PHYSICAL REVIEW D **81,** 085010 (2010)

Exact statement for Wilsonian and holographic renormalization group

E. T. Akhmedov* and E. T. Musaev†

B. Cheremushkinskaya, 25, ITEP, 117218, Moscow, Russia,
and Moscow Institute of Physics and Technology, Dolgoprudny, Russia
(Received 22 January 2010; published 9 April 2010)

We show that Polchinski equations in the D-dimensional matrix scalar field theory can be reduced at large N to the Hamiltonian equations in a $(D+1)$-dimensional theory. In the subsector of the $\mathrm{Tr}\phi^l$ (for all l) operators we find the exact form of the corresponding Hamiltonian. The relation to the holographic renormalization group is discussed.

DOI: 10.1103/PhysRevD.81.085010 PACS numbers: 11.10.Hi, 11.15.Tk, 11.25.Tq

I. INTRODUCTION

The Wilsonian renormalization group [1] appears to be a useful tool in the study of various phenomena in quantum field theory and statistical physics. A convenient form of the Wilsonian renormalization group is given by the Polchinski equations [2]. Usually these equations are formulated in the scalar field theory.

Some time ago, after the discovery of the AdS/CFT correspondence [3], it was recognized [4] that renormalization group equations on the conformal field theories side are represented by classical equations of motion on the anti–de Sitter side. The idea was further developed in [5–12]. And it became clear that the holography is the general phenomenon for the proper formulation of the Wilsonian renormalization group at large N [11,12].

In this picture an exact, simple, and easily testable statement was missing. The goal of this paper is to provide such a statement. We show that Polchinski equations in the D-dimensional matrix scalar field theory can be reduced at large N to the Hamiltonian equations in a $(D+1)$-dimensional theory. In the subsector of the $\mathrm{Tr}\phi^l$ (for all l) operators we find the exact form of the corresponding Hamiltonian. In the concluding section the relation to the holographic renormalization group is discussed.

II. FROM RENORMALIZATION GROUP TO THE HAMILTONIAN FLOW

In this section we consider the Euclidean D-dimensional matrix scalar field theory,

$$\mathcal{S}[\phi] = -\frac{N}{2}\int_p \mathrm{Tr}[\phi(p)(p^2+m^2)K_\Lambda^{-1}(p^2)\phi(-p)] + N\mathcal{S}_I[\phi], \tag{1}$$

whose action is written here in the Fourier transformed form. Here $\phi = ||\phi^{ij}||$, $i, j = 1, \ldots, N$, is the Hermitian matrix; the function K_Λ is

$$K_\Lambda(x) = \begin{cases} 1, & \text{when } x < \Lambda^2; \\ 0, & \text{when } x > \Lambda^2, \end{cases} \tag{2}$$

and is quickly changing near the point $x = \Lambda^2$; i.e. Λ is an UV cutoff in our theory; $\mathcal{S}_I$ is the interaction part of the action, which includes sources as well. In this paper we take

$$\mathcal{S}_I[\phi] = \sum_{l=0}^{\infty}\int_{k_1\ldots k_l}\mathrm{Tr}[\phi(k_1),\ldots,\phi(k_l)] \times J_l(-k_1-\cdots-k_l), \tag{3}$$

which is just the Fourier transform of $\sum_{l=0}^{\infty}\int J_l(x)\mathrm{Tr}\phi^l(x)$ representing the subspace of the complete operator product expansion (OPE) basis of the theory. The crucial observation for our further considerations is that in the Fourier transformed form J_l depends only on the sum of k's—arguments of ϕ's under the traces. If we were considering operators containing derivatives (e.g. $\mathrm{Tr}[(\partial_\mu\phi)\phi^l(\partial_\nu\phi)\phi^{n-l}]$), then the corresponding Fourier transformed sources (e.g. $J^n_{\mu\nu}(-\sum k)k_1^\mu k_{l+2}^\nu$) in general would depend on all k's separately.

The Polchinski equation for the theory in question is given in the Appendix [see Eq. (A4)]. Taking the quantum average of it, we arrive at

$$\left\langle \Lambda\frac{d\mathcal{S}_I[\phi]}{d\Lambda}\right\rangle = -\frac{1}{2}\int_p\frac{1}{p^2+m^2}\Lambda\frac{dK_\Lambda(p^2)}{d\Lambda} \times\left\langle\left[N^{-1}\frac{\delta^2\mathcal{S}_I[\phi]}{\delta\phi^{ij}(-p)\delta\phi^{ji}(p)} + \frac{\delta\mathcal{S}_I[\phi]}{\delta\phi^{ij}(p)}\frac{\delta\mathcal{S}_I[\phi]}{\delta\phi^{ji}(-p)}\right]\right\rangle. \tag{4}$$

The average is taken over the high-momentum modes only. It means that one should represent $\phi(p)$ as the sum of the high-momentum and low-momentum modes $\phi(p) = \phi_0(p) + \varphi(p)$ and integrate out the field $\varphi(p)$ [1]. Here $\phi_0(p)$ is the solution of the equations of motion following from the action (1). As we will see below, taking the expectation value in (4) is necessary to close the system of equations for the sources [12].

It is easy to verify the following relations:

*akhmedov@itep.ru
†musaev@itep.ru

$$\mathrm{Tr}\left[\frac{\delta \mathcal{S}_I}{\delta\phi(p)}\frac{\delta \mathcal{S}_I}{\delta\phi(-p)}\right] = \sum_{n,l=0}^{\infty}\int_{p_1\ldots p_n k_1\ldots k_l}(n\cdot l)\mathrm{Tr}[\phi(p_1),\ldots,\phi(p_n)\phi(k_1),\ldots,\phi(k_l)] \times J_\kappa(-p-p_1-\cdots-p_n)J_\kappa(p-k_1-\cdots-k_l);$$

$$\mathrm{Tr}\left[\frac{\delta^2 \mathcal{S}_I}{\delta\phi(p)\delta\phi(-p)}\right] = \sum_{n=0}^{\infty}\int_{p_1\ldots p_n} n\sum_{m=0}^{n}\mathrm{Tr}[\phi(p_1),\ldots,\phi(p_m)]\mathrm{Tr}[\phi(p_{m+1}),\ldots,\phi(p_n)]J_\kappa(-p_1-\cdots-p_n). \quad (5)$$

Now one should calculate the quantum average of these traces. As usual this is a complicated problem, but there is a way to simplify the final expressions.

First, let us introduce the following notations:

$$\int_{p_{(n)}} := \int_{p_1\ldots p_n};$$
$$T_n(p_1,\ldots,p_n) := \mathrm{Tr}[\phi_0(p_1),\ldots,\phi_0(p_n)];$$
$$J_l(-k_{(l)}) := J_l(-k_1-\cdots-k_l). \quad (6)$$

In these notations the action (3) takes a short form $\mathcal{S}_I[\phi_0] = \sum_{l=0}^{\infty}\int_{k_{(l)}} T_l(k_1,\ldots,k_l)J_l(-k_{(l)})$. The quantum average of the trace $\mathrm{Tr}[(\phi_{01}+\varphi_1)\ldots(\phi_{0n}+\varphi_n)]$ over the high-momentum modes can be reduced to the action of some operator on $T_l(k_1,\ldots,k_l)$:

$$\left\langle\int_{p_{(n)}}\mathrm{Tr}[(\phi_0(p_1)+\varphi(p_1)),\ldots,(\phi_0(p_n)+\varphi(p_n))]\right\rangle$$
$$\equiv \int_{p_{(n)}}\int\mathcal{D}\varphi e^{S_0}\mathrm{Tr}[(\phi_0(p_1)+\varphi(p_1)),\ldots,(\phi_0(p_n)+\varphi(p_n))]$$
$$= \int_{p_{(n)}}\int\mathcal{D}\varphi e^{S_0}\exp\left[\int_p \varphi_p\frac{\delta}{\delta\phi_0(p)}\right]\mathrm{Tr}[\phi_0(p_1),\ldots,\phi_0(p_n)]$$
$$= \hat{W}\left\{\int_{p_{(n)}}\mathrm{Tr}[\phi_{01},\ldots,\phi_{0n}]\right\}$$
$$= \hat{W}\left[\int_{p_{(n)}}T_n(p_1,\ldots,p_n)\right],$$

where $S_0 = -\frac{N}{2}\int_p \mathrm{Tr}[\phi(p)(p^2+m^2)K_\Lambda^{-1}(p^2)\phi(-p)]$ and

$$\hat{W} = \exp\left(\frac{1}{2N}\int_p \mathrm{Tr}\left[\frac{\delta}{\delta\phi_{0p}}\frac{\delta}{\delta\phi_{0-p}}\right]G_\Lambda(p)\right) \quad (7)$$

with $G_\Lambda(p) = K_\Lambda(p^2)/(p^2+m^2)$ being the free propagator.[1]

We work in the large N limit, where the following factorization property is in effect:

$$\left\langle\prod_n \mathrm{Tr}O_n\right\rangle = \prod_n\langle\mathrm{Tr}O_n\rangle. \quad (8)$$

Using this property and the notation $\tilde{T} = \hat{W}T$, we can write

$$\hat{W}[T_l(k_1,\ldots,k_l)T_n(p_1,\ldots,p_n)]$$
$$= \hat{W}[T_l(k_1,\ldots,k_l)]\hat{W}[T_n(p_1,\ldots,p_n)]$$
$$= \tilde{T}_l(k_1,\ldots,k_l)\tilde{T}_n(p_1,\ldots,p_n), \quad (9)$$

and the Polchinski equation for the theory (1) acquires the form

$$\sum_{l=1}^{\infty}\int_{k_{(l)}}\tilde{T}_l(\{k_l\})\dot{J}_l(-k_{(l)}) = -\frac{1}{2}\int_p\frac{1}{p^2+m^2}\dot{K}_\Lambda(p^2)\Bigg[N^{-1}\sum_{a=1}^{\infty}\sum_{s=0}^{a-1}\int_{k_{(a-1)}}(a+1)\tilde{T}_{a-s-1}(\{k_{s+1}\})\tilde{T}_s(\{k_s\})J_{a+1}(-k_{(a+1)}) + \sum_{l,j=1}^{\infty}\int_{q_{(j-1)}k_{(l-1)}}(l\cdot j)\tilde{T}_{l+j-2}(\{k_{l-1}\},\{q_{j-1}\})J_l(-k_{(l-1)}-p)J_j(-q_{(j-1)}+p)\Bigg], \quad (10)$$

where the overdot means the differentiation with respect to $d/d\log\Lambda$. Note that $\tilde{T}$ depends on Λ, because the $\hat{W}$ operator does depend on the cutoff.

[1] Using the equation (see e.g. [13])

$$\mathrm{Tr}\left[\frac{\delta}{\delta\phi_{0_p}}\frac{\delta}{\delta\phi_{0_{-p}}}\right] = \int_{k_{(n)}}\sum_{l,m=1}^{\infty}(l\cdot m)T_{l+m-2}(\{k_{l-1}\},\{q_{m-1}\})\frac{\delta^2}{\delta T_l(\{k_{l-1}\},p)\delta T_m(\{q_{m-1}\},-p)} + \int_{k_{(l-2)}}\sum_{l=2}^{\infty}\sum_{m=0}^{l-2}T_m(\{k\})T_{l-m-2}(\{k\}) \times\frac{\delta}{\delta T_l(\{k_{l-2}\},p,-p)},$$

the operator $\hat{W}$ can be written in terms of derivatives with respect to the natural variables $T_n(k_1,\ldots,k_n)$, which makes it obvious that such actions as (3) (with single-trace operators only) give rise to multitrace operators in the Polchinski equation.

It will become clear in a moment that the structure of the theory in question suggests the introduction of the momentum conjugate to $J_l(k)$ as follows:

$$\Pi_l(k) = N^{-1} \int_{k_{(l)}} \delta^{(D)}[k - k_{(l)}] \tilde{T}_l(k_1, \ldots, k_l). \qquad (11)$$

This definition reflects the fact that our sources depend only on the sum of the arguments of $\tilde{T}$'s. And the factor of N was included to make the sources J_l and the canonical momenta Π_l to be of the same order as $N \to \infty$. In these variables Eq. (10) reduces to

$$\begin{aligned}
&\int_q \sum_{l=0}^{\infty} \Pi_l(q) \dot{J}_l(-q) \\
&= -\frac{1}{2} \int_p \frac{\dot{K}_\Lambda(p^2)}{p^2 + m^2} \bigg[\int_{q_1 q_2} \sum_{l,s=0}^{\infty} (l + s + 2) \Pi_l(q_1) \\
&\quad \times \Pi_s(q_2) J_{l+s+2}(-q_1 - q_2) + \int_{q_1 q_2} \sum_{f,h=1}^{\infty} (f \cdot h) \\
&\quad \times \Pi_{f+h-2}(q_1 + q_2) J_f(-q_1) J_h(-q_2) \bigg]. \qquad (12)
\end{aligned}$$

Then the corresponding equations for the sources and for the momenta can be represented in the form of the Hamiltonian equations:

$$\frac{dJ_l(-q)}{dT} = \frac{\delta H}{\delta \Pi_l(q)}, \qquad \frac{d\Pi_l(q)}{dT} = -\frac{\delta H}{\delta J_l(-q)}, \qquad (13)$$

where the "time" $dT = d\log\Lambda \int_p \frac{\dot{K}_\Lambda(p^2)}{p^2+m^2}$ is related to the cutoff scale. The first equation in (13) follows from the Polchinski equation[2] (12). Recall that this equation imposes the condition that the functional integral of the theory in question is independent of the cutoff. The second equation in (13) similarly follows from the derivation of the vacuum expectation value (VEV) $\langle \mathrm{Tr}\phi^l(x) \rangle$ with respect to Λ. Or it may be obtained via the variation of the Polchinski equation with respect to $J_k(-q)$. The easiest way to see the latter fact is to recall that the effective actions expressed through the sources and through the VEV's are related to each other via the Legendre (functional Fourier) transformation [12].

The Hamiltonian can be calculated exactly and has a remarkably simple form:

$$\begin{aligned}
H = -\frac{1}{2} \int_{q_1 q_2} \sum_{l,s=0}^{\infty} &[(l + s + 2) \Pi_l(q_1) \\
&\times \Pi_s(q_2) J_{l+s+2}(-q_1 - q_2) + (l + 1)(s + 1) \\
&\times \Pi_{l+s}(q_1 + q_2) J_{l+1}(-q_1) J_{s+1}(-q_2)]. \qquad (14)
\end{aligned}$$

We emphasize that the momentum $\Pi_l(q)$ contains all powers of the traces since it is the result of the action of the $\hat{W}$ operator on $T_l(k)$.

The trivial observation here is that the summation over s and l in the Hamiltonian in question can be converted into the integration over the new artificial coordinate $\sigma \in [0, \pi]$. Such a conversion obviously could have been done in the original action (1)–(3).

III. DISCUSSION

Thus, we have managed to rewrite the Polchinski equations for the matrix scalar field theory at large N as the Hamiltonian equations. The configuration space of the obtained $(D + 1)$-dimensional theory consists of the single trace operators of the original D-dimensional theory. Our result does not depend on whether the theory in question is renormalizable, or even whether it is UV divergent.

Why is such a relation important? First of all, it clearly shows that if one keeps all sources for a subsector of the full OPE basis, then the renormalization group becomes holographic. Indeed, knowing the values of J's and Π's at some energy scale, one can find them, through the Hamiltonian equations, at any other scale.

Furthermore, despite the fact that we average over the Gaussian quadratic part of the action in the transformation from (4) to (13) and (14) we still have the complete knowledge of the renormalization group flow in the subsector of the theory in question. In particular, if one would like to know e.g. where the UV theory

$$\begin{aligned}
\mathcal{S}[\phi] = &-\frac{N}{2} \int_p \mathrm{Tr}[\phi(p)(p^2 + m^2) K_\Lambda^{-1}(p^2) \phi(-p)] \\
&- gN \int_{k_1,\ldots,k_4} \delta\bigg(\sum_{i=1}^{4} k_i\bigg) \mathrm{Tr}[\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)] \qquad (15)
\end{aligned}$$

($g = \mathrm{const}$) flows under the renormalization group, one just has to solve the Hamiltonian equations (13) with the initial conditions $J_4 = g$ and $J_n = 0$ for all $n \neq 4$ as $\Lambda \to \infty$.

Fortunately enough the subsector of the OPE basis, which we are considering in this paper, factors (at large N) under the renormalization group flow from the rest of the OPE basis. So far we did not find the closed form Hamiltonian if the other parts of the OPE basis are included in the renormalization group dynamics. This remains a challenge for future work. As well, it would have been interesting to reconcile our observations with the

[2]Note that to make the transformation from (12) to the first equation in (13) legal, one has to extend the collection of couplings $J_l \mathrm{Tr}\phi^l$ to the full OPE basis in the theory and then perform the same transformations as we did to arrive at (13) [12]. At the end one has to put all the additional sources to zero to obtain (13) in its present form.

information theory interpretation of the renormalization group flow [14].

ACKNOWLEDGMENTS

We would like to thank A. Mironov and A. Morozov for the valuable discussions. A. E. T. would like to specially thank A. Gerasimov for sharing his ideas and for the collaboration and initiation of this project. A. E. T. would like to thank MPI–AEI, Golm, where this work was completed, for the hospitality.

APPENDIX

Here we derive the Polchinski equation [2] in the case of the D-dimensional matrix scalar field theory (1). In the Wilsonian renormalization group one integrates out the high-momentum modes in ϕ so that the energy scale is reduced from the cutoff to a much lower scale Λ_R—the scale where we are probing our physics [1]. We assume that $m^2 \ll \Lambda_R^2$.

Consider generating a functional for the theory (1):

$$\mathcal{Z} = \int \mathcal{D}\phi e^{\mathcal{S}[\phi,\Lambda,\{J\}]}. \tag{A1}$$

Obviously one has to impose the condition that physics should not depend on the cutoff:

$$\Lambda \frac{d\mathcal{Z}}{d\Lambda} = 0. \tag{A2}$$

The result of the differentiation is

$$\Lambda \frac{d\mathcal{Z}}{d\Lambda} = \int \mathcal{D}\phi e^{\mathcal{S}[\phi,\Lambda,\{J\}]} \mathrm{Tr}\Big[\int_p \phi(-p)(p^2+m^2) \times \phi(p)\Lambda \frac{dK_\Lambda^{-1}(p^2)}{d\Lambda} + \Lambda \frac{d\mathcal{S}_I}{d\Lambda}\Big]. \tag{A3}$$

It is easy to verify that the expression under the integral on the right-hand side of (A3) becomes a full functional derivative if

$$\Lambda \frac{d\mathcal{S}_I}{d\Lambda} = -\frac{1}{2}\int_p \frac{1}{p^2+m^2}\Lambda \frac{dK_\Lambda(p^2)}{d\Lambda}\Big(\frac{\delta^2 \mathcal{S}_I}{\delta\phi^{ij}(-p)\delta\phi^{ji}(p)} + \frac{\delta\mathcal{S}_I}{\delta\phi^{ij}(p)}\frac{\delta\mathcal{S}_I}{\delta\phi^{ji}(-p)}\Big). \tag{A4}$$

Indeed substituting (A4) into (A3) we obtain

$$\Lambda \frac{d\mathcal{Z}}{d\Lambda} = \int_p \Lambda \frac{dK_\Lambda(p^2)}{d\Lambda} \int \mathcal{D}\phi \,\mathrm{Tr}\frac{\delta}{\delta\phi}\Big[\Big(\phi(p)K_\Lambda^{-1}(p^2) + \frac{1}{2}(p^2+m^2)^{-1}\frac{\delta}{\delta\phi}\Big)e^{\mathcal{S}[\phi,\Lambda,\{J\}]}\Big]. \tag{A5}$$

It is straightforward to see that (A5) and (A3) are equivalent because

$$\frac{\delta e^{\mathcal{S}[\phi,\Lambda]}}{\delta\phi(-p)} = \Big(-\phi(p)(p^2+m^2)K_\Lambda^{-1}(p^2) + \frac{\delta\mathcal{S}_I}{\delta\phi}\Big) \times e^{\mathcal{S}[\phi,\Lambda,\{J\}]};$$
$$\frac{\delta^2 e^{\mathcal{S}[\phi,\Lambda]}}{\delta\phi(-p)\delta\phi(p)} = \Big[-(p^2+m^2)K_\Lambda^{-1}(p^2) + \frac{\mathcal{S}_I}{\delta\phi(-p)\delta\phi(p)} + \Big(-\phi(p^2+m^2)K_\Lambda^{-1}(p^2) + \frac{\delta\mathcal{S}_I}{\delta\phi}\Big)^2\Big] \times e^{\mathcal{S}[\phi,\Lambda,\{J\}]}. \tag{A6}$$

Equation (A4) is referred to as the Polchinski equation.

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