

Non-Abelian confinement in $\mathcal{N} = 2$ supersymmetric QCD: Duality and kinks on confining strings

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Recently we observed a crossover transition (in the Fayet-Iliopoulos parameter) from weak to strong coupling in $\mathcal{N} = 2$ supersymmetric QCD with the $U(N)$ gauge group and $N_f > N$ quark flavors. At strong coupling this theory can be described by a dual non-Abelian weakly coupled supersymmetric QCD with the dual gauge group $U(N_f - N)$ and N_f light dyon flavors. Both theories support non-Abelian strings. We continue the study of confinement dynamics in these theories, in particular, metamorphoses of excitation spectra, from a different side. A number of results obtained previously are explained, enhanced and supplemented by analyzing the world-sheet dynamics on the non-Abelian confining strings. The world-sheet theory is the two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric weighted $CP(N_f - 1)$ model. We explore the vacuum structure and kinks on the world sheet, corresponding to confined monopoles in the bulk theory. We show that (in the equal quark mass limit) these kinks fall into the fundamental representation of the unbroken global $SU(N) \times SU(N_f - N) \times U(1)$ group. This result confirms the presence of extra stringy meson states in the adjoint representation of the global group in the bulk theory. The non-Abelian bulk duality is in one-to-one correspondence with a duality taking place in the $\mathcal{N} = (2, 2)$ supersymmetric weighted $CP(N_f - 1)$ model.

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I. INTRODUCTION

The standard scenario for color confinement suggested in the 1970s by Nambu, Mandelstam and 't Hooft [1] is based on the dual Meissner effect. In this scenario, upon condensation of monopoles, chromoelectric flux tubes (strings) of the Abrikosov-Nielsen-Olesen (ANO) type [2] are formed. This must lead to confinement of quarks attached to the endpoints of confining strings.

Much later it was shown by Seiberg and Witten [3,4] that this scenario is indeed realized in $\mathcal{N} = 2$ supersymmetric QCD in the monopole vacua. A more careful examination shows, however, that this confinement is Abelian¹ [6–10]. The reason is that the non-Abelian gauge group of underlying $\mathcal{N} = 2$ SQCD (say, $SU(N)$) is broken down to an Abelian $U(1)^{N-1}$ subgroup by condensation of adjoint scalars in the strongly coupled monopole vacua. Further condensation of monopoles occurs essentially in the $U(1)$ theory.

A non-Abelian mechanism for confinement in four dimensions was recently proposed in [11]. In this paper we considered $\mathcal{N} = 2$ SQCD with the $U(N)$ gauge group and N_f flavors of fundamental quark hypermultiplets, $N <$

$N_f < 2N$. This theory is endowed with the Fayet-Iliopoulos (FI) [12] term ξ which singles out a vacuum in which $r = N$ scalar quarks condense. At large ξ this theory is at weak coupling. In the limit of equal quark masses it supports non-Abelian strings [13–16] (see also the review papers [17–20]). Formation of these strings leads to confinement of monopoles. In fact, in the Higgsed $U(N)$ gauge theories the monopoles become junctions of two distinct elementary non-Abelian strings.

In [11] (see also [21,22]) we demonstrated that upon reducing the FI parameter ξ the theory goes through a crossover transition into a strongly coupled phase which can be described in the infrared in terms of weakly coupled dual $\mathcal{N} = 2$ SQCD with the $U(\tilde{N})$ gauge group and N_f flavors of light dyons,² where

$$\tilde{N} = N_f - N. \quad (1.1)$$

This non-Abelian $\mathcal{N} = 2$ duality is conceptually similar to Seiberg's duality in $\mathcal{N} = 1$ SQCD [24,25] where the emergence of the dual $SU(\tilde{N})$ group was first observed.

The dual theory also supports non-Abelian strings formed due to condensation of light dyons. Moreover, these latter strings still confine monopoles, rather than quarks [11]. Thus, the $\mathcal{N} = 2$ non-Abelian duality is *not* the electromagnetic duality. It is the monopoles that are confined both in the original and dual theories. The reason

¹By non-Abelian confinement we mean such a dynamical regime in which at distances of the flux tube formation all gauge bosons are equally important, while Abelian confinement occurs when the relevant gauge dynamics at such distances is Abelian. Note that Abelian confinement can take place in non-Abelian theories. The Seiberg-Witten solution is just one example. Another example is the Polyakov three-dimensional confinement [5] in the Georgi-Glashow model.

²This is in a perfect agreement with the results obtained in [23] where the $SU(\tilde{N})$ dual gauge group was identified at the root of a baryonic Higgs branch in the $SU(N)$ gauge theory with massless (s)quarks.

for this is as follows. Light dyons which condense in the dual theory (in addition to magnetic charges) carry weight-like electric charges, (i.e. the quark charges). Therefore, the strings formed through the condensation of these dyons can confine only the states with the rootlike magnetic charges, i.e. the monopoles, see [11] for more details.

Our mechanism of non-Abelian confinement works as follows. There is no confinement of color-electric charges, to begin with. The color-electric charges of quarks (or gauge bosons) are Higgs-screened. In the domain of small ξ (where the dual description is applicable) the quarks and gauge bosons of the original theory decay into the monopole-antimonopole pairs at the curves of marginal stability (CMS). At small but nonvanishing ξ the (anti) monopoles forming the pair cannot abandon each other because they are confined. In other words, the original quarks and gauge bosons evolve in the strong-coupling domain of small ξ into “stringy mesons” with two constituents being connected by two strings as shown in Fig. 1, see [19] for a detailed discussion of these stringy mesons. The color-magnetic charges are confined in the theory under consideration; the mesons they form are expected to lie on Regge trajectories.

One might think that this pattern of confinement has little to do with what we have in real life because the real-world mesons can be in the adjoint representation of the global flavor group, while the monopoles we discuss at first glance seem to be neutral under the flavor group. If so, the monopole-antimonopole pairs could produce only flavor-singlet mesons. In this paper we address this problem and show that this naive guess is incorrect. We demonstrate that deep in the non-Abelian quantum regime the confined monopoles are in the *fundamental representation* of the global group. Therefore, the monopole-antimonopole mesons can be both in the adjoint and singlet representation of the flavor group.

If in Ref. [11] we explored the non-Abelian duality and the evolution of spectra in the ξ transition from the standpoint of the four-dimensional bulk theories, in this work we will invoke a totally different and seemingly quite powerful tool. Our strategy is to expand and supplement the bulk theory analysis [11] by studies in the world-sheet theories on the non-Abelian strings supported by the bulk theory. As we know from our previous work, in the BPS sector there should exist a one-to-one correspondence between the bulk and world-sheet results. Therefore, metamorpho-

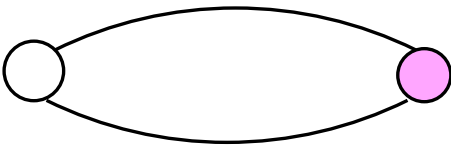


FIG. 1 (color online). Meson formed by monopole-antimonopole pair connected by two strings. Open and closed circles denote the monopole and antimonopole, respectively.

ses of spectra can be studied in the world-sheet theory, providing us with additional information. In particular, the latter should undergo its own crossover transition into a dual world-sheet theory. The obvious strategical advantage is a relative simplicity of two-dimensional theories compared to their four-dimensional progenitor.

The main feature of the non-Abelian strings is the presence of orientational zero modes. Dynamics of these orientational zero modes (uplifted to two-dimensional fields) can be described, at low energies, by an effective two-dimensional sigma model on the string world sheet. Particular details of this model depend on the bulk parameters. For instance, in the simplest case of $\mathcal{N} = 2$ SQCD with the $U(N)$ gauge group and $N_f = N$ (s)quarks, one obtains the $\mathcal{N} = (2, 2)$ supersymmetric $CP(N - 1)$ model on the world sheet [13–16]. If $N_f > N$ it is the weighted $CP(N_f - 1)$ model that we get.

In our previous works we revealed a number of “protected” quantities, such as the mass of the (confined) monopoles. These parameters are calculable both in the bulk theory and on the world sheet, with one and the same result. The first example of this remarkable correspondence was the explanation of the coincidence of the BPS spectrum of monopoles in four-dimensional theory in the $r = N$ vacuum on the Coulomb branch at $\xi = 0$ (given by the exact Seiberg-Witten solution [4]) with the BPS spectrum of kinks in the $\mathcal{N} = (2, 2)$ supersymmetric $CP(N - 1)$ model. This coincidence was noted in [26,27]. The explanation [15,16] is: (i) the confined monopoles of the bulk theory (represented by two-string junctions) are seen as kinks interpolating between two different vacua in the sigma model on the string world sheet; (ii) the masses of the BPS monopoles cannot depend on the nonholomorphic parameter ξ .

Later on various deformations of the bulk theory were considered and their responses in the sigma model on the string world sheet were found, for reviews see [19,20]. In all cases the bulk physics is “projected” onto the world-sheet physics. The protected quantities come out the same. However, technical/calculational aspects are much simpler in the two-dimensional world-sheet theory than in the four-dimensional bulk theory. Therefore, it is beneficial to use the above correspondence not only in the direction from four to two dimensions (the preferred direction in the past), but in the opposite direction too. In the present paper we exploit this idea and study confined monopoles of the bulk theory in terms of kinks of the effective theory on the string world sheet. Two-dimensional CP models are well understood even at strong coupling. We use this knowledge to extract information on confined monopoles of the bulk theory.

If $N_f > N$ the non-Abelian strings are semilocal (see [28] for a review), and the effective sigma model on their world sheet is the $\mathcal{N} = (2, 2)$ weighted $CP(N_f - 1)$ model on a toric manifold [13,29,30]. The bulk

duality observed in [11] must be in one-to-one correspondence with a two-dimensional duality in the weighted CP model.

At large ξ , internal dynamics of the semilocal non-Abelian strings is described by the sigma model of N orientational and \tilde{N} size moduli, while at small ξ the roles of orientational and size moduli interchange. Two dual weighted CP models transform into each other upon changing the sign of the coupling constant [11]. The BPS kink spectra in these two dual sigma models (describing the confined monopoles of the bulk theory) coincide.

In this paper we study the kinks in the weighted CP model in detail, calculate their spectra and show that, in the equal-mass limit of strong coupling (inside CMS), the kinks fall in fundamental representation of the global symmetry group (aka flavor group). This confirms our picture of confinement in the bulk theory. In particular, the mesons shown in Fig. 1 belong either to the adjoint or to singlet representation of the flavor group, as was expected.

The fact that the kinks in the quantum limit form a fundamental representation of the global group is not that surprising. Say, in the $\mathcal{N} = (2, 2)$ supersymmetric $CP(N-1)$ models it was known for a long time [31,32]. Here we generalize this result to the case of the $\mathcal{N} = (2, 2)$ weighted CP models and translate it in terms of the confined monopoles of the bulk theory.

The paper is organized as follows. In Sec. II we briefly review duality in the bulk four-dimensional theory and discuss evolution of excitation spectra in passing from one side of duality to another, through the crossover domain. Section III is devoted to the world-sheet theory on the non-Abelian strings—the weighted CP model. We outline its two versions related to each other by the world-sheet duality. In Sec. IV we treat the (semi)classical limit of the world-sheet theory, examining both components of the dual pair. In Sec. V we consider the exact superpotential and the semiclassical BPS spectrum at large and intermediate values of $|\Delta m_{A,B}|$, i.e. outside CMS. In Sec. VI we formulate and explore a mirror representation for both dual theories. In Sec. VII we study kinks using the mirror representation. We calculate their spectra and count the number of kinks at strong coupling, inside CMS. In Sec. VIII we translate our two-dimensional results in four-dimensional bulk theory, i.e. interpret them in terms of strings and confined monopoles of the bulk theory. Section IX summarizes our conclusions.

II. BULK DUALITY

This section presents a brief review of the bulk non-Abelian duality [11] and introduces all relevant notation (which is also summarized in [19]). The bulk theory is $\mathcal{N} = 2$ SQCD with the $U(N)$ gauge group and N_f flavors of fundamental quark hypermultiplets ($N < N_f < 2N$).

A. Bulk theory at large ξ

The field content is as follows. The $\mathcal{N} = 2$ vector multiplet consists of the $U(1)$ gauge field A_μ and the $SU(N)$ gauge field A_μ^a , where $a = 1, \dots, N^2 - 1$, and their Weyl fermion superpartners plus complex scalar fields a , and a^a and their Weyl superpartners. The N_f quark multiplets of the $U(N)$ theory consist of the complex scalar fields q^{kA} and \tilde{q}_{Ak} (squarks) and their fermion superpartners, all in the fundamental representation of the $SU(N)$ gauge group. Here $k = 1, \dots, N$ is the color index while A is the flavor index, $A = 1, \dots, N_f$. We will treat q^{kA} as a rectangular matrix with N rows and N_f columns.

This theory is endowed with the FI term ξ which singles out the vacuum in which $r = N$ squarks condense. Consider, say, the $(1, 2, \dots, N)$ vacuum in which the first N flavors develop vacuum expectation values (VEVs),

$$\begin{aligned} \langle q^{kA} \rangle &= \sqrt{\xi} \begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix}, \\ \langle \tilde{q}^{kA} \rangle &= 0, \quad k = 1, \dots, N, \quad A = 1, \dots, N_f. \end{aligned} \quad (2.1)$$

In this vacuum the adjoint fields also develop VEVs, namely,

$$\left\langle \left(\frac{1}{2} a + T^a a^a \right) \right\rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & m_N \end{pmatrix}, \quad (2.2)$$

where m_A are quark mass parameters.

For generic values of m_A 's, the VEVs (2.2) break the $SU(N)$ subgroup of the gauge group down to $U(1)^{N-1}$. However, in the special limit

$$m_1 = m_2 = \dots = m_{N_f}, \quad (2.3)$$

the $SU(N) \times U(1)$ gauge group remains unbroken by the adjoint field. In this limit the theory acquires a global flavor $SU(N_f)$ symmetry.

While the adjoint VEVs do not break the $SU(N) \times U(1)$ gauge group in the limit (2.3), the quark condensate (2.1) results in the spontaneous breaking of both gauge and flavor symmetries. A diagonal global $SU(N)$ combining the gauge $SU(N)$ and an $SU(N)$ subgroup (which rotates first N quarks) of the flavor $SU(N_f)$ group survives, however. Below we will refer to this diagonal global symmetry as to $SU(N)_{C+F}$. More exactly, the pattern of breaking of the color and flavor symmetry is as follows:

$$\begin{aligned} U(N)_{\text{gauge}} \times SU(N_f)_{\text{flavor}} &\rightarrow SU(N)_{C+F} \times SU(\tilde{N})_F \\ &\times U(1), \end{aligned} \quad (2.4)$$

where $\tilde{N} = N_f - N$. The phenomenon of color-flavor locking takes place in the vacuum. The global $SU(N)_{C+F}$ group is responsible for formation of the non-Abelian strings (see below). For unequal quark masses in (2.2) the global symmetry (2.4) is broken down to $U(1)^{N_f-1}$.

Since the global (flavor) $SU(N_f)$ group is broken by the quark VEVs anyway we can consider the following mass splitting:

$$m_P = m_{P'}, \quad m_K = m_{K'}, \quad m_P - m_K = \Delta m \quad (2.5)$$

where $P, P' = 1, \dots, N$ and $K, K' = N + 1, \dots, N_f$.³ This mass splitting respects the global group (2.4) in the $(1, 2, \dots, N)$ vacuum. This vacuum then becomes isolated. No Higgs branch develops. We will often use this limit below.

Now let us discuss the mass spectrum in our theory. Since both $U(1)$ and $SU(N)$ gauge groups are broken by squark condensation, all gauge bosons become massive. In fact, at nonvanishing ξ , both the quarks and adjoint scalars combine with the gauge bosons to form long $\mathcal{N} = 2$ supermultiplets [9], for a review see [19]. Note that all states come in representations of the unbroken global group (2.4), namely, the singlet and adjoint representations of $SU(N)_{C+F}$

$$(1, 1), \quad (N^2 - 1, 1), \quad (2.6)$$

and in the bifundamental representations

$$(\bar{N}, \bar{N}), \quad (N, \bar{N}), \quad (2.7)$$

where in (2.6) and (2.7) we mark representation with respect to two non-Abelian factors in (2.4). The singlet and adjoint fields are the gauge bosons, and the first N flavors of the squarks q^{kP} ($P = 1, \dots, N$), together with their fermion superpartners. The bifundamental fields are the quarks q^{kK} with $K = N + 1, \dots, N_f$. These quarks transform in the two-index representations of the global group (2.4) due to the color-flavor locking.

At large ξ this theory is at weak coupling. Namely, the condition

$$\xi \gg \Lambda, \quad (2.8)$$

ensures weak coupling in the $SU(N)$ sector because the $SU(N)$ gauge coupling does not run below the scale of the quark VEVs which is determined by $\sqrt{\xi}$. Here Λ is the dynamical scale of the $SU(N)$ gauge theory. More explicitly,

$$\frac{8\pi^2}{g_2^2(\xi)} = (N - \tilde{N}) \ln \frac{g_2 \sqrt{\xi}}{\Lambda} \gg 1, \quad (2.9)$$

where g_2^2 is the coupling constant of the $SU(N)$ sector.

³A generic mass difference $m_A - m_B$ (for all $A, B = 1, 2, \dots, N_f$) will be referred to as Δm_{AB} below, while Δm is reserved for $m_P - m_K$, ($P = 1, 2, \dots, N$, $K = N + 1, \dots, N_f$). In Ref. [22] the mass differences inside the first group (or inside the second group) were called ΔM_{inside} . The mass differences $m_P - m_K$ were referred to as $\Delta M_{\text{outside}}$.

B. Duality

As was shown in [11], at $\sqrt{\xi} \sim \Lambda$ the theory goes through a crossover transition to the strong-coupling regime. At small ξ ($\sqrt{\xi} \ll \Lambda$) this regime can be described in terms of weakly coupled dual $\mathcal{N} = 2$ SQCD, with the gauge group

$$U(\tilde{N}) \times U(1)^{N-\tilde{N}}, \quad (2.10)$$

and N_f flavors of light *dyons*. This non-Abelian $\mathcal{N} = 2$ duality is similar to Seiberg's duality in $\mathcal{N} = 1$ supersymmetric QCD [24,25]. Later a dual non-Abelian gauge group $SU(\tilde{N})$ was identified on the Coulomb branch at the root of a baryonic Higgs branch in the $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory with massless quarks [23].

Light dyons are in the fundamental representation of the gauge group $U(\tilde{N})$ and are charged under Abelian factors in (2.10). In addition, there are light dyons D^l ($l = \tilde{N} + 1, \dots, N$) neutral under the $U(\tilde{N})$ group, but charged under the $U(1)$ factors. A small but nonvanishing ξ triggers condensation of all these dyons,

$$\begin{aligned} \langle D^{lA} \rangle &= \sqrt{\xi} \begin{pmatrix} 0 & \dots & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix}, \\ \langle \bar{D}^{lA} \rangle &= 0, \quad l = 1, \dots, \tilde{N}, \quad \langle D^l \rangle = \sqrt{\xi}, \\ \langle \bar{D}^l \rangle &= 0, \quad l = \tilde{N} + 1, \dots, N. \end{aligned} \quad (2.11)$$

Now, consider either equal quark masses or the mass choice (2.5). Both, the gauge and flavor $SU(N_f)$ groups, are broken in the vacuum. However, the color-flavor locked form of (2.11) guarantees that the diagonal global $SU(\tilde{N})_{C+F}$ survives. More exactly, the unbroken global group of the dual theory is

$$SU(N)_F \times SU(\tilde{N})_{C+F} \times U(1). \quad (2.12)$$

Here $SU(\tilde{N})_{C+F}$ is a global unbroken color-flavor rotation, which involves the last \tilde{N} flavors, while the $SU(N)_F$ factor stands for the flavor rotation of the first N dyons. Thus, a color-flavor locking takes place in the dual theory too. Much in the same way as in the original theory, the presence of the global $SU(\tilde{N})_{C+F}$ group is the reason behind formation of the non-Abelian strings. For generic quark masses the global symmetry (2.4) is broken down to $U(1)^{N_f-1}$.

In the equal-mass limit or for the mass choice (2.5) the global unbroken symmetry (2.12) of the dual theory at small ξ coincides with the global group (2.4) present in the $r = N$ vacuum of the original theory at large ξ . Note however, that this global symmetry is realized in two distinct ways in two dual theories. As was already mentioned, the quarks and $U(N)$ gauge bosons of the original theory at large ξ come in the $(1, 1)$, $(N^2 - 1, 1)$, (\bar{N}, \bar{N}) , and (N, \bar{N}) representations of the global group (2.4), while the dyons and $U(\tilde{N})$ gauge bosons form $(1, 1)$, $(1, \tilde{N}^2 - 1)$,

(N, \tilde{N}) , and (\tilde{N}, \tilde{N}) representations of (2.12). We see that the adjoint representations of the $(C + F)$ subgroup are different in two theories. A similar phenomenon was detected in [21] for the Abelian dual theory (i.e. $\tilde{N} = 0$).

This means that the quarks and gauge bosons which form the adjoint $(N^2 - 1)$ representation of $SU(N)$ at large ξ and the dyons and gauge bosons which form the adjoint $(\tilde{N}^2 - 1)$ representation of $SU(\tilde{N})$ at small ξ are, in fact, *distinct* states. The $(N^2 - 1)$ adjoints of $SU(N)$ become heavy and decouple as we pass from large to small ξ along the line (2.5). Moreover, some composite $(\tilde{N}^2 - 1)$ adjoints of $SU(\tilde{N})$, which are heavy and invisible in the low-energy description at large ξ become light at small ξ and form the D^{IK} dyons ($K = N + 1, \dots, N_f$) and gauge bosons of $U(\tilde{N})$. The phenomenon of level crossing takes place (Fig. 2). Although this crossover is smooth in the full theory, from the standpoint of the low-energy description the passage from large to small ξ means a dramatic change: the low-energy theories in these domains are completely different; in particular, the degrees of freedom in these theories are different.

This logic leads us to the following conclusion. In addition to light dyons and gauge bosons included in the low-energy theory at small ξ we have heavy fields which form the adjoint representation $(N^2 - 1, 1)$ of the global symmetry (2.12). These are screened quarks and gauge bosons from the large ξ domain. Let us denote them as $M_P^{P'}$ ($P, P' = 1, \dots, N$).

As was already explained in Sec. I, at small ξ they decay into the monopole-antimonopole pairs on the curves of marginal stability (CMS).⁴ This is in accordance with results obtained for $\mathcal{N} = 2$ $SU(2)$ gauge theories [3,4,33] on the Coulomb branch at zero ξ (we confirm this result for the theory at hand in Sec. VIII). The general rule is that the only states which exist at strong coupling inside CMS are those which can become massless on the Coulomb branch [3,4,33]. For our theory these are light dyons shown in Eq. (2.11), gauge bosons of the dual $U(\tilde{N})$ theory and monopoles.

At small nonvanishing ξ the monopoles and antimonopoles produced in the decay process of adjoints $(N^2 - 1, 1)$ cannot escape from each other and fly off to separate because they are confined. Therefore, the quarks or gauge bosons in the strong-coupling domain of small ξ evolve into stringy mesons $M_P^{P'}$ ($P, P' = 1, \dots, N$)—the monopole-antimonopole pairs connected by two strings [11] as shown in Fig. 1.

⁴Strictly speaking, such pairs can be formed by monopole-antidyons and dyon-antidyons as well, the dyons carrying root-like electric charges. In this paper we will call all these states “monopoles.” This is to avoid confusion with dyons which appear in Eq. (2.11). The latter dyons carry weightlike electric charges and, roughly speaking, behave as quarks, see [11] for further details.

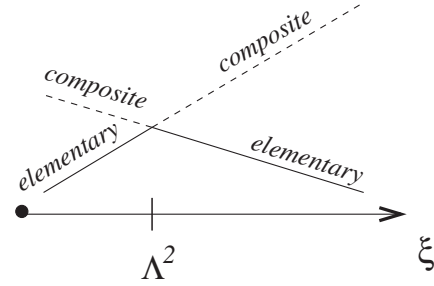


FIG. 2. Evolution of the $SU(N)$ and $SU(\tilde{N})$ gauge bosons and light quarks (dyons) vs ξ . On both sides of the level crossing at $\xi = \Lambda^2$ the global groups are $SU(N) \times SU(\tilde{N})$, however, above Λ^2 it is $SU(N)_{C+F} \times SU(\tilde{N})_F$ while below Λ^2 it is $SU(N)_F \times SU(\tilde{N})_{C+F}$.

By the same token, at large ξ , in addition to the light quarks and gauge bosons, we have heavy fields $M_K^{K'}$ ($K, K' = N + 1, \dots, N_f$), which form the adjoint $(\tilde{N}^2 - 1)$ representation of $SU(\tilde{N})$. This is schematically depicted in Fig. 2.

The $M_K^{K'}$ states are (screened) light dyons and gauge bosons of the dual theory. At large ξ they decay into monopole-antimonopole pairs and form stringy mesons [11] shown in Fig. 1.

In [11] we also conjectured that the fields $M_P^{P'}$ and $M_K^{K'}$ are Seiberg’s meson fields [24,25], which occur in the dual theory upon breaking of $\mathcal{N} = 2$ supersymmetry by the mass-term superpotential $\mu[A^2 + (A^a)^2]$ for the adjoint fields in the limit $\mu \rightarrow \infty$. In this limit our theory becomes $\mathcal{N} = 1$ SQCD.

We see that the picture of the non-Abelian confinement obtained in [11] is based on the presence of extra stringy meson states—the monopole-antimonopole pairs—bound by confining strings both in the weak and strong coupling domains of the bulk theory. These meson states fill representations $(N^2 - 1, 1)$ and $(1, \tilde{N}^2 - 1)$ of the global unbroken group at small and large ξ , respectively. Below we confirm the presence of these stringy mesons by studying the global quantum numbers of confined monopoles in both domains. To this end we explore kinks in the $\mathcal{N} = (2, 2)$ supersymmetric weighted CP model on the string world sheet.

We remind that the confined monopoles of the bulk theory are presented by the junctions of two elementary non-Abelian strings [15,16,34]. These elementary strings corresponds to different vacua of the effective sigma model on the world sheet. See also the review paper [19] for details.

III. WORLD-SHEET THEORY

In this section we briefly describe the world-sheet low-energy sigma models on the non-Abelian strings at large and small ξ . Non-Abelian strings in $\mathcal{N} = 2$ SQCD with $N_f = N$ were first found and studied in [13–16]. Then we

discuss how the bulk duality translates into the world-sheet duality [11].

A. World-sheet theory at large ξ

To warm up, we start from $N_f = N$. The Abelian Z_N -string solutions break the $SU(N)_{C+F}$ global group. As a result, the non-Abelian strings have orientational zero modes associated with rotations of their color flux inside the non-Abelian $SU(N)$ group. The global group is broken on the Z_N string solution down to $SU(N-1) \times U(1)$. Hence, the moduli space of the non-Abelian string is described by the coset space

$$\frac{SU(N)}{SU(N-1) \times U(1)} \sim CP(N-1), \quad (3.1)$$

in addition to C spanned by the translational modes. The translational moduli totally decouple. They are sterile free fields, and we can forget about them. Therefore, the low-energy effective theory on the non-Abelian string is the two-dimensional $\mathcal{N} = 2$ $CP(N-1)$ model [13–16].

Now we add extra quark flavors with degenerate masses, increasing N_f from N up to a certain value $N_f > N$. The strings emerging in such theory are semilocal. In particular, the string solutions on the Higgs branches (typical for multiflavor theories) usually are not fixed-radius strings, but, rather, possess radial moduli, aka size moduli, see [28] for a comprehensive review of the Abelian semilocal strings. The transverse size of such a string is not fixed.

Non-Abelian semilocal strings in $\mathcal{N} = 2$ SQCD with $N_f > N$ were studied in [13,16,29,30]. The orientational zero modes of the semilocal non-Abelian string are parametrized by a complex vector n^P ($P = 1, \dots, N$), while its $\tilde{N} = (N_f - N)$ size moduli are parametrized by a complex vector ρ^K ($K = N+1, \dots, N_f$). The effective two-dimensional theory which describes the internal dynamics of the non-Abelian semilocal string is the $\mathcal{N} = (2, 2)$ weighted CP model on a “toric” manifold, which includes both types of fields. The bosonic part of the action in the gauged formulation (which assumes taking the limit $e^2 \rightarrow \infty$) has the form⁵

$$\begin{aligned} S = \int d^2x & \left\{ |\nabla_\alpha n^P|^2 + |\tilde{\nabla}_\alpha \rho^K|^2 + \frac{1}{4e^2} F_{\alpha\beta}^2 + \frac{1}{e^2} |\partial_\alpha \sigma|^2 \right. \\ & + 2 \left| \sigma + \frac{m_P}{\sqrt{2}} \right|^2 |n^P|^2 + 2 \left| \sigma + \frac{m_K}{\sqrt{2}} \right|^2 |\rho^K|^2 \\ & \left. + \frac{e^2}{2} (|n^P|^2 - |\rho^K|^2 - 2\beta)^2 \right\}, \\ P = 1, \dots, N, \quad K = N+1, \dots, N_f. \end{aligned} \quad (3.2)$$

The fields n^P and ρ^K have charges $+1$ and -1 with respect to the auxiliary $U(1)$ gauge field; hence, the corresponding

⁵Equation (3.2) and similar expressions below are given in Euclidean notation.

covariant derivatives in (3.2) are defined as

$$\nabla_\alpha = \partial_\alpha - iA_\alpha, \quad \tilde{\nabla}_\alpha = \partial_\alpha + iA_\alpha, \quad (3.3)$$

respectively.

If only charge $+1$ fields were present, in the limit $e^2 \rightarrow \infty$ we would get a conventional twisted-mass deformed $CP(N-1)$ model. The presence of charge -1 fields ρ^K converts the $CP(N-1)$ target space into that of the a weighted $CP(N_f-1)$ model. As in the $CP(N-1)$ model, small mass differences $|m_A - m_B|$ lift orientational and size zero modes generating a shallow potential on the modular space. The D -term condition

$$|n^P|^2 - |\rho^K|^2 = 2\beta \quad (3.4)$$

is implemented in the limit $e^2 \rightarrow \infty$. Moreover, in this limit the gauge field A_α and its $\mathcal{N} = 2$ bosonic superpartner σ become auxiliary and can be eliminated.

The two-dimensional coupling constant β is related to the four-dimensional one as

$$\beta = \frac{2\pi}{g_2^2}. \quad (3.5)$$

This relation is obtained at the classical level [14,15]. In quantum theory both couplings run. In particular, the model (3.2) is asymptotically free [35] and develops its own scale Λ_σ . The ultraviolet cutoff in the sigma model on the string world sheet is determined by $g_2\sqrt{\xi}$. Equation (3.5) relating the two- and four-dimensional couplings is valid at this scale. At this scale the four-dimensional coupling is given by (2.9) while the two-dimensional one

$$4\pi\beta(\xi) = (N - \tilde{N}) \ln \frac{g_2\sqrt{\xi}}{\Lambda_\sigma} \gg 1. \quad (3.6)$$

Then Eq. (3.5) implies

$$\Lambda_\sigma = \Lambda, \quad (3.7)$$

and from now on we will omit the subscript σ . In the bulk, the running of the coupling constant is frozen at $g_2\sqrt{\xi}$, because of the VEVs of the squark fields. The logarithmic evolution of the coupling constant in the string world-sheet theory continues uninterrupted below this point, with the same running law. As a result, the dynamical scales of the bulk and world-sheet theories turn out to be the same, much in the same way as in the $N_f = N$ theory [15].

We remind that the scale $g_2\sqrt{\xi}$ determines the ultraviolet cutoff in the sigma model on the string world sheet. Therefore we can consider the theory (3.2) as an effective theory on the string only at energies below $g_2\sqrt{\xi}$. This is fulfilled if

$$\sqrt{\xi} \gg \max(|m_A - m_B|, \Lambda). \quad (3.8)$$

Summarizing, if the quark mass differences are small, the $(1, \dots, N)$ vacuum of the original $U(N)$ gauge theory supports non-Abelian semilocal strings. Their internal dy-

namics is described by the effective two-dimensional low-energy $\mathcal{N} = (2, 2)$ sigma model (3.2). The model has N orientational moduli n^P with the $U(1)$ charge $+1$ and masses $m_P = \{m_1, \dots, m_N\}$, plus \tilde{N} size moduli ρ^K , with the $U(1)$ charge -1 and masses $(-m_K) = -\{m_{N+1}, \dots, m_{N_f}\}$.

A final remark is in order here. The strict semilocality achieved at $\Delta m_{AB} = 0$ destroys confinement of monopoles [29,36]. The reason is that the string transverse size (determined by ρ^K 's) can grow indefinitely. When it becomes of the order of the distance L between sources of the magnetic flux (the monopoles), the linearly rising confining potential between these sources is replaced by a Coulomb-like potential. To have confinement of monopoles we should lift the size zero modes keeping Δm nonvanishing. That's exactly what we will do, eventually sticking to (2.5), preserving both confinement and the global symmetry.

B. Dual world-sheet theory

The dual bulk $U(\tilde{N})$ theory at small ξ also supports non-Abelian semilocal strings. The $(1, \dots, N)$ vacuum of the original theory (2.1) transforms into the vacuum (2.11) of the dual theory. Therefore, the internal string dynamics on the string world sheet is described by a similar $\mathcal{N} = (2, 2)$ sigma model. Now it has \tilde{N} orientational moduli with the $U(1)$ charge $+1$ and masses $m_K = \{m_{N+1}, \dots, m_{N_f}\}$. To make contact with (3.2) we call them $\tilde{\rho}^K$. In addition, it has N size moduli with the $U(1)$ charge -1 and masses $(-m_P) = -\{m_1, \dots, m_N\}$. We refer to these size moduli as \tilde{n}^P .

The bosonic part of the action of the world-sheet model in the gauged formulation (which assumes taking the limit $\tilde{e}^2 \rightarrow \infty$) has the form

$$S_{\text{dual}} = \int d^2x \left[|\nabla_\alpha \tilde{\rho}^K|^2 + |\tilde{\nabla}_\alpha \tilde{n}^P|^2 + \frac{1}{4e^2} F_{\alpha\beta}^2 \right. \\ \left. + \frac{1}{e^2} |\partial_\alpha \sigma|^2 + 2 \left| \sigma + \frac{m_P}{\sqrt{2}} \right|^2 |\tilde{n}^P|^2 + 2 \left| \sigma \right. \right. \\ \left. \left. + \frac{m_K}{\sqrt{2}} \right|^2 |\tilde{\rho}^K|^2 + \frac{e^2}{2} (|\tilde{\rho}^K|^2 - |\tilde{n}^P|^2 - 2\tilde{\beta})^2 \right], \\ P = 1, \dots, N, \quad K = N + 1, \dots, N_f, \quad (3.9)$$

where the covariant derivatives are defined in (3.3).

We see that the roles of the orientational and size moduli interchange in Eq. (3.9) compared with (3.2). As in the model (3.2), small mass differences Δm_{AB} lift orientational and size zero modes of the non-Abelian semilocal string generating a shallow potential on the moduli space. Much in the same way as in the model (3.2), the dual coupling constant $\tilde{\beta}$ is determined by the bulk dual coupling \tilde{g}_2^2 ,

$$4\pi\tilde{\beta}(\xi) = \frac{8\pi^2}{\tilde{g}_2^2}(\xi) = (N - \tilde{N}) \ln \frac{\Lambda}{\tilde{g}_2 \sqrt{\xi}}, \quad (3.10)$$

see Eqs. (3.5) and (3.6). The dual theory makes sense at $\tilde{g}_2 \sqrt{\xi} \ll \Lambda$ where $\tilde{\beta}$ is positive and

$$4\pi\tilde{\beta}(\xi) \gg 1$$

(weak coupling).

The bulk and world-sheet dual theories have identical β functions, with the first coefficient $(\tilde{N} - N) < 0$. They are both infrared (IR) free. As in the model (3.2), the coincidence of the β functions in the bulk and world-sheet theories implies that the scale of the dual model (3.9) is equal to that of the bulk theory,

$$\tilde{\Lambda}_\sigma = \Lambda,$$

cf. Eq. (3.7). Comparing (3.10) with (3.6) we see that

$$\tilde{\beta} = -\beta. \quad (3.11)$$

At $\xi \gg \Lambda^2$ the original theory is at weak coupling, and β is positive. Analytically continuing to the domain $\xi \ll \Lambda^2$, we formally make β negative, which signals, of course, that the low-energy description in terms of the original model is inappropriate. At the same time, $\tilde{\beta}$ satisfying (3.11) becomes positive, and the dual model assumes the role of the legitimate low-energy description (at weak coupling). A direct inspection of the dual theory action (3.9) shows that the dual theory can be interpreted as a continuation of the sigma model (3.2) to negative values of the coupling constant β .

Both world-sheet theories (3.2) and (3.9) give the *effective low-energy* descriptions of string dynamics valid at the energy scale below $g_2 \sqrt{\xi}$.

Let us note that the world-sheet duality between two-dimensional sigma models (3.2) and (3.9) was previously noted in Ref. [30]. In this paper two bulk theories, with the $U(N)$ and $U(\tilde{N})$ gauge groups, were considered (these theories were referred to as a dual pair in [30]). Two-dimensional sigma models (3.2) and (3.9) were presented as effective low-energy descriptions of the non-Abelian strings for these two bulk theories.

IV. SEMICLASSICAL DESCRIPTION OF THE WORLD-SHEET THEORIES

At $N < N_f < 2N$ the original model (3.2) is asymptotically free, see (3.6). Its coupling β continues running below $g_2 \sqrt{\xi}$ until it stops at the scale of the mass differences Δm_{AB} . If all mass differences are large, $|\Delta m_{AB}| \gg \Lambda$, the model is at weak coupling. From (3.2) we see that the model has N vacua (i.e. N strings from the standpoint of the bulk theory),

$$\sqrt{2}\sigma = -m_{P_0}, \quad |n^{P_0}|^2 = 2\beta, \quad n^{P \neq P_0} = \rho^K = 0, \quad (4.1)$$

where $P_0 = 1, \dots, N$.

In each vacuum there are $2(N_f - 1)$ elementary excitations, counting real degrees of freedom. The action (3.2)

contains N complex fields n^P and \tilde{N} complex fields ρ^K . The phase of n^{P_0} is eaten by the Higgs mechanism. The condition $|n^{P_0}|^2 = 2\beta$ eliminates one extra field. The physical masses of the elementary excitations

$$M_A = |m_A - m_{P_0}|, \quad A \neq P_0. \quad (4.2)$$

In addition to the elementary excitations, there are kinks (domain ‘‘walls’’ which are particles in two dimensions) interpolating between different vacua. Their masses scale as

$$M^{\text{kink}} \sim \beta M_A. \quad (4.3)$$

The kinks are much heavier than elementary excitations at weak coupling.⁶

Now we pass to the dual world-sheet theory (3.9). It is *not* asymptotically free (rather, IR-free) and, therefore, is at weak coupling at small mass differences, $|m_{AB}| \ll \Lambda$. From (3.9) we see that this model has \tilde{N} vacua

$$\sqrt{2}\sigma = -m_{K_0}, \quad |\rho^{K_0}|^2 = 2\tilde{\beta}, \quad n^P = \rho^{K \neq K_0} = 0, \quad (4.4)$$

where $K_0 = N + 1, \dots, N_f$. In each vacuum there are $2(N_f - 1)$ elementary excitations with the physical masses

$$M_A = |m_A - m_{K_0}|, \quad A \neq K_0. \quad (4.5)$$

The dual model has kinks too; their masses scale as (4.3).

It is important to understand that the dual theory (3.9) is not asymptotically free at energies much larger than the mass differences. At energies smaller than some mass differences certain fields decouple, and the theory may or may not become asymptotically free. Keeping in mind the desired limit (2.5) we will consider the following mass choice in the dual theory

$$m_P \sim m_{P'}, \quad m_K \sim m_{K'}, \quad m_P - m_K \sim \Delta m \quad (4.6)$$

where $P, P' = 1, \dots, N$ and $K, K' = N + 1, \dots, N_f$. Moreover, we will often consider the mass hierarchy

$$|\Delta m_{PP'}| \sim |\Delta m_{KK'}| \ll |\Delta m| \ll \Lambda, \quad (4.7)$$

where $\Delta m_{PP'} = m_P - m_{P'}$ and $\Delta m_{KK'} = m_K - m_{K'}$.

Clearly, the dual model is not asymptotically free only if the mass differences $\Delta m_{KK'}$ are not too small. If we take

$$|\Delta m_{KK'}| \ll |\Delta m| \ll \Lambda \quad (4.8)$$

the model becomes asymptotically free below $|\Delta m|$. In fact, the model then reduces to the $CP(\tilde{N} - 1)$ model with an effective scale

$$\tilde{\Lambda}_{\tilde{N}} \equiv \frac{(\Delta m)^{\tilde{N}}}{\Lambda^{N-\tilde{N}}}, \quad \tilde{\Lambda}_{\tilde{N}} \ll |\Delta m|. \quad (4.9)$$

⁶Note that they have nothing to do with Witten’s n solitons [31] identified as the n^P fields at strong coupling. In the next section we present a general formula for the kink spectrum outside CMS (at weak coupling).

In particular, if $|\Delta m_{KK'}| \lesssim \tilde{\Lambda}_{\tilde{N}}$, descending down to $\tilde{\Lambda}_{\tilde{N}}$ we enter (more exactly, the dual $CP(\tilde{N} - 1)$ enters) the strong-coupling regime.

Thus, there are *two* strong-coupling regimes in the dual model. One is at large mass differences $|m_{AB}| \gg \Lambda$ where the original model (3.2) is at weak coupling and provides an adequate description, while the other is at very small mass differences $\Delta m_{KK'} \lesssim \tilde{\Lambda}_{\tilde{N}}$ where the dual model effectively reduces to the strongly coupled $CP(\tilde{N} - 1)$ model.

V. EXACT SUPERPOTENTIAL

The $CP(N - 1)$ models are known to be described by an exact superpotential [26,35,37,38] of the Veneziano-Yankielowicz type [39]. This superpotential was generalized to the case of the weighted CP models in [27,40]. In this section we will briefly outline this method. Integrating out the fields n^P and ρ^K we can describe the original model (3.2) by the following exact twisted superpotential:

$$\begin{aligned} \mathcal{W}_{\text{eff}} = & \frac{1}{4\pi} \sum_{P=1}^N (\sqrt{2}\Sigma + m_P) \ln \frac{\sqrt{2}\Sigma + m_P}{\Lambda} \\ & - \frac{1}{4\pi} \sum_{K=N+1}^{N_f} (\sqrt{2}\Sigma + m_K) \ln \frac{\sqrt{2}\Sigma + m_K}{\Lambda} \\ & - \frac{N - \tilde{N}}{4\pi} \sqrt{2}\Sigma, \end{aligned} \quad (5.1)$$

where Σ is a twisted superfield [35] (with σ being its lowest scalar component). Minimizing this superpotential with respect to σ we get the vacuum field formula,

$$\prod_{P=1}^N (\sqrt{2}\sigma + m_P) = \Lambda^{(N-\tilde{N})} \prod_{K=N+1}^{N_f} (\sqrt{2}\sigma + m_K). \quad (5.2)$$

Note that the roots of this equation coincide with the double roots of the Seiberg-Witten curve of the bulk theory [26,27]. This is, of course, a manifestation of coincidence of the Seiberg-Witten solution of the bulk theory with the exact solution of (3.2) given by the superpotential (5.1). As was mentioned in Sec. I, this coincidence was observed in [26,27] and explained later in [15,16].

Now, let us consider the effective superpotential of the dual world-sheet theory (3.9). It has the form

$$\begin{aligned} \tilde{\mathcal{W}}_{\text{eff}} = & \frac{1}{4\pi} \sum_{K=N+1}^{N_f} (\sqrt{2}\Sigma + m_K) \ln \frac{\sqrt{2}\Sigma + m_K}{\Lambda} \\ & - \frac{1}{4\pi} \sum_{P=1}^N (\sqrt{2}\Sigma + m_P) \ln \frac{\sqrt{2}\Sigma + m_P}{\Lambda} \\ & - \frac{\tilde{N} - N}{4\pi} \sqrt{2}\Sigma. \end{aligned} \quad (5.3)$$

We see that it coincides with the superpotential (5.1) up to a sign. Clearly, both, the root equations and the BPS spectra,

are the same for the two sigma models, as was expected [11].

Although classically the dual pair of the weighted CP models at hand are given by different actions (3.2) and (3.9), in the quantum regime they reduce to one and the same theory. This is, of course, expected. Classically the couplings of both theories are determined by the ultraviolet (UV) cutoff scale $\sqrt{\xi}$, see (3.6) and (3.10). However, in quantum theory these couplings run and, in fact, are determined by the mass differences. Therefore, if $|\Delta m_{AB}| \gtrsim \Lambda$, the coupling β is positive and we use the original theory (3.2). On the other hand, if $|\Delta m_{AB}| \lesssim \Lambda$, the coupling β becomes negative, we use the dual theory (3.9) which has positive $\tilde{\beta}$, see Sec. IV. The bulk FI parameter ξ no longer plays a role. Only the values of the mass differences matter.

It is instructive to summarize the situation. The theory has three distinct regimes, namely,

- (i) The weak-coupling domain in the original description at large mass differences,

$$|\Delta m_{AB}| \gg \Lambda. \quad (5.4)$$

- (ii) The mixed regime in the dual description at intermediate masses,

$$\tilde{\Lambda}_{\text{LE}} \ll |\Delta m_{AB}| \ll \Lambda, \quad (5.5)$$

where all mass differences above are assumed to be of the same order. Certain vacua (namely, \tilde{N} vacua) are at weak coupling and can be seen classically, see (4.4), while $N - \tilde{N}$ other vacua are at strong coupling. If, instead, we assume the mass hierarchy (4.7) then in order to keep \tilde{N} vacua at weak coupling we have to impose the condition

$$|\Delta m_{KK'}| \gg \tilde{\Lambda}_{\text{LE}}, \quad (5.6)$$

see (4.9). This is the reason why we call this region the ‘‘intermediate mass’’ domain.

- (iii) The strong-coupling regime in the dual description at hierarchically small masses,

$$|\Delta m_{PP'}| \sim |\Delta m_{KK'}| \lesssim \tilde{\Lambda}_{\text{LE}} \ll |\Delta m| \ll \Lambda. \quad (5.7)$$

The masses of the BPS kinks interpolating between the vacua σ_I and σ_J are given by the appropriate differences of the superpotential (5.1) calculated at distinct roots [26,27,40],

$$\begin{aligned} M_{IJ}^{\text{BPS}} &= 2|\mathcal{W}_{\text{eff}}(\sigma_J) - \mathcal{W}_{\text{eff}}(\sigma_I)| \\ &= \left| \frac{N - \tilde{N}}{2\pi} \sqrt{2}(\sigma_J - \sigma_I) \right. \\ &\quad - \frac{1}{2\pi} \sum_{P=1}^N m_P \ln \frac{\sqrt{2}\sigma_J + m_P}{\sqrt{2}\sigma_I + m_P} \\ &\quad \left. + \frac{1}{2\pi} \sum_{K=N+1}^{N_f} m_K \ln \frac{\sqrt{2}\sigma_J + m_K}{\sqrt{2}\sigma_I + m_K} \right|. \quad (5.8) \end{aligned}$$

The masses obtained from (5.8) were shown [15,16] to coincide with those of monopoles and dyons in the bulk theory. The latter are given by the period integrals of the Seiberg-Witten curve [26,27].

Now we will consider the vacuum structure and the kink spectrum in more detail in two quasiclassical regions—at large mass differences (the original theory) and at intermediate mass differences (the dual theory).

A. Large $|\Delta m_{AB}|$

Consider the vacuum structure of the theory (5.2) in the weak-coupling regime $|\Delta m_{AB}| \gg \Lambda$. In this domain the model has N vacua which in the leading (classical) approximation are determined by Eq. (4.1). Equation (5.2) reproduces this vacuum structure. Namely, VEVs of σ in each of the N vacua (say, at $P = P_0$) are given by the corresponding mass m_{P_0} , plus a small correction,

$$\sqrt{2}\sigma_{P_0} \approx -m_{P_0} + \Lambda^{N-\tilde{N}} \frac{\prod_{K=N+1}^{N_f} (m_K - m_{P_0})}{\prod_{P \neq P_0} (m_P - m_{P_0})}. \quad (5.9)$$

The spectrum of kinks is given by Eq. (5.8). To be more specific, let us consider the kinks interpolating between the neighboring vacua⁷ P_0 and $(P_0 + 1)$. Then we have

$$\begin{aligned} m^{\text{kink}} &= |m_D^{P_0+1} - m_D^{P_0}| \\ &\approx \left| (m_{P_0} - m_{P_0+1}) \frac{N - \tilde{N}}{2\pi} \ln \frac{\bar{\Delta} m_{AB}}{\Lambda} \right|, \quad (5.10) \end{aligned}$$

where $\bar{\Delta} m_{AB}$ is a certain average value of the mass differences (it depends holomorphically on all mass differences in the problem). Here we use (5.9). If all mass differences are of the same order so is $\bar{\Delta} m_{AB}$, although $\bar{\Delta} m_{AB}$ does not coincide with any of the individual mass differences. For a generic choice of the mass differences $\bar{\Delta} m_{AB}$ has a non-vanishing phase.

We see that in the logarithmic approximation the kink mass is proportional to the mass difference $(m_{P_0} - m_{P_0+1})$

⁷If the mass parameters m_P are randomly scattered in the complex plane, how should one define the ‘‘neighboring vacua’’? In the regime under consideration, for all P the vacuum values σ_P are close to $-m_P/\sqrt{2}$. Assume σ_{P_0} is chosen. Then the neighboring vacuum σ_{P_0+1} is defined in such a way that the difference $|m_{P_0} - m_{P_0+1}|$ is the smallest in the set $|\Delta m_{P_0 P}|$.

times the coupling constant β , as one would expect at weak-coupling, cf. Equation (4.3).

This is not the end of the story, however. The logarithmic functions in (5.8) are multivalued, and we have to carefully choose their branches. Each logarithm in (5.1) can be written in the integral form as

$$\frac{m_A}{2\pi} \ln \frac{\sqrt{2}\sigma_{P_0+1} + m_A}{\sqrt{2}\sigma_{P_0} + m_A} = \frac{m_A}{2\pi} \int_{\sigma_{P_0}}^{\sigma_{P_0+1}} \frac{\sqrt{2}d\sigma}{\sqrt{2}\sigma + m_A}, \quad (5.11)$$

with the integration contour to be appropriately chosen. Distinct choices differ by pole contributions

$$\text{integer} \times im_A \quad (5.12)$$

for different A . These different mass predictions for the BPS states correspond to dyonic kinks. In addition to the topological charge, the kinks can carry Noether charges with respect to the global group (2.4) broken down to $U(1)^{N_f}$ by the mass differences. This produces a whole family of dyonic kinks.⁸ We stress that all these kinks with the imaginary part (5.12) in the mass formula interpolate between the same pair of vacua: P_0 and $(P_0 + 1)$. Our aim is to count their number and calculate their masses.

Generically there are way too many choices of the integration contours in (5.8). Not all of them are realized. Moreover, the kinks present in the quasiclassical domain decay on CMS or form new bound states, cf. e.g. [41,42]. Therefore the quasiclassical spectrum outside CMS and quantum spectrum inside CMS are different. We have to use an additional input on the structure of kink solutions to find out the correct form of the BPS spectrum. In this section we will summarize information on the classical spectrum while in the remainder of the present paper we will use the mirror representation [32] of the model at hand to obtain the quantum spectrum.

The general formula for the BPS spectrum can be written as follows [26]:

$$M^{\text{BPS}} = \left| \sum_I m_D^I T_I + i \sum_A m_A S_A \right|, \quad (5.13)$$

where the first term is a nonperturbative contribution and T_I is the topological charge N -vector, while the second term represents the dyonic (the Noether charge) ambiguity discussed above, with S_A describing a global $U(1)$ charge of the given BPS state with respect to the $U(1)^{N_f}$ group.

The topological charge is given by

$$T_P = \delta_{PP_0+1} - \delta_{PP_0} \quad (5.14)$$

for kinks interpolating between the vacua P_0 and $(P_0 + 1)$, while m_D 's are approximately given by the logarithmic terms in (5.10),

⁸They represent confined monopoles and dyons with the root-like electric charges in the bulk theory.

$$m_D^{P_0} \approx m_{P_0} \frac{N - \tilde{N}}{2\pi} \ln \frac{\bar{\Delta} m_{AB}}{\Lambda}. \quad (5.15)$$

Equation (5.10) corresponds to the kink with $S_A = 0$.

At weak coupling the BPS kinks can be studied using the classical solutions of the first-order equations. Each given kink solution breaks the global $U(1)^{N_f}$ group. Therefore, the kinks acquire zero modes associated with rotations in this internal group. Quantization of the corresponding dynamics gives rise to dyonic kinks which carry global charges S_A . This program was carried out for the $CP(N - 1)$ model in [26] and for the weighted CP model (3.2) in [27]. The result is

$$S_P = sT_P, \quad S_K = 0 \quad (5.16)$$

for $P = 1, \dots, N$ and $K = N + 1, \dots, N_f$, where s is an integer. Thus, at large $|\Delta m_{AB}|$ we have an infinite tower of the dyonic kinks with masses

$$M^{\text{kink}} \approx |(m_{P_0} - m_{P_0+1})| \left| \frac{N - \tilde{N}}{2\pi} \ln \frac{\bar{\Delta} m_{AB}}{\Lambda} - is \right|. \quad (5.17)$$

The expression in the second line under the sign of the absolute value has both real and imaginary parts. The real part is obtained from the logarithm by replacing $\bar{\Delta} m_{AB}$ under the logarithm by $|\bar{\Delta} m_{AB}|$. The imaginary part includes the phase of $\bar{\Delta} m_{AB}$, which, in principle, could be obtained for any given set of the mass differences, but in practice this is hard to do for generic mass choices. In addition, the imaginary part includes is , where s is an integer (positive, negative or zero). When we change s , we scan all possible values of the $U(1)$ charge (i.e. go through the entire set of dyons).

In addition to the above monopoles/dyons, in this domain of Δm_{AB} there are elementary excitations, see Eq. (4.2). These excitations are BPS-saturated too and can be described by the general formula (5.13) with $T = 0$ and $S_A = \delta_{AB} - \delta_{AP_0}$ for any $B = 1, \dots, N_f$ in the P_0 -vacuum.⁹

The above spectrum changes upon passing through CMS. In particular, we will see that elementary excitations do not exist inside CMS. All excitations that survive inside CMS are the kinks with nonvanishing topological charges. This is a two-dimensional counterpart of the bulk picture: the quarks and gauge bosons decay inside the strong-coupling domain giving rise to the monopole-antimonopole pairs, see Sec. II.

⁹The actual kink spectrum at weak coupling is more complicated than the one in (5.17). The kink states from the tower (5.17) can form bound states with different elementary states in certain special domains of the mass parameters [27,43].

B. Intermediate masses

Now let us consider the domain of intermediate mass differences, see Eq. (5.5). Then Eq. (5.2) has $(N - \tilde{N})$ solutions with

$$\sqrt{2}\sigma = \Lambda \exp\left(\frac{2\pi i}{N - \tilde{N}} l\right), \quad l = 0, \dots, (N - \tilde{N} - 1). \quad (5.18)$$

We will refer to these vacua as the Λ -vacua. They are at strong coupling, and will be studied later. In this section we consider other \tilde{N} vacua, which are at weak coupling and seen classically in the dual theory, see Eq. (4.4). For these vacua Eq. (5.2) gives

$$\sqrt{2}\sigma_{K_0} \approx -m_{K_0} + \frac{1}{\Lambda^{N-\tilde{N}}} \frac{\prod_{P=1}^N (m_P - m_{K_0})}{\prod_{K \neq K_0} (m_K - m_{K_0})}, \quad (5.19)$$

where $K_0 = N + 1, \dots, N_f$. These \tilde{N} vacua will be referred to as the *zero-vacua* since in these vacua, with small masses, the σ vacuum expectation values are much smaller than in the Λ -vacua.

Substituting this in Eq. (5.8) we get the spectrum of the kinks interpolating between the neighboring vacua K_0 and $K_0 + 1$,

$$M^{\text{kink}} \approx \left| (m_{K_0} - m_{K_0+1}) \frac{N - \tilde{N}}{2\pi} \ln \frac{\Lambda}{\tilde{\Delta} m_{AB}} + is(m_{K_0} - m_{K_0+1}) \right|, \quad (5.20)$$

where we take into account the U(1) charges parallelizing the derivation of Eq. (5.17) and applying the quantization procedure of [26,27] to the dual theory (3.9). As previously, all mass differences are assumed to be of the same order.

If, instead, we consider a stricter mass hierarchy (4.7) (still requiring that we are at weak coupling $|\Delta m_{KK'}| \gg \tilde{\Lambda}_{LE}$) then Eq. (5.20) must be modified. With this stricter hierarchy the product in the numerator of the second term in (5.19) reduces to $(\Delta m)^N$ to form $\tilde{\Lambda}_{LE}$, and the kink spectrum takes the form

$$M^{\text{kink}} \approx \left| (m_{K_0} - m_{K_0+1}) \frac{\tilde{N}}{2\pi} \ln \frac{\tilde{\Delta} m_{KK'}}{\tilde{\Lambda}_{LE}} + is(m_{K_0} - m_{K_0+1}) \right|, \quad (5.21)$$

where $\tilde{\Lambda}_{LE}$ is given by (4.9). This is just the kink spectrum in the $CP(\tilde{N} - 1)$ model at weak coupling.

In addition to the $T \neq 0$ kinks, there are elementary excitations with masses given in Eq. (4.5). They correspond to $T = 0$ and $S_A = \delta_{AB} - \delta_{AK_0}$ for any $B = 1, \dots, N_f$ in the K_0 -vacuum in (5.13).

Confronting Eqs. (5.17) and (5.20) we see that the kinks have different Noether charges in the domains of large and intermediate mass differences. At large masses they have

charges with respect to the first N factors of the global $U(1)^{N_f}$ group, while the kinks at the intermediate masses are charged with respect to the last \tilde{N} factors (this would correspond to $SU(N)$ and $SU(\tilde{N})$ factors of the global group (2.4) in the limit (2.5)). Therefore, they are, in fact, absolutely distinct states. The BPS states decay/form new bound states upon passing from one domain to another. The restructuring happens on CMS which are surfaces located at $|\Delta m_{AB}| \sim \Lambda$ in the mass parameter space.¹⁰ As we will see shortly, in the weighted CP model at hand we have another set of CMS at much smaller mass differences $|\Delta m_{KK'}| \sim \tilde{\Lambda}_{LE}$. This additional CMS separates the domain of intermediate masses from that at strong coupling, see (5.7).

VI. MIRROR DESCRIPTION

Now we turn to the study of the quantum BPS spectrum inside CMS. We will determine the BPS spectrum in the Λ -vacua (5.18) at intermediate and small masses, as well as the spectrum in the zero-vacua in the strong-coupling domain at hierarchically small masses (5.7).

A. Mirror superpotential

To this end we will rely on the mirror formulation [32] of the weighted CP model (3.2). In this formulation one describes the CP model as a Coulomb gas of instantons (see [44] where it was first done in the nonsupersymmetric $CP(1)$ model). In supersymmetric setting this description leads to an affine Toda theory with an exact superpotential. The exact mirror superpotentials were found for the $\mathcal{N} = (2, 2)$ $CP(N - 1)$ model and its various generalizations with toric target spaces in [32]. For the model (3.2) the mirror superpotential has the form

$$W_{\text{mirror}} = -\frac{\Lambda}{4\pi} \left\{ \sum_{P=1}^N X_P - \sum_{K=N+1}^{N_f} Y_K - \sum_{P=1}^N \frac{m_P}{\Lambda} \ln X_P + \sum_{K=N+1}^{N_f} \frac{m_K}{\Lambda} \ln Y_K \right\} \quad (6.1)$$

supplemented by the constraint

$$\prod_{P=1}^N X_P = \prod_{K=N+1}^{N_f} Y_K. \quad (6.2)$$

This representation can be checked by a straightforward calculation. Indeed, add the term

$$\frac{\Lambda}{4\pi} \sqrt{2} \Sigma \left(\sum_{P=1}^N \ln X_P - \sum_{K=N+1}^{N_f} \ln Y_K \right) \quad (6.3)$$

to the superpotential (6.1), which takes into account the

¹⁰Of course, this restructuring is a reflection of the same phenomenon in the bulk theory, see Sec. II.

constraint (6.2). Here Σ plays a role of the Lagrange multiplier. Then integrate over X_P and Y_K ignoring their kinetic terms. In this way one arrives at

$$X_P = \frac{1}{\Lambda}(\sqrt{2}\sigma + m_P), \quad Y_K = \frac{1}{\Lambda}(\sqrt{2}\sigma + m_K). \quad (6.4)$$

Substituting (6.4) back into (6.1) one gets the superpotential (5.1).

Clearly for the dual model (3.9) the mirror superpotential coincides with that in (6.1) up to an (irrelevant) sign.

Below we will find the critical points of the superpotential (6.1) and discuss the vacuum structure of the model in the mirror representation. Since our goal is to study the domain of intermediate or hierarchically small masses, see (5.5) and (5.7), respectively, we will assume that

$$|\Delta m_{AB}| \ll \Lambda \quad (6.5)$$

and keep only terms linear in $|\Delta m_{AB}|/\Lambda$. As a warm-up exercise we will start with the $CP(N-1)$ model.

B. $CP(N-1)$ model

For $CP(N-1)$ model $\tilde{N} = 0$ and the superpotential (6.1) reduces to

$$W_{\text{mirror}}^{CP(N-1)} = -\frac{\Lambda}{4\pi} \left\{ \sum_{P=1}^N X_P - \sum_{P=1}^N \frac{m_P}{\Lambda} \ln X_P \right\}, \quad (6.6)$$

while the constraint (6.2) reads

$$\prod_{P=1}^N X_P = 1. \quad (6.7)$$

Expressing, say X_1 in terms of X_P with $P = 2, 3, \dots, N$ by virtue of this constraint and substituting the result in (6.6), we get the vacuum equations,

$$X_P = X_1 + \frac{m_P - m_1}{\Lambda} = X_1 + \frac{\Delta m_{P1}}{\Lambda}, \quad P = 2, \dots, N. \quad (6.8)$$

Substituting this in (6.7) we obtain X_1 . This procedure leads us to the following VEVs of the X_P fields:

$$X_P \approx \exp\left(\frac{2\pi i}{N} l\right) + \frac{1}{\Lambda}(m_P - m), \quad \forall P, \quad (6.9)$$

where we ignore quadratic in mass differences terms,

$$m \equiv \frac{1}{N} \sum_{P=1}^N m_P, \quad (6.10)$$

and each of the N vacua of the model is labeled by a value of l , namely $l = 0$ in the first vacuum, $l = 1$ in the second, and so on until we arrive at $l = (N-1)$.

C. Λ -vacua

Now we turn to the strong-coupling vacua in the weighted CP model at intermediate or small masses (see

Eq. (5.7)), using the mirror representation (6.1). These vacua were defined as Λ -vacua in Sec. VB, Eq. (5.18).

Again expressing, say, X_1 in terms of other fields by virtue of the constraint (6.2) we get the vacuum equations

$$X_P = X_1 + \frac{\Delta m_{P1}}{\Lambda}, \quad Y_K = X_1 + \frac{\Delta m_{K1}}{\Lambda}. \quad (6.11)$$

Substituting this in the constraint (6.2) and resolving for X_1 we get the following VEVs:

$$\begin{aligned} X_P &\approx \exp\left(\frac{2\pi i}{N-\tilde{N}} l\right) + \frac{1}{\Lambda}(m_P - \hat{m}), & P = 1, \dots, N, \\ Y_K &\approx \exp\left(\frac{2\pi i}{N-\tilde{N}} l\right) + \frac{1}{\Lambda}(m_K - \hat{m}), \\ K &= N+1, \dots, N_f, & l = 0, \dots, (N-\tilde{N}-1) \end{aligned} \quad (6.12)$$

for $(N-\tilde{N})$ vacua of the theory. Here

$$\hat{m} \equiv \frac{1}{N-\tilde{N}} \left(\sum_{P=1}^N m_P - \sum_{K=N+1}^{N_f} m_K \right). \quad (6.13)$$

As in (6.9), in Eq. (6.12) we neglect the quadratic in mass difference terms, cubic, and so on.

D. Zero-vacua

Now consider other \tilde{N} vacua of the model (3.9) (which were termed zero-vacua in Sec. VB) using the mirror description (6.1). At intermediate masses, VEVs of σ are given, in the classical approximation, by the mass parameters m_K in the dual theory, see (4.4). The corrections are given by (5.19). Since the mirror representation is particularly useful at strong coupling, in this section we will focus on a very small hierarchical region of the mass parameters (5.7).

It is convenient to express one of Y_K 's, say, Y_{N_f} in terms of other fields using (6.2). Then vacuum equations take the form

$$X_P = Y_{N_f} + \frac{\Delta m_{PN_f}}{\Lambda}, \quad Y_K = Y_{N_f} + \frac{\Delta m_{KN_f}}{\Lambda}. \quad (6.14)$$

From the first equation we see that, given the hierarchical masses (5.7), all X_P fields are equal to each other to the leading order,

$$X_P^{(0)} \approx \frac{\Delta m}{\Lambda}, \quad P = 1, \dots, N. \quad (6.15)$$

With these X_P 's the constraint (6.2) takes the form

$$\prod_{K=N+1}^{N_f} Y_K = \left(\frac{\tilde{\Lambda}_{\text{LE}}}{\Lambda}\right)^{\tilde{N}}, \quad (6.16)$$

where $\tilde{\Lambda}_{\text{LE}}$ is the scale of the effective low-energy $CP(\tilde{N}-1)$ model (4.9). Substituting the fields Y_K from the second equation in (6.14) we get

$$Y_K \approx \frac{\tilde{\Lambda}_{\text{LE}}}{\Lambda} \left\{ \exp\left(\frac{2\pi i}{\tilde{N}} l\right) + \frac{1}{\tilde{\Lambda}_{\text{LE}}} (m_K - \tilde{m}) \right\}, \quad (6.17)$$

where $l = 0, \dots, \tilde{N} - 1$,

$$\tilde{m} \equiv \frac{1}{\tilde{N}} \sum_{K=N+1}^{N_f} m_K. \quad (6.18)$$

As usual, we neglect quadratic, cubic, etc. mass-difference terms.

Finally, we are ready to solve the vacuum equations. Substituting (6.17) in the first expression in (6.14) we get $O(\tilde{\Lambda}_{\text{LE}}/\Lambda)$ corrections to (6.15),

$$X_P \approx \frac{1}{\Lambda} (m_P - \tilde{m}) + \frac{\tilde{\Lambda}_{\text{LE}}}{\Lambda} \exp\left(\frac{2\pi i}{\tilde{N}} l\right), \quad (6.19)$$

$$l = 0, \dots, \tilde{N} - 1.$$

We see that there are exactly \tilde{N} vacua with very small values of Y_K 's. The VEV structure of Y_K 's reduces to that of the $CP(\tilde{N} - 1)$ model with the scale parameter $\tilde{\Lambda}_{\text{LE}}$, see (6.9).

VII. KINKS INSIDE CMS

In this section we use the mirror representation to find the kink spectrum inside CMS in the weighted CP model on the string world sheet. First, we briefly review the kink solutions and their spectrum [32] in the $CP(N - 1)$ model and only then turn to the weighted CP model (3.2).

A. Kinks in the $CP(N - 1)$ model

As was shown in [32], in the strong-coupling regime (inside CMS) the number of kinks interpolating between the vacua P and $P + k$ of $\mathcal{N} = (2, 2)$ supersymmetric $CP(N - 1)$ model is

$$\nu(N, k) = C_N^k \equiv \frac{N!}{k!(N - k)!}. \quad (7.1)$$

In particular, the number of kinks interpolating between the neighboring vacua ($k = 1$) is N , and they form a fundamental representation of the $SU(N)$ group. They carry the minimal charge with respect to the gauge $U(1)$ and, therefore, were identified [31] as n^P fields in terms of the original description, see (3.2) for $\tilde{N} = 0$.

Consider a kink interpolating between the neighboring $l = 0$ and $l = 1$ vacua, see (6.9). The kink solution has the following structure [32]. All X_P 's start in the vacuum with $l = 0$ and end in the vacuum with $l = 1$. Moreover, all X_P with $P \neq P_0$ (where P_0 is fixed for a given kink solution) are equal to each other and have phases which wind by the angle $2\pi/N$ in the anticlockwise direction. Then the constraint (6.7) ensures that X_{P_0} winds in the opposite (clockwise) direction by the angle $[-2\pi(N - 1)/N]$, see Fig. 3. (If one considers nonelementary kinks interpolating between non-neighboring vacua, say $l = 0$ and $l = 2$, then

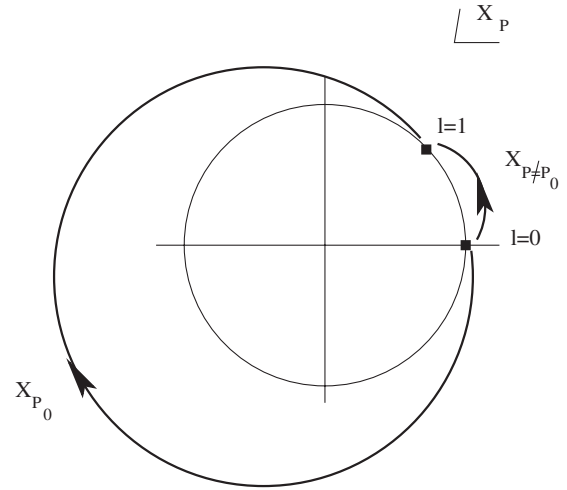


FIG. 3. Windings of the fields X_P for the kink interpolating between the $l = 0$ and $l = 1$ vacua.

two fields X_{P_0} and X_{P_0} would have opposite windings with respect to all others, and so on [32].)

The kink mass is given by

$$M^{\text{BPS}} = 2 \left| \mathcal{W}_{\text{mirror}}^{CP(N-1)}(l=1) - \mathcal{W}_{\text{mirror}}^{CP(N-1)}(l=0) \right|$$

$$\approx \left| \frac{N}{2\pi} \Lambda (e^{2\pi/N} - 1) + i(m_{P_0} - m) \right|, \quad (7.2)$$

where we use (6.6) and neglect quadratic in mass differences terms. The parameter m is defined in Eq. (6.10).

If we look at the absolute values of the fields X_P rather than at their phases, we will see that, generically, their absolute values differ from unity. This is discussed in Appendix A, cf. also [45]. The explicit profile functions of the kink solutions are irrelevant for determination of the kink spectrum, since the latter is given by central charges—the differences of the superpotential on the initial and final vacua. The phases of X_P are important, however, because logarithms in (6.6) are multivalued functions. In particular, the term im_{P_0} appeared in (7.2) because X_{P_0} has the relative winding angle (-2π) with respect to other X_P 's.

We see that we have exactly N dyonic kinks associated for the given choice of P_0 and its neighbor. (In addition, P_0 can be chosen arbitrarily from the set $P_0 = 1, \dots, N$). The above dyonic kinks have different charges with respect to the global $U(1)^N$ and are split at generic masses, but become degenerate in the equal-mass limit. Clearly, they form a fundamental representation of the global $SU(N)$ in this limit. We stress again that all N kinks here interpolate between two fixed vacua, $l = 0$ and $l = 1$.

The BPS spectrum inside CMS, see (7.2), is very different from that outside CMS, see (5.17) with $\tilde{N} = 0$. The weak-coupling spectrum has an infinite tower of dyonic kinks, all associated with the same mass difference $(m_{P_0} - m_{P_0+1})$. Also, the weak-coupling spectrum has elementary

states with $T_P = 0$. The quantum spectrum has only a finite number of states (N), with masses which depend on all mass differences present in the theory. Moreover, all these states are topological (they are kinks), no elementary states are present. The majority of the BPS states present at weak coupling (in particular, all elementary excitations) decay on CMS and are absent at strong coupling.

In conclusion we note that Eq. (5.8) is exact and, in principle, can be used to calculate the BPS spectrum in any domain of the parameter space of the theory. We can put $\tilde{N} = 0$ in this formula (descending down to the $CP(N - 1)$ model) and apply (5.8) to the kinks which interpolate between two vacua $l = 0$ and $l = 1$ with VEVs of σ given by

$$\sqrt{2}\sigma \approx \Lambda \exp\left(\frac{2\pi i}{N}l\right), \quad l = 0, \dots, N \quad (7.3)$$

at small masses, cf. (5.18) with $\tilde{N} = 0$.

Say, the main contribution in (7.2) proportional to Λ comes from the first nonlogarithmic term in the second line in (5.8). Moreover, now the result in (7.2) shows how ambiguities related to the choice of the logarithm branches in (5.8) should be resolved at strong coupling. Namely, we get exactly the same BPS spectrum as in (7.2) from (5.8) (with $\tilde{N} = 0$) if we choose the integration contours in (5.11) as shown in Fig. 4. For the P_0 -th dyonic kink, the integration contour should pick up exactly one pole contribution located at $\sqrt{2}\sigma = -m_{P_0}$ in the clockwise direction. This shows, in fact, how the kink solutions look in terms of the field σ .

The above recipe was obtained in the $CP(N - 1)$ model in [40] using a brane construction, see also [27]. Here we confirm it directly in field theory using the mirror description of the model. See also [45].

B. Kinks in the Λ -vacua

In this section we work out a similar procedure to obtain the BPS spectrum in the weighted CP model (3.9). We will focus on the Λ -vacua at intermediate or small masses, see (5.5) and (5.7).

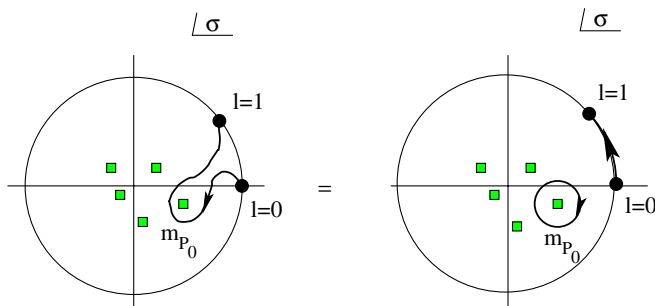


FIG. 4 (color online). Integration contour in the σ plane. Dots denote two vacua $l = 0$ and $l = 1$, while filled squares denote poles located at $\sqrt{2}\sigma = -m_P$ for $P = 1, \dots, N$.

Consider kinks interpolating between the neighboring $l = 0$ and $l = 1$ vacua (6.12). Much in the same way as in the $CP(N - 1)$ model (Sec. VII A), all X_P 's and Y_K 's are equal to each other and wind by the angle $2\pi/(N - \tilde{N})$, except one field whose winding angle is determined by the constraint (6.2). There are two different types of kinks depending on the choice of the latter variable: X_{P_0} or Y_{K_0} , ($P_0 = 1, \dots, N$ and $K_0 = (N + 1), \dots, N_f$). We refer to these two types of solutions as to the P - and K -kinks, respectively.

1. P -kinks

Solutions for P -kinks are very similar to those for kinks in the $CP(N - 1)$ model. In a given P -kink the variable X_{P_0} winds in the clockwise direction by the angle $[-2\pi(N - \tilde{N} - 1)/(N - \tilde{N})]$, see Fig. 3. All other fields, i.e. $X_{P \neq P_0}$ and Y_K , are equal to each other and wind counterclockwise by the angle $2\pi/(N - \tilde{N})$.

The superpotential (6.1) implies that the mass of this kink is

$$M_{P_0}^{\text{BPS}} \approx \left| \frac{N - \tilde{N}}{2\pi} \Lambda (e^{(2\pi)/(N - \tilde{N})i} - 1) + i(m_{P_0} - \hat{m}) \right|, \quad (7.4)$$

where \hat{m} is given in Eq. (6.13).

We see that in the transition $l = 0 \rightarrow l = 1$ we have exactly N kinks associated with the arbitrary choice of the contour (with the loop around the pole $P_0 = 1, \dots, N$). They are split for generic masses, but become degenerate in the limit (2.5) we are interested in. They form the fundamental representation of the global $SU(N)$ in this limit.

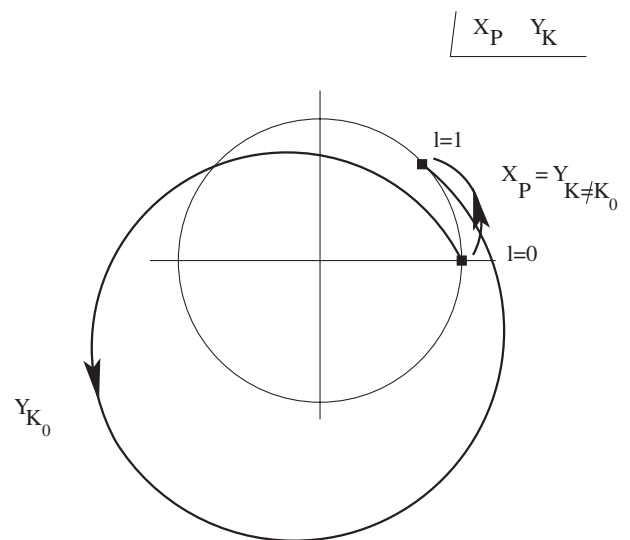


FIG. 5. Windings of fields X_P and Y_K for the K -kink interpolating between the $l = 0$ and $l = 1$ Λ -vacua.

2. *K*-kinks

In the *K*-kink solutions all fields X_P and $Y_{K \neq K_0}$ are equal to each other and have the winding angle $2\pi/(N - \tilde{N})$, while the field Y_{K_0} winds in the anticlockwise direction by the angle $[2\pi(N - \tilde{N} + 1)/(N - \tilde{N})]$, see Fig. 5, as dictated¹¹ by the constraint (6.2). Thus, the Y_{K_0} field has the relative winding $+2\pi$ with respect to all other fields.

Substituting this information in the superpotential (6.1) we obtain the *K*-kink mass,

$$M_{K_0}^{\text{BPS}} \approx \left| \frac{N - \tilde{N}}{2\pi} \Lambda (e^{(2\pi/(N - \tilde{N}))i} - 1) + i(m_{K_0} - \hat{m}) \right|. \tag{7.5}$$

Clearly, we have \tilde{N} kinks of this type due to the possibility of choosing the contour encompassing any of the \tilde{N} poles, $K_0 = (N + 1), \dots, N_f$. They are split at generic masses, but become degenerate in the limit (2.5). They form the fundamental representation of the global $SU(\tilde{N})$ group in this limit.

Altogether we have N_f kinks interpolating between each pair of neighboring Λ -vacua. They form the fundamental representation of the global group (2.12) in the limit (2.5). More exactly, they form the $(N, 1) + (1, \tilde{N})$ representation of this group.

Much in the same way as in the $CP(N - 1)$ model, we can verify that Eq. (5.8) reproduces this spectrum with the appropriate choice of the integration contours. Namely, for the P_0 -kink the contour in the σ plane encircles the pole at $\sqrt{2}\sigma = -m_{P_0}$ in the clockwise direction, see Fig. 4. For the K_0 -kink the contour encircles the pole at $\sqrt{2}\sigma = -m_{K_0}$ in the anticlockwise direction.

C. Kinks in the zero-vacua

Now we consider kinks in the zero-vacua in the domain of small hierarchical masses (5.7). These vacua of the weighted CP model are most interesting since they correspond to \tilde{N} non-Abelian strings of the dual bulk theory. Clearly, the number of kinks does not depend on which pair of neighboring vacua we pick up. Thus, we expect to have altogether N_f kinks, as was the case in the Λ -vacua. We check this explicitly below.

Much in the same way as in the Λ -vacua, the kinks interpolating between the neighboring $l = 0$ and $l = 1$ zero-vacua (see (6.17)) fall into two categories: the *P*- and *K*-kinks, respectively, depending on the choice of the

¹¹In fact, for these solutions to exist $(N - \tilde{N})$ is required to be large enough. If $(N - \tilde{N})$ is not large enough, the functions of $|X_P|$ and $|Y_K|$ develop singularities; the singular solutions must be discarded, cf. Appendix A. The reason behind this phenomenon is that for $(N - \tilde{N})$ not large enough the vacuum which is the closest neighbor to $l = 0$ is, in fact, one of the zero-vacua (Sec. VID) rather than the $l = 1$ Λ -vacuum.

particular X_{P_0} or Y_{K_0} field with an opposite winding with respect to other fields.

1. *K*-kinks

Let us start from the *K*-kinks. The kink solution looks very similar to that in the $CP(\tilde{N} - 1)$ model. All $Y_{K \neq K_0}$ fields are approximately equal to each other and have the winding angles $2\pi/\tilde{N}$, while the Y_{K_0} field has the winding angle $-2\pi(\tilde{N} - 1)/\tilde{N}$, see (6.17). Correction terms in X_P , proportional to $\tilde{\Lambda}_{\text{LE}}/\Lambda$ also all have the same windings by the angle $2\pi/\tilde{N}$, see (6.19). This gives the following expression for the mass of the K_0 -kink:

$$M_{K_0}^{\text{BPS}} \approx \left| \frac{N - \tilde{N}}{2\pi} \tilde{\Lambda}_{\text{LE}} (e^{(2\pi/\tilde{N})i} - 1) - i(m_{K_0} - \tilde{m}) \right|. \tag{7.6}$$

The factor $(N - \tilde{N})$ in the first term appears from two first terms in (6.1) due to winding of both X_P and Y_K fields. The im_{K_0} term is due to the relative winding -2π of the Y_{K_0} field.

We have \tilde{N} kinks of this type associated with the arbitrary choice of $K_0 = (N + 1), \dots, N_f$. They are split with generic masses, but become degenerate in the limit (2.5). In this limit they form the fundamental representation of the global $SU(\tilde{N})$ group.

2. *P*-kinks

In this case all Y_K fields are approximately equal to each other and have the winding angles $2\pi/\tilde{N}$. The fields $X_{P \neq P_0}$ are all equal to $\Delta m/\Lambda$, to the leading order, and do not wind; however, they have windings $2\pi/\tilde{N}$ in the correction terms, see (6.19). The field X_{P_0} does wind. Its absolute value is $\Delta m/\Lambda$. The winding angle is 2π , as enforced by the constraint (6.2), see Fig. 6. A more detailed description of the kink solutions is presented in Appendix B. The

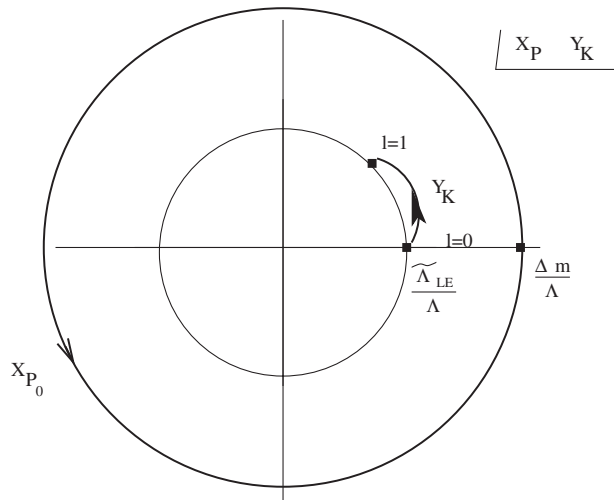


FIG. 6. Windings of fields X_{P_0} and Y_K for the P_0 -kink interpolating between $l = 0$ and $l = 1$ zero-vacua.

windings above imply the following mass of the P_0 -kink:

$$\begin{aligned} M_{P_0}^{\text{BPS}} &\approx \left| \frac{N - \tilde{N}}{2\pi} \tilde{\Lambda}_{\text{LE}} (e^{(2\pi/\tilde{N})i} - 1) - i(m_{P_0} - \tilde{m}) \right| \\ &= \left| -i\Delta m + \frac{N - \tilde{N}}{2\pi} \tilde{\Lambda}_{\text{LE}} (e^{(2\pi/\tilde{N})i} - 1) \right. \\ &\quad \left. - i(m_{P_0} - m) \right|, \end{aligned} \quad (7.7)$$

where m and \tilde{m} are given by (6.10) and (6.18), while $\Delta m = m - \tilde{m}$, see (4.6). In the last expression the first term is the leading contribution, the second one is a correction, while the third term accounts for still smaller splittings.

There are N kinks of this type associated with the arbitrary choice of $P_0 = 1, \dots, N$. They are split at generic masses, but become degenerate in the limit (2.5). They form the fundamental representation of the global $SU(N)$ group in this limit.

Much in the same way as in the Λ -vacua, the total number of the kinks interpolating between each pair of the neighboring zero-vacua is N_f . They form the $(N, 1) + (1, \tilde{N})$ representation of the global group (2.12) in the limit (2.5). Note, that the P -kinks in the zero-vacua are heavier than the K -kinks. Their masses are given by Δm (to the leading order) while the K -kink masses are of the order of $\tilde{\Lambda}_{\text{LE}}$. Still, given small hierarchical masses (5.7), all kinks in the zero-vacua are lighter than those in the Λ -vacua which have masses of the order of Λ , see Eqs. (7.4) and (7.5).

In parallel with the Λ -vacua, we can verify that Eq. (5.8) reproduces the above BPS spectrum with the appropriate choice of the integration contours. Namely, for the P_0 -kink the contour in the σ plane encircles the pole at $\sqrt{2}\sigma = -m_{P_0}$ in the anticlockwise direction, cf. Fig. 4. For the K_0 -kink the contour encircles the pole at $\sqrt{2}\sigma = -m_{K_0}$ in the clockwise direction.

VIII. LESSONS FOR THE BULK THEORY

This section carries a special weight and is, in a sense, central for the present investigation, since here we translate our results for the kink spectrum in the weighted two-dimensional CP model (3.2) in the language of strings and confined monopoles of the bulk four-dimensional theory.

We start from the most interesting strong-coupling domain

$$\xi \ll \Lambda \quad (8.1)$$

which can be described in terms of weakly coupled dual bulk theory [11], see Sec. II B. At this point we take the limit (2.5) to ensure the presence of the unbroken global group (2.12).

As was mentioned previously, the elementary non-Abelian strings of the bulk theory correspond to various

vacua of the world-sheet two-dimensional theory, see, e.g. the review paper [19] for a detailed discussion. The weighted CP model (3.9) has two types of vacua, namely: $(N - \tilde{N})$ Λ -vacua and \tilde{N} zero-vacua. The former are not-so-interesting from the standpoint of the bulk theory. Indeed, they yield just Abelian $Z_{N-\tilde{N}}$ strings associated with the winding of the $(N - \tilde{N})$ singlet dyons D^I charged with respect to $U(1)$ factors of the gauge group of the dual theory (2.10) [11], see (2.11). Moreover, the weighted CP model (3.9) is, in fact, inapplicable in the description of these strings. This model is an effective low-energy theory which can be used below the scale $\sqrt{\xi}$. However, the energy scale in the Λ -vacua is of order of Λ , i.e. much larger than $\sqrt{\xi}$ in the domain (8.1).

Below we focus on \tilde{N} zero-vacua which correspond to \tilde{N} elementary non-Abelian strings associated with the winding of the light dyons D^{IA} of the dual bulk theory. The latter are charged with respect to both Abelian and non-Abelian factors [11] of the dual gauge group (2.10). The energy scale in these vacua of the world-sheet theory is of the order of

$$\max(\Delta m_{KK'}, \tilde{\Lambda}_{\text{LE}}),$$

which we assume to be much less than $\sqrt{\xi}$. Thus, in these vacua the weighted CP model (3.9) can be applied to describe the internal dynamics of the non-Abelian strings of the bulk theory.

The confined monopoles of the bulk theory are seen as kinks in the world-sheet theory. The results presented in Sec. VII demonstrate that in the weighted CP model there are N_f elementary kinks interpolating between the neighboring zero-vacua. More exactly, we found N P -kinks with masses (7.7) and \tilde{N} K -kinks with masses (7.6). In the limit (2.5) they form the $(N, 1) + (1, \tilde{N})$ representations of the global group (2.12).

This means that the total number of stringy mesons M_A^B formed by the monopole-antimonopole pairs connected by two different elementary non-Abelian strings (Fig. 1) is N_f^2 . The mesons $M_P^{P'}$ form the singlet and $(N^2 - 1, 1)$ adjoint representations of the global group (2.12), the mesons M_P^K and M_K^P form bifundamental representations (N, \tilde{N}) and (\tilde{N}, \tilde{N}) , while the mesons $M_K^{K'}$ form the singlet and $(1, \tilde{N}^2 - 1)$ adjoint representations. (Here as usual, $P = 1, \dots, N$ and $K = (N + 1), \dots, N_f$.) All these mesons have masses of the order of $\sqrt{\xi}$, determined by the string tension

$$T = 2\pi\xi. \quad (8.2)$$

They are heavier than the elementary states, namely, dyons and dual gauge bosons which form the $(1, 1)$, (N, \tilde{N}) , (\tilde{N}, \tilde{N}) , and $(1, \tilde{N}^2 - 1)$ representations and have masses $\sim \tilde{g}_2 \sqrt{\xi}$.

Therefore, the $(1, 1)$, (N, \tilde{N}) , (\tilde{N}, \tilde{N}) , and $(1, \tilde{N}^2 - 1)$ stringy mesons decay into elementary states, and we are left with $M_P^{P'}$ stringy mesons in the representation $(N^2 - 1,$

1). This is exactly what was predicted in [11] from the bulk perspective, see Sec. II B. Thus, our world-sheet picture nicely matches the bulk analysis.

Note also that the $M_p^{P'}$ stringy mesons with strings corresponding to the Λ -vacua of the weighted CP model (the “ Λ -strings”) are heavy and decay into $M_p^{P'}$ stringy mesons with strings corresponding to the zero-vacua (the “zero-strings”). To see that this is indeed the case, observe that the confined monopoles (i.e. kinks of the weighted CP model) on the Λ -string have masses of the order of Λ , see (7.4). Therefore, the Λ -stringy mesons also have masses of the order of Λ . The $M_p^{P'}$ mesons with the zero-strings are much lighter in the domain (8.1). Their masses are of the order of $\max(\Delta m, \sqrt{\xi})$.

Now, let us discuss yet another match of the world-sheet and bulk pictures. In Sec. V A we saw that there are elementary excitations ($T = 0$ and $S_A = \delta_{AP_1} - \delta_{AP_2}$) on the string at weak coupling (this is attainable with large $|\Delta m_{AB}|$). These excitations would form the adjoint representation $(N^2 - 1, 1)$ of the global group if the limit (2.5) could be taken. However, in the strong-coupling domain of hierarchically small masses (5.7) the kink spectrum is very different, see Sec. VII. In particular, no elementary excitations are left: these states decay on CMS into a P_1 -kink plus a P_2 -antikink, see Eqs. (7.4) and (7.7).

Since the BPS spectra of the $\mathcal{N} = (2, 2)$ two-dimensional theory on the string and the $\mathcal{N} = 2$ four-dimensional bulk theory on the Coulomb branch (at zero ξ) coincide [15,16,26,27], the decay process above is in one-to-one correspondence with the decay of the bulk states identified in [11]. Namely, the quarks q^{kP_1} (with $k = P_2$ due to the color-flavor locking) and the gauge bosons present in the bulk theory at weak coupling decay into monopole-antimonopole pairs. If ξ is small but nonvanishing, the monopoles and antimonopoles cannot move apart: they are bound together by pairs of confining strings and form [11] the $M_{P_2}^{P_1}$ mesons shown in Fig. 1. Thus, our results from two dimensions confirm the decay of the quarks and gauge bosons in the strong-coupling domain of the bulk theory. This decay process is a crucial element of our mechanism of non-Abelian confinement.

To explain this in more detail we present a “phase diagram” of the bulk theory, see Fig. 7. The vertical and horizontal solid lines in this figure schematically represent the bulk theory CMS. The horizontal axis gives the masses $\Delta m_{PP'} \sim \Delta m_{KK'}$ which we force to be of the same order, while Δm is fixed, $\Delta m \ll \Lambda$. The vertical axis gives the FI parameter ξ . The vertical dashed lines depict CMS of the world-sheet theory. On these lines the spectrum of the stringy mesons of the bulk theory rearranges itself. On the solid lines the “perturbative” spectrum of the bulk theory rearranges itself. Different domains inside CMS (where the spectra change continuously) are denoted by capital letters A, B, \dots, F .

The domains A, B and C are at weak coupling in the original $U(N)$ gauge theory. The elementary (perturbative)

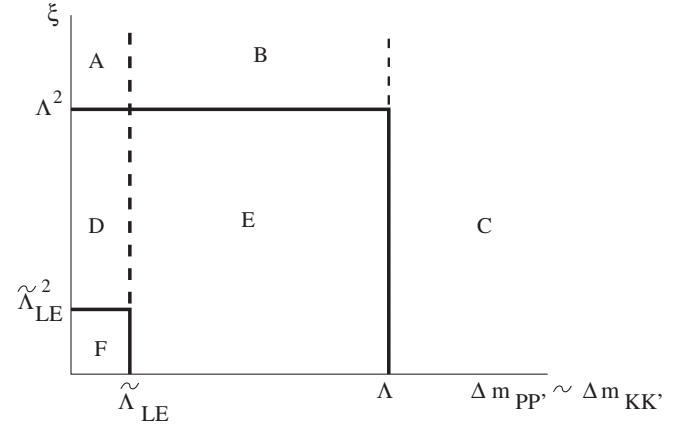


FIG. 7. “Phase diagram” of the bulk theory. Various domains are separated by CMS on which the physical spectrum is rearranged.

states in these domains are the q^{kA} quarks and gauge bosons of the $U(N)$ gauge group. The domains D and E are at weak coupling in the dual theory, see Sec. II B. The elementary (perturbative) states in these domains are the light dyons D^{IA} and D^I in Eq. (2.11) plus the gauge bosons of the dual gauge group (2.10). Masses of all these states smoothly evolve with the change of parameters inside these domains. The domain F is at strong coupling in the dual theory. With small mass differences in the domain (5.7), N flavors of dyons in Eq. (2.11) decouple, the dual theory becomes asymptotically free, and at $\xi \ll \tilde{\Lambda}_{LE}$ passes into the strong-coupling regime. The dual gauge group (2.10) gets broken down to $U(1)^N$ by the Seiberg-Witten mechanism.

Keeping in mind the limit (2.5), our task is to pass from the domain A into the domain D and prove that the quarks and gauge bosons of the former domain decay into the monopole-antimonopole pairs connected by confining strings in the domain D . We do it in three steps. First, we pass from the domain A to C at large ξ and then move towards small ξ (preserving large mass differences) inside C . The original bulk theory is in the weak-coupling regime in these domains and the quarks and gauge boson spectra evolve smoothly.

Next, we pass from the domain C to the domain E at small (or vanishing) ξ . Here we use correspondence between the BPS spectra of the bulk and world-sheet theories. The q^{kP_1} quarks (with $k = P_2$ due to the color-flavor locking) and gauge bosons of the domain C correspond to elementary states with $T = 0$ and $S_A = \delta_{AP_1} - \delta_{AP_2}$, see (4.2). As was already explained, these states decay on CMS of the two-dimensional theory into a P_1 -kink and a P_2 -antikink, interpolating between the Λ -vacua, see (7.4). Since the spectra of the massive BPS states in the bulk (at $\xi = 0$) on the one hand, and in the world-sheet theory on the other hand, are identical, both the quarks and gauge bosons from domain C do not exist in the domain E . They decay into a P_1 -monopole and a P_2 -antimonopole. At

small but nonvanishing ξ the latter states are confined by Λ -strings which gives rise to $M_{P_2}^{P_1}$ stringy mesons.

Furthermore, as we pass from the domain E to our final destination—the domain D —these $M_{P_2}^{P_1}$ mesons “glued” by Λ -strings, decay on the world-sheet theory CMS located at $\Delta m_{PP'} \sim \Delta m_{KK'} \sim \tilde{\Lambda}_{LE}$. They decay into lighter $M_{P_2}^{P_1}$ mesons glued by zero-strings. The latter stringy mesons were absent in the domain E at intermediate masses (see (5.20)), but emerge in the domain D at small masses, see (7.7). This completes our proof.

In conclusion it is worth noting that the stringy mesons $M_P^{P'}$ in the adjoint representation $(N^2 - 1, 1)$ of the global group are metastable, strictly speaking. An extra monopole-antimonopole pair can be created on the string, making the $(N^2 - 1, 1)$ meson to decay into a pair of stringy mesons in the bifundamental representation, (N, \tilde{N}) and (\tilde{N}, N) . During the subsequent stage these stringy bifundamentals decay into elementary bifundamental dyons. One can suppress the rate of this decay, however. If we keep $\Delta m \ll \sqrt{\xi}$ and take a limit of large N , while \tilde{N} is fixed this decay rate is of the order of $\tilde{N}/N \ll 1$. To see that this indeed the case, note that the above decay process goes through creation of a monopole-antimonopole pair from the fundamental representation of $SU(\tilde{N})$ on the string which selects \tilde{N} channels out of N_f .

Similar considerations can be applied to the weak-coupling domain of large ξ ,

$$\sqrt{\xi} \gg \Lambda,$$

of the bulk theory. We still have kinks in the $(N, 1) + (1, \tilde{N})$ representations of the global group (2.4) in the limit (2.5). Thus, all types of mesons M_B^A are formed with masses $\sim \sqrt{\xi}$. However, in this regime the $(1, 1)$, (N, \tilde{N}) , (\tilde{N}, N) , and $(N^2 - 1, 1)$ stringy mesons can decay into quarks and gauge bosons with the same quantum numbers and masses $\sim g_2^2 \sqrt{\xi}$. Therefore, we are left with elementary states and stringy adjoint mesons $(1, \tilde{N}^2 - 1)$. The latter are metastable and decay in pairs of bifundamental states.

IX. CONCLUSIONS

Both $\mathcal{N} = 2$ four-dimensional theories belonging to the dual pair discussed above support non-Abelian strings. The world-sheet theory on the strings is given by the weighted CP model which appears in two varieties depending on which side of duality we are. These two weighted CP models also form a dual pair. We explore the kink spectra (which represent confined monopoles of the bulk theory) and their evolution in passing through the crossover domain. Of most interest is small- ξ dynamics. In fact, at small ξ we find *two* weak-coupling subdomains and a strong-coupling one depending on the values of the differences of the mass parameters Δm_{AB} .

We have shown that in the limit (2.5) where the global group (2.4) is unbroken confined monopoles form the

fundamental representation of the global group. Therefore, stringy mesons (shown in Fig. 1) formed by pairs of monopoles and antimonopoles belong to the adjoint or singlet representations of the global group. This nicely matches global quantum numbers of mesons in the “real world.”

We proved the statement proposed in [11] that quarks and gauge bosons present in the original theory at large ξ decay on CMS into monopole-antimonopole pairs confined by non-Abelian strings as we enter the small- ξ domain. This result is a crucial element of our mechanism of non-Abelian confinement.

In summary, in this paper we used the world-sheet theory to confirm the picture of non-Abelian confinement obtained in [11] from the bulk perspective. Non-Abelian confinement is *not* associated with formation of chromoelectric strings connecting quarks, as a naive extrapolation of the Abelian confinement picture suggests. Rather, it is due to the decay on CMS of the Higgs-screened quarks and gauge bosons into monopole-antimonopole pairs confined by non-Abelian strings in the strong-coupling domain of small ξ . We stress again that the non-Abelian strings confine monopoles both in the original and dual theories.¹²

Analysis of the mass spectra presented in the bulk of the paper raises a number of intriguing questions. One of them refers to typical sizes of the objects considered. At the moment, in the absence of a detailed analysis, one can address this issue only at the qualitative level. At small ξ and $\Delta m \ll \sqrt{\xi} \ll \Lambda$ we expect that the smallest size $\sim \Lambda^{-1}$ is that of the elementary (perturbative) states, namely, dyons from Eq. (2.11) and dual gauge bosons. The sizes of the stringy mesons in the representation $N^2 - 1$ are expected to be of the order $(\tilde{g} \sqrt{\xi})^{-1}$, i.e. of order of the thickness of the non-Abelian strings. Finally, the largest sizes $(\Delta m)^{-1}$ and $\sim \tilde{\Lambda}_{LE}^{-1}$ belong to the P - and K -kinks (confined monopoles on a string), respectively, provided that the mass difference $\Delta m_{KK'} \rightarrow 0$.

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APPENDIX A: KINK SOLUTIONS IN $CP(N - 1)$ MODEL

In this appendix we discuss in more detail the kink solutions in the $CP(N - 1)$ model (see Sec. VII A). For simplicity we set $\Delta m_{PP'} = 0$. Small mass differences will

¹²Similar results were recently obtained in [46] for $\mathcal{N} = 2$ supersymmetric QCD with $N_f > 2N$.

slightly deform the kink profile functions but will not change the very fact of its existence. The kink masses are given by differences of the mirror superpotential evaluated at the initial and final vacua, see (7.2).

We look for the kink solution interpolating between $l = 0$ and $l = 1$ vacua using the *ansatz*

$$X_{P \neq P_0} = r e^{i\theta}, \quad X_{P_0} = \frac{e^{-i(N-1)\theta}}{r^{(N-1)}}, \quad (\text{A1})$$

where $r(z)$ and $\theta(z)$ are kink profile functions, subject to boundary conditions

$$\begin{aligned} \theta(z = -\infty) &= 0, & r(z = -\infty) &= 1, \\ \theta(z = \infty) &= \frac{2\pi}{N}, & r(z = \infty) &= 1, \end{aligned} \quad (\text{A2})$$

see (6.9). The last expression in (A1) is dictated by (6.7).

The explicit form of the kink profile functions depends on the form of the kinetic term which is not known. Therefore, both profile functions $r(z)$ and $\theta(z)$ cannot be determined. Still we can obtain the kink solution up to a single unknown profile function $\theta(z)$ expressing r as a function of θ . To this end we exploit the fact that the kink trajectory in the complex plane of superpotential goes along the straight line connecting the points $W_{\text{mirror}}(l = 0)$ with $W_{\text{mirror}}(l = 1)$ [38]. The difference

$$\begin{aligned} W_{\text{mirror}}(l = 1) - W_{\text{mirror}}(l = 0) &= -\frac{N\Lambda}{4\pi} (e^{(2\pi/N)i} - 1) \\ &= -\frac{N\Lambda}{4\pi} e^{(\pi/N)i} 2i \sin\left(\frac{\pi}{N}\right), \end{aligned} \quad (\text{A3})$$

where we used (6.6) and (6.9).

On the other hand,

$$\begin{aligned} W_{\text{mirror}}(\theta) - W_{\text{mirror}}(\theta = 0) &= -\frac{\Lambda}{4\pi} \left[(N-1) r e^{i\theta} + \frac{1}{r^{N-1}} e^{-i(N-1)\theta} - N \right] \\ &= -\frac{r\Lambda}{4\pi} e^{(\pi/N)i} \left\{ (N-1) e^{i(\theta - (\pi/N))} \right. \\ &\quad \left. + \frac{1}{r^N} e^{-i(N-1)\theta - i(\pi/N)} - \frac{N}{r} e^{-\pi/N} \right\}. \end{aligned} \quad (\text{A4})$$

Comparing Eqs. (A3) and (A4) we see that for the kink trajectory to go along the straight line, the real part of the expression in the curly brackets in (A4) must vanish. As a result,

$$\begin{aligned} (N-1) \cos\left(\theta - \frac{\pi}{N}\right) - \frac{N}{r} \cos\frac{\pi}{N} \\ + \frac{1}{r^N} \cos\left[(N-1)\theta + \frac{\pi}{N}\right] = 0. \end{aligned} \quad (\text{A5})$$

This equation determines the function $r(\theta)$. It has a non-vanishing nonsingular solution at $0 \leq \theta \leq 2\pi/N$ which

satisfies the boundary conditions $r(2\pi/N) = r(0) = 1$. Say, for $N = 2$, $r = 1$. For large N the profile function r is approximately given by

$$r(\theta) \approx 1 + \frac{1}{N} \left\{ 1 - \cos\left[(N-1)\theta + \frac{\pi}{N}\right] \right\} + \dots \quad (\text{A6})$$

APPENDIX B: KINK SOLUTIONS IN THE ZERO-VACUA

Here we consider the kink solutions interpolating between the zero-vacua with $l = 0$ and $l = 1$, see (6.17), at hierarchically small masses (5.7). The K -kink solutions in the zero-vacua are quite similar to kinks in the $CP(N-1)$ model. Therefore, we focus on P -kinks.

Analyzing the P -kinks we assume the limit (2.5) (for simplicity). The *ansatz* for the kink solution takes the form

$$Y_K = r e^{i\theta} \frac{\tilde{\Lambda}_{\text{LE}}}{\Lambda}, \quad X_{P \neq P_0} \approx \frac{\Delta m}{\Lambda}, \quad X_{P_0} \approx \frac{\Delta m}{\Lambda} r^{\tilde{N}} e^{i\tilde{N}\theta}, \quad (\text{B1})$$

where we restrict ourselves to the leading order contributions ($\sim \Delta m/\Lambda$) ignoring next-to-leading $O(\tilde{\Lambda}_{\text{LE}}/\Lambda)$ terms. The profile functions $r(z)$ and $\theta(z)$ are subject to the boundary conditions

$$\begin{aligned} \theta(z = -\infty) &= 0, & r(z = -\infty) &= 1, \\ \theta(z = \infty) &= \frac{2\pi}{N}, & r(z = \infty) &= 1, \end{aligned} \quad (\text{B2})$$

see (6.17) and (6.15). The last expression in (B1) is dictated by (6.2).

The superpotential (6.1) then gives

$$\begin{aligned} W_{\text{mirror}}(\theta) - W_{\text{mirror}}(\theta = 0) \approx -\frac{\Delta m}{4\pi} \{ r^{\tilde{N}} e^{i\tilde{N}\theta} \\ - \tilde{N}(i\theta + \ln r) - 1 \}, \end{aligned} \quad (\text{B3})$$

while the difference of the superpotential in the initial and final vacua is

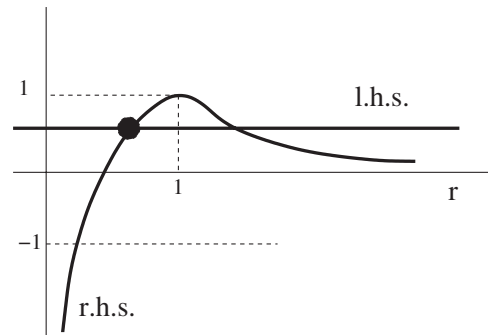


FIG. 8. The right- and left-hand sides of Eq. (B5). The closed circle denotes the regular solution.

$$W_{\text{mirror}}(l=1) - W_{\text{mirror}}(l=0) \approx \frac{\Delta m}{4\pi} \times 2\pi i. \quad (\text{B4})$$

For the kink trajectory to go along the straight line in the complex plane of superpotential, the real part of the expression in the curly brackets in (B3) must vanish. This requirement implies

$$r^{\tilde{N}} \cos(\tilde{N}\theta) - \tilde{N} \ln r = 1. \quad (\text{B5})$$

The latter equation always has a finite nonvanishing solution in the interval $0 \leq \theta \leq 2\pi/N$, subject to the boundary conditions $r(2\pi/N) = r(0) = 1$. To check that this is in-

deed the case we rewrite it as

$$\cos(\tilde{N}\theta) = \frac{1 + \tilde{N} \ln r}{r^{\tilde{N}}}. \quad (\text{B6})$$

The right- and left-hand sides of (B6) are schematically plotted in Fig. 8. For any $-1 < \cos(\tilde{N}\theta) < 1$ we have only one nonsingular solution $r(\theta)$. In particular, at $\tilde{N} \gg 1$

$$r(\theta) \approx 1 - \frac{1}{\tilde{N}} [1 - \cos(\tilde{N}\theta)] + \dots \quad (\text{B7})$$

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