1}{(\sqrt{M^2-\Box})^a}=\frac{1}{\Gamma(a/2)}\int_0^\infty ds s^{a/2-1}e^{-s(M^2-\Box)}, \quad (50)$$

we need a similar representation for the new operator of the form $\sqrt{M^2-\Box}-\tilde{m}$ and its logarithm. It can be derived with the aid of the integral representation of the cylindrical function of a half-integer order

$$\frac{1}{(\sqrt{M^2-\Box}-\tilde{m})^\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^\infty dx x^{\alpha-1}e^{\tilde{m}x}e^{-x\sqrt{M^2-\Box}}$$
$$=\frac{\alpha}{2\sqrt{\pi}\Gamma(1+\alpha)}\int_0^\infty ds s^{-3/2}e^{-s(M^2-\Box)}$$
$$\times\int_0^\infty dx x^\alpha e^{\tilde{m}x}e^{-(x^2/(4s))}. \quad (51)$$

Differentiating it with respect to α at $\alpha=0$ one gets

$$\ln(\sqrt{M^2-\Box}-\tilde{m})=-\frac{1}{2\sqrt{\pi}}\int_0^\infty ds s^{-3/2}e^{-s(M^2-\Box)}$$
$$\times\int_0^\infty dx e^{\tilde{m}x}e^{-(x^2/(4s))}$$
$$=-\frac{1}{2}\int_0^\infty\frac{ds}{s}w(-\tilde{m}\sqrt{s})e^{-s(M^2-\Box)}, \quad (52)$$

where the function $w(-\tilde{m}\sqrt{s})$ is given in terms of the complementary error function erfc(z),

$$w(-\sigma)\equiv\frac{2}{\sqrt{\pi}}\int_0^\infty dx e^{2\sigma x}e^{-x^2}=e^{\sigma^2}\mathrm{erfc}(-\sigma), \quad (53)$$

$$\mathrm{erfc}(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty dt e^{-t^2}. \quad (54)$$

A multiple differentiation of (52) with respect to m then gives

$$\frac{1}{(\sqrt{M^2-\Box}-\tilde{m})^a}=\frac{1}{2\Gamma(a)}\int_0^\infty ds s^{a/2-1}\frac{d^a w(-\sigma)}{d\sigma^a}\bigg|_{\sigma=\tilde{m}\sqrt{s}}$$
$$\times e^{-s(M^2-\Box)}. \quad (55)$$

Using (50), (52), and (55) in (47) we finally come to the following expressions for the weights in Eqs. (43) and (44)

$$w_{00}=\frac{1}{2}(w(-m_+\sqrt{s})+w(-m_-\sqrt{s})), \quad (56)$$

$$w_{kl}=\sum_{a=1}^{k}\frac{B_{kl}^a(m_+,m_-)}{2\Gamma(a/2)}s^{a/2}+\sum_{a=1}^{l}\frac{D_{kl}^a(m_+,m_-)}{2\Gamma(a)}s^{a/2}$$
$$\times\frac{d^a w(-\sigma)}{d\sigma^a}\bigg|_{\sigma=m_+\sqrt{s}}+\{m_+\leftrightarrow m_-\}. \quad (57)$$

In terms of these weights the action (37) takes the form

$$\frac{1}{2}\,\mathrm{Tr}\ln\boldsymbol{F}^{\mathrm{brane}}\equiv-\frac{1}{2}\sum_{k,l=0}^{\infty}\int_0^\infty\frac{ds}{s}w_{kl}(s)e^{-sM^2}$$
$$\times\mathrm{Tr}(W_{kl}(\nabla)e^{s\Box}), \quad (58)$$

where we disregarded the contribution of the local measure $-\mathrm{Tr}\ln(2m)\sim\delta^{(d)}(0)$ and, in addition to W_{kl} of (37), introduced

$$W_{00}=1,\qquad W_{k0}=0,\qquad k\geq 1. \quad (59)$$

V. GENERALIZED SCHWINGER-DEWITT EXPANSION

The calculation of (58) is based on the heat kernel expansion (13) for the covariant d'Alembertian $\Box$ acting in a curved d-dimensional space without boundaries. The Schwinger-DeWitt coefficients $a_n(x,x')$ in this expansion represent brane curvature invariants of the growing background dimensionality $2n$, $a_n(x,x')=O(\mathfrak{M}^{2n})$. With this expansion the functional traces in (58) take the form

$$\mathrm{Tr}(W_{kl}(\nabla)e^{s\Box})=\frac{1}{(4\pi s)^{d/2}}\sum_{n=0}^{\infty}s^n\int_{\mathbf{b}}d^dx\sqrt{g}W_{kl}(\tilde{\nabla}(s))$$
$$\times a_n(x,x')|_{x'=x},$$

where the new generalized covariant derivative

$$\tilde{\nabla}_\alpha(s)\equiv\nabla_\alpha-\frac{1}{2s}\partial_\alpha^x\sigma(x,x')+\frac{1}{2}\partial_\alpha^x\ln D(x,x')$$

originates from the commutation of ∇_α with the exponential factor $\exp(-\sigma(x,x')/2s)$ and the Van Vleck-Morette determinant $D(x,x')$ in the kernel of $\exp(s\Box)$. In addition to

$$\nabla_{\mu_1}\ldots\nabla_{\mu_p}a_n(x,x')|_{x'=x}$$

this lengthening of the covariant derivatives also generates the coincidence limits

$$\nabla_{\mu_1}\ldots\nabla_{\mu_p}\sigma(x,x')|_{x'=x},\qquad\nabla_{\mu_1}\ldots\nabla_{\mu_p}D(x,x')|_{x'=x}$$

easily calculable by the DeWitt recurrence procedure [16,17]. Moreover, this leads to extra negative powers of the proper time, so that

$$\int_{\mathbf{b}}d^dx\sqrt{g}W_{kl}(\tilde{\nabla}^x)a_n(x,x')|_{x=x'}$$
$$=\sum_{p=0}^{\max\{0,l-2\}}\sum_{c=0}^{[(2k+2l+p)/6]}\frac{(\mathcal{A}_n)_{kl,p}^c}{s^c m^p}, \quad (60)$$

where $(\mathcal{A}_n)_{kl,p}^c$ represents the set of integrals of bulk and brane curvature invariants, powers of potential terms P and p/m and their derivatives of the growing dimensionality $k+l+p+2n-2c$ (which now coincides with their background dimensionality, because they are no longer the differential operators),

$$(\mathcal{A}_n)_{kl,p}^c=O(\mathfrak{M}^{k+l+p+2n-2c}). \quad (61)$$

The highest power of $1/s$ in (60) is determined by (the integer part of) half the order of $W_{kl,p}(\nabla)$ in derivatives (the latter equals the difference between the total dimensionality of this operator and its background dimensionality). As W_{kl} is a sum over the powers of the inverse DGP mass scale (39), the quantities $(\mathcal{A}_n)^c_{kl,p}$ above represent the relevant coefficients of $1/m^p$.

Thus

$$\frac{1}{2}\,\mathrm{Tr}\ln\boldsymbol{F}^{\mathrm{brane}} = -\frac{1}{2}\frac{1}{(4\pi)^{d/2}}\sum_{\{k,l,n,p,c\}}\int_0^\infty \frac{ds}{s}s^{n-c-d/2} \times w_{kl}(s)e^{-sM^2}\frac{(\mathcal{A}_n)^c_{kl,p}}{m^p}, \tag{62}$$

where the domain of summation over all indices is given by

$$\sum_{\{k,l,n,p,c\}} = \sum_{k,l,n=0}^{\infty}\sum_{p=0}^{\max\{0,l-2\}}\sum_{c=0}^{[(2k+2l+p)/6]} \tag{63}$$

The quantities $(\mathcal{A}_n)^c_{kl,p}$ play the role of integrated generalized Schwinger-DeWitt coefficients. For a scalar field in the approximation $O(\mathfrak{M}^2)$ only the following coefficients contribute to the brane effective action

$$\begin{aligned}
&(\mathcal{A}_0)^0_{00,0} = \int_{\mathbf{b}} d^dx\sqrt{g}, \qquad (\mathcal{A}_1)^0_{00,0} = \int_{\mathbf{b}} d^dx\sqrt{g}\frac{1}{6}{}^{\mathbf{b}}R(g) = \int_{\mathbf{b}} d^dx\sqrt{g}\Big(\frac{1}{6}{}^{\mathbf{B}}R - \frac{1}{3}{}^{\mathbf{B}}R_{nn} + \frac{1}{6}k^2 - \frac{1}{6}k^2_{\alpha\beta}\Big),\\
&(\mathcal{A}_0)^0_{01,0} = \int_{\mathbf{b}} d^dx\sqrt{g}\Big(-\frac{1}{2}k - \frac{p}{2m}\Big), \qquad (\mathcal{A}_0)^0_{11,0} = \int_{\mathbf{b}} d^dx\sqrt{g}\Big(\frac{1}{4}{}^{\mathbf{B}}R_{nn} + \frac{1}{4}k^2_{\mu\nu} - \frac{1}{2}P - \frac{1}{8}k^2\Big),\\
&(\mathcal{A}_0)^0_{02,0} = \int_{\mathbf{b}} d^dx\sqrt{g}\Big(\frac{1}{8}k^2 + \frac{1}{2}k\frac{p}{2m} + \frac{1}{2}\Big(\frac{p}{2m}\Big)^2\Big), \qquad (\mathcal{A}_0)^1_{21,0} = \int_{\mathbf{b}} d^dx\sqrt{g}\frac{1}{4}k,\\
&(\mathcal{A}_0)^1_{31,0} = \int_{\mathbf{b}} d^dx\sqrt{g}\Big(-\frac{1}{8}{}^{\mathbf{B}}R_{nn} + \frac{3}{8}k^2_{\alpha\beta}\Big), \qquad (\mathcal{A}_0)^1_{22,0} = \int_{\mathbf{b}} d^dx\sqrt{g}\Big(-\frac{1}{8}k^2 - \frac{1}{4}k\frac{p}{2m}\Big),\\
&(\mathcal{A}_0)^2_{51,0} = \int_{\mathbf{b}} d^dx\sqrt{g}\Big(\frac{5}{16}k^2_{\alpha\beta} + \frac{5}{32}k^2\Big).
\end{aligned} \tag{64}$$

VI. LARGE MASS EXPANSION AND ITS CUTOFF SCALES

Integration over s in (62) gives

$$\frac{1}{2}\,\mathrm{Tr}\ln\boldsymbol{F}^{\mathrm{brane}} = -\frac{1}{2}\frac{M^d}{(4\pi)^{d/2}}\sum_{\{k,l,n,p,c\}} C^{n-c}_{kl}\frac{(\mathcal{A}_n)^c_{kl,p}}{M^{2n-2c+k+l}m^p}, \tag{65}$$

where C^j_{kl} (with $j = n - c$) are the following functions of the mass parameters of the model

$$C^j_{kl} = M^{2j-d+k+l}\int_0^\infty \frac{ds}{s}s^{j-d/2}w_{kl}(s)e^{-sM^2}. \tag{66}$$

The behavior of these functions for $M \to \infty$ is important for the determination of the efficiency of the expansion (65) and of the range of its validity—the cutoff M_{cutoff} below which, $\mathfrak{M} \ll M_{\mathrm{cutoff}}$, this expansion makes sense. This behavior easily follows from a simple observation that the functions C^j_{kl} can be directly obtained from the nonlocal form factors (44) by integration over their argument $\Box = -\lambda$ with the weight $\lambda^{d/2-j-1}$. In the domain of convergence of this integral in the complex plane of d we have

$$\int_0^\infty \frac{d\lambda}{\lambda}\lambda^{d/2-j}\frac{1}{(\sqrt{M^2-\Box})^k}\frac{1}{\boldsymbol{F}_0{}^l}\bigg|_{\Box=-\lambda} = \frac{\Gamma(d/2-j)}{M^{2j-d+k+l}}C^j_{kl}. \tag{67}$$

With the replacement of the integration variable $x = \sqrt{1+\lambda/M^2} - 1$ the integral representation for C^j_{kl} takes the form

$$C^j_{kl} = \frac{1}{\Gamma(\nu)}\Big(\frac{2m}{M}\Big)^l\int_0^\infty dx x^{\nu-1}(x+\varepsilon_+)^{-l}\varphi(x)|_{\nu=d/2-j},$$
$$\varphi(x) = \frac{(x+2)^{\nu-1}}{(x+1)^{k-1}(x+\varepsilon_-)^l}, \tag{68}$$

where $\varepsilon_\pm = 1 - m_\pm/M$. For $M \to \infty$ the parameter $\varepsilon_+ \to 0$ ($\varepsilon_- \to 2$), and the integral here has a nonanalytic in ε_+ part because of the singularity of its integrand at $x = 0$ [14],

$$\int_0^\infty dx x^{\nu-1}(x+\varepsilon)^{-l}\varphi(x) = \varepsilon^{\nu-l}\frac{\Gamma(\nu)\Gamma(l-\nu)}{\Gamma(l)}\varphi(0) + O(1). \tag{69}$$

Since $\varepsilon_+ \to m/M$ in this limit, we have

$$C^j_{kl} = C_1\Big(\frac{m}{M}\Big)^{d/2-j} + C_2\Big(\frac{m}{M}\Big)^l, \qquad M \to \infty. \tag{70}$$

Thus, in view of the background dimensionality of $(\mathcal{A}_n)^c_{kl,p} = O(\mathfrak{M}^{k+l+p+2n-2c})$ the local expansion (65) for the effective action takes the form

$$\frac{1}{2}\,\mathrm{Tr}\ln\boldsymbol{F}^{\mathrm{brane}} = \left(\frac{Mm}{4\pi}\right)^{d/2}\sum_{\{k,l,n,p,c\}}\frac{m^{c-n-p}O(\mathfrak{M}^{k+l+p+2n-2c})}{M^{n-c+k+l}} + \frac{M^d}{(4\pi)^{d/2}}\sum_{\{k,l,n,p,c\}}\frac{m^{l-p}O(\mathfrak{M}^{k+l+p+2n-2c})}{M^{2n-2c+k+2l}}. \tag{71}$$

By introducing the summation index $N = n - c + k + l$—an overall power of $1/M$—in the first sum and correspondingly the summation index $L = 2n - 2c + k + 2l$ in the second sum, one can rewrite this series in the form (15) presented in the introduction

$$\frac{1}{2}\,\mathrm{Tr}\ln\boldsymbol{F}^{\mathrm{brane}} = \left(\frac{Mm}{4\pi}\right)^{d/2}\sum_{N=0}^{\infty}\frac{1}{M^N}\sum_{i\le N}\frac{O(\mathfrak{M}^{2N-i})}{m^{N-i}} + \frac{M^d}{(4\pi)^{d/2}}\sum_{L=0}^{\infty}\frac{1}{M^L}\sum_{i\le L}m^i O(\mathfrak{M}^{L-i}), \tag{72}$$

where the coefficient of any power of M turns out to be a *finite* sum of terms of a limited order in $\mathfrak{M}$—the background dimensionality of relevant field invariants.

The finiteness of such sums over $i = k + l - p$ in the first series of (72) follows from the following simple argumentation. The range of summation over $c \le [(2k + 2l + p)/6]$ in (71) is not greater than $[(2k + 3l)/6]$ because $p \le \max\{0, l-2\}$. Therefore,

$$k + l = N + c - n \le N + c \le N + \left[\frac{k}{3} + \frac{l}{2}\right] \le N + \frac{1}{3}k + \frac{1}{2}l,$$

so that $2k/3 + l/2 \le N$, thus only limited number of curvature structures (from W_{kl}) can contribute to term with fixed N. As can be easily seen the ranges of summation over c, p and $i \equiv k + l - p$ indeed turn out to be limited at least by N.

Similarly, the finiteness of sums over $i = l - p$ in the second series of (72) is based on the following chain of inequalities

$$k + 2l = L + 2c - 2n \le L + 2c \le L + \left[\frac{2k}{3} + l\right] \le L + \frac{2k}{3} + l,$$

so that $k/3 + l \le L$, and the ranges of summation over c, p and $i \equiv l - p$ are again restricted from above for any given L.

This property is very important for the efficiency of the perturbation theory with the cutoff scale M, because otherwise any given order in $1/M$ would require an infinite series in $\mathfrak{M}/m$—the price one could have paid for the presence of the second scale m. Fortunately, for any N only a finite order $O(\mathfrak{M}^{2N})$ of perturbation theory is required. This follows from a special asymptotic behavior of the coefficient (70) which brings extra powers of $1/M$ to (65). The form of the asymptotics (70) is responsible for the two series in the expansion (72), having qualitatively different analytic behavior in M and m,—the property recently discovered for the effective potential of the toy DGP model [14]. Whereas the second part is analytic in a small DGP scale $m \to 0$, the first "nonanalytic" part is formally singular in this limit, and this leads to the redefinition of the cutoff M_{cutoff} of the theory, below which $\mathfrak{M} \ll M_{\mathrm{cutoff}}$ the local expansion remains valid.

Indeed, despite the efficiency of the obtained expansion, in the first series of (72) it contains negative powers of the DGP scale m and blows up for small $m \to 0$. This is a typical situation of the presence of a strong-coupling scale [6]. In fact, for $m < M$ the actual cutoff is lower than M and is given by the expression (16), $M_{\mathrm{cutoff}} = \sqrt{Mm}$, presented in Introduction (the condition of smallness of the strongest $i = 0$ term in the first series of (72), $\mathfrak{M}^2/Mm \ll 1$).

The actual calculation of the functions (66) can be done by using the proper time weights (56) and (57). Since these weights imply explicit symmetrization with respect to $m_\pm$, they take the form

$$C^j_{kl} = \tilde{C}^j_{kl}(M, m_+, m_-) + (m_+ \leftrightarrow m_-). \tag{73}$$

In the $O(\mathfrak{M}^2)$-approximation the relevant $\tilde{C}^j_{kl}$ turn out to be

$$\begin{aligned}
&\tilde{C}^j_{00} = \Phi_{(0)}(2j,1), \qquad \tilde{C}^j_{01} = \left(\frac{m}{\triangle}\right)\Phi_{(0)}(2j+1,2), \qquad \tilde{C}^j_{11} = \left(\frac{m}{\triangle}\right)\Phi_{(0)}(2j+2,1),\\
&\tilde{C}^j_{02} = \left(\frac{m}{\triangle}\right)^2\left(-\frac{2M}{\triangle}\Phi_{(0)}(2j+1,2) + \Phi_{(0)}(2j+2,3)\right), \qquad \tilde{C}^j_{21} = 2\left(\frac{m}{\triangle}\right)\Phi_{(1)}(2j+2,1),\\
&\tilde{C}^j_{31} = 2\left(\frac{m}{\triangle}\right)\Phi_{(2)}(2j+2,1), \qquad \tilde{C}^j_{22} = 2\left(\frac{m}{\triangle}\right)^2\left(-\frac{2M}{\triangle}\Phi_{(1)}(2j+2,1) - \Phi_{(2)}(2j+2,1) + \Phi_{(1)}(2j+3,2)\right),\\
&\tilde{C}^j_{51} = 2\left(\frac{m}{\triangle}\right)\Phi_{(4)}(2j+2,1), \qquad \tilde{C}^j_{42} = 2\left(\frac{m}{\triangle}\right)^2\left(-\frac{2M}{\triangle}\Phi_{(3)}(2j+2,1) - 3\Phi_{(4)}(2j+2,1) + \Phi_{(3)}(2j+3,2)\right).
\end{aligned} \tag{74}$$

Here

$$\triangle \equiv \frac{m_+ - m_-}{2}, \tag{75}$$

the basic function $\Phi_{(0)}(a, b)$ is given by the regularized Gauss hypergeometric function

$$\Phi_{(0)}(a,b) = \Phi_{(0)}(a,b|\sigma)|_{\sigma=m_+/2M} \equiv \frac{\Gamma(a-d)\Gamma(b)}{\Gamma(\frac{a+b+1-d}{2})}\,{}_2F_1\Big(a-d, b; \frac{a+b+1-d}{2}; \sigma+\frac{1}{2}\Big)\Big|_{\sigma=m_+/2M} \tag{76}$$

and

$$\Phi_{(n)}(a,b) = \frac{1}{\sigma^n}\Big(\Phi_{(0)}(a,b|\sigma) - \sum_{k=0}^{n-1}\frac{1}{k!}[d^k\Phi_{(0)}(a,b|\sigma)/d\sigma^k]_{\sigma=0}\sigma^k\Big)\Big|_{\sigma=m_+/2M} \tag{77}$$

is the function $\Phi_{(0)}(a, b|\sigma)/\sigma^n$ with the singular at $\sigma = 0$ part subtracted, also taken at $\sigma = m_+/2M$.

The transformation property of the hypergeometric function from the argument z to $1 - z$ allows one to rewrite (76) in the form which reveals the structure of the expansion (72)

$$\begin{aligned}\Phi_{(0)}(a,b) &= \Big(\frac{\varepsilon_+}{2}\Big)^{(d+1-a-b)/2}\Gamma\Big(\frac{a+b-1-d}{2}\Big)\,{}_2F_1\Big(\frac{d+b-a+1}{2}, \frac{a-b+1-d}{2}; \frac{d+3-a-b}{2}; \frac{\varepsilon_+}{2}\Big)\\ &\quad + \frac{\Gamma(a-d)\Gamma(b)\Gamma(\frac{d-a-b+1}{2})}{\Gamma(\frac{d+b-a+1}{2})\Gamma(\frac{a-b+1-d}{2})}\,{}_2F_1\Big(a-d, b; \frac{a+b-1-d}{2}; \frac{\varepsilon_+}{2}\Big),\end{aligned} \tag{78}$$

where $\varepsilon_+ = 1 - m_+/M$ as in (70). In view of $\varepsilon_+ \sim m/M$ the first term here generates the first nonanalytic in m series of (72) and the second term is responsible for the analytic part because the hypergeometric function is expandable in Taylor series in $\varepsilon_+/2 \to 0$. The $\sigma = m_-/2M$ part of the action originating from the second term of (73) contributes only to the analytic part, because $m_-/2M \to -1/2$ and the relevant large M expansion originates directly from the representation (76) which does not give rise to nonanalytic terms.

The brane effective action in the $O(\mathfrak{R}^2)$ approximation and UV divergences

Substituting the curvature invariants of (64) into (65) we get the lowest orders of the brane effective action in terms of the curvature invariants (17)–(19) listed in Introduction,

$$\begin{aligned}\frac{1}{2}\,\mathrm{Tr}\ln\boldsymbol{F}^{\mathrm{brane}} &= -\frac{1}{2}\frac{M^d}{(4\pi)^{d/2}}\int_{\mathbf{b}} dx\sqrt{g}C^0_{00} - \frac{1}{2}\frac{M^{d-1}}{(4\pi)^{d/2}}\int_{\mathbf{b}} dx\sqrt{g}\Big\{\Big(-\frac{1}{2}C^0_{01} + \frac{1}{4}C^{-1}_{21}\Big)k - C^0_{01}\frac{p}{2m}\Big\}\\ &\quad -\frac{1}{2}\frac{M^{d-2}}{(4\pi)^{d/2}}\int_{\mathbf{b}} dx\sqrt{g}\Big\{\frac{1}{6}C^1_{00}{}^{\mathbf{B}}R - \frac{1}{2}C^0_{11}P + \Big(-\frac{1}{3}C^1_{00} + \frac{1}{4}C^0_{11} - \frac{1}{8}C^{-1}_{31}\Big)^{\mathbf{B}}R_{nn}\\ &\quad + \Big(-\frac{1}{6}C^1_{00} + \frac{1}{4}C^0_{11} + \frac{3}{8}C^{-1}_{31} + \frac{5}{16}C^{-2}_{51} + \frac{1}{16}C^{-2}_{42}\Big)k^2_{\alpha\beta}\\ &\quad + \Big(\frac{1}{6}C^1_{00} - \frac{1}{8}C^0_{11} + \frac{1}{8}C^0_{02} - \frac{1}{8}C^{-1}_{22} + \frac{5}{32}C^{-2}_{51} + \frac{1}{32}C^{-2}_{42}\Big)k^2 + \Big(\frac{1}{2}C^0_{02} - \frac{1}{4}C^{-1}_{22}\Big)\frac{kp}{2m} + \frac{1}{2}C^0_{02}\Big(\frac{p}{2m}\Big)^2\Big\}\\ &\quad + O(\mathfrak{R}^3).\end{aligned} \tag{79}$$

Here the coefficient functions C^j_{kl} are given by Eqs. (73) and (74) and represent a set of very complicated functions of M, m and μ. One can check that for $\mu = 0$ the C^0_{00} term given by $C^0_{00} = \Phi_{(0)}(0, 1) + (m_+ \leftrightarrow m_-)$ coincides with the effective potential calculated for a toy DGP model in [14].

More instructive are the ultraviolet divergences of the brane action which we present here for the four-dimensional case in the dimensional regularization $d \to 4$. They read

$$\begin{aligned}\frac{1}{2}\,\mathrm{Tr}\ln\boldsymbol{F}^{\mathrm{brane}}|^{\mathrm{div}} &= \frac{1}{32\pi^2(4-d)}\int_{\mathbf{b}} dx\sqrt{g}(-4m^2(M^2+2m^2-2\mu^2)-\mu^4)\\ &\quad + \frac{1}{32\pi^2(4-d)}\int_{\mathbf{b}} dx\sqrt{g}\Big(\frac{1}{2}m(M^2+12m^2-3\mu^2)k + 2(\mu^2-4m^2)p\Big)\\ &\quad + \frac{1}{32\pi^2(4-d)}\int_{\mathbf{b}} dx\sqrt{g}\Big(\frac{1}{3}(-2m^2+\mu^2)^{\mathbf{B}}R + \Big(\frac{17}{6}m^2 - \frac{2}{3}\mu^2\Big)^{\mathbf{B}}R_{nn} + \Big(\frac{9}{2}m^2 - \frac{1}{3}\mu^2\Big)k^2_{\alpha\beta}\\ &\quad + \Big(-2m^2 + \frac{1}{3}\mu^2\Big)k^2 - 4m^2P - \frac{3}{2}mkp - p^2\Big) + O(\mathfrak{R}^3).\end{aligned} \tag{80}$$

This result confirms the general properties of ultraviolet divergences in any dimension d. These divergences are contained in both series of the expansion (72). For an even d they are analytic and polynomial in both M and m, for an odd d they have a structure $\sqrt{Mm}$ times a finite polynomial in M and m. Finally, their background dimensionality is always bounded by $O(\mathfrak{M}^d)$. These general properties follow from the property of the integral (68) which is UV divergent at the upper limit only for $d \geq 2j + 2l + k$. Therefore, the background dimensionality of the relevant terms in (71) (with $j = n - c$) satisfies the bound $k + l + p + 2n - 2c \leq d + p - l \leq d$, because $p \leq \max\{l - 2, 0\}$. The relevant overall powers of m and M in the nonanalytic part of (71) are also positive, because for the same reasons $p + n - c < d/2$ and $n - c + k + l < d/2$. Finally, in the analytic part of (71) the overall power of M in the divergent terms is again nonnegative, because $2n - 2c + k + 2l \leq d$.

VII. THE NEUMANN AND DIRICHLET LIMITS

As we see, the curvature expansion in brane induced gravity models is essentially more complicated than for pure Dirichlet and Neumann (Robin) boundary conditions. Even the conformity of our results with these two limiting cases [corresponding, respectively, to $m \to 0$ and $m \to \infty$ in (10)] requires nontrivial calculations. Here we present these calculations and check the consistency of the Neumann limit to the $O(\mathfrak{M}^2)$ order in the curvature, and verify the Dirichlet limit to all orders in $\mathfrak{M}$.

First we present the known results for a local inverse mass expansion for pure Dirichlet, $\Phi|_{\mathbf{b}} = 0$, and Robin, $(\partial_n - S)\Phi|_{\mathbf{b}} = 0$, boundary conditions (see [19] and references therein). In these two cases labeled, respectively, by D and N this expansion for the effective action in the $(d+1)$-dimensional bulk reads

$$\frac{1}{2}\mathrm{Tr}^{(d+1)}_{D/N}\ln F = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\frac{1}{(4\pi s)^{(d+1)/2}}e^{-sM^2}\sum_{n=0}^{\infty} s^{n/2}A_n^{D/N}$$
$$= -\frac{1}{2}\frac{M^d}{(4\pi)^{(d+1)/2}}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{-d-1+n}{2})}{M^{n-1}}A_n^{D/N} \quad (81)$$

where $A_n^{D/N}$ represent the bulk and boundary integrals of the relevant Schwinger-DeWitt coefficients. The first four of them for the Dirichlet case read

$$A_0^D = \int_{\mathbf{B}} d^{d+1}X\sqrt{G}, \qquad A_1^D = -\frac{\sqrt{\pi}}{2}\int_{\mathbf{b}} d^dx\sqrt{g},$$
$$A_2^D = \int_{\mathbf{B}} d^{d+1}X\sqrt{G}\Big(-P + \frac{1}{6}{}^{\mathbf{B}}R\Big) + \int_{\mathbf{b}} d^dx\sqrt{g}\Big(-\frac{1}{3}k\Big),$$
$$A_3^D = \sqrt{\pi}\int_{\mathbf{b}} d^dx\sqrt{g}\Big(+\frac{1}{2}P - \frac{1}{12}{}^{\mathbf{B}}R + \frac{1}{24}{}^{\mathbf{B}}R_{nn} - \frac{7}{192}k^2 + \frac{5}{96}k^2_{\alpha\beta}\Big). \quad (82)$$

For the Neumann (Robin) case together with the bulk curvature and the extrinsic curvature of the boundary they involve the coefficient function S from the Robin boundary condition,

$$A_0^N = \int_{\mathbf{B}} d^{d+1}X\sqrt{G}, \qquad A_1^N = \frac{\sqrt{\pi}}{2}\int_{\mathbf{b}} d^dx\sqrt{g},$$
$$A_2^N = \int_{\mathbf{B}} d^{d+1}X\sqrt{G}\Big(-P + \frac{1}{6}{}^{\mathbf{B}}R\Big) + \int_{\mathbf{b}} d^dx\sqrt{g}\Big(-\frac{1}{3}k - 2S\Big),$$
$$A_3^N = \sqrt{\pi}\int_{\mathbf{b}} d^dx\sqrt{g}\Big(-\frac{1}{2}P + \frac{1}{12}{}^{\mathbf{B}}R - \frac{1}{24}{}^{\mathbf{B}}R_{nn} + \frac{13}{192}k^2 + \frac{1}{96}k^2_{\alpha\beta} + \frac{1}{2}kS + S^2\Big). \quad (83)$$

A. The Neumann (Robin) limit

The Robin limit of the boundary condition (10) corresponds to $m \to +\infty$ with $M/m = 0$, $\mu/m = 0$ and the finite limiting value of $p/2m \to S$. This implies the following limits for the auxiliary mass parameters

$$m_+ \to 0, \qquad m_- \to -\infty, \qquad \triangle \to +\infty, \qquad \frac{m}{\triangle} \to 1. \quad (84)$$

Therefore the contribution of $\sigma = m_-/2M \to -\infty$ terms in the coefficients $C^j_{kl}(M, m_\pm)$ given by (73)–(77) vanishes in virtue of the asymptotic behavior of the hypergeometric function ${}_2F_1(a, b; c; z)$ at $z \to -\infty$. Moreover, only the functions $\Phi_{(n)}(a, b)$ with a special combination of the hypergeometric function's indices (76) arise, and due to the relation

$${}_2F_1\Big(a, b; \frac{a}{2} + \frac{b}{2} + \frac{1}{2}; \frac{1}{2}\Big) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{a}{2} + \frac{b}{2} + \frac{1}{2})}{\Gamma(\frac{a}{2} + \frac{1}{2})\Gamma(\frac{b}{2} + \frac{1}{2})}$$

they equal

$$\Phi_{(n)}(a, b) = \frac{\Gamma(\frac{1}{2})}{n!}\frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(\frac{a}{2} + \frac{n}{2} + \frac{1}{2})\Gamma(\frac{b}{2} + \frac{n}{2} + \frac{1}{2})}$$
$$= \frac{2^{a+b+2n-2}}{n!\Gamma(\frac{1}{2})}\Gamma\Big(\frac{a}{2} + \frac{n}{2}\Big)\Gamma\Big(\frac{b}{2} + \frac{n}{2}\Big),$$

where we used the gamma function identity $\Gamma(2z)\Gamma(1/2) = 2^{2z-1}\Gamma(z)\Gamma(z + 1/2)$. As a result the coefficient functions C^j_{kl} reduce to

$$C^j_{00} = \frac{1}{2}\Gamma(j - d/2), \qquad C^j_{kl} = \frac{\Gamma(j + \frac{k+l-d}{2})}{\Gamma(\frac{k+l}{2})}, \qquad k + l > 0, \quad (85)$$

and the expansion (79) for the case of the Neumann boundary conditions takes the form

$$\frac{1}{2}\,\mathrm{Tr}\ln \boldsymbol{F}^{\mathrm{brane}} = -\frac{1}{2}\frac{M^d}{(4\pi)^{d/2}}\sum_{n=0}^{\infty}\frac{\Gamma(-\frac{d}{2} + \frac{n}{2})}{M^n}\mathcal{A}_n, \quad (86)$$

where the first three coefficients equal

$$\mathcal{A}_0 = \frac{1}{2}\int_{\mathbf{b}} d^d x\sqrt{g}, \qquad \mathcal{A}_1 = -\frac{1}{\sqrt{\pi}}\int_{\mathbf{b}} d^d x\sqrt{g}\,\frac{p}{2m},$$

$$\mathcal{A}_2 = \int_{\mathbf{b}} d^d x\sqrt{g}\Big(\frac{1}{12}{}^{\mathbf{B}}R - \frac{1}{24}{}^{\mathbf{B}}R_{nn} - \frac{1}{48}k^2_{\alpha\beta} + \frac{5}{96}k^2 - \frac{1}{2}P + \frac{1}{4}k\Big(\frac{p}{2m}\Big) + \frac{1}{2}\Big(\frac{p}{2m}\Big)^2\Big). \tag{87}$$

The consistency of this result with the Robin case is based on the duality relation (6) which implies the following identities for the coefficients of the Schwinger-DeWitt expansion (81)

$$\mathcal{A}_n = \frac{1}{\sqrt{4\pi}}(A^N_{n+1} - A^D_{n+1}). \tag{88}$$

These identities can be directly checked for (82) and (83) with $S = p/2m$ for $n = 0, 1, 2$.

B. The Dirichlet case

The Dirichlet case can be extracted from the generalized Neumann effective action (6) by taking the opposite limit $m \to 0$. Indeed, the Dirichlet effective action is determined by the path integral

$$\exp\Big(-\frac{1}{2}\,\mathrm{Tr}_D^{(d+1)}\ln F\Big) = \int D\Phi \exp(-S_{\mathbf{B}}[\Phi])\delta(\varphi(x)), \tag{89}$$

where $S_{\mathbf{B}}[\Phi]$ is the bulk part of the action (1) and the expression for the delta function of $\varphi(x) = \Phi(X)|_{\mathbf{B}}$ can be viewed as the factor

$$\delta(\varphi(x)) = \lim_{m\to 0}(\mathrm{Det}\varkappa)^{1/2}\exp\Big(-\frac{1}{2}\int_{\mathbf{b}} d^d x\sqrt{g}\,\varphi\varkappa(\nabla)\varphi\Big) \tag{90}$$

regularized by $m \to 0$ in $\varkappa(\nabla) = (-\Box + \mu^2 + p)/2m$. Comparing with the definition of the generalized Neumann case,

$$\exp\Big(-\frac{1}{2}\,\mathrm{Tr}_N^{(d+1)}\ln F\Big) = \int D\Phi \exp\Big(-S_{\mathbf{B}}[\Phi] - \frac{1}{2}\int_{\mathbf{b}} d^d x\sqrt{g}\,\varphi\varkappa(\nabla)\varphi\Big), \tag{91}$$

one finds that under the $m \to 0$ limit the Neumann-to-Dirichlet reduction (6) leads to

$$\lim_{m\to 0}\tfrac{1}{2}\mathrm{Tr}_N^{(d+1)}\ln F = \tfrac{1}{2}\mathrm{Tr}_D^{(d+1)}\ln F + \tfrac{1}{2}\mathrm{Tr}\ln\varkappa, \tag{92}$$

where the last term originates from the preexponential factor of the regularized delta function, $(\mathrm{Det}\varkappa)^{1/2} = \exp(\mathrm{Tr}\ln\varkappa/2)$. Thus, the brane contribution is nonzero in this limit and coincides with the d-dimensional effective action for the massive operator (9)

$$\lim_{m\to 0}\tfrac{1}{2}\mathrm{Tr}^{(d)}\ln \boldsymbol{F}^{\mathrm{brane}} = \tfrac{1}{2}\mathrm{Tr}\ln\varkappa(\nabla). \tag{93}$$

Within the inverse mass expansion (14) it reads

$$\frac{1}{2}\mathrm{Tr}\ln\varkappa(\nabla) = \frac{1}{2}\mathrm{Tr}\ln(-\Box + \mu^2 + p) = -\frac{1}{2}\frac{\mu^d}{(4\pi)^{d/2}}\sum_{j=0}^{\infty}\frac{\Gamma(j-d/2)}{\mu^{2j}}A_{2j} \tag{94}$$

(we of course disregard the volume divergent $\mathrm{Tr}\ln 2m$ part canceled by the local measure).

To verify (93) one cannot however take the limit $m \to 0$ directly in the expansion (72) because of its obvious nonanalyticity at $m = 0$. The reason is that for small m the behavior (70) of C^j_{kl} used in the original curvature expansion series (65) no longer applies. Indeed, when $mM < \mu^2$ the parameter ε_+ in the integral (68) has another asymptotics $\mu^2/2M^2$ independent of m, and $C^j_{kl} \sim m^l$ for $m \to 0$. Therefore, in this range of m the DGP scale arises in (65) only in positive powers $l - p > 0$, because $p \le l - 2$, and the curvature expansion takes the form qualitatively different from (72). The only nonvanishing term in the sum over k and l is the one with $k = l = 0$ [cf. Eq. (59)]. The relevant coefficients (74) are

$$C^j_{00}(M, m_\pm) = \frac{1}{2}\int_0^\infty \frac{ds}{s}s^\nu(w(-m_+\sqrt{s}) + w(-m_-\sqrt{s}))e^{-sM^2} = \frac{\Gamma(j-d/2)}{\mu^{2j-d}}, \tag{95}$$

because in the Dirichlet limit ${m_+}^2 = {m_-}^2 = M^2 - \mu^2$ and the error functions in the definition of the proper time weight $w(-m_+\sqrt{s})$ satisfy $\mathrm{erfc}(-m_\pm\sqrt{s}) = 1 \pm \mathrm{erf}(\sqrt{M^2-\mu^2}\sqrt{s})$. These are exactly the weights of the integrated Schwinger-DeWitt coefficients $A_{2j} = \int d^d x a_j(x,x)$ in (94), which confirms (93). This relation can be seen already at the level of W-expansion (36). In the limit $m \to 0$ all perturbation terms in (36) with $1/\boldsymbol{F}_0 = 2m/(\mu^2 - \Box)$ vanish except the contribution of $U_1/\boldsymbol{F}_0 = -p/(\mu^2 - \Box)$, which complements this expansion to the one-loop action of the full brane operator with a potential (94).

VIII. CONCLUSIONS

Thus we have constructed the covariant curvature expansion in massive brane induced gravity models, found its peculiar structure (15) nonanalytic in the DGP scale and derived a nontrivial cutoff (16) of this general expansion. Finally, we calculated several lowest orders of this expansion for a quantum scalar field in a curved bulk spacetime with a kinetic term on the brane to a quadratic order in background dimensionality and found its ultraviolet divergences for the case of a 4-dimensional brane.

These results might find important applications. Although a comparison of our massive model with the massless DGP model of [1] is not straightforward, we can observe a common feature in their cutoff properties. In both theories their cutoff (16) is different from the bulk one M and is modified by the DGP scale m. For the tree-level DGP model with the Planck mass $M = M_P$, playing the role of the bulk cutoff, this cutoff equals $M_{\rm cutoff} = (m^2 M)^{1/3}$ [6]. With m identified with the cosmological horizon scale, this is about $(1000\ {\rm km})^{-1}$ which is much below the submillimeter scale capable of featuring the infrared modifications of the Einstein theory [21]. As we see, the situation with the local expansion for the *quantum* action is much better—the cutoff (16) is a geometric average of M and m, which is much higher,

$$(mM)^{1/2} \gg (m^2 M)^{1/3}, \tag{96}$$

and comprises $(0.1\ {\rm mm})^{-1}$. This supports the conjecture [22] that the replacement of the weak field perturbation theory by a derivative expansion, as is the case of the local Schwinger-DeWitt series (probably with the nonperturbative resummation of powers of a local potential term of the operator (8) [23]), might improve the range of validity of the calculational scheme.

Obviously, the Schwinger-DeWitt technique in brane induced gravity models turns out to be much more complicated than in models without spacetime boundaries or in case of boundaries with local Dirichlet and Neumann boundary conditions. It does not reduce to a simple bookkeeping of local surface terms like the one reviewed in [19]. Nevertheless it looks complete and self-contained, because it provides in a systematic way a manifestly covariant calculational procedure for a wide class of boundary conditions including tangential derivatives (in fact of any order). On the other hand, the calculational strategy of the above type requires a further extension, because there is still a large set of issues and possible generalizations to be resolved in concrete problems.

One important generalization is a physically most interesting limit of a vanishing bulk mass M^2, whose rigorous treatment should justify a qualitative comparison of the above type for the cutoff scales in our expansion (72) and the weak field expansion of the DGP model. The local curvature expansion is perfect and nonsingular for nonvanishing M^2 and is applicable within its cutoff scale (16). However, for $M^2 \to 0$ it obviously breaks down, because the proper time integrals start diverging at the upper limit and all UV finite terms of (15) blow up. These infrared divergences can be avoided by a nonlocal curvature expansion of the heat kernel of [24]. Up to the cubic order in curvatures this expansion explicitly exists for ${\rm Tr} e^{s\Box}$ [25], but for the structure involving a local differential operator ${\rm Tr} W(\nabla) e^{s\Box}$ it still has to be developed.

Another important generalization is the extension of these calculations to the cases when already the lowest order approximation involves a curved spacetime background (i.e. dS or AdS bulk geometry, de Sitter rather than flat brane, etc.). The success of the above technique is obviously based on the exact knowledge of the y-dependence in the lowest order Green's function in the bulk and the possibility to perform exactly (or asymptotically for large M^2) the integration over y. All of these generalizations and open issues are currently under study.

To summarize, we developed a new scheme of calculating quantum effective action for the braneworld DGP-type system in curved spacetime. This scheme gives a systematic curvature expansion by means of a manifestly covariant technique. Combined with the method of fixing the background covariant gauge for diffeomorphism invariance developed in [12,26] this gives the universal background field method of the Schwinger-DeWitt type in gravitational brane systems.

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