Gravity-Yang-Mills-Higgs unification by enlarging the gauge group

Alexander Torres-Gomez and Kirill Krasnov
School of Mathematical Sciences, University of Nottingham, Nottingham, NG7 2RD, United Kingdom
(Received 19 November 2009; published 5 April 2010)

We revisit an old idea that gravity can be unified with Yang-Mills theory by enlarging the gauge group of gravity formulated as gauge theory. Our starting point is an action that describes a generally covariant gauge theory for a group G. The Minkowski background breaks the gauge group by selecting in it a preferred gravitational SU(2) subgroup. We expand the action around this background and find the spectrum of linearized theory to consist of the usual gravitons plus Yang-Mills fields charged under the centralizer of the SU(2) in G. In addition, there is a set of Higgs fields that are charged both under the gravitational and Yang-Mills subgroups. These fields are generically massive and interact with both gravity and the Yang-Mills sector in the standard way. The arising interaction of the Yang-Mills sector with gravity is also standard. Parameters such as the Yang-Mills coupling constant and Higgs mass arise from the potential function defining the theory. Both are realistic in the sense explained in the paper.

DOI: 10.1103/PhysRevD.81.085003 PACS numbers: 12.10.−g

I. INTRODUCTION

There have been numerous attempts to unify Einstein's theory of gravity with gauge fields describing other interactions. One such unification proposal is that of Kaluza-Klein, where the metric and gauge fields arise from a higher-dimensional metric tensor upon compactification of extra dimensions. This scenario has become an indispensable part of string theory, which also provides another unifying perspective by viewing gravity and Yang-Mills as excitations of closed and open strings, respectively. For more details on string-inspired unification schemes see a recent exposition [1].

There have also been attempts to unify gravity with gauge theory without introducing extra dimensions. There is, however, a very strong no-go theorem [2] that shows that at least one type of such unification is impossible. The theorem states that the symmetry group of the S-matrix of a consistent quantum field theory (in Minkowski spacetime) is the product of the Poincaré and internal gauge group. In other words, the spacetime and internal symmetries do not mix. The only way to go around this statement is via supersymmetric extensions of the Poincaré group [3].

Now, since gravity can be (at least loosely) viewed as a gauge theory for the diffeomorphism group, and the latter contains the Poincaré group as that of rigid global transformations, the Coleman-Mandula theorem [2] is sometimes interpreted as saying that no unification of gravity and gauge theory that puts together diffeomorphisms and gauge transformations is possible. In this discussion, however, one must be careful to distinguish between local gauge invariances of a theory and global symmetries whose presence or absence depends on a particular state one works with; see [4] that emphasizes this point.

While it may be difficult or impossible to "unify" diffeomorphisms and gauge transformations into a single gauge group, this is not the only possible way to approach the unification problem. To understand how a different type of unification might be possible, let us recall that in the so-called first-order formalism gravity becomes a theory of metrics as well as Lorentz group spin connections. The "internal" Lorentz group acts by rotating the frame and has no effect on the metric. Thus, the physical dynamical variable is still the metric; one simply added some gauge variables and enlarged the gauge group, which in this formulation is a (semi-) direct product of the diffeomorphism group and $SO(1, 3)$. Further, in the Hamiltonian formulation this theory can be easily cast into one on the Yang-Mills phase space. This is done by adding to the action a term that vanishes on shell [5]. The phase space is then that of pairs SU(2) connection plus the canonically conjugate "electric" field. Thus, after the trick of adding an on shell unimportant term, gravity becomes a generally covariant theory of an SU(2) connection. The spacetime metric (tetrad) is still a dynamical variable, but in this formulation it receives the interpretation of the momentum canonically conjugate to the connection.

Yang-Mills theory, on the other hand, after it is written for a general spacetime metric, also becomes a generally covariant theory of a connection and spacetime metric. One could then attempt to put the two generally covariant gauge theories together in some way that combines the internal gauge groups, while leaving the total gauge group to be a (semi-) direct product of diffeomorphisms and internal symmetries. This would not be in any conflict with the no-go theorem [2] for what is unified is not the Poincaré and internal symmetry groups. This might not be what can legitimately be called a unification, for the end gauge group is not simple, but this idea does lead to some interesting "unified" theories, as we hope to be able to demonstrate in this paper.

The first proposal of this type was put forward by Einstein and Mayer in [6] and later developed by, e.g., Hoffmann [7]. A more recent version of the same proposal appeared in [8,9], with the idea being precisely to extend the gauge group of gravity formulated in tetrad first-order formalism as a theory of the Lorentz connection. This proposal was later pushed forward in [10,11]; see also [12] for the most recent development. The key point of this proposal is that it is a nondegenerate metric that breaks the gauge symmetry of the unified theory down to a smaller group consisting of SO(1, 3) for gravity and some internal group for Yang-Mills fields.

A similar in spirit, but very different in the realization, idea was proposed in [13] and further developed in [14–16]. This approach stems from the fact that Einstein's general relativity (GR) can be reformulated as a theory on the Yang-Mills phase space. At the time of writing [13] it was achieved in Ashtekar's Hamiltonian formulation of GR [17] that interprets gravity as a special generally covariant (complexified) SU(2) gauge theory. The fact that gravity in this formulation becomes a theory of connection suggests that a gauge group larger than SU(2) can be considered. This is what was attempted in [13–16], with the main result of [16] being that Yang-Mills theory arises in an expansion of the theory around the de Sitter background. Another relevant reference is [18]; this gave a formulation of a unified Einstein-Maxwell theory based on a generalization of Plebański formalism [19].

The idea to put together the internal gauge groups of gravity and gauge theory is an interesting one. However, its particular realizations available in the literature are not without problems. Thus, the approach reviewed and further developed in [12] does a very good job of describing the fermionic content of the theory. Bosons, on the other hand, are described less convincingly in that many new propagating degrees of freedom (DOF) are introduced. The other approach [16] is also not very convincing since it works at the phase space level, and it is generally very difficult to approach a theory if no action principle is prescribed. Another aspect of the particular realization given in [16] is that it naturally describes a complexified GR put together with complexified Yang-Mills. No natural reality conditions that would convert this into a physical theory were given.

The unification by enlarging the internal gauge group proposal was recently revisited in [20], where the new action principle [21] for a class of modified gravity theories [22], extended to a larger gauge group, was used. This work also avoided the reality conditions problem by extending the gauge group of an explicitly real formulation of gravity that works with the Lorentz, not with the complexified rotation group. Specifically, it was suggested in [20] that the action of the type proposed in [21] considered for a general Lie group G describes gravity in its SO(4) part plus Yang-Mills fields in the remaining quotient $G/\mathrm{SO}(4)$. As in [16], the Yang-Mills coupling constant is related in [20] to the cosmological constant. As in the approach of [8,9], in [20] it is a nondegenerate metric that breaks the symmetry down to a smaller gauge group. The approach of [20] is also similar to that of [8,9] in that many new bosonic degrees of freedom are introduced. Thus, it was shown in [23] that the BF-type action of [21] for $G = \mathrm{SO}(4)$ no longer describes pure gravity theory but now describes six new DOF.

In this paper we take the described unification idea one step further. Our approach is similar in spirit to [20] in that we start from an action principle of the type first proposed in [21]. However, unlike in [20], we interpret only a (complexified) SU(2) subgroup of the gauge group G as that corresponding to gravity. The part of the gauge group that commutes with this gravitational SU(2) is then seen to describe Yang-Mills fields, and the part that does not commute with SU(2) describes charged scalar, i.e. Higgs, fields. We note that the suggestion that in unifications of this type the "off-diagonal" part of the Lie algebra that corresponds to Higgs fields is contained already in [20].

Our approach is also similar to the original proposal [16] that enlarged the SU(2) gravitational gauge group. However, in contrast to [16] that worked at the phase space level, our starting point is an action principle that makes a much more systematic analysis possible. Also the details of our proposal differ significantly from that of [16] in that a semirealistic (more on this below) unification is achieved without the need for a cosmological constant. Thus, the Yang-Mills (YM) coupling constant in our scheme is related not to the cosmological constant, which we set to zero, but to a certain other parameter of the theory. These features of our proposal also make it different from that of [20]. We also note that some details of our proposal are quite similar to that of [18], e.g., the fact that the reality conditions play an important role; our Lagrangian is, however, different from the one studied in this reference.

More specifically, we start from a generally covariant gauge theory for a (complex) semisimple Lie group G, with certain reality conditions later imposed to select real physical configurations. A particularly simple solution of the theory describes Minkowski spacetime. This solution breaks G down to a (complexified) SU(2) times the centralizer of SU(2) in G. The spectrum of linearized theory around the Minkowski background is then shown to consist of the usual gravitons with their two propagating DOF, gauge bosons charged under the centralizer of SU(2) in G, and a set of scalar Higgs fields. The Higgs fields are in general massive, with the mass being related to a certain parameter of the potential defining the theory. After the reality conditions are imposed, all sectors of the theory have a positive-definite Hamiltonian. We also work out interactions to cubic order and show that all interactions are precisely as expected. That is, all nongravitational fields interact with gravity via their stress-energy tensor, and the interactions in the nongravitational sector are also standard and are as expected for Higgs fields. Thus, our unification scheme passes the zeroth order test of being not in any obvious contradiction with observations. However, to obtain a truly realistic unification model, many problems have to be solved. In particular, fermionic DOF are not considered in this paper at all. Thus, our results provide only one of the first steps along this potentially interesting research direction. We return to open questions of our approach in the discussion section.

In this paper we have illustrated the general G case by considering the simplest nontrivial example of $G = \mathrm{SU}(3)$. This example is rather generic, and the same technology that we develop for $G = \mathrm{SU}(3)$ can be used for any Lie group. We could have presented a general semisimple case treatment phrased in terms of the root basis in the Lie algebra. However, at this stage of the development of the theory it is not clear whether there is any added value in doing things in full generality. We thus decided to keep our discussion as simple as possible and treat one example that, if necessary, is easily extendible to the general situation.

Another general remark on this paper is as follows. As the reader will undoubtedly notice, a sizable part of our paper is occupied by the Hamiltonian analysis of various sectors, or of the full theory. We also always give the Lagrangian treatment in which things are much more transparent. Thus, it might at first sight seem that the Hamiltonian formulation only clutters the exposition. We, however, believe that some aspects of the theory are much clearer precisely in the Hamiltonian formulation. For instance, our treatment of the reality conditions heavily uses the Hamiltonian analysis, and it would be very hard to arrive at the correct conditions without it. This is our main reason for carrying out such an analysis in all cases that are discussed.

Our final remark is concerning our strategy of dealing with the reality conditions. As we have already mentioned, we start with a complexified theory, and only at the end are the reality conditions imposed. In this paper appropriate reality conditions are deduced and dealt with at the linearized level, i.e. are imposed on the perturbation fields only. This is sufficient for both the classical linearized theory and the quantum theory if the latter is considered perturbatively. It would be very interesting to formulate the reality conditions nonperturbatively as well [at least classically such a formulation exists for the SU(2) gravitational sector], but we do not consider this problem in the present paper.

The organization of the paper is as follows. In Sec. II we define the class of generally covariant gauge theories that is the subject of this paper. Section III performs a Legendre transformation that introduces the two-form field as the main dynamical variable and rewrites the action of our theory in a form most useful for practical computations. In Sec. IV we sketch the Hamiltonian analysis and count the number of propagating DOF. Section V contains a general discussion on the problem of linearization. In Sec. VI we warm up by considering the case of pure gravity corresponding to $G = \mathrm{SU}(2)$. The Minkowski space background that we expand about is described here. Section VII deals with an example of a nontrivial group for which we take $G = \mathrm{SU}(3)$. It is here that we obtain a Lagrangian describing the YM and Higgs sectors of our model. In Sec. VIII we deduce interactions between various sectors of our model and show that they are the standard interactions expected from such fields. In Sec. IX we consider a more general set of defining potentials and show how Higgs masses are generated. We conclude with a summary and discussion.

II. A CLASS OF GENERALLY COVARIANT GAUGE THEORIES

We start by giving the most compact formulation of our class of theories. This is not the formulation that is most suited for practical computations, but it is conceptually the simplest.

According to our proposal, a theory that unifies gravity with gauge fields is simply the most general generally covariant group G gauge theory. Thus, consider a connection A^I in the principal G bundle over the spacetime manifold M. As is usual in physics literature, the bundle is assumed to be trivial, so the connection can be viewed as a Lie-algebra-valued one-form on M. The group G that we consider is a general semisimple complex Lie group. Reality conditions will later need to be imposed to select a sector of the theory that corresponds to a particular metric signature. Note, however, that at this point there is no metric; the only dynamical variable of our theory is the connection A^I.

As we have said, the idea is to consider the most general gauge and diffeomorphism invariant action that can be constructed from A^I. The following simple construction, generalizing verbatim considerations [24] for the case of pure gravity, provides a Lagrangian with the required properties. Being gauge invariant, it must involve only the curvature two-form $F^I = dA^I + (1/2)[A, A]^I$, where $[\cdot, \cdot]^I$ is the Lie bracket and the wedge product of forms is assumed. Consider the four-form $F^I \wedge F^J$. This is a four-form valued in the space of symmetric bilinear forms in $\mathfrak{g}$, the Lie algebra of G. Choosing an arbitrary volume four-form (vol) we can write $F^I \wedge F^J = (\mathrm{vol})\Omega^{IJ}$, where now Ω^{IJ} is a symmetric $n \times n$ matrix, where $n = \dim(\mathfrak{g})$. Since (vol) is defined only modulo rescalings $(\mathrm{vol}) \to \alpha(\mathrm{vol})$, so is the matrix Ω^{IJ} that under such rescalings transforms as $\Omega^{IJ} \to (1/\alpha)\Omega^{IJ}$. Let us now introduce a function $f(X)$ of symmetric $n \times n$ matrices X^{IJ} with the following properties. First, the function has to be gauge invariant: $f(\mathrm{ad}_g X) = f(X)$, where ad_g is the adjoint action of the gauge group on the space of symmetric bilinear forms on the Lie algebra. Second, the function must be holomorphic (we work with complex-valued quantities). Third, and most important, the function must be homogeneous of degree one $f(\alpha X) = \alpha f(X)$. This property allows it to be applied to the four-form $F^I \wedge F^J$, with the result being again a four-form. Indeed, we have $f(F^I \wedge F^J) = (\mathrm{vol}) f(\Omega^{IJ})$, and it is easy to see that due to the homogeneity of $f(\cdot)$, the resulting four-form does not depend on which particular volume form (vol) is chosen. Thus, the quantity $f(F^I \wedge F^J)$ is an invariantly defined four-form, and it can be integrated over the spacetime manifold to produce an action:

$$S[A] = \int_M f(F^I \wedge F^J). \tag{1}$$

As we have already said, the action is complex, so later certain reality conditions will be imposed.

The presented formulation (1) is conceptually nice, but it is very difficult to deal with in practice. One of the main reasons for this is that there is no natural background around which the theory can be expanded to produce a physically meaningful perturbation theory. This can be seen as follows. The first variation of the action (1) is given by

$$\delta S = \int \frac{\partial f}{\partial F^I} \wedge D_A \delta A^I, \tag{2}$$

where the derivative of $f(\cdot)$ with respect to F^I can be shown to make sense and is a certain $\mathfrak{g}$-valued two-form. The second variation is given by

$$\delta^2 S = \int \frac{1}{2} \frac{\partial f}{\partial F^I} \wedge [\delta A, \delta A]^I + \frac{\partial^2 f}{\partial F^I \partial F^J} D_A \delta A^I \wedge D_A \delta A^J, \tag{3}$$

where the second derivative of $f(\cdot)$ is a zero-form. Now, the most natural "vacuum" of the theory seems to be

$$F^I = 0, \qquad \frac{\partial f}{\partial F^I} = 0, \qquad \frac{\partial^2 f}{\partial F^I \partial F^J} \neq 0. \tag{4}$$

Indeed, this would indeed be a vacuum of the theory in the sense that the first derivative of the "potential" function vanishes, which then automatically satisfies the field equations $D_A(\partial f/\partial F^I) = 0$, and only the second derivative is nontrivial. From (3) we see that in this case the first "mass" term is absent, and there is only the "kinetic" term for the connection. Thus, it seems like the perfect vacuum to expand about. However, an immediate problem with this vacuum is that in the absence of any background structure the second derivative in (4) can be proportional only to the Killing form g^{IJ}, which then gives a degenerate kinetic term. So, there does not seem to be any way to build a meaningful perturbation theory around (4).

As an aside remark, we mention that the fact that the kinetic form in (3) is necessarily degenerate is very important for the possibility to describe gravity as a gauge theory. Indeed, as work [25] showed, general relativity can be put in the form (1) for $G = \mathrm{SU}(2)$ and a very special choice of the function $f(\cdot)$. At the same time, it is known to be impossible to describe gravity that is mediated by a spin two particle in terms of a gauge field that corresponds to a spin one particle. The resolution of this seeming paradox lies in the fact that the pure connection formulation (1) of gravity does not allow for a well-defined perturbation theory around the Minkowski background, and so the particles that it describes are not spin one as would be the case in any other gauge theory. Below we shall see how the usual spin two graviton arises via a certain "duality" trick.

A conventional perturbative treatment for theory (1) is possible, but requires a rather strange, at least from the pure connection point of view, choice of vacuum. Thus, as we shall see in details in the following sections, the usual perturbative expansion around a flat metric corresponds in the pure connection language to an expansion around the following point:

$$F^I = 0, \qquad \frac{\partial f}{\partial F^I} \neq 0. \tag{5}$$

This is a very strange point to expand the theory about, for one seems to be sitting at a point that is not a minimum of the potential. However, the nonzero right-hand side of the first derivative of the potential receives the interpretation of essentially the Minkowski metric, and a usual expansion then results. It might seem that this choice introduces a mass term for the connection, but this is not so. In fact, the second kinetic term is still a total derivative and plays no role, and there is only the mass term. However, as we shall see, the connection is no longer a natural variable in this case, and one works with a certain new two-form field B^I via which the connection is expressed as $A^I \sim \partial B^I$, so what appears as a mass term is, in fact, the usual kinetic one but for the two-form field.

This discussion motivates introduction of a new set of dynamical fields. These are originally introduced via the standard "Legendre transform" trick so that integrating them out one gets an original action (1). However, one can then also integrate out the original connection field and obtain a theory for the new fields only. This point of view turns out to be very profitable, and we develop it in the next section.

III. TWO-FORM FIELD FORMULATION

There are at least two different ways to arrive at the new formulation. One of them is via a Legendre transform from (1), the other one by thinking about generalizations of BF theory.

A. Legendre transform

As we have already explained, we introduce a new set of fields, given by a $\mathfrak{g}$-valued two-form B^I. The action that we would like to consider is then of BF-type and is given by

$$S[A, B] = \int_M g_{IJ} B^I \wedge F^J - \frac{1}{2} V(B^I \wedge B^J). \tag{6}$$

Here $V(\cdot)$ is again a G-invariant, holomorphic, and homogeneous order one function of symmetric $n \times n$ matrices, and as such it can be applied to the four-form $B^I \wedge B^J$, with the result again being a four-form. The quantity g_{IJ} is the Killing-Cartan form on $\mathfrak{g}$.

Integrating out B^I by solving its field equation,

$$F^I = \frac{1}{2}\frac{\partial V}{\partial B^I}, \tag{7}$$

which is algebraic in B^I, we get back the formulation (1) with $f(\cdot)$ being an appropriate Legendre transform of $V(\cdot)$. However, the formulation (6) is much more powerful in that we can now choose a constant B^I background and obtain a well-defined perturbation theory. We will later see how both gravity and Yang-Mills theory appear in such a perturbative expansion.

While the theories (1) and (6) are obviously classically equivalent, it may appear that this equivalence does not extend to the quantum theory. Indeed, with the action (6) depending on B^I in an essentially nonlinear way, the result of integration out of the two-form field in quantum theory is much more involved than in the classical one, where one simply solves for B^I from its field equation and substitutes the result back into the action. In contrast, in quantum theory the resulting "partition function" as a function of F^I contains both the classical terms, which, if computation is carried out via perturbation theory appear as tree-level diagrams, as well as additional terms coming from loop diagrams. Thus, it appears that what (6) produces once B^I is integrated out is much more involved than the theory (1). However, this conclusion misses an important point. It turns out that in the theory (6) there are second-class constraints. For this reason, the integration measure in the space of B^I fields is nontrivial and needs to be corrected by the determinant of the matrix of commutators of constraints. It can be shown that the correcting determinant is just that of the matrix $\partial^2 V/\partial B^I \partial B^J$ of second derivatives of the potential. The effect of this determinant can be taken into account by introducing ghost variables. One can then see that, once the ghost loops are allowed, all loop diagrams cancel, and the result of path integration over B^I with the correct measure is exactly given by (the exponent of) (1). This shows that the theories (6) and (1) are, in fact, equivalent as quantum theories as well, once it is taken into account that the integration measure over B^I is nontrivial.

An alternative viewpoint on the "Legendre transform" described is as follows. As we shall see below, the new two-form field that we have introduced is essentially the momentum canonically conjugate to the connection A^I. Thus, a meaningful analogy for the relation between (1) and (6) is the relation between Lagrangian and Hamiltonian formulation of mechanics. The former one uses only position variables as dynamical variables, but leads to second-derivative equations of motion. The latter contains an independent variable—momentum—and leads to first-order equations of motion. Thus, loosely speaking, the action (6) can be referred to as (1) written in the "Hamiltonian form" in which the momentum variable becomes an independent dynamical field.

Before we proceed with an analysis of properties of the theory (6), we would like to present an alternative derivation of this action.

B. Generalization of BF theory

An alternative way to arrive at (6) is to consider possible ways to generalize the topological BF theory. For the case of $G = \mathrm{SU}(2)$ this was done in [26], and here we generalize this analysis to a semisimple Lie group. Following this reference we begin with the action

$$S[A, B] = \int g_{IJ} B^I \wedge F^J - \frac{1}{2}\Phi_{IJ} B^I \wedge B^J, \tag{8}$$

where B^I is a two-form valued in $\mathfrak{g}$, F^I is the curvature $F^I = dA^I + \frac{1}{2} f^I_{JK} A^J \wedge A^K$ of A^I, f^I_{JK} are the structure constants, and Φ^{IJ} is a function (zero-form) valued in the symmetric product of two copies of $\mathfrak{g}$. At this stage this quantity is undetermined. But we should say now that it is not to be thought of as an independent field to be varied with respect to, for it will later be fixed by Bianchi identities. Note that only the symmetric part of Φ^{IJ} enters the action, and this is why it is assumed symmetric from the beginning. Our conventions are that we raise and lower indices with the Killing-Cartan metric g_{IJ} and its inverse g^{IJ}. We also note that for a semisimple Lie algebra we can always find a basis in which the metric is diagonal, i.e. $g_{IJ} = \delta_{IJ}$, where δ_{IJ} is the Kronecker delta.

Varying this action with respect to the connection A^L and the field B^L we get, respectively,

$$D_A B^I \equiv dB^I + f^I_{JK} A^J \wedge B^K = 0, \tag{9}$$

$$F^I = \Phi^I_J B^J. \tag{10}$$

We see that the idea of the above action ansatz is to generalize BF theory in such a way that Eq. (9) relating B and A is unchanged, while we now allow for a nonzero curvature. As we have already said, we do not consider a variation with respect to Φ^{IJ} because we will later show that the Bianchi identities fix this quantity in terms of certain components of the two-form field B^I.

Let us now take the covariant exterior derivative of (10) and use (9) together with the Bianchi identity $D_A F^I = 0$. We obtain

$$D_A \Phi^I_J \wedge B^J = 0. \tag{11}$$

Now, the covariant exterior derivative of $D_A B^I$ is

$$D_A(D_A B^I) = f^I_{JK} dA^J \wedge B^K + f^I_{JK} f^K_{LM} A^J \wedge A^L \wedge B^M. \tag{12}$$

Using the Jacobi identity $f^N_{IJ} f^L_{NK} + f^N_{JK} f^L_{NI} + f^N_{KI} f^L_{NJ} = 0$, the equation above can be rewritten as

$$D_A(D_A B^I) = f^I_{JL} F^J \wedge B^L, \tag{13}$$

and using Eqs. (9) and (10) we get

$$f^I_{JL} \Phi^J_K B^K \wedge B^L = 0. \tag{14}$$

Let us now compute the wedge product between (11) and the one-form $\iota_\xi B^I$, which has components $(\iota_\xi B^I)_\mu = \xi^\alpha B^I_{\alpha\mu}$, where ξ is an arbitrary vector field. We get

$$D\Phi_{IJ} \wedge \iota_\xi B^{(I} \wedge B^{J)} = 0. \tag{15}$$

But using $\iota_\xi B^{(I} \wedge B^{J)} = \frac{1}{2}\iota_\xi(B^I \wedge B^J)$, we can rewrite this as

$$D\Phi_{IJ} \wedge \iota_\xi(B^I \wedge B^J) = 0. \tag{16}$$

Let us now define the internal metric h^{IJ} by means of the following relation:

$$B^I \wedge B^J = h^{IJ}(\mathrm{vol}), \tag{17}$$

where (vol) is an arbitrary volume four-form. We can then rewrite (16) as

$$h_{IJ} D\Phi^{IJ} \wedge \iota_\xi(\mathrm{vol}) = 0. \tag{18}$$

Using the definition of h^{IJ}, we can also rewrite (14) as

$$f^I_{JK} \Phi^J_L h^{LK} = 0. \tag{19}$$

Now, computing $h_{IJ} D\Phi^{IJ}$,

$$h_{IL} D\Phi^{IL} = h_{IL}(d\Phi^{IL} + 2f^I_{JK} A^J \Phi^{KL}), \tag{20}$$

we can see that the second term in the right-hand side vanishes because of (19) and the condition that the Lie algebra is semisimple. The latter is used because for a semisimple Lie algebra it is possible to define a Killing-Cartan metric, in our case δ_{IJ}, with respect to which the object $f_{IJK} = \delta_{IL} f^L_{JK}$ is completely antisymmetric. Our final result is

$$h_{IJ} \partial_\mu \Phi^{IJ} \xi^\mu = 0, \tag{21}$$

which implies

$$h_{IJ} \partial_\mu \Phi^{IJ} = 0, \tag{22}$$

since ξ is an arbitrary vector.

The above equation implies that the quantities h^{IJ} and Φ^{IJ} are not independent. Let us define the "potential function" $V := h^{IJ}\Phi_{IJ}$. Then,

$$dV = \Phi_{IJ} dh^{IJ} + h_{IJ} d\Phi^{IJ} = \Phi_{IJ} dh^{IJ}, \tag{23}$$

where we have used (22). This means the following: (a) the potential V is only a function of h^{IJ}, i.e., $V = V(h^{IJ})$; (b) the quantities Φ^{IJ} are given,

$$\Phi_{IJ} = \frac{\partial V}{\partial h^{IJ}}; \tag{24}$$

and (c) the potential V is a homogeneous function of order one in h^{IJ} since

$$V = h^{IJ} \frac{\partial V}{\partial h^{IJ}}. \tag{25}$$

Thus, using the above definition of h^{IJ}, and the fact that $V(\cdot)$ is homogeneous, we can rewrite the action (8) as

$$S = \int g_{IJ} B^I \wedge F^J - \frac{1}{2} V(B^I \wedge B^J), \tag{26}$$

which is exactly the action (6) we have obtained in the previous subsection.

C. parametrizations of the potential

As defined so far, the theory is specified by the potential function $V(\cdot)$. In the action (6) it is applied to a four-form, which makes things rather inconvenient in practice, since we do not have much experience with functions of forms. Thus, it is desirable to rewrite it as a usual function of a matrix. We have already discussed how to do it by introducing an auxiliary volume form, but it would be nice if we could avoid any arbitrariness such as that of rescalings of (vol). A possible way to do this is as follows. With our choice of conventions $dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = -\tilde{\epsilon}^{\mu\nu\rho\sigma} d^4x$, and we have

$$\begin{aligned} B^I \wedge B^J &= \tfrac{1}{4} B^I_{\mu\nu} B^J_{\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \\ &= -\tfrac{1}{4} \tilde{\epsilon}^{\mu\nu\rho\sigma} B^I_{\mu\nu} B^J_{\rho\sigma} d^4x, \end{aligned} \tag{27}$$

where $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ is a density weight one object that does not require a metric for its definition. Thus, if we now define a densitized "internal metric"

$$\tilde{h}^{IJ} = \tfrac{1}{4} B^I_{\mu\nu} B^J_{\rho\sigma} \tilde{\epsilon}^{\mu\nu\rho\sigma}, \tag{28}$$

we can write the action as

$$S[B, A] = \int g_{IJ} B^I \wedge F^J + \frac{1}{2} V(\tilde{h}) d^4x. \tag{29}$$

Thus, the potential function is now applied to an $n \times n$ matrix (densitized), and its derivatives can be computed via the usual partial differentiation. For example, the first variation of this action can be seen to be given by

$$\delta S = \int \delta B^I \wedge \left(g_{IJ} F^J - \frac{\partial V(\tilde{h})}{\partial \tilde{h}^{IJ}} B^J \right) - g_{IJ} D_A B^I \wedge \delta A^J. \tag{30}$$

Indeed, the variation of the last, potential term is given by

$$\frac{1}{2} \int \frac{\partial V(\tilde{h})}{\partial \tilde{h}^{IJ}} \frac{1}{2} \delta B^I_{\mu\nu} B^J_{\rho\sigma} \tilde{\epsilon}^{\mu\nu\rho\sigma} d^4x = -\int \frac{\partial V(\tilde{h})}{\partial \tilde{h}^{IJ}} \delta B^I \wedge B^J, \tag{31}$$

where the matrix of first derivatives $(\partial V(\tilde{h})/\partial \tilde{h}^{IJ})$ is an object of density weight zero. Then, the field equations of our theory can be written as

$$F_I = \frac{\partial V(\tilde{h})}{\partial \tilde{h}^{IJ}} B^J, \tag{32}$$

$$DB^I \equiv dB^I + f^I_{JK} A^J \wedge B^K = 0. \tag{33}$$

In the literature on this class of theories a different parametrization of the potential is sometimes used; see e.g. the original paper [21], and also the unification paper [20]. Thus, to avoid having to take a function of forms, and/or having to work with a homogeneous function, one can parametrize the potential so that an ordinary function of one less variable arises. This can be done via a Legendre transform trick. Thus, we introduce a new variable Ψ^{IJ} that is required to be tracefree $g_{IJ}\Psi^{IJ} = 0$. The idea is that the matrix Ψ^{IJ} is the trace-free part of the matrix of first derivatives $\Phi^{IJ} = (\partial V/\partial \tilde{h}^{IJ})$. In other words, let us write

$$\Phi_{IJ} = \Psi_{IJ} - \frac{\Lambda}{n} g_{IJ}, \tag{34}$$

where Ψ_{IJ} is traceless. With Φ^{IJ} being a function of $\tilde{h}^{IJ}$, so is the trace part Λ. However, we can also declare Λ to be a function of Ψ^{IJ}, make Ψ^{IJ} an independent variable, and write the action in the form

$$S[B, A, \Psi] = \int g_{IJ} B^I \wedge F^J - \frac{1}{2}\left(\Psi_{IJ} - \frac{\Lambda(\Psi)}{n} g_{IJ}\right) B^I \wedge B^J. \tag{35}$$

Varying the action with respect to Ψ^{IJ} one gets an equation for this matrix, which, after being solved and substituted into the action, gives back (29) with $V(\cdot)$ being an appropriate Legendre transform of $\Lambda(\Psi)$. In the formulation (35) the function $\Lambda(\Psi)$ is an arbitrary function of a trace-free matrix Ψ^{IJ}, so there is no complication of having to require $V(\cdot)$ to be homogeneous. This formulation was used in the first papers on this class of theories, but it was later realized that the formulation that works solely with the two-form field B^I is more convenient. Thus, we do not use (35) in this paper.

IV. HAMILTONIAN ANALYSIS

To exhibit the physical content of the above theory, it is useful to perform the canonical analysis. After the 3 + 1 decomposition the action reads, up to an unimportant overall numerical factor,

$$S = \int dt \int_\Sigma d^3x (\tilde{P}^{aI} \dot{A}^I_a - H), \tag{36}$$

where

$$\tilde{P}^{aI} := \tilde{\epsilon}^{abc} B^I_{bc}, \tag{37}$$

and the Hamiltonian H is

$$-\tilde{H} = A^I_0 D_a \tilde{P}^{aI} + B^I_{0a} \tilde{\epsilon}^{abc} F^I_{bc} - V(B^{(I}_{0a} \tilde{P}^{aJ)}). \tag{38}$$

If we dealt with the pure BF theory, the last potential term would be absent and all the quantities B^I_{0a} would be Lagrange multipliers. However, now the Lagrangian is not linear in B^I_{0a}, and, as we shall see, all but four of these quantities are no longer Lagrange multipliers and should be solved for. The equations one obtains by varying the Lagrangian with respect to B^I_{0a} are

$$\tilde{\epsilon}^{abc} F^I_{bc} = V^{IJ}_{(1)} \tilde{P}^{aJ}, \tag{39}$$

where $V^{IJ}_{(1)}$ denotes the matrix of first partial derivatives of the function $V(\cdot)$ with respect to its arguments:

$$V^{IJ}_{(1)} := \frac{\partial V(\tilde{h})}{\partial \tilde{h}^{IJ}}. \tag{40}$$

Equation (39) can be solved in quite a generality by finding a convenient basis in the Lie algebra. Thus, consider the momenta $\tilde{P}^{aI}$. There are at least $n-3$ vectors N^I_α, $\alpha = 1, \ldots, n-3$, that are orthogonal to the momenta:

$$\tilde{P}^{aI} N^I_\alpha = 0, \qquad \forall a, \alpha. \tag{41}$$

These vectors can be chosen [uniquely up to $SO(n-3)$ rotations] by requiring

$$N^I_\alpha N^I_\beta = \delta_{\alpha\beta}. \tag{42}$$

We can then use the quantities $\tilde{P}^{aI}$, $a = 1, 2, 3$, N^I_α, $\alpha = 1, \ldots, n-3$, as a basis in the Lie algebra.

We can now decompose the quantity B^I_{0a} as

$$B^I_{0a} = \tilde{P}^{bI} \underset{\sim}{B}_{ab} + N^I_\alpha B^\alpha_a, \tag{43}$$

where $\underset{\sim}{B}_{ab}$ and B^α_a are components of B^I_{0a} in this basis. There are in total $3n$ components of B^I_{0a}, and they are represented here as nine quantities $\underset{\sim}{B}_{ab}$ as well as $3(n-3)$ quantities B^α_a. The argument of the function $V(\cdot)$ is now given by

$$B^{(I}_{0a} \tilde{P}^{aJ)} = \tilde{P}^{b(I} \tilde{P}^{aJ)} \underset{\sim}{B}_{ab} + N^{(I}_\alpha B^\alpha_a \tilde{P}^{aJ)}. \tag{44}$$

It is clear that this depends only on the symmetric part $\underset{\sim}{B}_{ab}$ of the components $\underset{\sim}{B}_{ab}$. Thus, the antisymmetric part of this 3×3 matrix cannot be determined from Eq. (39), and thus N^a in $\underset{\sim}{B}_{[ab]} := (1/2)\epsilon_{abc} N^c$ remain Lagrange multipliers. It is also clear that due to the homogeneity of $V(\cdot)$ one more component of B^I_{0a} cannot be solved for. This can be chosen, for example, to be the trace part $B^I_{0a}\tilde{P}^{aI}$, which will then play the role of the lapse function. All other $6 + 3(n-3) - 1$ components of B^I_{0a} can be solved for a generic function $V(\cdot)$, i.e. under the condition that the matrix of second derivatives of $V(\cdot)$ is nondegenerate. We are not going to demonstrate this in full generality, but will verify it in the linearized theory below.

After the quantities B^I_{0a} are solved for, we substitute them into (38) and obtain the following Hamiltonian:

$$-\tilde{H} = A^I_0 D_a \tilde{P}^{aI} + N^a \tilde{P}^{bI} F^I_{ab} + \tilde{N} \Lambda(F, P), \tag{45}$$

where $\tilde{N}$ is the lapse function and $\Lambda(F, P)$ is an appropriate

Legendre transform of $V(\cdot)$ that now becomes a function of the curvature F^I_{ab} and momentum $\tilde{P}^{aI}$. Thus, there are n Gauss as well as four diffeomorphism constraints in the theory. It should be possible to check by an explicit computation that they are first class, as was done, for example, for the case of $G = \mathrm{SU}(2)$ in [27], but we shall not attempt this here, postponing such an analysis till the linearized case considerations. The above arguments allow a simple count of the degrees of freedom described by the theory: we have $3n$ configurational degrees of freedom minus n Gauss constraints minus 4 diffeomorphisms, thus leading to $2n - 4$ DOF. Thus, when $G = K \times \mathrm{SU}(2)$, the above count of DOF gives the right number for a gravity plus K Yang-Mills theory. For a general G one might suspect that the centralizer of the gravitational SU(2) describes Yang-Mills, while another part of the Lie algebra corresponds to some new kind of fields. Below we will unravel their nature by considering the linearized theory. We also note that the above count of degrees of freedom agrees with the one presented in [23] for the case $G = \mathrm{SO}(4)$. Thus, it was seen there that the theory describes in total $2 \cdot 6 - 4 = 8$ DOF, which were interpreted as those corresponding to two graviton polarizations plus six new DOF.

V. THE LINEARIZED THEORY: GENERAL CONSIDERATIONS

As we have seen in the previous section, the mechanism that selects the gravitational SU(2) in G is that the momentum variable $\tilde{P}^{aI}$ provides a map from the (co-) tangent space to the spatial slice into $\mathfrak{g}$. This selects a three-dimensional subspace in $\mathfrak{g}$ that plays the role of the gravitational gauge group. Below we are going to see this mechanism at play at the level of the Lagrangian formulation by studying the linearization of the action (6). In this section it will be convenient to introduce a certain numerical factor in front of this action so that the normalization of the graviton kinetic term in the case of gravity will come out right. Thus, we shall from now on consider the following action:

$$S[A, B] = 4\mathrm{i} \int_M g_{IJ} B^I \wedge F^J - \frac{1}{2} V(B^I \wedge B^J), \qquad (46)$$

where $\mathrm{i} = \sqrt{-1}$.

A. Kinetic term

In this section we present some general considerations that apply to any background. We specialize to the Minkowski spacetime background in the next section. Let us call the first term in (46) S_{BF} and the second potential term S_{BB}. Then, the second variation of S_{BF} is given by

$$\delta^2 S_{\mathrm{BF}} = 4\mathrm{i} \int 2\delta B^I \wedge D_A \delta A^I + B^I \wedge [\delta A, \delta A]^I, \qquad (47)$$

and the action linearized around B_0, A_0 is obtained by evaluating this on B_0, A_0.

As we have already mentioned, we are to view our theory as that of the two-form field B^I, with the connection A^I to be eliminated (whenever possible; see below) by solving its field equations. Thus, let us assume that we are given a background two-form B^I_0. The linearized connection is then to be determined from the linearized Eq. (9) that reads:

$$D_0 \delta B^I + [\delta A, B_0]^I = 0, \qquad (48)$$

where D_0 is the covariant derivative with respect to the background connection A^I_0. Now the background two-form B^I_0 is a map from the six-dimensional space of bivectors to $\mathfrak{g}$, and thus selects in $\mathfrak{g}$ at most a six-dimensional preferred subspace. Let us denote this subspace by $\mathfrak{k}$. This subspace may or may not be closed under Lie brackets, but for simplicity, in this paper we shall assume that our background B^I_0 is such that $\mathfrak{k}$ is a Lie subalgebra (below we shall make an even stronger assumption about $\mathfrak{k}$). It is then clear that the part of δA^I that lies in the centralizer of $\mathfrak{k}$ in $\mathfrak{g}$ drops from Eq. (48) and cannot be solved for. As we shall later see, this will be the part of the group that is to describe Yang-Mills fields. The other part of δA^I can in general be found. For this part of the connection both terms in (47) are of the same form due to (48), and the linearized action can be written compactly as

$$\delta^2 S_{\mathrm{BF}} = 4\mathrm{i} \int \delta B^I \wedge D_0 \delta A^I, \qquad (49)$$

where δA^I has to be solved for from (48). On the other hand, for the subgroup of $\mathfrak{g}$ that centralizes $\mathfrak{k}$ the last term in (47) is absent, and we have

$$\delta^2 S_{\mathrm{BF}} = 8\mathrm{i} \int \delta B^I \wedge D_0 \delta A^I. \qquad (50)$$

Thus, our analysis of the kinetic term is going to be different for different parts of the Lie algebra.

B. Potential term

In this subsection we compute the second variation of the potential term S_{BB} and discuss how it can be evaluated on a given background. We have

$$\delta^2 S_{\mathrm{BB}} = 4\mathrm{i} \int 2 \frac{\partial^2 V(\tilde{h})}{\partial \tilde{h}^{KL} \partial \tilde{h}^{IJ}} (B_0 \delta B)^{IJ} (B_0 \delta B)^{KL} + \frac{\partial V(\tilde{h})}{\partial \tilde{h}^{IJ}} (\delta B \delta B)^{IJ}, \qquad (51)$$

where the integration measure d^4x is implied, and we have introduced notations

$$(B_0 \delta B)^{IJ} = \tfrac{1}{4} \tilde{\epsilon}^{\mu\nu\rho\sigma} B^{(I}_{0\mu\nu} \delta B^{J)}_{\rho\sigma}, \qquad (\delta B \delta B)^{IJ} = \tfrac{1}{4} \tilde{\epsilon}^{\mu\nu\rho\sigma} \delta B^I_{\mu\nu} \delta B^J_{\rho\sigma}, \qquad (52)$$

where the matrix of second derivatives is of density weight minus one.

Let us now discuss how the derivatives of the potential can be computed. In general, with the potential function $V(\tilde{h})$ being the homogeneous order one function of an $n \times n$ matrix, it can be reduced to a function of ratios of its invariants. A subset of invariants is obtained by considering traces of powers of $\tilde{h}^{IJ}$. However, in general these are not all invariants, and other invariants will be introduced and discussed below in section IX. But for now, to simplify the discussion, let us consider a special class of potentials that depend only on the invariants obtained as the traces of powers of $\tilde{h}^{IJ}$. Many aspects of our theory can be seen already for this special choice. Thus, consider the potential of the form

$$V = \frac{\mathrm{Tr}\tilde{h}}{n} f\left(\frac{\mathrm{Tr}\tilde{h}^2}{(\mathrm{Tr}\tilde{h})^2}, \ldots, \frac{\mathrm{Tr}\tilde{h}^n}{(\mathrm{Tr}\tilde{h})^n}\right), \tag{53}$$

where f is now an arbitrary function of its $n-1$ arguments, $\mathrm{Tr}\tilde{h} = g_{IJ}\tilde{h}^{IJ}$, and

$$\mathrm{Tr}\,\tilde{h}^p = \tilde{h}^{M_1}{}_{M_2}\tilde{h}^{M_2}{}_{M_3}\cdots\tilde{h}^{M_p}{}_{M_1}, \tag{54}$$

for $p \geq 2$. In view of the fact that the rank of $\tilde{h}^{IJ}$ is at most six, not all the invariants are independent, so we could consider only 5 first arguments of $f(\cdot)$. Note that $f(\cdot)$ here is distinct from the function used in the action (1) in the pure connection formulation of our theory: it is now an arbitrary function of its arguments, while this symbol in (1) stands for a homogeneous order one function.

The parametrization given allows derivatives to be computed. Thus, the first derivative of the potential function with respect to $\tilde{h}^{IJ}$ is

$$\frac{\partial V(\tilde{h})}{\partial \tilde{h}^{IJ}} = \frac{g_{IJ}}{n} f + \frac{\mathrm{Tr}\tilde{h}}{n}\frac{\partial f}{\partial \tilde{h}^{IJ}}, \tag{55}$$

with $(\partial f/\partial \tilde{h}^{IJ})$ given by

$$\frac{\partial f}{\partial \tilde{h}^{IJ}} = \sum_{p=2}^{n} f'_p \frac{\partial}{\partial \tilde{h}^{IJ}}\left(\frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^p}\right) = \sum_{p=2}^{n} p f'_p\left(\frac{\tilde{h}^{p-1}_{IJ}}{(\mathrm{Tr}\tilde{h})^p} - \frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^{p+1}} g_{IJ}\right), \tag{56}$$

where f'_p is the derivative of f with respect to its argument $(\mathrm{Tr}\tilde{h}^p/(\mathrm{Tr}\tilde{h})^p)$ and $\tilde{h}^p_{IJ}$ is

$$\tilde{h}^{\,p}_{IJ} = \tilde{h}_{IM_1}\tilde{h}^{M_1}{}_{M_2}\cdots\tilde{h}^{M_{p-1}}{}_J. \tag{57}$$

The second derivative of $V(\tilde{h})$ is given by

$$\frac{\partial^2 V(\tilde{h})}{\partial \tilde{h}^{KL}\partial \tilde{h}^{IJ}} = \frac{g_{IJ}}{n}\frac{\partial f}{\partial \tilde{h}^{KL}} + \frac{g_{KL}}{n}\frac{\partial f}{\partial \tilde{h}^{IJ}} + \frac{\mathrm{Tr}\tilde{h}}{n}\frac{\partial^2 f}{\partial \tilde{h}^{KL}\partial \tilde{h}^{IJ}}, \tag{58}$$

with $(\partial^2 f/\partial \tilde{h}^{KL}\partial \tilde{h}^{IJ})$ given by

$$\frac{\partial^2 f}{\partial \tilde{h}^{KL}\partial \tilde{h}^{IJ}} = \sum_{p=2}^{n}\sum_{q=2}^{n} f''_{pq}\frac{\partial}{\partial \tilde{h}^{IJ}}\left(\frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^p}\right)\frac{\partial}{\partial \tilde{h}^{KL}}\left(\frac{\mathrm{Tr}\tilde{h}^q}{(\mathrm{Tr}\tilde{h})^q}\right) + \sum_{p=2}^{n} f'_p \frac{\partial^2}{\partial \tilde{h}^{KL}\partial \tilde{h}^{IJ}}\left(\frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^p}\right), \tag{59}$$

where f''_{pq} stands for the derivative of f'_p with respect to its q argument and

$$\frac{\partial^2}{\partial \tilde{h}^{KL}\partial \tilde{h}^{IJ}}\left(\frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^p}\right) = \frac{p}{(\mathrm{Tr}\tilde{h})^p}\frac{\partial \tilde{h}^{p-1}_{IJ}}{\partial \tilde{h}^{KL}} - \frac{p^2\tilde{h}^{p-1}_{IJ}}{(\mathrm{Tr}\tilde{h})^{p+1}} g_{KL} - \frac{p^2\tilde{h}^{p-1}_{KL}}{(\mathrm{Tr}\tilde{h})^{p+1}} g_{IJ} + \frac{p(p+1)\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^{p+2}} g_{IJ}g_{KL}, \tag{60}$$

with

$$\frac{\partial \tilde{h}^{p-1}_{IJ}}{\partial \tilde{h}^{KL}} = g_{I(K}\tilde{h}_{L)M_1}\cdots\tilde{h}^{M_{p-3}}{}_J + \tilde{h}_{I(K}\tilde{h}_{L)M_1}\cdots\tilde{h}^{M_{p-4}}{}_J + \cdots + \tilde{h}_{IM_1}\cdots\tilde{h}^{M_{p-3}}{}_{(K}g_{L)J}. \tag{61}$$

With the above formulas for the first and second derivatives of the potential, it is relatively easy to find the linearized action for any semisimple Lie group.

VI. THE $G = \mathrm{SU}(2)$ CASE: GRAVITY

As we have already mentioned, the case $G = \mathrm{SU}(2)$ describes (complexified) gravity theory. A particular choice of the potential function (see below) gives general relativity, while a general potential corresponds to a family of deformations of GR. In this section, as a warm-up to the general G case, we shall study the corresponding linearized theory. Such an analysis has already appeared in [28]. However, our method and goals here differ significantly from that reference.

A. The metric

To understand how the $G = \mathrm{SU}(2)$ case can describe gravity, we need to see how the spacetime metric described by the theory is encoded. The answer to this is very simple: there is a unique (conformal) metric that makes the triple B^i, where i is the $\mathfrak{su}(2)$ index, into a set of self-dual two-forms. This is the so-called Urbantke metric [29]

$$\sqrt{-g}\, g_{\mu\nu} \sim \epsilon^{ijk} B^i_{\mu\alpha} B^j_{\nu\beta} B^k_{\rho\sigma}\tilde{\epsilon}^{\alpha\beta\rho\sigma} \tag{62}$$

that is defined modulo an overall factor. We remind the reader that at this stage all our fields are complex, and later reality conditions will be imposed to select physical real Lorentzian signature metrics.

Alternatively, given a metric $g_{\mu\nu}$ one can easily construct a "canonical" triple of self-dual two-forms that encode all information about $g_{\mu\nu}$. This proceeds via introducing tetrad one-forms θ^I, with $I = 0, 1, 2, 3$ here. One then constructs the two-forms $\Sigma^{IJ} := \theta^I \wedge \theta^J$ and takes the self-dual part of Σ^{IJ} with respect to IJ. The resulting two-forms are automatically self-dual. They can be explicitly constructed by decomposing $I = (0, a)$ and then writing

$$\Sigma^a = \mathrm{i}\theta^0 \wedge \theta^a - \tfrac{1}{2}\epsilon^{abc}\theta^b \wedge \theta^c. \qquad (63)$$

Here $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. Its presence in this formula has to do with the fact that self-dual quantities in a spacetime of Lorentzian signature are necessarily complex. Thus, even though at this stage there is no well-defined signature (all quantities are complex), it is convenient to introduce i here so that later appropriate reality conditions are easily imposed. We note that internal Lorentz rotations of the tetrad θ^I at the level of Σ^a boil down to (complexified) SU(2) rotations of Σ^a.

A general $\mathfrak{su}(2)$-valued two-form field B^i carries more information than just that about a metric. Indeed, one needs 3×6 numbers to specify it, while only 10 are necessary to specify a metric. A very convenient description of the other components is obtained by introducing a metric defined by B^i via (62) and then using the "metric" self-dual two-forms (63) as a basis and decomposing

$$B^i = b^i_a \Sigma^a. \qquad (64)$$

The quantities b^i_a give nine components, the metric gives ten, and the choice of internal frame for Σ^a adds three more components. There is also a freedom of rescalings $b^i_a \to \Omega^{-2} b^i_a$ and $\Sigma^a \to \Omega^2 \Sigma^a$, as well as freedom of SO(3) rotations, acting simultaneously on Σ^a and b^i_a, overall producing 18 independent components of B^i.

When one substitutes the parametrization (64) into the action (6), one finds that the fields b^i_a are nonpropagating and should be integrated out. Once this is done, one obtains an "effective" Lagrangian for the metric described by Σ^a. Below we shall see how this works in the linearized theory. However, we first need to choose a background.

B. Minkowski background

The Minkowski background is described in our framework by a collection of metric two-forms (63) constructed from the Minkowski metric. Thus, we choose an arbitrary time plus space split and write

$$\Sigma^a_0 = \mathrm{i}dt \wedge dx^a - \tfrac{1}{2}\epsilon^{abc} dx^b \wedge dx^c, \qquad (65)$$

where dt, dx^a, $a = 1, 2, 3$, form a tetrad for the Minkowski metric $ds^2 = -dt^2 + \sum_a (dx^a)^2$. Our two-form field background is then chosen to be

$$B^i_0 = \delta^i_a \Sigma^a_0, \qquad (66)$$

where δ^i_a is an arbitrary SO(3) matrix that for simplicity can be chosen to be the identity matrix.

In what follows we will also need a triple of anti-self-dual metric forms that, together with (63), form a basis in the space of two-forms. A convenient choice is given by

$$\bar{\Sigma}{}^a_0 = \mathrm{i}dt \wedge dx^a + \tfrac{1}{2}\epsilon^{abc} dx^b \wedge dx^c. \qquad (67)$$

The following formulas, which can be shown to follow directly from definitions (65) and (67), are going to be very useful:

$$\Sigma^a_{0\mu\sigma}\Sigma_0^{b\sigma}{}_{\nu} = -\delta^{ab}\eta_{\mu\nu} + \epsilon^{abc}\Sigma^c_{0\mu\nu}, \qquad (68)$$

$$\Sigma_0^{a\mu\nu}\Sigma^b_{0\mu\nu} = 4\delta^{ab}, \qquad (69)$$

$$\epsilon^{abc}\Sigma^a_{0\mu\sigma}\Sigma_0^{b\sigma}{}_{\lambda}\Sigma^c_{0\lambda\mu} = -4!, \qquad (70)$$

$$\epsilon^{abc}\Sigma^a_{0\mu\nu}\Sigma^b_{0\rho\sigma}\Sigma_0^{d\nu\sigma} = -2\delta^{cd}\eta_{\mu\rho}, \qquad (71)$$

$$\Sigma^a_{0\mu\nu}\Sigma^a_{0\rho\sigma} = \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho} - \mathrm{i}\epsilon_{\mu\nu\rho\sigma}, \qquad (72)$$

where $\eta_{\mu\nu}$ is the Minkowski metric. We are going to refer to them as the algebra of Σ's.

The first of the relations above, namely (68), is central, for all others [apart from (72)] can be derived from it. It is useful to develop some basis-independent understanding of this relation. We are working with the Lie algebra $\mathfrak{su}(2)$ and are considering a basis X^a in it in which the structure constants read $[X^a, X^b] = \epsilon^{abc}X^c$. This is the basis given by $X^a = -(\mathrm{i}/2)\sigma^a$, where σ^a are Pauli matrices. The metric $g^{ab} = \delta^{ab}$ on the Lie algebra can be obtained as $g^{ab} = -2\,\mathrm{Tr}(X^a X^b)$. Then (68) can be understood as follows: the product of two Σ's is given by minus the metric plus the structure constants times Σ. We will see that in this form the relations (68) persist to any basis in $\mathfrak{su}(2)$.

C. Linearized action

We are now going to linearize the $G = \mathrm{SU}(2)$ theory around the background (66). Thus, we take

$$B^i = B^i_0 + b^i. \qquad (73)$$

As we have already discussed, to linearize the kinetic BF term of the action we need to solve for the linearized connection if we can. This is certainly possible for the case at hand, as we shall now see.

If we denote the linearized connection by a^i, we have to solve the following system of equations:

$$db^i + \epsilon^i_{jk} a^j \wedge B^k_0 = 0, \qquad (74)$$

where we have used the fact that the background connection is zero. It is convenient at this stage to replace all i indices by a ones, which we can do using the background object δ^i_a that provides such an identification. We can now use the self-duality $\epsilon^{\mu\nu\rho\sigma}\Sigma^a_{0\mu\nu} = 2i\Sigma_0^{a\mu\nu}$ of the background to rewrite this equation as

$$\frac{1}{2\mathrm{i}}\epsilon^{\mu\nu\rho\sigma}\partial_\nu b^a_{\rho\sigma} + \epsilon^{abc} a^b_\nu \Sigma_0^{c\mu\nu} = 0. \qquad (75)$$

We now multiply this equation by $\Sigma_0^{a\alpha\beta}\Sigma^d_{0\alpha\mu}$ and use the identity (71) to get

$$a^a_\beta = \frac{1}{2}\Sigma^b_{0\beta}{}^{\alpha}\Sigma^a_{0\alpha\mu}\frac{1}{2\mathrm{i}}\epsilon^{\mu\nu\rho\sigma}\partial_\nu b^b_{\rho\sigma}, \quad \text{or}$$
$$a^a_\beta = \frac{1}{4\mathrm{i}}\Sigma^b_{0\beta}{}^{\alpha}\Sigma^a_{0\alpha\mu}(\partial b^b)^\mu, \tag{76}$$

where we have introduced a compact notation:

$$(\partial b^b)^\mu := \epsilon^{\mu\nu\rho\sigma}\partial_\nu b^b_{\rho\sigma} \tag{77}$$

for a multiple of the Hodge dual of the exterior derivative of the perturbation two-form.

The BF part of the linearized action was obtained in (49). We need to divide the second variation given in this formula by 2 to get the correct action quadratic in the perturbation. Thus, we have

$$S^{(2)}_{\mathrm{BF}} = 2\mathrm{i}\int b^a \wedge da^a = -\mathrm{i}\int a^a_\mu(\partial b^a)^\mu, \tag{78}$$

where we have written everything in index notations and integrated by parts to put the derivative on $b^a_{\mu\nu}$, and used the definition (77). Now substituting (76) we get

$$S^{(2)}_{\mathrm{BF}} = \frac{1}{4}\int \eta^{\alpha\beta}\Sigma^a_{0\alpha\mu}(\partial b^b)^\mu\Sigma^b_{0\beta\nu}(\partial b^a)^\nu. \tag{79}$$

Let us now linearize the potential term. For this we need to know the background $\tilde{h}^{ij}$ as well as the matrices of first and second derivatives for the background. Using (65), it is easy to see that $\tilde{h}^{ij}_0 = 2\mathrm{i}\delta^{ij}$. Since the background volume form is just the identity, we can now safely remove the density weight symbol from the matrix $\tilde{h}^{ij}_0$. Also, as before, let us replace all i indices by a indices using δ^i_a. Using (55) and the fact that the first derivatives $(\partial f/\partial h^{ab})$ vanish on this background, we immediately get

$$\frac{\partial V}{\partial h^{ab}}\bigg|_{h_0} = \frac{\delta_{ab}}{3}f_0, \tag{80}$$

where f_0 is the background value of the function f in the parametrization (53). It is not hard to see that this value plays the role of the cosmological constant of the theory, so in our Minkowski background it is necessarily zero by the background field equations. The matrix of second derivatives of the potential is easily evaluated using (58), and we find

$$\frac{\partial^2 V}{\partial h^{cd}\partial h^{ab}}\bigg|_{h_0} = \frac{g}{2\mathrm{i}}\left(\delta_{a(c}\delta_{d)b} - \frac{1}{3}\delta_{ab}\delta_{cd}\right), \tag{81}$$

where we have introduced

$$g := \sum_{p=2,3}\frac{(f'_p)_0 p(p-1)}{3^p}. \tag{82}$$

This is a constant of dimensions of the cosmological constant $1/L^2$. It is going to play a role of a parameter determining the strength of gravity modifications.

We can now write the linearized potential term (51). We must divide it by two to get the correct action for the perturbation. This gives

$$S^{(2)}_{\mathrm{BB}} = -\frac{g}{2}\int\left(\delta_{a(c}\delta_{d)b} - \frac{1}{3}\delta_{ab}\delta_{cd}\right)(\Sigma^{a\mu\nu}_0 b^b_{\mu\nu}) \times (\Sigma^{c\rho\sigma}_0 b^d_{\rho\sigma}). \tag{83}$$

Note that the tensor in brackets here is just the projector on the trace-free part. This fact will be important in our Hamiltonian analysis below. Our total linearized action is thus (79) plus (83).

D. Symmetries

The quadratic form obtained above is degenerate, and its degenerate directions correspond to the symmetries of the theory. These are not hard to write down. An obvious symmetry is that under (complexified) SO(3) rotations of the fields. Considering an infinitesimal gauge transformation of the background $\Sigma^a_{0\mu\nu}$, we find that the action must be invariant under the following set of transformations:

$$\delta_\omega b^a_{\mu\nu} = \epsilon^{abc}\omega^b\Sigma^c_{0\mu\nu}, \tag{84}$$

where ω^a are infinitesimal generators of the transformation. It is clear that (83) is invariant since it involves only the ab-symmetric part of $(\Sigma^{a\mu\nu}_0 b^b_{\mu\nu})$, and the transformation (84) affects the antisymmetric part. Let us check the invariance of the kinetic term (79). We have the following expression for the variation:

$$\frac{1}{2}\int \eta^{\alpha\beta}\Sigma^a_{0\alpha\mu}(\partial\delta_\omega b^b)^\mu\Sigma^b_{0\beta\nu}(\partial b^a)^\nu. \tag{85}$$

Substituting here the expression (84) for the variation we find

$$\eta^{\alpha\beta}\Sigma^a_{0\alpha\mu}(\partial\delta_\omega b^b)^\mu\Sigma^b_{0\beta\nu} = 2\mathrm{i}\eta^{\alpha\beta}\Sigma^a_{0\alpha\mu}\epsilon^{bcd}\partial_\rho\omega^c\Sigma^{d\mu\rho}_0\Sigma^b_{0\beta\nu} = 4\mathrm{i}\partial_\nu\omega^i, \tag{86}$$

where we have used the self-duality of $\Sigma^a_{0\mu\nu}$ and applied the identity (71) once. Substituting this into (85) and integrating by parts to move the derivative from ω^a to b^a, we get under the integral $\epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\nu b^a_{\rho\sigma} = 0$, since the partial derivatives commute. This proves the invariance under gauge transformations.

Another set of symmetries of the action is that of diffeomorphisms. These are given by

$$\delta_\xi b^a = d\iota_\xi\Sigma^a_0, \tag{87}$$

where ι_ξ is the operator of the interior product with a vector field ξ^μ. It is not hard to compute this explicitly in terms of derivatives of the components of the vector field. However, we do not need all the details of this two-form. Indeed, let us first note that the first kinetic term of the action is, in fact, invariant under a larger symmetry:

$$\delta_\eta b^a = d\eta^a, \tag{88}$$

where η^a is an arbitrary Lie-algebra valued one-form. Indeed, this is obvious given that the kinetic term is constructed from the components of the three-form db^a given by the exterior derivative of the perturbation two-form. Thus, (88) indeed leaves the kinetic term invariant. Then, since (87) is of the form (88) with $\eta^a = \iota_\xi \Sigma_0^a$, we have the invariance of the first term. To see that the potential term (83) is invariant, we should simply show that the symmetric trace-free part of the matrix $(\Sigma_0 \delta_\xi b)^{ab}$ is zero. Let us compute the symmetric part explicitly. We have

$$\Sigma_0^{(a\mu\nu}\partial_\mu \xi^\rho \Sigma_{0\rho\nu}^{b)} = \delta^{ab}\partial_\rho \xi^\rho, \tag{89}$$

where we have used (68). Thus, there is only the trace symmetric part, so the part that enters into the variation of the action (85) is zero. This proves the invariance under diffeomorphisms. Note that the second potential term is not invariant under all transformations (88), since for such a transformation that is not a diffeomorphism the matrix $(\Sigma_0 \delta_\eta b)^{ab}$ contains a nontrivial symmetric trace-free part, as can be explicitly checked.

We will see that these are the only symmetries when we perform the Hamiltonian analysis. However, before we do this, let us show how the usual linearized GR appears from our theory.

E. Relation to GR

In this subsection we would like to describe how general relativity (linearized) with its usual gravitons appears from the linearized Lagrangian described above. We shall see that to get GR we must take the limit when the mass parameter g for the components $(\Sigma_0 b)^{ab}_{tf}$, where tf stands for the trace-free part, is sent to infinity. Indeed, the potential part (83) depends precisely on these components, and when the parameter g is sent to infinity these components are effectively set to zero. We shall now see that this gives GR.

It is not hard to show that in general the trace-free part $h^{tf}_{\mu\nu} := h_{\mu\nu} - (1/4)\eta_{\mu\nu}h^\rho_\rho$ of the metric perturbation $h_{\mu\nu}$ defined via $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ corresponds in our language of two-forms to the anti-self-dual part of the two-form perturbation:

$$(b^a_{\mu\nu})_{asd} = \Sigma^a_{0[\mu}{}^\rho h^{tf}_{\nu]\rho}. \tag{90}$$

The fact that this two-form is anti-self-dual can be easily checked by contracting it with $\Sigma_0^{b\mu\nu}$ and using the algebra (68). The result is zero, as appropriate for an anti-self-dual two-form. In addition to (90), there is in general also the self-dual part of the two-form perturbation. However, in the limit $g \to \infty$ all but the trace part of this gets set to zero by the potential term. The trace part, on the other hand, is proportional to the trace part $\eta^{\mu\nu}h_{\mu\nu}$ of the metric perturbation. To simplify the analysis, it is convenient to set this to zero $\eta^{\mu\nu}h_{\mu\nu} = 0$. This is allowed since in pure gravity the trace of the perturbation does not propagate. Then (90) is the complete two-form perturbation, and we can drop the tf symbol.

To simplify the analysis further, instead of deriving the full linearized action for the metric perturbation $h_{\mu\nu}$, let us work in the gauge where the perturbation is transverse $\partial^\mu h_{\mu\nu} = 0$. Let us then compute the quantity $(\partial b^a)^\mu$ in this gauge. Using anti-self-duality of $b^a_{\mu\nu}$ given by (90) we have

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu b^a_{\rho\sigma} = -2\mathrm{i}\partial_\nu b^{a\mu\nu}. \tag{91}$$

Substituting here the explicit expression (90) and using the transverse gauge condition, we get

$$(\partial b^a)^\mu = \mathrm{i}\Sigma_0^{a\nu\rho}\partial_\nu h^\mu_\rho. \tag{92}$$

We can now substitute this into the action (79) to get

$$\begin{aligned} S^{(2)} &= -\frac{1}{4}\int \eta^{\alpha\beta}\Sigma^a_{0\alpha\mu}\Sigma_0^{b\rho\sigma}\partial_\rho h^\mu_\sigma \Sigma^b_{0\beta\nu}\Sigma_0^{a\gamma\delta}\partial_\gamma h^\nu_\delta \\ &= -\frac{1}{4}\int \eta^{\alpha\beta}(\delta^\gamma_\alpha\delta^\delta_\mu - \delta^\delta_\alpha\delta^\gamma_\mu - \mathrm{i}\epsilon_{\alpha\mu}{}^{\gamma\delta}) \\ &\quad \times (\delta^\rho_\beta\delta^\sigma_\nu - \delta^\sigma_\beta\delta^\rho_\nu - \mathrm{i}\epsilon_{\beta\nu}{}^{\rho\sigma})\partial_\rho h^\mu_\sigma \partial_\gamma h^\nu_\delta, \end{aligned} \tag{93}$$

where we have used (72) to get the second line. We can now contract the indices and take into account the trace-free as well as the transverse condition on $h_{\mu\nu}$. We get the following simple action as the result:

$$S^{(2)} = -\frac{1}{2}\int \partial_\mu h_{\rho\sigma}\partial^\mu h^{\rho\sigma}, \tag{94}$$

which is the correctly normalized transverse traceless graviton action. Note that in the passage to GR we have secretly assumed that $h_{\mu\nu}$ in (90) is a real metric perturbation. Below we will see how to impose the reality conditions on our theory that this comes out. Also note that the sign in front of (94) is correct for our choice of the signature being $(-, +, +, +)$.

F. Hamiltonian analysis of the linearized theory

For a finite g our theory describes a deformation of GR. Since not all components of the two-form perturbation $b^a_{\mu\nu}$ are dynamical, the nature of this deformation is most clearly seen in the Hamiltonian framework. This is what this subsection is about.

We note that the outcome of this rather technical subsection is that at "low" energies $E^2 \ll g$ the modification can be ignored and one can safely work with the usual linearized GR. Thus, it may be advisable to skip this subsection on the first reading. Let us start by analyzing the kinetic BF part.

Kinetic term.—Expanding the product of two Σ matrices in (79) and using (68), we can write the linearized Lagrangian density for the BF part as

$$\mathcal{L}_{\rm BF} = \tfrac{1}{4}(\partial b^a)^\mu(\partial b^b)^\nu(\epsilon^{abc}\Sigma^c_{0\mu\nu} + \delta^{ab}\eta_{\mu\nu}). \tag{95}$$

Let us now perform the space plus time decomposition. Thus, we split the spacetime index as $\mu = (0, a)$, where $a = 1, 2, 3$. Note that we have denoted the spatial index by the same lowercase Latin letter from the beginning of the alphabet that we are already using to denote the internal $\mathfrak{su}(2)$ index. This is allowed since we can use spatial projection of the $\Sigma^a_{0\mu\nu}$ two-form to provide such an identification. Thus, from (63) we have

$$\Sigma^a_{0bc} = -\epsilon^a{}_{bc}, \tag{96}$$

and

$$\Sigma^a_{00b} = i\delta^a_b. \tag{97}$$

Let us now use these simple relations to obtain the space plus time decomposition of the Lagrangian. First, we need to know components of the $(\partial b^a)^\mu$ vector. The time component is given by

$$(\partial b^a)^0 = \epsilon^{0bcd}\partial_b b^a_{cd} = -\partial_b t^{ab}, \tag{98}$$

where our conventions are $\epsilon^{0abc} = -\epsilon^{abc}$ and we have introduced

$$t^{ab} := \epsilon^{bcd}b^a_{cd}. \tag{99}$$

The spatial component of $(\partial b^a)^\mu$ is given by

$$\begin{aligned}(\partial b^a)^b &= \epsilon^{b0cd}\partial_0 b^a_{cd} + 2\epsilon^{bc0d}\partial_c b^a_{0d}\\ &= \partial_0 t^{ab} - 2\epsilon^{bcd}\partial_c b^a_{0d}.\end{aligned} \tag{100}$$

Now, the Lagrangian (95) is given by

$$\begin{aligned}\mathcal{L}_{\rm BF} = &-\tfrac{1}{4}(\partial b^a)^0(\partial b^a)^0 + \tfrac{1}{2}(\partial b^a)^0(\partial b^b)^d\epsilon^{abc}\Sigma^c_{0d}\\ &+ \tfrac{1}{4}(\partial b^a)^e(\partial b^b)^f(\epsilon^{abc}\Sigma^c_{ef} + \delta^{ab}\delta_{ef}).\end{aligned} \tag{101}$$

Substituting the above expressions we get

$$\begin{aligned}\mathcal{L}_{\rm BF} = &-\frac{1}{4}\partial_b t^{ab}\partial_c t^{ac} - \frac{i}{2}\partial_d t^{ad}(\partial_0 t^{bc} - 2\epsilon^{cef}\partial_e b^b_{0f})\epsilon^{abc}\\ &-\frac{1}{4}(\partial_0 t^{ae} - 2\epsilon^{emn}\partial_m b^a_{0n})(\partial_0 t^{bf} - 2\epsilon^{fpq}\partial_p b^b_{0q})\\ &\times(\epsilon^{abc}\epsilon^c{}_{ef} - \delta^{ab}\delta_{ef}).\end{aligned} \tag{102}$$

Our fields are now therefore b^a_{0b} and t^{ab}. There will also be another, potential part to this Lagrangian, but it does not contain time derivatives, so the conjugate momenta can be determined already at this stage. Thus, it is clear that the field b^a_{0b} is nondynamical since the Lagrangian does not depend on its time derivatives. The momentum conjugate to t^{ab}, on the other hand, is given by

$$\begin{aligned}\pi^{ab} &:= \frac{\partial\mathcal{L}_{\rm BF}}{\partial(\partial_0 t^{ab})}\\ &= -\frac{i}{2}\epsilon^{abc}\partial_d t^{cd} - \frac{1}{2}(\partial_0 t^{ef} - 2\epsilon^{fpq}\partial_p b^e_{0q})\\ &\quad\times(\epsilon^{aec}\epsilon^{cbf} - \delta^{ae}\delta^{bf}).\end{aligned} \tag{103}$$

It is not hard to check that the momentum variable is simply related to the spatial projection of the connection (76) as

$$\pi^a_b = -2ia^a_b. \tag{104}$$

To rewrite the Lagrangian in the Hamiltonian form, one must solve for the velocities $\partial_0 t^{ab}$ in terms of the momenta π^{ab}. However, it is clear that not all the velocities can be solved for—there are constraints. A subset of these constraints is given by the $\mu = 0$ component of the (75) equation that, when written in terms of π^{ab}, becomes

$$\mathcal{G}^a := \epsilon^{abc}\pi^{bc} + i\partial_b t^{ab} = 0. \tag{105}$$

These are primary constraints that must be added to the Hamiltonian with Lagrange multipliers.

Thus, the expression for velocities in terms of momenta will contain undetermined functions. These functions are simply the a^a_0 components of the connection, as well as (at this stage undetermined) b^a_{0b} components of the two-form field. The expression for velocities is given by the spatial components of Eq. (75). After some algebra it gives

$$\partial_0 t^{ab} = 2\epsilon^{bef}\partial_e b^a_{0f} - 2\epsilon^{abc}a^c_0 - \epsilon^{aed}\epsilon^{dbf}\pi^{ef}. \tag{106}$$

Let us now obtain a slightly more convenient expression for the Lagrangian. Indeed, recall that using the compatibility equation between the connection and the two-form perturbation, we could have chosen to write our linearized action (78) as

$$S^{(2)}_{\rm BF} = -2i\int\epsilon^{abc}\Sigma^a_0\wedge a^b\wedge a^c = -2\int\Sigma^{a\mu\nu}\epsilon^{abc}a^b_\mu a^c_\nu. \tag{107}$$

Introducing the time plus space split and writing the result in terms of the momentum variable (104), we get the following Lagrangian:

$$\mathcal{L}_{\rm BF} = -2\epsilon^{abc}\pi^{ab}a^c_0 - \tfrac{1}{2}\epsilon^{aef}\epsilon^{abc}\pi^{be}\pi^{cf}. \tag{108}$$

We can now easily find the BF part of the Hamiltonian:

$$\begin{aligned}\mathcal{H}_{\rm BF} &= \pi^{ab}\partial_0 t^{ab} - \mathcal{L}_{\rm BF}\\ &= 2\pi^{ab}\epsilon^{bef}\partial_e b^a_{0f} - \tfrac{1}{2}\epsilon^{aef}\epsilon^{abc}\pi^{be}\pi^{cf}.\end{aligned} \tag{109}$$

We need to add to this the primary constraints (105) with Lagrange multipliers. Thus, the total Hamiltonian coming from the BF part of the action is

$$\mathcal{H}^{\rm total}_{\rm BF} = 2\pi^{ab}\epsilon^{bef}\partial_e b^a_{0f} - \tfrac{1}{2}\epsilon^{aef}\epsilon^{abc}\pi^{be}\pi^{cf} + \omega^a\mathcal{G}^a. \tag{110}$$

This is, of course, the standard result for the linearized BF Hamiltonian. If not for the potential term, the Hamiltonian would be a sum of terms generating the topological constraint $\partial_{[b}\pi^a_{c]} = 0$ and the Gauss constraint (105). Let us now consider the other "BB" part of the Lagrangian.

Potential part.—We can rewrite the linearized Lagrangian density for the BB part (83) as

$$\mathcal{L}_{\rm BB} = -\frac{g}{2}(b^{(a}_{\mu\nu}\Sigma_0^{b)\mu\nu})_{tf}(b^{(a}_{\rho\sigma}\Sigma_0^{b)\rho\sigma})_{tf}, \tag{111}$$

where tf stands for the trace-free parts of the matrices. Splitting the space and time indices gives

$$(b^{(a}_{\mu\nu}\Sigma_0^{b)\mu\nu})_{tf} = -(2{\rm i}b_0^{(ab)} + t^{(ab)})_{tf}, \tag{112}$$

and so

$$\mathcal{L}_{\rm BB} = -\frac{g}{2}(2{\rm i}b_0^{(ab)} + t^{(ab)})_{tf}(2{\rm i}b_0^{(ab)} + t^{(ab)})_{tf}. \tag{113}$$

Analysis of the constraints.—Thus, the total linearized Hamiltonian density $\mathcal{H} = \mathcal{H}^{\rm total}_{\rm BF} - \mathcal{L}_{\rm BB}$ is given by

$$\mathcal{H} = 2\pi^{ab}\epsilon^{bef}\partial_e b_0^{af} - \frac{1}{2}\epsilon^{aef}\epsilon^{abc}\pi^{be}\pi^{cf} + \omega^a\mathcal{G}^a + \frac{g}{2}(2{\rm i}b_0^{(ab)} + t^{(ab)})_{tf}(2{\rm i}b_0^{(ab)} + t^{(ab)})_{tf}.$$

It is now clear that only the antisymmetric part and trace parts of b_0^{ab} remain Lagrange multipliers in the full theory. These are the generators of the diffeomorphisms. The other part of b_0^{ab}, namely, the symmetric traceless, is clearly nondynamical and should be solved for from its field equations. Varying the Hamiltonian with respect to this symmetric trace-free part we get

$$(2{\rm i}b_0^{(ab)} + t^{(ab)})_{tf} = \frac{\rm i}{g}(\epsilon^{ef(a}\partial_e\pi^{b)}_f)_{tf}. \tag{114}$$

Now writing

$$b_0^{ab} = {\rm i}N\delta^{ab} + \tfrac{1}{2}\epsilon^{abc}N^c + (b_0^{(ab)})_{tf} \tag{115}$$

and substituting the symmetric trace-free part from (114), we get the following Hamiltonian:

$$\mathcal{H} = -2N{\rm i}\epsilon^{abc}\partial_a\pi_{bc} - 2\partial_{[a}\pi^a_{b]}N^b + \omega^a\mathcal{G}^a - \frac{1}{2}\epsilon^{aef}\epsilon^{abc}\pi^{be}\pi^{cf} + {\rm i}(\epsilon^{ef(a}\partial_e\pi^{b)}_f)_{tf}(t^{(ab)})_{tf} + \frac{1}{2g}(\epsilon^{ef(a}\partial_e\pi^{b)}_f)_{tf}(\epsilon^{pq(a}\partial_p\pi^{b)}_q)_{tf}. \tag{116}$$

The reason why we introduced a factor of i in front of the lapse function will become clear below. One can recognize in the first line the usual Hamiltonian, diffeomorphism, and Gauss linearized constraints of Ashtekar's Hamiltonian formulation of general relativity [17]. The terms in the second and third lines comprise the Hamiltonian. Finally, the last term is due to the modification and goes away in the limit $g \to \infty$.

It is not hard to show that the reduced phase space for the above system is obtained by considering π^{ab}, t^{ab} that are symmetric, traceless, and transverse $\partial_a\pi^{ab} = 0$, $\partial_a t^{ab} = 0$. On such configurations the matrix $\epsilon^{efa}\partial^e\pi^{fb}$ is automatically symmetric traceless and transverse. The reduced phase space Hamiltonian density is then given by

$$\mathcal{H}^{\rm phys} = \frac{1}{2}(\pi^{ab})^2 + {\rm i}\epsilon^{efa}\partial^e t^{fb}\pi^{ab} + \frac{1}{2g}(\partial^a\pi^{bc})^2, \tag{117}$$

where we have integrated by parts and put the derivative on t^{ab} in the second term. This Hamiltonian is complex, so we need to discuss the reality conditions.

Reality conditions.—So far our discussion was in terms of complex-valued fields. Thus, the reduced phase space obtained above after imposing the constraints and quotienting by their action was complex dimension 2 + 2. Reality conditions need to be imposed to select the physical phase space corresponding to Lorentzian signature gravity.

In the case of GR that corresponds to $g \to \infty$ the reality condition could be guessed from the form of the Hamiltonian (117). Indeed, we can write it as

$$\mathcal{H}^{\rm phys}_{\rm GR} = \tfrac{1}{2}(\pi^{ab} + {\rm i}\epsilon^{efa}\partial^e t^{fb})^2 + \tfrac{1}{2}(\partial^a t^{bc})^2. \tag{118}$$

Thus, it is clear that we just need to require t^{ab} and $\pi^{ab} + {\rm i}\epsilon^{efa}\partial^e t^{fb}$ to be real. This procedure, however, does not work for the full Hamiltonian because of the last term in (117).

Let us now note that the last term in (117), when written in momentum space, behaves as E^2/M^2, where E is the energy and $M^2 = g$ is the modification parameter. Thus, for energies $E \ll M$ the modification term is much smaller than the term π^2 and can be dropped. It is natural to expect that gravity is only modified close to the Planck scale, so it is natural to expect $M^2 \approx M_p^2$, where M_p is the Planck mass. With this assumption the last term in (117) is unimportant for "ordinary" energies and can be dropped. Thus, if we are to work at energies much smaller than the Planck scale's ones, then we do not need to go beyond GR described by the first two terms in (117).

The above discussion shows that a discussion of the reality conditions for the full Hamiltonian (117), even though possible and necessary if one is interested in the behavior of the theory close to the Planck scale, is not needed if one only wants to work with much smaller energies. For this reason, and in order not to distract the reader from the main line of the argument, a somewhat technical reality conditions discussion for the full theory is placed in the Appendix.

Now that we understand how the simplest case $G = SU(2)$ gives rise to gravity, we can apply the same procedure to more interesting cases of a larger gauge group. We consider the example of SU(3) that well illustrates the general pattern.

VII. THE $G = \mathrm{SU}(3)$ CASE: GRAVITY-MAXWELL SYSTEM

In this section we perform an analysis analogous to that in the previous section but taking a larger gauge group. As before, we first consider the complex theory, and only at the end do we impose the reality conditions. The end result of this section is a description of the 12 DOF that, as the analysis of Sec. IV shows, our theory must have in this case. These split as follows: we have 2 DOF for the gravity sector and 2 more for the U(1) YM, and the remaining 8 propagating DOF are those of the Higgs sector. Let us start by reviewing some basic facts about the $\mathfrak{su}(3)$ Lie algebra.

A. Lie algebra of SU(3)

The standard matrix representation of the Lie algebra of SU(3) consists of all traceless anti-Hermitian 3×3 complex matrices. The standard basis for $\mathfrak{su}(3)$ is given by the imaginary unit times a generalization of Pauli matrices, known as Gell-Mann matrices. These Hermitian matrices are given by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{119}$$

However, in our computations the Cartan-Weyl basis is going to be more convenient. Let us recall that in the Cartan-Weyl formalism one starts with the maximally commuting Cartan subalgebra, which in our case is spanned by two elements λ_3, λ_8. One then selects basis vectors that are eigenstates of the elements of Cartan under the adjoint action. This leads to the following basis (see [30,31]):

$$T_\pm = \frac{1}{\sqrt{2}}(T_x \pm \mathrm{i}T_y), \qquad V_\pm = \frac{1}{\sqrt{2}}(V_x \pm \mathrm{i}V_y),$$
$$W_\pm = \frac{1}{\sqrt{2}}(W_x \pm \mathrm{i}W_y), \qquad T_z = \frac{1}{2}\lambda_3, \qquad Y = \frac{1}{2}\lambda_8, \tag{120}$$

where $T_x = \frac{1}{2}\lambda_1$, $T_y = \frac{1}{2}\lambda_2$, $V_x = \frac{1}{2}\lambda_4$, $V_y = \frac{1}{2}\lambda_5$, $W_x = \frac{1}{2}\lambda_6$, and $W_y = \frac{1}{2}\lambda_7$. Then the Cartan subalgebra is $H_i = \mathrm{Span}(T_z, Y)$, and the commutator between any of the H_i's and the rest of the elements of the basis E_α, $E_\alpha = \{T_+, T_-, T_z, V_+, V_-, W_+, W_-\}$, is a multiple of E_α, i.e. $[H_i, E_\alpha] = \alpha_i E_\alpha$. One considers the α_i's, for $i = 1, 2$, as the components of a vector, called a root of the system. In this case we have six roots, i.e. $\{1, 0\}$, $\{-1, 0\}$, $\{\frac{1}{2}, \frac{\sqrt{3}}{2}\}$, $\{-\frac{1}{2}, -\frac{\sqrt{3}}{2}\}$, $\{-\frac{1}{2}, \frac{\sqrt{3}}{2}\}$, and $\{\frac{1}{2}, -\frac{\sqrt{3}}{2}\}$. The Lie brackets between elements of this basis are given in Table I. We also need to know the metric $g_{IJ} = -2\,\mathrm{Tr}(T_I T_J)$ in this basis. It is given in Table II.

B. Background

Let us now discuss how a background to expand around can be chosen. A background two-form field B_0^I is a map from the space of bivectors, which is six-dimensional, to the Lie algebra in question. Thus, its image is at most a six-dimensional subspace in $\mathfrak{su}(3)$. There are many different subspaces one can consider. In this paper we study the simplest possibility. Thus, we choose B_0^I such that the image of the space of two-forms that it produces in $\mathfrak{su}(3)$ is three-dimensional. Moreover, we choose this image to be an $\mathfrak{su}(2)$ Lie subalgebra. Even further, we choose this subalgebra to be that spanned by $\{T_+, T_-, T_z\}$. Clearly, this is not the only $\mathfrak{su}(2)$ subalgebra in $\mathfrak{su}(3)$. Other possibilities include $\{V_+, V_-, \frac{1}{2}(\sqrt{3}Y + T_z)\}$ and $\{W_+, W_-, \frac{1}{2} \times (\sqrt{3}Y - T_z)\}$. In this paper we do not study these different possibilities, leaving a more thorough investigation to fur-

TABLE I. Commutators among T_+, T_-, T_z, V_+, V_-, W_+, W_-, and Y.

$[\downarrow, \rightarrow]$	T_+	T_-	T_z	V_+	V_-	W_+	W_-	Y
T_+	0	T_z	$-T_+$	0	$-\frac{1}{\sqrt{2}}W_-$	$\frac{1}{\sqrt{2}}V_+$	0	0
T_-	$-T_z$	0	T_-	$\frac{1}{\sqrt{2}}W_+$	0	0	$-\frac{1}{\sqrt{2}}V_-$	0
T_z	T_+	$-T_-$	0	$\frac{1}{2}V_+$	$-\frac{1}{2}V_-$	$-\frac{1}{2}W_+$	$\frac{1}{2}W_-$	0
V_+	0	$-\frac{1}{\sqrt{2}}W_+$	$-\frac{1}{2}V_+$	0	$\frac{1}{2}(\sqrt{3}Y + T_z)$	0	$\frac{1}{\sqrt{2}}T_+$	$-\frac{\sqrt{3}}{2}V_+$
V_-	$\frac{1}{\sqrt{2}}W_-$	0	$\frac{1}{2}V_-$	$-\frac{1}{2}(\sqrt{3}Y + T_z)$	0	$-\frac{1}{\sqrt{2}}T_-$	0	$\frac{\sqrt{3}}{2}V_-$
W_+	$-\frac{1}{\sqrt{2}}V_+$	0	$\frac{1}{2}W_+$	0	$\frac{1}{\sqrt{2}}T_-$	0	$\frac{1}{2}(\sqrt{3}Y - T_z)$	$-\frac{\sqrt{3}}{2}W_+$
W_-	0	$\frac{1}{\sqrt{2}}V_-$	$-\frac{1}{2}W_-$	$-\frac{1}{\sqrt{2}}T_+$	0	$-\frac{1}{2}(\sqrt{3}Y - T_z)$	0	$\frac{\sqrt{3}}{2}W_-$
Y	0	0	0	$\frac{\sqrt{3}}{2}V_+$	$-\frac{\sqrt{3}}{2}V_-$	$\frac{\sqrt{3}}{2}W_+$	$-\frac{\sqrt{3}}{2}W_-$	0

TABLE II. Components for the internal metric in the base $\{T_+, T_-, T_z, V_+, V_-, W_+, W_-, Y\}$.

$\langle\downarrow\mid\rightarrow\rangle$	T_+	T_-	T_z	V_+	V_-	W_+	W_-	Y
T_+	0	−1	0	0	0	0	0	0
T_-	−1	0	0	0	0	0	0	0
T_z	0	0	−1	0	0	0	0	0
V_+	0	0	0	0	−1	0	0	0
V_-	0	0	0	−1	0	0	0	0
W_+	0	0	0	0	0	0	−1	0
W_-	0	0	0	0	0	−1	0	0
Y	0	0	0	0	0	0	0	−1

ther research. We believe that the example we choose to study is sufficiently illustrating.

Thus, our background is essentially the same as the one we considered in the previous section. This is motivated by our desire to have the usual gravity theory arising as the part of the larger theory we are now considering. Since in the general Lie-algebra context it is convenient to work with the Cartan-Weyl basis, we need to change the basis of basic two-forms (65) as well. This can be worked out as follows. In the previous section we were using a basis in the Lie algebra in which the structure constants were given by ϵ_{abc}. If we denote the corresponding generators by X_a, then $[X_a, X_b] = \epsilon_{abc} X_c$. On the other hand, for generators T_a used in (120) we have $[T_a, T_b] = \mathrm{i}\epsilon_{abc} T_c$. The relation between these two bases is $X_a = -\mathrm{i}T_a$. We can then define a new set of self-dual two-forms $\Sigma^\pm$, Σ^z via

$$\Sigma \equiv \sum_{a=1,2,3} \Sigma^a X_a = \Sigma^+ T_+ + \Sigma^- T_- + \Sigma^z T_z. \tag{121}$$

This gives

$$\Sigma^+ = \frac{-\mathrm{i}}{\sqrt{2}}(\Sigma^1 - \mathrm{i}\Sigma^2), \qquad \Sigma^- = \frac{-\mathrm{i}}{\sqrt{2}}(\Sigma^1 + \mathrm{i}\Sigma^2), \qquad \Sigma^z = -\mathrm{i}\Sigma^3. \tag{122}$$

The $\mathfrak{su}(3)$-valued two-form Σ is our background to expand about.

C. Linearization: Kinetic term

As before, the first step of the linearization procedure is to solve for those components of the connection for which this is possible. As we have discussed in Sec. V, this is in general possible for the components of the connection in the directions in the Lie algebra that do not commute with the directions spanned by the background two-forms. In our case these are the directions spanned by $T_\pm$, T_z and $V_\pm$, $W_\pm$. We already know how to solve for the connection components in the directions $T_\pm$, T_z. Indeed, the solution is given by (76), which we just have to rewrite in the different basis. It is, however, more practical to solve the equations once more by working in the different basis from the very beginning.

The $\mathfrak{su}(2)$ part.—The $\mathfrak{su}(2)$ sector equations in the Cartan-Weyl basis are

$$\begin{aligned} db^+ + a^z \wedge \Sigma^+ - a^+ \wedge \Sigma^z &= 0, \\ db^- + a^- \wedge \Sigma^z - a^z \wedge \Sigma^- &= 0, \\ db^z + a^+ \wedge \Sigma^- - a^- \wedge \Sigma^+ &= 0. \end{aligned} \tag{123}$$

We rewrite them in spacetime notations, take the Hodge dual, and use the self-duality of the $\Sigma^\pm$, Σ^z matrices to get

$$\begin{aligned} \frac{1}{2\mathrm{i}}(\partial b^+)^\mu + a^z_\nu \Sigma^{+\mu\nu} - a^+_\nu \Sigma^{z\mu\nu} &= 0, \\ \frac{1}{2\mathrm{i}}(\partial b^-)^\mu + a^-_\nu \Sigma^{z\mu\nu} - a^z_\nu \Sigma^{-\mu\nu} &= 0, \\ \frac{1}{2\mathrm{i}}(\partial b^z)^\mu + a^+_\nu \Sigma^{-\mu\nu} - a^-_\nu \Sigma^{+\mu\nu} &= 0, \end{aligned} \tag{124}$$

where the notation is, as before $(\partial b)^\mu = \epsilon^{\mu\nu\rho\sigma}\partial_\nu b_{\rho\sigma}$. We now need the algebra of the new Σ matrices. It can be worked out from the relations (122) and the algebra (68). We get

$$\begin{aligned} &\Sigma^+_{\mu\sigma}\Sigma^{-\sigma}{}_\nu = \eta_{\mu\nu} + \Sigma^z_{\mu\nu}, \qquad \Sigma^z_{\mu\sigma}\Sigma^{+\sigma}{}_\nu = \Sigma^+_{\mu\nu}, \\ &\Sigma^z_{\mu\sigma}\Sigma^{-\sigma}{}_\nu = -\Sigma^-_{\mu\nu}, \qquad \Sigma^z_{\mu\sigma}\Sigma^{z\sigma}{}_\nu = \eta_{\mu\nu}, \\ &\Sigma^+_{\mu\sigma}\Sigma^{+\sigma}{}_\nu = 0, \qquad \Sigma^-_{\mu\sigma}\Sigma^{-\sigma}{}_\nu = 0. \end{aligned} \tag{125}$$

For purposes of the calculation it is very convenient to rewrite these relations in the schematic form, by viewing them as matrix algebra. Our matrix multiplication convention for the two-forms is $(XY)_\mu{}^\nu = X_\mu{}^\rho Y_\rho{}^\nu$. We have

$$\begin{aligned} &\Sigma^+\Sigma^- = \eta + \Sigma^z, \qquad \Sigma^z\Sigma^+ = \Sigma^+, \\ &\Sigma^z\Sigma^- = -\Sigma^-, \qquad \Sigma^z\Sigma^z = \eta, \\ &\Sigma^+\Sigma^+ = 0, \qquad \Sigma^-\Sigma^- = 0. \end{aligned} \tag{126}$$

This is precisely the relations (68), just written in terms of metric and the structure constants on $\mathfrak{su}(2)$ for a different basis.

In matrix product conventions, Eqs. (124) take the following transparent form:

$$\begin{aligned} \frac{1}{2\mathrm{i}}(\partial b^+) + \Sigma^+ a^z - \Sigma^z a^+ &= 0, \\ \frac{1}{2\mathrm{i}}(\partial b^-) + \Sigma^z a^- - \Sigma^- a^z &= 0, \\ \frac{1}{2\mathrm{i}}(\partial b^z) + \Sigma^- a^+ - \Sigma^+ a^- &= 0, \end{aligned} \tag{127}$$

where the convention is that the second spacetime index of Σ is contracted with the spacetime index of a.

We can now solve (127) by using the algebra (126). To this end we multiply the first equation by Σ^+ and the second one by Σ^-. This leads to two equations involving only $a^\pm$ but not a^z. We can obtain another two equations of the same sort by multiplying the last equation in (124) by $\Sigma^\pm$. Then adding and/or subtracting the resulting equations we get

$$a^+ = -\frac{1}{4i}(\Sigma^-\Sigma^+(\partial b^+) + \Sigma^+(\partial b^z)),$$
$$a^- = -\frac{1}{4i}(\Sigma^+\Sigma^-(\partial b^-) - \Sigma^-(\partial b^z)). \quad (128)$$

To obtain the last component of the connection, we multiply the first equation in (127) by Σ^- and the second by Σ^+, and then subtract the resulting equations. We find $\Sigma^- a^+ - \Sigma^+ a^- = -(1/2i)(\partial b^z)$ using (128). We get

$$a^z = -\frac{1}{4i}((\partial b^z) + \Sigma^-(\partial b^+) - \Sigma^+(\partial b^-)). \quad (129)$$

It is now easy to write the $\mathfrak{su}(2)$ part of the linearized BF part of the action. Using the metric components given in Table II, from (78) we have

$$S_{\rm BF}^{\mathfrak{su}(2)} = -\frac{1}{4}\int(\partial b^+)(\Sigma^+\Sigma^-(\partial b^-) - \Sigma^-(\partial b^z)) + (\partial b^-)(\Sigma^-\Sigma^+(\partial b^+) + \Sigma^+(\partial b^z)) + (\partial b^z)((\partial b^z) + \Sigma^-(\partial b^+) - \Sigma^+(\partial b^-)), \quad (130)$$

where again our convenient schematic form of the notation is used. This is simplified to give

$$S_{\rm BF}^{\mathfrak{su}(2)} = -\frac{1}{2}\int(\partial b^+)(\eta + \Sigma^z)(\partial b^-) + (\partial b^-)\Sigma^+(\partial b^z) - (\partial b^+)\Sigma^-(\partial b^z) + \frac{1}{2}(\partial b^z)(\partial b^z). \quad (131)$$

We could now use this as the starting point of the Hamiltonian analysis similar to the one in the previous section. However, it is clear that its results are basis independent, so we do not need to repeat it. Still, the above considerations are quite useful as a warm-up for the more involved analysis that now follows.

The part that does not commute with $\mathfrak{su}(2)$.—Let us denote the four directions $V_\pm$, $W_\pm$ collectively by index $\alpha = 4, 5, 6, 7$. We have to solve the following system of equations:

$$db^\alpha + f^\alpha{}_{\beta a} a^\beta \wedge \Sigma^a = 0, \quad (132)$$

where the terms $f^\alpha_{ab} a^a \wedge \Sigma^b$ are absent since the corresponding structure constants are zero. Explicitly, using Table I we have

$$db^4 - \frac{1}{\sqrt{2}}a^6 \wedge \Sigma^+ - \frac{1}{2}a^4 \wedge \Sigma^z = 0, \quad (133)$$

$$db^5 + \frac{1}{\sqrt{2}}a^7 \wedge \Sigma^- + \frac{1}{2}a^5 \wedge \Sigma^z = 0, \quad (134)$$

$$db^6 - \frac{1}{\sqrt{2}}a^4 \wedge \Sigma^- + \frac{1}{2}a^6 \wedge \Sigma^z = 0, \quad (135)$$

$$db^7 + \frac{1}{\sqrt{2}}a^5 \wedge \Sigma^+ - \frac{1}{2}a^7 \wedge \Sigma^z = 0. \quad (136)$$

We can solve this system using the same technology that we used above for the $\mathfrak{su}(2)$ sector. Thus, we take the Hodge dual of the above equations, use the self-duality of the Σ's, and rewrite everything in the schematic matrix form. We get

$$\frac{1}{2i}(\partial b^4) - \frac{1}{\sqrt{2}}\Sigma^+ a^6 - \frac{1}{2}\Sigma^z a^4 = 0,$$
$$\frac{1}{2i}(\partial b^5) + \frac{1}{\sqrt{2}}\Sigma^- a^7 + \frac{1}{2}\Sigma^z a^5 = 0,$$
$$\frac{1}{2i}(\partial b^6) - \frac{1}{\sqrt{2}}\Sigma^- a^4 + \frac{1}{2}\Sigma^z a^6 = 0,$$
$$\frac{1}{2i}(\partial b^7) + \frac{1}{\sqrt{2}}\Sigma^+ a^5 - \frac{1}{2}\Sigma^z a^7 = 0. \quad (137)$$

We can now manipulate these equations using the algebra (126). Thus, let us multiply the third equation by $\sqrt{2}\Sigma^+$ and subtract the result from the first equation. This gives

$$\frac{1}{2i}(\partial b^4) - \frac{\sqrt{2}}{2i}\Sigma^+(\partial b^6) + \left(\eta + \frac{1}{2}\Sigma^z\right)a^4 = 0. \quad (138)$$

It is now easy to find a^4 by noting that $(\eta + (1/2)\Sigma^z)^{-1} = (4/3)(\eta - (1/2)\Sigma^z)$. Thus, we have

$$a^4 = \frac{1}{3i}(\sqrt{2}\Sigma^+(\partial b^6) - (2\eta - \Sigma^z)(\partial b^4)). \quad (139)$$

Similarly, we multiply the last equation by $\sqrt{2}\Sigma^-$ and add it to the second equation. Multiplying then by the inverse of $(\eta - (1/2)\Sigma^z)$, we get

$$a^5 = -\frac{1}{3i}(\sqrt{2}\Sigma^-(\partial b^7) + (2\eta + \Sigma^z)(\partial b^5)). \quad (140)$$

To find a^6 we multiply the first equation by $\sqrt{2}\Sigma^-$ and subtract the result from the third equation. We then multiply the result by the inverse of $(\eta - (1/2)\Sigma^z)$. We get

$$a^6 = \frac{1}{3i}(\sqrt{2}\Sigma^-(\partial b^4) - (2\eta + \Sigma^z)(\partial b^6)). \quad (141)$$

Finally, to find a^7 we multiply the second equation by $\sqrt{2}\Sigma^+$ and add the result to the last equation. Multiplying the result by the inverse of $(\eta + (1/2)\Sigma^z)$, we get

$$a^7 = -\frac{1}{3i}(\sqrt{2}\Sigma^+(\partial b^5) + (2\eta - \Sigma^z)(\partial b^7)). \quad (142)$$

We should now substitute the above results into the relevant part of the action. This is again obtained from (78) by taking into account the expression for the metric. We shall refer to this part of the action as "Higgs" in view of its interpretation to be developed later. We have

$$S_{\rm BF}^{\rm Higgs} = {\rm i}\int a^4(\partial b^5) + a^5(\partial b^4) + a^6(\partial b^7) + a^7(\partial b^6), \tag{143}$$

where we took into account the extra minus sign that comes from the metric. Substituting here the above connections, we get, after some simple algebra,

$$S_{\rm BF}^{\rm Higgs} = \frac{2}{3}\int \sqrt{2}(\partial b^5)\Sigma^+(\partial b^6) - \sqrt{2}(\partial b^4)\Sigma^-(\partial b^7) - (\partial b^4)(2\eta + \Sigma^z)(\partial b^5) - (\partial b^6)(2\eta - \Sigma^z)(\partial b^7). \tag{144}$$

A more illuminating way to write this action is by introducing two two-component fields:

$$\begin{pmatrix} b^4 \\ b^6 \end{pmatrix}, \qquad \begin{pmatrix} b^5 \\ b^7 \end{pmatrix}. \tag{145}$$

It is not hard to see that this split of the Higgs sector part of the Lie algebra is just the split into two irreducible representation spaces with respect to the action of the gravitational SU(2). In terms of these columns the above action takes the following form:

$$S_{\rm BF}^{\rm Higgs} = \frac{2}{3}\int ((\partial b^5)(\partial b^7))\begin{pmatrix} -2\eta + \Sigma^z & \sqrt{2}\Sigma^+ \\ \sqrt{2}\Sigma^- & -2\eta - \Sigma^z \end{pmatrix} \times \begin{pmatrix} (\partial b^4) \\ (\partial b^6) \end{pmatrix}. \tag{146}$$

Below we will use this action as the starting point for an analysis that will eventually exhibit the physical DOF propagating in this sector.

Centralizer U(1) *part.*—We cannot solve for the components of the connection in the part that commutes with $\mathfrak{su}(2)$. In our case this is the direction Y of the Lie algebra. We shall refer to this part of the action as "YM." Thus, the action remains of BF-type:

$$S_{\rm BF}^{\rm YM} = -4{\rm i}\int b^8 \wedge da^8, \tag{147}$$

where the extra minus sign is the one in the metric.

D. Linearization: Potential term

As in the SU(2) case our background internal metric $\tilde{h}_0^{IJ}$ is just $2{\rm i}g^{ab}$ in the $\mathfrak{su}(2)$ directions and zero in all other directions. Since the background metric is flat, we shall drop the tilde from $\tilde{h}^{IJ}$ in this section. We compute the matrix of first derivatives of the potential using (55). We get

$$\frac{\partial V}{\partial h^{ab}}\bigg|_0 = \frac{f_0}{8}g_{ab}, \tag{148}$$

$$\frac{\partial V}{\partial h^{a\alpha}}\bigg|_0 = 0, \tag{149}$$

$$\frac{\partial V}{\partial h^{\alpha\beta}}\bigg|_0 = \left(\frac{f_0}{8} - \frac{1}{8}\sum_{p=2}^{6}(f'_p)_0\frac{p}{3^{p-1}}\right)g_{\alpha\beta}. \tag{150}$$

Here f_0, $(f'_p)_0$ are the value of the function and its derivatives at the background, and index α stands for all directions in the Lie algebra that are not in $\mathfrak{su}(2)$. The quantity f_0 can be identified with a multiple of the cosmological constant. More specifically,

$$\Lambda = -\frac{3f_0}{8}. \tag{151}$$

Let us also define another constant of dimensions $1/L^2$:

$$\kappa \equiv \frac{1}{8}\sum_{p=2}^{6}(f'_p)_0\frac{p}{3^{p-1}}. \tag{152}$$

Then we have

$$\frac{\partial V}{\partial h^{\alpha\beta}}\bigg|_0 = -(\Lambda/3 + \kappa)g_{\alpha\beta}. \tag{153}$$

The sum here and in the previous formula is taken over $p = 2, \ldots, 6$, because the function f can at most depend on five ratios of six invariants of the matrix h^{IJ}. It has at most only six independent invariants since it is constructed from the map $B^I_{\mu\nu}$ that has the rank at most six. Since we want to work with the Minkowski spacetime background, we should set $\Lambda = 0$, which we do in what follows.

We now need to compute the matrix of second derivatives. Let us first obtain its $\mathfrak{su}(2)$ part. Using (58) we get

$$\frac{\partial^2 V}{\partial h^{cd}\partial h^{ab}} = \frac{g}{2{\rm i}}\left(g_{a(c}g_{d)b} - \frac{1}{3}g_{ab}g_{cd}\right), \tag{154}$$

where we have defined

$$g = \frac{1}{8}\sum_{p=2}^{6}(f'_p)_0\frac{p(p-1)}{3^{p-1}}. \tag{155}$$

As in the SU(2) case this constant is going to measure strength of gravity modifications. Both κ and g constants have dimensions of $1/L^2$ and are, in general, independent parameters of our linearized theory, related to first derivatives $(f'_p)_0$ of the function f of the ratios.

Let us now compute the matrix of second derivatives in its part not in $\mathfrak{su}(2)$. We only need its mixed components $a\alpha$ and $b\beta$. The computation is easy, and using (58) we get

$$\frac{\partial^2 V}{\partial h^{a\alpha}\partial h^{b\beta}}\bigg|_0 = \frac{\kappa}{4{\rm i}}g_{ab}g_{\alpha\beta}. \tag{156}$$

We note that in this computation only one of the terms in (61) survives, and this is the reason why it is the constant κ that appears in this formula.

We can now compute all the potential parts. We use (51), which we have to divide by two to get the correct quadratic action. For the $\mathfrak{su}(2)$ gravitational part the result is unchanged from that in the previous section, and we have

$$S_{\rm BB}^{\rm grav} = -\frac{g}{2}\int\left(g_{a(c}g_{d)b} - \frac{1}{3}g_{ab}g_{cd}\right)(\Sigma_0^{a\mu\nu}b^b_{\mu\nu}) \times (\Sigma_0^{c\rho\sigma}b^d_{\rho\sigma}). \tag{157}$$

The Higgs and YM parts of the potential term are both given by

$$S_{\rm BB}^{\rm Higgs\text{-}YM} = -\frac{\kappa}{4}\int g_{ab}g_{\alpha\beta}(\Sigma^{a\mu\nu}b^\alpha_{\mu\nu})(\Sigma^{b\rho\sigma}b^\beta_{\rho\sigma}) + 2{\rm i}g_{\alpha\beta}\epsilon^{\mu\nu\rho\sigma}b^\alpha_{\mu\nu}b^\beta_{\rho\sigma}, \tag{158}$$

so the indices α, β here take values 4, 5, 6, 7, 8. We can further simplify this using (72). We get

$$S_{\rm BB}^{\rm Higgs\text{-}YM} = -\kappa\int g_{\alpha\beta}b^{\alpha\mu\nu}b^{\beta\rho\sigma}P^-_{\mu\nu\rho\sigma}, \tag{159}$$

where

$$P^- = \frac{1}{2}\left(\eta_{\mu[\rho}\eta_{\sigma]\nu} + \frac{\rm i}{2}\epsilon_{\mu\nu\rho\sigma}\right) \tag{160}$$

is the anti-self-dual projector.

E. Symmetries

We have seen that the $\mathfrak{su}(2)$ sector of the theory is completely unchanged from what we have obtained in the $G = \mathrm{SU}(2)$ case. One can moreover see that diffeomorphisms still act only within this sector. Indeed, the action of a diffeomorphism in the direction of a vector field ξ^μ is still given by (87) and changes only the $\mathfrak{su}(2)$ part of the two-form field. Similarly, the SU(2) gauge transformations act only on the $\mathfrak{su}(2)$ sector. Thus, the gravity story that we have considered in the previous section is unchanged.

Let us now consider what happens in directions not in $\mathfrak{su}(2)$. Let us first consider the Higgs sector spanned by $V_\pm$, $W_\pm$. A gauge transformation with the gauge parameter ω valued in this sector acts as $\delta_\omega b = [\omega, \Sigma]$. In components this reads:

$$\begin{aligned}
\delta_\omega b^4 &= -\frac{1}{\sqrt{2}}\omega^6\Sigma^+ - \frac{1}{2}\omega^4\Sigma^z,\\
\delta_\omega b^5 &= \frac{1}{\sqrt{2}}\omega^7\Sigma^- + \frac{1}{2}\omega^5\Sigma^z,\\
\delta_\omega b^6 &= -\frac{1}{\sqrt{2}}\omega^4\Sigma^- + \frac{1}{2}\omega^6\Sigma^z,\\
\delta_\omega b^7 &= \frac{1}{\sqrt{2}}\omega^5\Sigma^+ - \frac{1}{2}\omega^7\Sigma^z.
\end{aligned} \tag{161}$$

The remaining part of the Lie algebra is that spanned by Y. The corresponding gauge transformation has no effect on the two-form field b^8 (nor on b^α, $\alpha = 4, 5, 6, 7$) since it commutes with the background. However, this gauge transformation does act on the connection a^8 by the usual U(1) gauge transformation $a^8 \to a^8 + d\omega^8$. The kinetic part (147) clearly remains invariant, and the potential part is invariant since it depends only on b^8 that does not transform.

F. Low-energy limit of the Higgs sector

Our analysis of the YM sector presented below will show that the parameter κ that appeared in the "Higgs-YM" part of the potential (158) must be taken to be of the order M_p^2, where M_p is the Planck mass. This will follow from the fact that the YM coupling constant should be of order unity in a realistic unification scheme, which then immediately implies $\kappa \sim M_p^2$. Another way to reach the same conclusion is to note that M_p is the only scale in our problem, so all dimensionful quantities must be of the Planck size; see more on this in the last discussion section. If this is the case, then the role of the potential term (158) for the Higgs sector is to make the anti-self-dual components of the two-forms $b^\alpha_{\mu\nu}$ have Planckian mass and thus effectively set them to zero. This is completely analogous to what happened in the gravitational sector in the limit $E^2 \ll g$ with the b^{ab}_{tf} components. Thus, we see that in the low-energy limit $E^2 \ll \kappa$ the two-forms $b^\alpha_{\mu\nu}$ can be effectively assumed to be self-dual. As such they can be expanded in the background self-dual two-forms $\Sigma^a_{0\mu\nu}$. After such an ansatz is substituted into the action (146), the result simplifies considerably. However, in order to exhibit the physical modes, we need to introduce some convenient gauge fixing. Inspecting (161) we see that it is possible to set to zero the following components of the $b^{\alpha a}$:

$$b^4_+ = 0, \qquad b^5_- = 0, \qquad b^6_- = 0, \qquad b^7_- = 0. \tag{162}$$

This gauge turns out to be very convenient. We now write the gauge-fixed two-forms $b^\alpha_{\mu\nu}$ as follows:

$$\begin{aligned}
b^4_{\mu\nu} &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}b^4_-\Sigma^-_{\mu\nu} + \frac{\sqrt{3}}{2}b^4_z\Sigma^z_{\mu\nu}\right),\\
b^5_{\mu\nu} &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}b^5_+\Sigma^+_{\mu\nu} + \frac{\sqrt{3}}{2}b^5_z\Sigma^z_{\mu\nu}\right),\\
b^6_{\mu\nu} &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}b^6_+\Sigma^+_{\mu\nu} + \frac{\sqrt{3}}{2}b^6_z\Sigma^z_{\mu\nu}\right),\\
b^7_{\mu\nu} &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}b^7_-\Sigma^-_{\mu\nu} + \frac{\sqrt{3}}{2}b^7_z\Sigma^z_{\mu\nu}\right),
\end{aligned} \tag{163}$$

where the independent fields are now b^4_-, b^5_+, b^6_+, b^7_-, and b^α_z and the "strange" normalization coefficients are chosen in order for the Lagrangian to be obtained to have the canonical form.

Substituting (163) into (146) and using the algebra of Σ matrices, we get the following simple effective low-energy action:

$$S^{\rm Higgs}_{\rm eff} = -\int \partial^\mu b^5_+ \partial_\mu b^4_- + \partial^\mu b^7_- \partial_\mu b^6_+ + \partial^\mu b^5_z \partial_\mu b^4_z + \partial^\mu b^7_z \partial_\mu b^6_z. \tag{164}$$

This form of the Lagrangian makes the reality conditions necessary to get a real theory obvious. Indeed, it is clear that the reality conditions are

$$(b^5_+)^* = b^4_-, \qquad (b^7_-)^* = b^6_+, \\ (b^5_z)^* = b^4_z, \qquad (b^7_z)^* = b^6_z. \tag{165}$$

These conditions can be compactly stated by introducing the following $\mathfrak{su}(2) \otimes \mathfrak{g}$ valued object:

$$\mathbf{b} := (b^4_- T_+ + b^4_z T_z) \otimes V_+ + (b^5_+ T_- + b^5_z T_z) \otimes V_- + (b^6_+ T_- + b^6_z T_z) \otimes W_+ + (b^7_- T_+ + b^7_z T_z) \otimes W_- \tag{166}$$

and requiring it to be Hermitian:

$$\mathbf{b}^\dagger = \mathbf{b}. \tag{167}$$

The action can also be written quite compactly in terms of $\mathbf{b}$. Indeed, using the pairing given by the $\langle \cdot, \cdot \rangle$ metric in the Lie algebra, we get

$$\mathcal{L}^{\rm Higgs}_{\rm eff} = -\langle \partial^\mu \mathbf{b}^\dagger, \partial_\mu \mathbf{b} \rangle \tag{168}$$

for the low-energy $E^2 \ll \kappa$ effective Higgs sector Lagrangian. It is thus clear that, at least in the low-energy regime, the Higgs sector of our theory consists just of four complex massless scalar fields with the usual Lagrangian. It is not hard to show [see expression (180) for the Hamiltonian in the next subsection] that in the finite κ limit the content of this sector does not change and is still given by massless fields.

G. Hamiltonian formulation for the Higgs sector

In this subsection we obtain the Hamiltonian formulation of the sector spanned by $V_\pm$, $W_\pm$. After the analysis performed in the previous subsection, such an analysis is not really necessary as we know what the propagating DOF described by this sector are like, and we even know the correct reality conditions. However, we decided to perform such an analysis for completeness, and also to confirm the reality conditions found from the Hamiltonian perspective. One finds the Hamiltonian analysis to be exactly parallel to that in the gravitational case, with even the final expression for the Hamiltonian being analogous. This subsection is quite technical, and the reader is advised to skip it on the first reading. As in the case of gravity, we start by performing the space plus time split of the kinetic BF part.

BF part.—From (144) our Lagrangian density is

$$\mathcal{L}^{\rm Higgs}_{\rm BF} = \frac{2\sqrt{2}}{3}((\partial b^5)^\mu (\partial b^6)^\nu \Sigma^+_{\mu\nu} - (\partial b^4)^\mu (\partial b^7)^\nu \Sigma^-_{\mu\nu}) - \frac{2}{3}((\partial b^4)^\mu (\partial b^5)^\nu (2\eta_{\mu\nu} + \Sigma^z_{\mu\nu}) + (\partial b^6)^\mu (\partial b^7)^\nu (2\eta_{\mu\nu} - \Sigma^z_{\mu\nu})). \tag{169}$$

Now, denoting the indices 4, 5, 6, 7 collectively by α, we have

$$(\partial b^\alpha)^0 = -\partial_b t^{\alpha b}, \qquad (\partial b^\alpha)^a = \partial_0 t^{\alpha a} - 2\epsilon^{abc}\partial_b b^\alpha_{0c}, \tag{170}$$

where we have introduced the configurational variables

$$t^{\alpha a} := \epsilon^{abc} b^\alpha_{bc}. \tag{171}$$

We do not need an expression for the expanded Lagrangian (169) because a more compact expression in terms of the conjugate momenta will be obtained below. For now let us compute the momenta conjugate to the configurational variables $t^{\alpha a}$. It is sufficient to compute just one of the momenta to see the pattern. We have

$$\pi_{4a} := \frac{\partial \mathcal{L}^{\rm Higgs}_{\rm BF}}{\partial \partial_0 t^{4a}} = -\frac{2\sqrt{2}}{3}(\Sigma^-_{ab}(\partial_0 t^{7b} - 2\epsilon^{bef}\partial_e b^7_{0f}) + \Sigma^-_{0a}\partial_b t^{7b}) - \frac{4}{3}(\partial_0 t^5_a - 2\epsilon_a^{\ ef}\partial_e b^5_{0f}) - \frac{2}{3}(\Sigma^z_{ab}(\partial_0 t^{5b} - 2\epsilon^{bef}\partial_e b^5_{0f}) + \Sigma^z_{0a}\partial_b t^{5b}). \tag{172}$$

Comparing it to (140), we see that $\pi_{4a} = 2{\rm i}a^5_a$. This is precisely analogous to the relation (104) we had in the case of gravity. Indeed, the above relation can be rewritten as $\pi_{4a} = -2{\rm i}g_{4\alpha}a^\alpha_a$, which generalizes (104). The other momenta are obtained as follows:

$$\pi_{\alpha a} = -2{\rm i}g_{\alpha\beta}a^\beta_a. \tag{173}$$

We now need to solve for the velocities in terms of the momenta and substitute the result into the Lagrangian. Similar to the case of gravity, the velocities can be obtained by taking the spatial component of the Eqs. (137). We get

$$\partial_0 t^4_a - 2\epsilon_a{}^{bc}\partial_b b^4_{0c} = \mathrm{i}\sqrt{2}\Sigma^+_{0a} a^6_0 + \frac{1}{\sqrt{2}}\Sigma^{+b}_a \pi_{7b} + \mathrm{i}\Sigma^z_{0a} a^4_0 + \frac{1}{2}\Sigma^{z\,b}_a \pi_{5b},$$
$$\partial_0 t^5_a - 2\epsilon_a{}^{bc}\partial_b b^5_{0c} = -\mathrm{i}\sqrt{2}\Sigma^-_{0a} a^7_0 - \frac{1}{\sqrt{2}}\Sigma^{-b}_a \pi_{6b} - \mathrm{i}\Sigma^z_{0a} a^5_0 - \frac{1}{2}\Sigma^{z\,b}_a \pi_{4b},$$
$$\partial_0 t^6_a - 2\epsilon_a{}^{bc}\partial_b b^6_{0c} = \mathrm{i}\sqrt{2}\Sigma^-_{0a} a^4_0 + \frac{1}{\sqrt{2}}\Sigma^{-b}_a \pi_{5b} - \mathrm{i}\Sigma^z_{0a} a^6_0 - \frac{1}{2}\Sigma^{z\,b}_a \pi_{7b},$$
$$\partial_0 t^7_a - 2\epsilon_a{}^{bc}\partial_b b^7_{0c} = -\mathrm{i}\sqrt{2}\Sigma^+_{0a} a^5_0 - \frac{1}{\sqrt{2}}\Sigma^{+b}_a \pi_{4b} + \mathrm{i}\Sigma^z_{0a} a^7_0 + \frac{1}{2}\Sigma^{z\,b}_a \pi_{6b}. \qquad (174)$$

The time projections of Eqs. (137) are then the Gauss constraints.

For the last step we start from a convenient expression for the Lagrangian. This is given by an analog of (107), which reads:

$$\mathcal{L}^{\text{Higgs}}_{\text{BF}} = -2g_{ab}\Sigma^{a\mu\nu} f^b_{\alpha\beta} a^\alpha_\mu a^\beta_\nu = -2\sqrt{2}\Sigma^{+\mu\nu} a^5_\mu a^6_\nu + 2\sqrt{2}\Sigma^{-\mu\nu} a^4_\mu a^7_\nu + 2\Sigma^{z\mu\nu} a^4_\mu a^5_\nu - 2\Sigma^{z\mu\nu} a^6_\mu a^7_\nu, \qquad (175)$$

where $f^a_{\alpha\beta}$ are the structure constants. Expanding it and converting the spatial components of the connection into momenta, we get

$$\mathcal{L}^{\text{Higgs}}_{\text{BF}} = \frac{1}{\sqrt{2}}\Sigma^{+ab}\pi_{4a}\pi_{7b} - \frac{1}{\sqrt{2}}\Sigma^{-ab}\pi_{5a}\pi_{6b} - \frac{1}{2}\Sigma^{zab}\pi_{5a}\pi_{4b} + \frac{1}{2}\Sigma^{zab}\pi_{7a}\pi_{6b} - \mathrm{i}\sqrt{2}\Sigma^{+a}_0(a^5_0\pi_{7a} - \pi_{4a}a^6_0) + \mathrm{i}\sqrt{2}\Sigma^{-a}_0(a^4_0\pi_{6a} - \pi_{5a}a^7_0) + \mathrm{i}\Sigma^{za}_0(a^4_0\pi_{4a} - \pi_{5a}a^5_0) - \mathrm{i}\Sigma^{za}_0(a^6_0\pi_{6a} - \pi_{7a}a^7_0).$$

We can now compute the Hamiltonian:

$$\mathcal{H}^{\text{Higgs}}_{\text{BF}} = \pi_{\alpha a}\partial_0 t^{\alpha a} - \mathcal{L}^{\text{Higgs}}_{\text{BF}} = 2\pi_{\alpha a}\epsilon^{abc}\partial_b b^\alpha_{0c} + \tfrac{1}{2}g_{ab}g^{\alpha\gamma}g^{\beta\delta}f^a_{\alpha\beta}\Sigma^{bef}\pi_{\gamma e}\pi_{\delta f}. \qquad (176)$$

The obtained expression is not the full Hamiltonian. To obtain the latter we need to add four Gauss constraints that are obtained as the time components of the compatibility Eqs. (137). We will not need an explicit form of the Gauss constraints since we already know from (161) what is generated by them.

The BB part.—Let us now consider the potential part (159). The corresponding Lagrangian density reads:

$$\mathcal{L}^{\text{Higgs}}_{\text{BB}} = -\kappa P^-_{\mu\nu\rho\sigma} g_{\alpha\beta} b^{\alpha\mu\nu} b^{\beta\rho\sigma}. \qquad (177)$$

Expanding the spacetime index we get

$$\mathcal{L}^{\text{Higgs}}_{\text{BB}} = \kappa g_{\alpha\beta}(b^{\alpha a}_0 b^\beta_{0a} - \tfrac{1}{4}t^{\alpha a}t^\beta_a) + \mathrm{i}\kappa g_{\alpha\beta} b^\alpha_{0a} t^{\beta a}. \qquad (178)$$

Total Hamiltonian.—We now form the total Hamiltonian $\mathcal{H}^{\text{Higgs}} = \mathcal{H}^{\text{Higgs}}_{BF} - \mathcal{L}^{\text{Higgs}}_{BB}$ and integrate out the nondynamical fields b^α_{0a}. We get the following expressions for these fields by solving their field equations:

$$b^\alpha_{0a} = \frac{1}{\kappa} g^{\alpha\beta}\epsilon_{abc}\partial^b\pi^c_\beta - \frac{\mathrm{i}}{2}t^\alpha_a. \qquad (179)$$

This should be compared with (114) that we have in the gravitational sector. We now substitute this back to get the Hamiltonian with second-class constraints solved for

$$\mathcal{H}^{\text{Higgs}} = \frac{1}{2}g_{ab}g^{\alpha\gamma}g^{\beta\delta}f^a_{\alpha\beta}\Sigma^{bef}\pi_{\gamma e}\pi_{\delta f} - \mathrm{i}(\epsilon^{abc}\partial_b\pi_{\alpha c})t^\alpha_a + \frac{1}{\kappa}g^{\alpha\beta}(\epsilon^{abc}\partial_b\pi_{\alpha c})(\epsilon^{aef}\partial_e\pi_{\beta f}), \qquad (180)$$

plus Gauss constraints with their corresponding Lagrange multipliers. Note also that the Hamiltonian we have obtained is analogous to the one in the case of gravity (116). Indeed, there is similarly the π^2 term and a $(\epsilon\partial\pi)t$ term with an imaginary unit in front. There is also a $\partial^2\pi^2$ term with a parameter of dimensions $1/M^2$ as a coefficient. Note that for any value of the parameter κ this Hamiltonian describes modes that are massless. To rewrite this Hamiltonian in terms of physical propagating modes, we need to understand the gauge fixing.

Gauge fixing.—To choose a convenient gauge fixing that eliminates the gauge transformation freedom, let us discuss what the two-forms $\Sigma^\pm$, Σ^z become after they get projected onto the spatial hypersurface. Thus, let us find analogs of relations (97) and (96). Let us introduce the following three spatial vectors:

$$\Sigma^+_{0a} := m_a, \qquad \Sigma^-_{0a} := \bar{m}_a, \qquad \Sigma^z_{0a} := n_a. \qquad (181)$$

Then, taking various projections of (126), it is easy to check that the following relations hold:

$$m^a m_a = \bar{m}^a\bar{m}_a = 0, \qquad m^a n_a = \bar{m}^a n_a = 0, \qquad m^a\bar{m}_a = 1, \qquad n^a n_a = 1. \qquad (182)$$

Taking different projections of (126), one finds the spatial pullbacks of the two-forms in terms of the vectors introduced:

$$\Sigma^+_{ab} = n_a m_b - m_a n_b = -\mathrm{i}\epsilon_{abc} m^c,$$
$$\Sigma^-_{ab} = \bar{m}_a n_b - n_a \bar{m}_b = -\mathrm{i}\epsilon_{abc} \bar{m}^c, \qquad (183)$$
$$\Sigma^z_{ab} = m_a \bar{m}_b - \bar{m}_a m_b = -\mathrm{i}\epsilon_{abc} n^c.$$

We now use (161) to fix the gauge as in (162). In terms of the configurational variables $t^{\alpha a}$ the gauge conditions read:

$$t^{4-} = 0, \qquad t^{5+} = 0, \qquad t^{6+} = 0, \qquad t^{7-} = 0, \qquad (184)$$

where our convention is that $t^{\alpha +} := m^a t^\alpha_a$, $t^{\alpha -} = \bar{m}^a t^\alpha_a$, and $t^{\alpha z} = n^a t^\alpha_a$.

Let us now find the consequences of the Gauss constraints. In terms of the introduced vectors m^a, $\bar{m}^a$, and n^a these read:

$$\partial_a t^{4a} - \frac{1}{\sqrt{2}} m_a \pi^a_7 - \frac{1}{2} n_a \pi^a_5 = 0,$$
$$\partial_a t^{5a} + \frac{1}{\sqrt{2}} \bar{m}_a \pi^a_6 + \frac{1}{2} n_a \pi^a_4 = 0,$$
$$\partial_a t^{6a} - \frac{1}{\sqrt{2}} \bar{m}_a \pi^a_5 + \frac{1}{2} n_a \pi^a_7 = 0, \qquad (185)$$
$$\partial_a t^{7a} + \frac{1}{\sqrt{2}} m_a \pi^a_4 - \frac{1}{2} n_a \pi^a_6 = 0.$$

Introducing more compact notations $\pi^+_\alpha := m_a \pi^a_\alpha$, $\pi^-_\alpha = \bar{m}_a \pi^a_\alpha$, and $\pi^z_\alpha = n_a \pi^a_\alpha$ and passing to the momentum space, we have

$$\mathrm{i}|k| t^{4z} - \frac{1}{\sqrt{2}} \pi^+_7 - \frac{1}{2} \pi^z_5 = 0,$$
$$\mathrm{i}|k| t^{5z} + \frac{1}{\sqrt{2}} \pi^-_6 + \frac{1}{2} \pi^z_4 = 0,$$
$$\mathrm{i}|k| t^{6z} - \frac{1}{\sqrt{2}} \pi^-_5 + \frac{1}{2} \pi^z_7 = 0, \qquad (186)$$
$$\mathrm{i}|k| t^{7z} + \frac{1}{\sqrt{2}} \pi^+_4 - \frac{1}{2} \pi^z_6 = 0.$$

We now use these constraints to find the components of the momenta that are conjugate to the gauge-fixed variables (184). We have

$$\pi^+_4(k) = -\mathrm{i}\sqrt{2}|k| t^{7z} + \frac{1}{\sqrt{2}} \pi^z_6,$$
$$\pi^-_5(k) = \mathrm{i}\sqrt{2}|k| t^{6z} + \frac{1}{\sqrt{2}} \pi^z_7,$$
$$\pi^-_6(k) = -\mathrm{i}\sqrt{2}|k| t^{5z} - \frac{1}{\sqrt{2}} \pi^z_4, \qquad (187)$$
$$\pi^+_7(k) = \mathrm{i}\sqrt{2}|k| t^{4z} - \frac{1}{\sqrt{2}} \pi^z_5.$$

Let us now substitute these expressions into the π^2 part of the Hamiltonian. Thus, we have for the first term in (180)

$$-\frac{3}{4}(\pi^z_4(-k)\pi^z_5(k) + \pi^z_7(-k)\pi^z_6(k)) - \frac{1}{2}(\pi^-_4(-k)\pi^+_5(k) + \pi^-_7(-k)\pi^+_6(k)) + \frac{\mathrm{i}|k|}{2}(\pi^z_4(-k)t^{4z}(k) + \pi^z_7(-k)t^{7z}(k) - \pi^z_5(-k)t^{5z}(k) - \pi^z_6(-k)t^{6z}(k)) - |k|^2(t^{5z}(-k)t^{4z}(k) + t^{6z}(-k)t^{7z}(k)). \qquad (188)$$

Let us now work out the second term in (180). We use

$$-\mathrm{i}\epsilon^{abc} = n^a(m^b\bar{m}^c - \bar{m}^b m^c) + m^a(\bar{m}^b n^c - n^b \bar{m}^c) + \bar{m}^a(n^b m^c - m^b n^c), \qquad (189)$$

which can be easily derived from (183) to write the second term in (180) as

$$\mathrm{i}\,|k|(\pi^-_\alpha(-k)t^{\alpha +}(k) - \pi^+_\alpha(-k)t^{\alpha -}(k)). \qquad (190)$$

Here we again passed to the momentum space and used

$$\partial_a(e^{\mathrm{i}kx} t^\alpha_b(k)) = \mathrm{i}k_a e^{\mathrm{i}kx} t^\alpha_b(k), \qquad (191)$$

where $k^a = |k| n^a$ is a vector in the direction of n^a. This makes only two of the terms from (189) survive. Expanding and using the gauge-fixing conditions (184), we get for this term

$$\mathrm{i}|k|(\pi^-_4(-k)t^{4+}(k) + \pi^-_7(-k)t^{7+}(k) - \pi^+_5(-k)t^{5-}(k) - \pi^+_6(-k)t^{6-}(k)). \qquad (192)$$

The total Hamiltonian in the $E^2 \ll \kappa$ low-energy limit is given by the sum of two terms, i.e., (188) and (192).

Reality conditions.—Let us now discuss the reality conditions that are appropriate in the $E^2 \ll \kappa$ low-energy limit. It is clear that they can be determined by "completing the square," similar to what we have seen in the Hamiltonian formulation of the gravitational sector (in the low-energy limit). Thus, let us write the total Hamiltonian as

$$\begin{aligned}\mathcal{H}^{\text{Higgs}} = &-\frac{3}{4}\left(\pi_4^z(-k) - \frac{2\mathrm{i}|k|}{3}t^{5z}(-k)\right)\left(\pi_5^z(k) - \frac{2\mathrm{i}|k|}{3}t^{4z}(k)\right) - \frac{4}{3}|k|^2 t^{5z}(-k)t^{4z}(k)\\ &-\frac{3}{4}\left(\pi_7^z(-k) - \frac{2\mathrm{i}|k|}{3}t^{6z}(-k)\right)\left(\pi_6^z(k) - \frac{2\mathrm{i}|k|}{3}t^{7z}(k)\right) - \frac{4}{3}|k|^2 t^{6z}(-k)t^{7z}(k)\\ &-\frac{1}{2}(\pi_4^-(-k) - 2\mathrm{i}|k|t^{5-}(-k))(\pi_5^+(k) - 2\mathrm{i}|k|t^{4+}(k)) - 2|k|^2 t^{5-}(-k)t^{4+}(k)\\ &-\frac{1}{2}(\pi_7^-(-k) - 2\mathrm{i}|k|t^{6-}(-k))(\pi_6^+(k) - 2\mathrm{i}|k|t^{7+}(k)) - 2|k|^2 t^{6-}(-k)t^{7+}(k). \end{aligned} \tag{193}$$

The form of the reality conditions is now obvious. Indeed, we introduce new momenta variables:

$$\begin{aligned}\tilde\pi_4^z(k) &= \pi_4^z(k) + \frac{2\mathrm{i}|k|}{3}t^{5z}(k),\\ \tilde\pi_5^z(k) &= \pi_5^z(k) - \frac{2\mathrm{i}|k|}{3}t^{4z}(k),\\ \tilde\pi_6^z(k) &= \pi_6^z(k) - \frac{2\mathrm{i}|k|}{3}t^{7z}(k),\\ \tilde\pi_7^z(k) &= \pi_7^z(k) + \frac{2\mathrm{i}|k|}{3}t^{6z}(k),\\ \tilde\pi_4^-(k) &= \pi_4^-(k) + 2\mathrm{i}|k|t^{5-}(k),\\ \tilde\pi_5^+(k) &= \pi_5^+(k) - 2\mathrm{i}|k|t^{4+}(k),\\ \tilde\pi_6^+(k) &= \pi_6^+(k) - 2\mathrm{i}|k|t^{7+}(k),\\ \tilde\pi_7^-(k) &= \pi_7^-(k) + 2\mathrm{i}|k|t^{6-}(k),\end{aligned} \tag{194}$$

and then require the following reality conditions:

$$\begin{aligned} &\tilde\pi_4^z(-k) = -(\tilde\pi_5^z(k))^*, &&\tilde\pi_7^z(-k) = -(\tilde\pi_6^z(k))^*,\\ &\tilde\pi_4^-(-k) = -(\tilde\pi_5^+(k))^*, &&\tilde\pi_7^-(-k) = -(\tilde\pi_6^+(k))^*\\ &t^{5z}(-k) = -(t^{4z}(k))^*, &&t^{6z}(-k) = -(t^{7z}(k))^*,\\ &t^{5-}(-k) = -(t^{4+}(k))^*, &&t^{6-}(-k) = -(t^{7+}(k))^*.\end{aligned} \tag{195}$$

It is not hard to see that these conditions are the same as we have derived earlier in the Lagrangian framework; see (165). Indeed, the extra minus present in (195) is due to the following transformation properties of the basic two-forms:

$$(\Sigma_{ab}^+)^* = -\Sigma_{ab}^-, \qquad (\Sigma_{ab}^z)^* = -\Sigma_{ab}^z \tag{196}$$

that directly follow from (183). The obtained real positive-definite Hamiltonian is that of four complex massless scalar fields, so we have full agreement with our Lagrangian analysis above. Reality conditions and the Hamiltonian for the full finite κ theory can be obtained via precisely the same method as in the gravitational sector case treated in the Appendix. We refrain from giving such an analysis in this work, as it becomes even more technical.

H. Yang-Mills sector

In this subsection we work out the Lagrangian for the remaining part of the theory, which lives in the part of the gauge group that commutes with the background $\mathfrak{su}(2)$. The total Lagrangian we start with is a sum of kinetic term (147) and the potential term (159), with an extra sign in the potential term coming from the metric component $g_{88} = -1$. This gives

$$\mathcal{L}^{\text{YM}} = 2\mathrm{i}\epsilon^{\mu\nu\rho\sigma}b_{\mu\nu}^8\partial_\rho a_\sigma^8 + \kappa P^{-\mu\nu\rho\sigma}b_{\mu\nu}^8 b_{\rho\sigma}^8. \tag{197}$$

The further analysis is greatly simplified by making use of the reality condition for the $b_{\mu\nu}^8$ two-form from the outset. Thus, as we will also confirm by the Hamiltonian analysis in the next subsection, the two-form $b_{\mu\nu}^8$ needs to be purely imaginary:

$$b_{\mu\nu}^8 := -\mathrm{i}\tilde b_{\mu\nu}^8, \qquad (\tilde b_{\mu\nu}^8)^* = \tilde b_{\mu\nu}^8. \tag{198}$$

This immediately leads to simplifications as the real part of the Lagrangian (197) is then given simply by

$$\mathcal{L}_{\text{real}}^{\text{YM}} = 2\epsilon^{\mu\nu\rho\sigma}\tilde b_{\mu\nu}^8\partial_\rho a_\sigma^8 - \frac{\kappa}{2}\tilde b^{8\mu\nu}\tilde b_{\mu\nu}^8. \tag{199}$$

Taking a variation with respect to $\tilde b_{\mu\nu}^8$, we learn that

$$\tilde b^{\,8}_{\mu\nu} = \frac{1}{\kappa}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{200}$$

where $F_{\mu\nu} = \partial_\mu a_\nu^8 - \partial_\nu a_\mu^8$ is the curvature of our U(1) gauge field, which is therefore, for real κ, real. Substituting the result back into the Lagrangian, we get

$$\mathcal{L}^{\text{YM}} = -\frac{2}{\kappa}(F_{\mu\nu})^2. \tag{201}$$

This is the standard YM Lagrangian with the coupling constant:

$$g_{\text{YM}}^2 = \frac{\kappa}{8}. \tag{202}$$

To convert this into a physical coupling constant, we recall that we need to multiply the Lagrangian by $1/(32\pi G)$, as this is exactly the prefactor that converts the canonically normalized graviton Lagrangian (94) into the Einstein-Hilbert one. Thus, the physical coupling constant in our arising YM theory is given by

$$g_{\text{YM}}^2 = 4\pi G\kappa. \tag{203}$$

Realistic particle physics coupling constants are of the order of unity (and smaller), so we learn that the parameter κ must be of the order M_p^2, which is what we have been using in the previous subsections.

I. Reality conditions for the YM sector

In this subsection we perform the Hamiltonian analysis of the YM sector with the main aim being to obtain the reality conditions used above. As in all other cases considered, the reality conditions become obvious once the Hamiltonian is written down.

We start from the Lagrangian (197). Expanding

$$\epsilon^{\mu\nu\rho\sigma} b^8_{\mu\nu}\partial_\rho a^8_\sigma = -2\epsilon^{abc} b^8_{0a}\partial_b a^8_c - t^{8a}(\partial_0 a^8_a - \partial_a a^8_0), \tag{204}$$

where $t^{8a} := \epsilon^{abc} b^8_{bc}$, we see that the momentum conjugate to the connection a^8_a is

$$\pi^{8a} := \frac{\partial \mathcal{L}^{\rm YM}}{\partial \partial_0 a^8_a} = -2\mathrm{i} t^{8a}. \tag{205}$$

The Hamiltonian is then

$$\mathcal{H}^{\rm YM} = 4\mathrm{i}\epsilon^{abc} b^8_{0a}\partial_b a^8_c - a^8_0\partial_a\pi^{8a} + \kappa\left(b^{8a}_0 b^8_{0a} + \frac{1}{16}\pi^{8a}\pi^8_a\right) - \frac{\kappa}{2} b^8_{0a}\pi^{8a}. \tag{206}$$

We find the nondynamical fields b^8_{0a} via their field equations and get

$$b^8_{0a} = -\frac{2\mathrm{i}}{\kappa}\epsilon_{abc}\partial^b a^{8c} + \frac{1}{4}\pi^8_a. \tag{207}$$

Substituting this back into (206), we get the "physical" Hamiltonian

$$\mathcal{H}^{\rm YM}_{\rm phys} = \frac{4}{\kappa}\left(\epsilon^{abc}\partial_b a^8_c + \frac{\mathrm{i}\kappa}{8}\pi^{8a}\right)^2 + \frac{\kappa}{16}\pi^{8a}\pi^8_a. \tag{208}$$

It is now clear that the "correct" reality conditions that give rise to a real positive-definite Hamiltonian is

$$\mathrm{Im}\,(\epsilon^{abc}\partial_b a^8_c) + \frac{\kappa}{8}\mathrm{Re}(\pi^{8a}) = 0, \qquad \mathrm{Im}(\pi^{8a}) = 0. \tag{209}$$

From (207) and (205) it is easy to see that these reality conditions are equivalent to the condition that the $b^8_{\mu\nu}$ two-form is purely imaginary, which is what we have used in the previous subsection.

Passing to the real phase space and imposing the Gauss constraint $\partial_a\pi^{8a} = 0$ as well as the transverse gauge condition $\partial^a a^8_a = 0$, we get the following simple expression for the real Hamiltonian:

$$\mathcal{H}^{\rm YM}_{\rm real} = \frac{4}{\kappa}(\partial_a a^{8\,\rm real}_b)^2 + \frac{\kappa}{16}(\pi^{8a})^2, \tag{210}$$

which again confirms that the parameter $\kappa/8$ plays the role of $g^2_{\rm YM}$.

VIII. INTERACTIONS

In this section we work out (some of the) cubic order interactions for our theory. Our main goal is to verify that the YM and Higgs sectors interact with the gravitational field in the usual way, and that the YM-Higgs interaction is also standard. We start with general considerations on the cubic order expansion of our theory.

A. General considerations

The third variation of the BF term is

$$\delta^3 S_{\rm BF} = 4\mathrm{i}\int 3\delta B^I\wedge[\delta A, \delta A]^I, \tag{211}$$

and the third variation of the BB term is

$$\delta^3 S_{\rm BB} = 4\mathrm{i}\int d^4x\left(4\frac{\partial^3 V(\tilde h)}{\partial\tilde h^{MN}\partial\tilde h^{KL}\partial\tilde h^{IJ}}\times(B_0\delta B)^{IJ}(B_0\delta B)^{KL}(B_0\delta B)^{MN} + 6\frac{\partial^2 V(\tilde h)}{\partial\tilde h^{KL}\partial\tilde h^{IJ}}(B_0\delta B)^{IJ}(\delta B\delta B)^{KL}\right). \tag{212}$$

As in the case of the quadratic order expansion, it is most laborious to compute the derivatives of the potential. We have already computed the second derivative above. The third derivative of $V(\tilde h)$ is given by

$$\frac{\partial^3 V(\tilde h)}{\partial\tilde h^{MN}\partial\tilde h^{KL}\partial\tilde h^{IJ}} = \frac{g_{IJ}}{n}\frac{\partial^2 f}{\partial\tilde h^{MN}\partial\tilde h^{KL}} + \frac{g_{KL}}{n}\frac{\partial^2 f}{\partial\tilde h^{MN}\partial\tilde h^{IJ}} + \frac{g_{MN}}{n}\frac{\partial^2 f}{\partial\tilde h^{KL}\partial\tilde h^{IJ}} + \frac{\mathrm{Tr}\tilde h}{n}\frac{\partial^3 f}{\partial\tilde h^{KL}\partial\tilde h^{KL}\partial\tilde h^{IJ}}, \tag{213}$$

where the third derivative of the function of the ratios is given by

$$\frac{\partial^3 f}{\partial\tilde{h}^{MN}\partial\tilde{h}^{KL}\partial\tilde{h}^{IJ}} = \sum_{p=2}^{n}\sum_{q=2}^{n}\sum_{p=2}^{n} f'''_{pqr}\frac{\partial}{\partial\tilde{h}^{IJ}}\left(\frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^p}\right)\frac{\partial}{\partial\tilde{h}^{KL}}\left(\frac{\mathrm{Tr}\tilde{h}^q}{(\mathrm{Tr}\tilde{h})^q}\right)\frac{\partial}{\partial\tilde{h}^{MN}}\left(\frac{\mathrm{Tr}\tilde{h}^r}{(\mathrm{Tr}\tilde{h})^r}\right)$$
$$+ \sum_{p=2}^{n}\sum_{q=2}^{n} f''_{pq}\frac{\partial}{\partial\tilde{h}^{IJ}}\left(\frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^p}\right)\frac{\partial^2}{\partial\tilde{h}^{MN}\partial\tilde{h}^{KL}}\left(\frac{\mathrm{Tr}\tilde{h}^q}{(\mathrm{Tr}\tilde{h})^q}\right) + \sum_{p=2}^{n}\sum_{q=2}^{n} f''_{pq}\frac{\partial}{\partial\tilde{h}^{KL}}\left(\frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^p}\right)\frac{\partial^2}{\partial\tilde{h}^{MN}\partial\tilde{h}^{IJ}}\left(\frac{\mathrm{Tr}\tilde{h}^q}{(\mathrm{Tr}\tilde{h})^q}\right)$$
$$+ \sum_{p=2}^{n}\sum_{q=2}^{n} f''_{pq}\frac{\partial}{\partial\tilde{h}^{MN}}\left(\frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^p}\right)\frac{\partial^2}{\partial\tilde{h}^{KL}\partial\tilde{h}^{IJ}}\left(\frac{\mathrm{Tr}\tilde{h}^q}{(\mathrm{Tr}\tilde{h})^q}\right) + \sum_{p=2}^{n} f'_p\frac{\partial^3}{\partial\tilde{h}^{MN}\partial\tilde{h}^{KL}\partial\tilde{h}^{IJ}}\left(\frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^p}\right), \tag{214}$$

where f'''_{pqr} stands for the derivative of f''_{pq} with respect to its r argument and

$$\frac{\partial^3}{\partial\tilde{h}^{KL}\partial\tilde{h}^{KL}\partial\tilde{h}^{IJ}}\left(\frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^p}\right) = \frac{p}{(\mathrm{Tr}\tilde{h})^p}\frac{\partial^2\tilde{h}^{p-1}_{IJ}}{\partial\tilde{h}^{MN}\partial\tilde{h}^{KL}} - \frac{p^2}{(\mathrm{Tr}\tilde{h})^{p+1}}\left(g_{IJ}\frac{\partial\tilde{h}^{p-1}_{KL}}{\partial\tilde{h}^{MN}} + g_{KL}\frac{\partial\tilde{h}^{p-1}_{IJ}}{\partial\tilde{h}^{MN}} + g_{MN}\frac{\partial\tilde{h}^{p-1}_{IJ}}{\partial\tilde{h}^{KL}}\right)$$
$$+ \frac{p^2(p+1)}{(\mathrm{Tr}\tilde{h})^{p+2}}(g_{IJ}g_{KL}\tilde{h}^{p-1}_{MN} + g_{IJ}g_{MN}\tilde{h}^{p-1}_{KL} + g_{KL}g_{MN}\tilde{h}^{p-1}_{IJ})$$
$$- \frac{p(p+1)(p+2)\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^{p+3}}g_{IJ}g_{KL}g_{MN}. \tag{215}$$

The first derivative of a power of $\tilde{h}^{IJ}$ is given by (61). We have not found a sufficiently simple general expression for the second derivative of $\tilde{h}^{p-1}_{IJ}$ with respect to $\tilde{h}^{MN}\tilde{h}^{KL}$, but the expression (61) can be easily differentiated for any given p. The above expressions can be used to obtain the third derivatives of the potential for our background. The results are given in the next subsection.

B. Interactions with gravity

In this paper we shall not consider gravitational sector self-interactions. They are easily computable, but since the main emphasis of this work is on unification, it is of much more interest to compute the interactions of other fields with gravity and their self-interactions. In this subsection we consider the coupling of nongravitational fields to gravity.

Thus, at least one of the perturbation fields δB^I is to be taken to lie in the gravitational sector. It is then easy to see that this is the only interaction in the cubic order. Indeed, where two of the three perturbation fields lie in the gravitational sector and there is only one nongravitational perturbation, there is no interaction coming from the potential part since

$$\left.\frac{\partial^2 V(\tilde{h})}{\partial\tilde{h}^{a\alpha}\partial\tilde{h}^{bc}}\right|_0 = 0, \qquad \left.\frac{\partial^3 V(\tilde{h})}{\partial\tilde{h}^{e\alpha}\partial\tilde{h}^{cd}\partial\tilde{h}^{ab}}\right|_0 = 0, \tag{216}$$

where α stands for the nongravitational part of the Lie algebra. There is also no interaction coming from the kinetic part of the action for the structure constant f^I_{JK} is zero when two of the indices are in the $\mathfrak{su}(2)$ part and only one index is in the nongravitational part. Thus we need to consider only the interaction that is linear in the graviton perturbation. It is natural to expect that this coupling is that to the stress-energy tensor of our nongravitational fields, and this will be confirmed below.

The interaction coming from the kinetic term is nontrivial only for the Higgs sector fields (since the structure constant with two of its indices in the YM part of the Lie algebra and one in $\mathfrak{su}(2)$ is zero since the YM and the gravitational parts commute). This interaction is of the schematic type $h(\partial b)^2$, which is as expected for scalar fields coupled to gravity. We are not going to work out this term, even though it is not hard to do it using the explicit formulas for the connections worked out above.

Let us concentrate on the interactions coming from the potential part of the action as being the most interesting one. The relevant derivatives of the potential are as follows:

$$\left.\frac{\partial^2 V(\tilde{h})}{\partial\tilde{h}^{\alpha\beta}\partial\tilde{h}^{ab}}\right|_0 = 0,$$
$$\left.\frac{\partial^2 V(\tilde{h})}{\partial\tilde{h}^{b\beta}\partial\tilde{h}^{a\alpha}}\right|_0 = \frac{\kappa}{4\mathrm{i}}g_{\alpha\beta}g_{ab},$$
$$\left.\frac{\partial^3 V(\tilde{h})}{\partial\tilde{h}^{d\beta}\partial\tilde{h}^{c\alpha}\partial\tilde{h}^{ab}}\right|_0 = \frac{g_{\alpha\beta}}{2(2\mathrm{i})^2}\left((\kappa - g)\left(g_{a(c}g_{d)b} - \frac{1}{3}g_{ab}g_{cd}\right) - \frac{\kappa}{3}g_{ab}g_{cd}\right). \tag{217}$$

Note that the fact that the first quantity is zero is not completely trivial, as it involves a precise cancellation of two otherwise nonzero terms.

We can now compute the relevant interaction terms using (212). We need to divide this expression by 3! to remove the extra multiplicity introduced by taking the third variation of the action. An additional simplification comes from the fact that in the first term in the third derivative in (217) we have a matrix projecting onto the trace-free part of the gravitational two-form perturbation matrix $\Sigma_0^{a\mu\nu}b^b_{\mu\nu}$. This part is zero when the parameter $g \to \infty$, which is the limit of the usual GR that we are considering.

Thus, this part drops out, and we have for the gravity-nongravity interaction term coming from the potential

$$\mathcal{L}^{(3)} = \frac{4\mathrm{i}}{3!}\bigg(4\cdot 3\frac{1}{2(2\mathrm{i})^2}\Big(-\frac{\kappa}{3}\Big)\Big(\frac{\mathrm{i}}{2}\Big)^3(\Sigma_0^{a\mu\nu}b^a_{\mu\nu})(\Sigma_0^{c\rho\sigma}b^\alpha_{\rho\sigma}) \times(\Sigma_0^{d\rho\sigma}b^\beta_{\rho\sigma})g_{cd}g_{\alpha\beta} + 6\cdot 2\frac{\kappa}{4\mathrm{i}}\Big(\frac{\mathrm{i}}{2}\Big)(\Sigma_0^{a\mu\nu}b^\alpha_{\mu\nu})\frac{1}{4} \times(\epsilon^{\rho\sigma\tau\lambda}b^b_{\rho\sigma}b^\beta_{\tau\lambda})g_{ab}g_{\alpha\beta}\bigg). \tag{218}$$

Here the extra factors of 2 in the first term and 2 in the second come from expanding the general Lie-algebra indices in (212), and the factors of $\mathrm{i}/2$ come by using the self-duality of the background forms $\Sigma^a_{0\mu\nu}$. To understand this expression, it is useful to separate the coupling to the trace of the graviton perturbation and to the trace-free part. Let us consider the trace first. Thus, we take

$$b^a_{\mu\nu} = \frac{h}{3}\Sigma^a_{0\mu\nu}, \tag{219}$$

with the field h being proportional to the trace of the metric perturbation $h_{\mu\nu}$. It is then easy to see that the expression (218) vanishes on such gravitational perturbations. This is, of course, as expected, for both our YM and Higgs sectors are expected to be conformally invariant (classically). Indeed, this is standard for the YM fields, and for the Higgs sector this expectation follows from the fact that the fields are (up to now) massless. Using (218) it is not hard to check that there is indeed no coupling to the trace part of the metric, which confirms our expectation. Note that this also provides quite a nontrivial check of our scheme, for the whole scheme would be invalidated if we had found that our YM fields couple to the trace of the metric.

We now confirm that the coupling to the trace-free part of the metric perturbation is also as expected. We need to consider only the second term in (218), as the first term involves only the trace part of the metric perturbation. Let us consider the YM sector first. We now substitute

$$b^a_{\mu\nu} = \Sigma^a_{0[\mu}{}^{\rho}h_{\nu]\rho}, \tag{220}$$

and use the anti-self-duality of this two-form to get

$$\mathcal{L}^{(3)}_{\text{grav-YM}} = -\frac{\kappa}{2}\Sigma_0^{a\mu\nu}b^8_{\mu\nu}\Sigma_0^{a\rho\lambda}h^\sigma_\lambda b^8_{\rho\sigma} = -2\kappa P^{+\mu\nu\rho\lambda}b^8_{\mu\nu}b^8_{\rho\sigma}h^\sigma_\lambda. \tag{221}$$

Here an extra minus is due to the metric on the Lie algebra. The physical Lagrangian is obtained from here by taking the real part. This makes only the term in the self-dual projector $P^{+\mu\nu\rho\lambda}$ that contains the metric to survive. Substituting (200) we get

$$\mathcal{L}^{(3)}_{\text{grav-YM}} = \frac{1}{\kappa}\epsilon^{\lambda\mu\nu}_{\rho}F_{\mu\nu}\epsilon^{\rho\sigma\alpha\beta}F_{\alpha\beta}h_{\lambda\sigma}. \tag{222}$$

Expanding the product of two ϵ's here we get

$$\mathcal{L}^{(3)}_{\text{grav-YM}} = \frac{4}{\kappa}F_{\mu\rho}F_{\nu\sigma}h^{\mu\nu}\eta^{\rho\sigma}, \tag{223}$$

in which expression we recognize precisely the coupling to the stress-energy tensor that arises from the YM Lagrangian (201). The sign in front is different from that in (201) because the variation of the metric with two upper indices is given by $-h^{\mu\nu}$. Thus, the arising coupling of our YM fields to the gravitational sector is correct.

Let us now discuss the coupling of the Higgs sector to gravity. It is easy to see that in the low-energy approximation in which $E^2 \ll \kappa$ and the two-forms $b^\alpha_{\mu\nu}$ are self-dual there is no coupling coming from the potential term. Indeed, we have already discussed that there is no coupling to the trace part of the metric perturbation. Thus, there is only the second term in (218) that can contribute. However, it contains a factor of $(\epsilon^{\rho\sigma\tau\lambda}b^b_{\rho\sigma}b^\beta_{\tau\lambda})$, which is the contraction of a self-dual Higgs two-form and an anti-self-dual gravitational one. So, it is zero, and the only interaction term in the Higgs sector comes from the kinetic term of the action. As we have already discussed, it is of the $h(\partial b)^2$ form, which is just the coupling of the metric perturbation to the stress-energy tensor of our set of massless fields. We are not going to work out the details as they are slightly messy, but we hope that the discussion given is sufficient to show that the interaction is as expected.

C. Interactions in the nongravitational sector

Let us now concentrate on the interactions in the nongravitational sector, most interestingly those between the YM and Higgs sectors.

First, we note that there are no cubic interactions in the nongravitational sector that come from the potential term. Indeed, such an interaction term involves three perturbation two-forms $b^\alpha_{\mu\nu}$ with the Lie-algebra index outside of $\mathfrak{su}(2)$. It is not hard to see that the corresponding derivatives of the potential vanish:

$$\frac{\partial^2 V(\tilde h)}{\partial\tilde h^{\beta\gamma}\partial\tilde h^{a\alpha}}\bigg|_0 = 0, \qquad \frac{\partial^3 V(\tilde h)}{\partial\tilde h^{c\gamma}\partial\tilde h^{b\beta}\partial\tilde h^{a\alpha}}\bigg|_0 = 0. \tag{224}$$

Thus, at cubic order we need to consider only the interactions coming from the kinetic term. It is not hard to see that there are no self-interactions in the Higgs or YM sectors, but there are two possible types of interaction between these sectors. One of them comes from the term $g_{\alpha\beta}b^\alpha f^\beta_{\gamma 8}a^\gamma a^8$, the other comes from $b^8 f^8_{\alpha\beta}a^\alpha a^\beta$, where α now stands for the Higgs sector index. The second of this is an interaction of the type $(1/\kappa)F(\partial b)^2$ and is thus suppressed at low energies by E^2/κ. However, the first interaction is nontrivial and important even at low energies. In fact, it is not hard to show that this is the standard interaction of the gauge field a^8 with the conserved U(1) current of the Higgs sector that is charged under the YM subgroup. We are not going to spell out the details that are again slightly messy, but the important point is that the YM-

Higgs sectors' interaction is also as expected for a set of scalar fields charged under the YM gauge group (Higgs fields).

IX. MORE GENERAL POTENTIALS: MASS GENERATION

Up to now we have for simplicity considered a very special class of potentials that depend only on the invariants constructed from the internal metric $\tilde{h}^{IJ}$ using the Killing-Cartan metric g_{IJ}. It is not hard to show that due to the fact that the rank of $\tilde{h}^{IJ}$ is at most six, there are at most six such independent invariants, and thus only at most five ratios, to be considered as the arguments of the function $f(\cdot)$ in (53). However, it is clear that these are not the only possible invariants. Indeed, the most general gauge-invariant function of $\tilde{h}^{IJ}$ can also involve invariants constructed using the structure constants f^I_{JK}. For instance, let us consider

$$ff\tilde{h}\,\tilde{h}\,\tilde{h} := f^{PQR}f^{STU}\tilde{h}_{PS}\tilde{h}_{QT}\tilde{h}_{RU}, \tag{225}$$

where the indices on the structure constants are raised using the metric on the group. More generally, one can construct a matrix

$$(ff\tilde{h}\,\tilde{h})^{IJ} := f^{IQR}f^{JTU}\tilde{h}_{QT}\tilde{h}_{RU}, \tag{226}$$

and build more complicated invariants from traces of powers of $\tilde{h}^{IJ}$ and $(ff\tilde{h}\,\tilde{h})^{IJ}$. This leads to a much more general set of gauge-invariant functions. In this section we shall study implications of much more general potentials. Our main point in this section is that these more general potential functions lead naturally to Higgs fields becoming massive. This is very important for phenomenology, for massless Higgs fields interacting with the "visible" YM sector in the standard way are obviously inconsistent with observations.

A. Potential with an extra invariant

For simplicity, in this paper we shall consider only one additional invariant given by (225). We shall see that such a potential is sufficient to generate masses for the Higgs sector particles. It is not hard to consider even more general potentials, but we refrain from doing it in this already lengthy paper.

Thus, let us consider the potential depending on one more invariant:

$$V(\tilde{h}) = \frac{\mathrm{Tr}\tilde{h}}{n}F\left(\frac{\mathrm{Tr}\tilde{h}^2}{(\mathrm{Tr}\tilde{h})^2},\ldots,\frac{\mathrm{Tr}\tilde{h}^n}{(\mathrm{Tr}\tilde{h})^n},\frac{ff\tilde{h}\,\tilde{h}\,\tilde{h}}{(\mathrm{Tr}\tilde{h})^3}\right), \tag{227}$$

where we have divided (225) by $(\mathrm{Tr}\tilde{h})^3$ to make the potential homogeneous degree one. Then, the first derivative with respect to $\tilde{h}$ is

$$\frac{\partial V(\tilde{h})}{\partial \tilde{h}^{IJ}} = \frac{g_{IJ}}{n}F + \frac{\mathrm{Tr}\tilde{h}}{n}\frac{\partial F}{\partial \tilde{h}^{IJ}}, \tag{228}$$

with $(\partial F/\partial \tilde{h}^{IJ})$ given by

$$\frac{\partial F}{\partial \tilde{h}^{IJ}} = \sum_{p=2}^{n} F'_p \frac{\partial}{\partial \tilde{h}^{IJ}}\left(\frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^p}\right) + F'_{n+1}\frac{\partial}{\partial \tilde{h}^{IJ}}\left(\frac{ff\tilde{h}\,\tilde{h}\,\tilde{h}}{(\mathrm{Tr}\tilde{h})^3}\right), \tag{229}$$

where F'_p is the derivative of F with respect to its argument $(\mathrm{Tr}\tilde{h}^p/(\mathrm{Tr}\tilde{h})^p)$, F'_{n+1} is the derivative of F with respect to its last argument, and

$$\frac{\partial}{\partial \tilde{h}^{IJ}}\left(\frac{ff\tilde{h}\,\tilde{h}\,\tilde{h}}{(\mathrm{Tr}\tilde{h})^3}\right) = \frac{3f^{PQ}_{(I}f^{RS}_{J)}\tilde{h}_{PR}\tilde{h}_{QS}}{(\mathrm{Tr}\tilde{h})^3} - \frac{3ff\tilde{h}\,\tilde{h}\,\tilde{h}}{(\mathrm{Tr}\tilde{h})^4}g_{IJ}. \tag{230}$$

Now, let us compute the second derivative of V with respect to $\tilde{h}$. We get

$$\frac{\partial^2 V(\tilde{h})}{\partial \tilde{h}^{KL}\partial \tilde{h}^{IJ}} = \frac{g_{IJ}}{n}\frac{\partial F}{\partial \tilde{h}^{KL}} + \frac{g_{KL}}{n}\frac{\partial F}{\partial \tilde{h}^{IJ}} + \frac{\mathrm{Tr}\tilde{h}}{n}\frac{\partial^2 F}{\partial \tilde{h}^{KL}\partial \tilde{h}^{IJ}}, \tag{231}$$

with $(\partial^2 F/\partial \tilde{h}^{KL}\partial \tilde{h}^{IJ})$ given by

$$\begin{aligned}\frac{\partial^2 F}{\partial \tilde{h}^{KL}\partial \tilde{h}^{IJ}} = {} & \sum_{p=2}^{n} F'_p \frac{\partial^2}{\partial \tilde{h}^{KL}\partial \tilde{h}^{IJ}}\left(\frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^p}\right) + F'_{n+1}\frac{\partial^2}{\partial \tilde{h}^{KL}\partial \tilde{h}^{IJ}}\left(\frac{ff\tilde{h}\,\tilde{h}\,\tilde{h}}{(\mathrm{Tr}\tilde{h})^3}\right)\\ & + \sum_{p=2}^{n}\sum_{q=2}^{n}\left(F''_{pq}\frac{\partial}{\partial \tilde{h}^{KL}}\left(\frac{\mathrm{Tr}\tilde{h}^q}{(\mathrm{Tr}\tilde{h})^q}\right) + F''_{p(n+1)}\frac{\partial}{\partial \tilde{h}^{KL}}\left(\frac{ff\tilde{h}\,\tilde{h}\,\tilde{h}}{(\mathrm{Tr}\tilde{h})^3}\right)\right)\frac{\partial}{\partial \tilde{h}^{IJ}}\left(\frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^p}\right)\\ & + \sum_{p=2}^{n}\left(F''_{(n+1)p}\frac{\partial}{\partial \tilde{h}^{KL}}\left(\frac{\mathrm{Tr}\tilde{h}^p}{(\mathrm{Tr}\tilde{h})^p}\right) + F''_{(n+1)(n+1)}\frac{\partial}{\partial \tilde{h}^{KL}}\left(\frac{ff\tilde{h}\,\tilde{h}\,\tilde{h}}{(\mathrm{Tr}\tilde{h})^3}\right)\right)\frac{\partial}{\partial \tilde{h}^{IJ}}\left(\frac{ff\tilde{h}\,\tilde{h}\,\tilde{h}}{(\mathrm{Tr}\tilde{h})^3}\right),\end{aligned} \tag{232}$$

where F''_{pq} stands for the derivative of F'_p with respect to its q argument and similar for $F''_{p(n+1)}$ and $F''_{(n+1)(n+1)}$. It is easy to show that

$$\frac{\partial^2(ff\tilde{h}\,\tilde{h}\,\tilde{h})}{\partial\tilde{h}^{IJ}\partial\tilde{h}^{KL}} = -6f^P{}_{K)(I}f_{J)(L}{}^Q\tilde{h}_{PQ}. \tag{233}$$

Using the equations above, we obtain the following expressions:

$$\frac{\partial V}{\partial\tilde{h}^{ab}}\bigg|_0 = 0, \tag{234}$$

$$\frac{\partial V}{\partial\tilde{h}^{\alpha\beta}}\bigg|_0 = -\left(\kappa + \frac{2\lambda}{3}\right)g_{\alpha\beta}, \tag{235}$$

$$\frac{\partial^2 V}{\partial\tilde{h}^{b\beta}\partial\tilde{h}^{a\alpha}}\bigg|_0 = \frac{\kappa}{4\mathrm{i}}g_{ab}g_{\alpha\beta} + \frac{\lambda}{6\mathrm{i}}g_{cd}f^c_{ab}f^d_{\alpha\beta}, \tag{236}$$

where we have set $(F)_0 = 0$ and defined

$$\lambda = \frac{(F'_{n+1})_0}{8}. \tag{237}$$

The parameter κ is as before [see (152)] with the function $F(\cdot)$ of one more invariant in place of $f(\cdot)$.

B. Higgs sector masses

In this subsection we show that the new parameter λ introduced above receives the interpretation of mass squared of the Higgs sector scalar fields. To this end, let us work out the quadratic part of the action that comes from the potential, concentrating only on the λ-dependent part. The κ-dependent part was already taken care of by setting the Higgs sector perturbation two-forms $b^\alpha_{\mu\nu}$ to be self-dual, and this is unchanged for our more general potential. Dividing (51) by 2, using the self-duality of $b^\alpha_{\mu\nu}$ in the second term, and simplifying, we get

$$S^{(2)}_\lambda = -\frac{2\lambda}{3}\int\frac{1}{4}g_{cd}f^c_{ab}f^d_{\alpha\beta}(\Sigma_0^{a\mu\nu}b^\alpha_{\mu\nu})(\Sigma_0^{b\mu\nu}b^\beta_{\mu\nu}) - g_{\alpha\beta}b^{\alpha\mu\nu}b^\beta_{\mu\nu}. \tag{238}$$

We now substitute in this expression the expansions (163) for our two-forms (in a specific gauge). It is not hard to see that only the term $f^z_{ab}f^z_{\alpha\beta}$ contributes and we get

$$S^{(2)}_\lambda = \lambda\int b^{4-}b^{5+} + b^{6+}b^{7-} + b^{4z}b^{5z} + b^{6z}b^{7z} = -m^2_{\mathrm{Higgs}}\langle\mathbf{b}^\dagger, \mathbf{b}\rangle, \tag{239}$$

where

$$m^2_{\mathrm{Higgs}} = -\lambda. \tag{240}$$

Thus, as all other physical parameters arising in our theory, the mass of the Higgs sector particles also comes from the defining potential.

X. DISCUSSION

In view of the length of this paper it is probably appropriate to recap our logic and emphasize the main results that we have obtained. Thus, we have started with a generally covariant gauge theory for a group G, with the action given by (1). At this stage all fields are complex and reality conditions are later imposed to select the physical, real sector of the theory. We then perform the Legendre transform and pass to the two-form field formulation (6). Our phase space analysis in Sec. IV is only needed to get a better idea of what should be expected for the number of propagating DOF of the theory. It does not form an essential part of our argument. The main analysis starts in Sec. VI where we analyze the simplest case $G = \mathrm{SU}(2)$ and show how it describes the usual gravity in the limit when a certain parameter of the potential is taken to be large, or, alternatively, for low energies. For a finite value of the parameter (or for Planckian energies) one gets a modified gravity theory with two propagating DOF. However, as the low-energy limit of our theory is still given by GR, we do not need to understand the nature of this modification for purposes of this paper.

We start with the analysis of the pure gravity case by describing how the Minkowski spacetime looks in the language of two-forms; see (65). The action is then expanded to quadratic order, and the field equations for the connection field are solved for, with the solution given by (76). After the solution is substituted into the action, one gets the linearized kinetic term (79) as a functional of only the two-form perturbation. This is supplemented with the potential term part (83). After the parameter g is taken to infinity, one gets GR written in terms of two-forms, with a very compact linearized action (79). This action is considerably simpler than the one in terms of the metric perturbation, and the relation between the two arises via (90). We also perform the Hamiltonian analysis of the linearized theory, to show how the usual two polarizations of the graviton arise in this language. In the $g \to \infty$ limit this analysis reproduces Ashtekar's Hamiltonian formulation of GR, in its linearized version. The main purpose of this analysis is to select the reality conditions for the gravitational sector. These are particularly clear in the Hamiltonian formulation, and later in the paper the same strategy of deducing the reality conditions from the form of the Hamiltonian is used for other fields. In this section we discuss only the rather simple reality conditions appropriate in the GR limit $g \to \infty$. The finite g case reality conditions are deduced in the Appendix, for completeness.

Once the SU(2) case is understood, we enlarge the gauge group to $G = \mathrm{SU}(3)$. We take the same set of two-forms (65) for the background, which thus selects in the $\mathfrak{su}(3)$ Lie algebra a preferred gravitational $\mathfrak{su}(2)$ subalgebra. The analysis of the gravitational part is unchanged, but we have carried it out once more using a different basis in the Lie algebra (root basis), in preparation for the analysis of the nongravitational sectors. These split into a part that commutes with $\mathfrak{su}(2)$ and that will later be identified with the YM sector, and a part that does not commute with $\mathfrak{su}(2)$ and becomes the Higgs sector.

Let us start with the Higgs sector. As in the case of gravity, we first solve the equations for the connections a_μ^α in terms of the perturbation two-forms $b_{\mu\nu}^\alpha$ and then substitute the result back into the action. The resulting kinetic part of the action as a functional of the two-forms $b_{\mu\nu}^\alpha$ is given by (146). There is also the part (159) coming from the potential. Similar to the case of gravity, the role of the potential part, in the low-energy limit, is to set certain components of the two-form field $b_{\mu\nu}^\alpha$ to zero. After this is done, the perturbation two-forms $b_{\mu\nu}^\alpha$ becomes self-dual and can be expanded in the basis of self-dual two-forms (65). The coefficients in this expansion become our Higgs fields. They can be seen to be charged under the gravitational SU(2) subgroup, comprising two irreducible representations of spin $1/2$ of SU(2). They also transform nontrivially under the part of the gauge group that does not commute with SU(2), and so they are not all physical. A convenient gauge is given by (163). Finally, our Higgs fields are charged under the part of the gauge group that commutes with SU(2), i.e. under the YM subgroup, which in the case of $G = \mathrm{SU}(3)$ is U(1). After a gauge is fixed, one obtains a Lagrangian for the physical fields, and this is found to be just the usual one for a set of eight massless fields. We then determined the reality conditions needed to make it into a real Lagrangian with positive-definite Hamiltonian. These can be read off either from the Lagrangian we have obtained or from the Hamiltonian formulation that is also developed. The end result is a set of four complex (and at this stage massless) scalar fields with the usual real Lagrangian (168). These fields are later made massive by considering a slightly more general set of defining potentials.

We then analyze the YM sector, both in the Lagrangian and Hamiltonian frameworks. As usual in this paper, the Hamiltonian framework considerations are most useful for determining the reality conditions that need to be imposed. After these are deduced, the derivation of the Lagrangian becomes straightforward, with the result given by (201). The YM coupling constant arises as (203), with the parameter κ related to the first derivatives of the potential function via (152).

We then discuss (cubic) interactions between the various sectors of our theory and confirm that they are as expected for such fields. Namely, the interactions of all fields with gravity are via their stress-energy tensor, and interactions of the Higgs sector with the YM fields are via the Higgs conserved current.

Finally, we consider potentials more general than has been the case before and show how the first derivative (237) of the potential with respect to the new invariant becomes (minus) the mass squared (240) of the Higgs sector fields. The parameter $\lambda = -m_{\mathrm{Higgs}}^2$ can be both positive and negative, so we have the possibility of the Higgs potential pointing both up and down, depending on the form of the defining potential. For negative m_{Higgs}^2 and thus positive λ the configuration $\mathbf{b} = 0$ is unstable and a new vacuum to expand about should be chosen, as in the standard Higgs mechanism. This finishes our demonstration of the fact that the content of the theory expanded around the Minkowski spacetime background is as desired.

Let us now discuss whether the unification scheme described in this paper can be deemed "natural" in the sense that it naturally produces "realistic" values of the parameters such as masses and coupling constants. To this end let us see what dimensionful parameters are present in our theory. When the action is written in the form (1), the integrand has the mass dimension 4 (assuming that the connection has the mass dimension 1), and there are no dimensionful parameters in the theory at all. After the Legendre transform to (6) the two-form field has the mass dimension 2, and there are still no dimensionful parameters. However, since a part of this field is to be interpreted as the spacetime metric, it needs to be made dimensionless, and this is when a dimensionful parameter is introduced into the story. Rescaling the two-form field to give it the mass dimension 0 introduces a parameter of the mass dimension 2 in front of the action (interpreted as $1/G$, where G is the Newton's constant), as well as makes the potential function to have the mass dimension 2. This introduces a length (or mass) scale into the theory, and it is clear that there is only one natural mass scale given by M_p.

Various parameters of the theory are then obtained as derivatives of the potential function evaluated at the background. These have mass dimension 2, or, after being multiplied by G, are dimensionless as in the case of YM coupling (203). It is thus clear that the natural values for mass parameters arising in our theory are M_p, and for the dimensionless parameters such as the coupling constant $g_{\mathrm{YM}} \sim 1$. However, these are precisely the values that are realistic. Indeed, as our Higgs fields interact with the visible YM sector, we need to explain why they are not observed. This is explained by their very high mass that makes them essentially irrelevant for the low-energy physics. Second, the realistic values of the YM coupling constants of particle physics are order one, and precisely such values are natural in our unification scheme. Overall, our unification model is realistic in the sense that it reproduces everything that could be desired from such a simple setup.

An important ingredient that is missing from our simpleminded model is that of the usual symmetry breaking mechanism of particle physics. Such a breaking, if present, would introduce additional mass scales into the theory and make it much richer. The model considered in this paper in which the background broke only the G symmetry down to the gravitational and YM ones did not break the YM gauge group. However, it is clear that our model naturally allows for such further breaking of symmetry. Indeed, we could take the background to be more nontrivial and give to some of our Higgs fields a nontrivial vacuum expectation value.

Since our Higgs fields interact with the YM sector in the standard way, the effect of such a nontrivial vacuum expectation value (VEV) is also going to be standard—the YM symmetry is going to be broken, with some of the gauge fields becoming massive. It is then very interesting that in our scheme this standard particle physics symmetry breaking mechanism receives a new interpretation. Indeed, a nontrivial VEV for the Higgs is now on the same footing as a nonzero value for the metric. In other words, in our unification scenario the Higgs fields and the metric are just different parts of a single two-form field multiplet $B^I_{\mu\nu}$. Details of, for example, Hamiltonian analysis of the gravitational and Higgs sectors also confirm a very close analogy between the two. Thus, in a sense, it is the Higgs fields and the metric that become truly unified in our scenario. It is of considerable interest to study such more involved symmetry breaking scenarios. The goal would be to see if a truly realistic unification that puts together some GUT gauge group, a set of Higgs fields required to break it to the gauge group of the standard model, as well as gravity is possible. This question is, however, beyond the scope of this paper.

Yet another very important ingredient that is missing from our scenario is fermions. These are usually unproblematic for any scenario that operates in Minkowski spacetime. However, we start with a generally covariant theory with no metric in it, so it is not at all clear how and if fermions can be added. At the moment, this is probably the most serious objection against our scenario, but we remain hopeful that fermions can be described in our framework. The only possibility for this seems to be to further enlarge the connection field in (1) and make it "fermionic." This might also require a "generalized" connection that is no longer a one-form, as fermions that we would like to obtain are not forms. We leave investigation of all these difficult but very interesting questions to further research.

Finally, let us briefly touch on the question of quantization. The theory we have considered was classical, but, of course, it has to be quantized. It is then clear that our action (1) is nonrenormalizable in the usual sense of the word. Indeed, expanding the theory around Minkowski spacetime, we have obtained a Lagrangian consisting of some renormalizable pieces—in the Higgs and YM sectors—as well as gravity with its nonrenormalizable interactions. However, there are also higher order interactions that are nonrenormalizable, and the full action is given by an expansion containing an infinite number of nonrenormalizable terms. Thus, the full theory is nonrenormalizable. This is, of course, as expected, for we cannot hope to bring together a nonrenormalizable theory (gravity) with renormalizable other interactions in a renormalizable unified theory. At best, we can hope for a nonrenormalizable unified theory, and this is what is happening in our scenario.

At the same time, what our starting action (1) describes is just the most general generally covariant gauge theory. For this reason it can be expected that the class of theories (1) obtained by considering all possible potentials $f(\cdot)$ is closed under renormalization. Indeed, all terms that could arise as counterterms are already included in (1), and so the only effect of renormalization should be in renormalization of the defining function $f(\cdot)$. If this expectation is realized and the sole effect of renormalization is a flow in the space of potentials, the nonrenormalizability of our theory becomes much less of a problem. Indeed, it is then possible to invoke the asymptotic safety scenario and hope that in the UV the theory flows to some nonproblematic UV fixed point (corresponding to some very special potential) and that the dimension of the corresponding critical surface is finite. A Lagrangian with this potential would then provide a UV completion of our theory.

In this context it is interesting to remark that, since the gauge coupling is known to flow to zero value in the UV (asymptotic freedom), and such coupling in our scheme is on the same footing with e.g. the parameter g describing the strength of gravity modifications, it is possible that g flows to zero in the UV as well. However, it is not hard to see that this corresponds to the defining potential $V(\cdot)$ flowing toward the one of the topological BF theory. Thus, at least prior to any concrete analysis, it seems that the sought UV completion may be given by the topological BF theory, something that in the past has been suggested in the literature in other contexts. All in all, the absence of the usual "finite number of counterterms" renormalizability of our theory may not be a problem as the theory may possibly be renormalizable in the sense of Weinberg [32] as containing all possible counterterms; see also [33] for a more modern exposition of the notion of "effective renormalizability."

To summarize, there are many open problems of our scenario, notably questions of whether a realistic grand unification is possible, whether fermions can be described in the same framework, and whether the expectation of effective renormalizability is realized. However, it appears to us that in spite of all the open problems the scenario described already suggests some very interesting new interpretations and is thus worthy of further exploration.

ACKNOWLEDGMENTS

A. T. G. was supported by the University of Nottingham School of Mathematical Sciences and K. K. by EPSRC.

APPENDIX: REALITY CONDITIONS FOR MODIFIED GRAVITY

The correct reality conditions for the full modified gravity theory can be worked out from the condition $B^i \wedge (B^j)^* = 0$. In linearized theory this becomes

$$\Sigma^a \wedge (b^b)^* = \bar{\Sigma}^b \wedge b^a, \quad \text{or} \quad \Sigma^{a\mu\nu}(b^b_{\mu\nu})^* + \bar{\Sigma}^{b\mu\nu} b^a_{\mu\nu} = 0, \tag{A1}$$

where $(b^a)^*$ is the complex conjugate two-form perturbation and $\bar{\Sigma}$ is given by (67). We now rewrite this reality condition using the space plus time split. We get

$$\mathrm{i}\,(t^{ab} - (t^{ba})^*) + 2({b^{ab}}_0 + ({b^{ba}}_0)^*) = 0. \tag{A2}$$

To get this condition we have used $\bar{\Sigma}^a_{bc} = \epsilon^a_{bc}$, $\bar{\Sigma}^a_{0b} = \mathrm{i}\delta^a_b$ and recalled the definition (99) of the configurational variable. We should now analyze this condition together with the already known solution (115) and (114) for the components ${b^{ab}}_0$.

Let us first consider the trace and antisymmetric parts of (A2). Then in the trace-free symmetric gauge for t^{ab} these conditions simply state that the lapse and shift functions N and N^a are real. This explains why the factor of i was introduced in (115) in front of the lapse.

Consider now the symmetric trace-free part of (A2). The corresponding components of ${b^{ab}}_0$ are known from (114), and we arrive at the following condition on the phase space variables:

$$\frac{1}{2g}\,\mathrm{Re}(\epsilon^{ef(a}\partial_e \pi^{b)}_f)_{tf} = \mathrm{Im}(t^{ab})_{tf}. \tag{A3}$$

In the case $g \to \infty$ that corresponds to GR this implies that $(t^{ab})_{tf}$ is real, but in the modified case the situation is more interesting.

In addition to (A3), there is another condition that is obtained by requiring that (A3) is preserved under the evolution. Thus, we need to compute the Poisson bracket of (A3) with the Hamiltonian and impose the resulting condition as well. The computation is a bit technical, but at this phase space level there is no way to avoid it. Indeed, even in the case of GR it is clear from the form of the Hamiltonian (116) that the relevant condition cannot be that the momentum is real, for the Hamiltonian would be complex due to the presence of the second term in the second line. The computation of the Poisson bracket can be done as follows. First, we introduce the real and imaginary parts of the phase space variables:

$$t^{ab} = t^{ab}_1 + \mathrm{i}t^{ab}_2, \qquad \pi^{ab} = \pi^{ab}_1 + \mathrm{i}\pi^{ab}_2. \tag{A4}$$

Second, we substitute this decomposition into the action written in the Hamiltonian form. The resulting action has real and imaginary parts. It is not hard to convince oneself that any one of these two parts can be used as an action for the system, the resulting equations are the same due to Riemann-Cauchy equations that follow from the fact that the original action was holomorphic. We choose to work with the real part of the action. The relevant Poisson brackets are easily seen to be

$$\{\pi^{ab}_1(x), t_{1cd}(y)\} = \delta^{(a}_c \delta^{b)}_d \delta^3(x-y), \quad \{\pi^{ab}_2, (x) t_{2cd}(y)\} = -\delta^{(a}_c \delta^{b)}_d \delta^3(x-y), \tag{A5}$$

with all the other ones being zero. The real part of the Hamiltonian (with the constraint part already imposed and dropped) reads:

$$\mathcal{H}^{\text{real}} = \frac{1}{2}(\pi^{ab}_1)^2 - \frac{1}{2}(\pi^{ab}_2)^2 - \epsilon^{efa}\partial^e \pi^{bf}_1 t^{ab}_2 - \epsilon^{efa}\partial^e \pi^{bf}_2 t^{ab}_1 + \frac{1}{2g}(\partial^a \pi^{bc}_1)^2 - \frac{1}{2g}(\partial^a \pi^{bc}_2)^2.$$

We can now compute the Poisson bracket with the reality condition (A3) that becomes

$$\frac{1}{2g}\epsilon^{efa}\partial_e \pi^{fb}_1 = t^{ab}_2. \tag{A6}$$

The Poisson bracket with the left-hand side is

$$\left\{\mathcal{H}^{\text{real}}, \frac{1}{2g}\epsilon^{efa}\partial^e \pi^{bf}_1\right\} = -\frac{1}{2g}\Delta\pi^{ab}_2. \tag{A7}$$

The Poisson bracket with the right-hand side is

$$\{\mathcal{H}^{\text{real}}, t^{ab}_2\} = \pi^{ab}_2 + \epsilon^{efa}\partial^e t^{bf}_1 - \frac{1}{g}\Delta\pi^{ab}_2. \tag{A8}$$

Thus, the sought condition that guarantees the consistency of (A6) is

$$\pi^{ab}_2 + \epsilon^{efa}\partial^e t^{bf}_1 - \frac{1}{2g}\Delta\pi^{ab}_2 = 0. \tag{A9}$$

We now need to solve this for π^{ab}_2, which gives

$$\pi^{ab}_2 = -\frac{\epsilon^{efa}\partial^e t^{bf}_1}{1 - \Delta/2g}, \tag{A10}$$

where the denominator should be understood as a formal power series in powers of Δ/g. When $g \to \infty$, we reproduce the GR result reviewed in the beginning of this subsection.

We now have to substitute this, as well as the expression (A6) for t^{ab}_2 into the action. This is a simple exercise with the result being

$$S^{\text{real}} = \int dt d^3x \left(\pi^{ab}_{\text{GR}}\partial_0 t^{ab}_{\text{GR}} - \frac{1}{2}((\pi^{ab}_{\text{GR}})^2 + (\partial^a t^{bc}_{\text{GR}})^2)\right), \tag{A11}$$

where we have defined

$$\pi^{ab}_{\text{GR}} = \pi^{ab}_1, \qquad t^{ab}_{\text{GR}} = \frac{t^{ab}_1}{1 - \Delta/2g}. \tag{A12}$$

These are the phase space variables in terms of which the Hamiltonian takes the standard GR form. This shows how an explicitly real formulation with a positive definite Hamiltonian can be obtained. We also see that for any finite value of g the graviton is unmodified.

[1] C. Vafa, arXiv:0911.3008.
[2] S.R. Coleman and J. Mandula, Phys. Rev. **159**, 1251 (1967).
[3] R. Haag, J.T. Lopuszanski, and M. Sohnius, Nucl. Phys. **B88**, 257 (1975).
[4] R. Percacci, J. Phys. A **41**, 335403 (2008).
[5] S. Holst, Phys. Rev. D **53**, 5966 (1996).
[6] A. Einstein and W. Mayer, Sitz. der Preuss. Akad. (Berlin) **XXV**, 541 (1931); **XII**, 130 (1932); also available in a reprint of Einstein's works available as DOI: 10.1002/3527608958.
[7] B. Hoffmann, Phys. Rev. **43**, 615 (1933).
[8] R. Percacci, Phys. Lett. **144B**, 37 (1984).
[9] R. Percacci, Nucl. Phys. **B353**, 271 (1991).
[10] F. Nesti, Eur. Phys. J. C **59**, 723 (2009).
[11] F. Nesti and R. Percacci, J. Phys. A **41**, 075405 (2008).
[12] F. Nesti and R. Percacci, Phys. Rev. D **81**, 025010 (2010).
[13] P. Peldan, Phys. Rev. D **46**, R2279 (1992).
[14] P. Peldan, Nucl. Phys. **B395**, 239 (1993).
[15] S. Chakraborty and P. Peldan, Phys. Rev. Lett. **73**, 1195 (1994).
[16] S. Chakraborty and P. Peldan, Int. J. Mod. Phys. D **3**, 695 (1994).
[17] A. Ashtekar, Phys. Rev. D **36**, 1587 (1987).
[18] D.C. Robinson, Classical Quantum Gravity **11**, L157 (1994).
[19] J.F. Plebanski, J. Math. Phys. (N.Y.) **18**, 2511 (1977).
[20] L. Smolin, Phys. Rev. D **80**, 124017 (2009).
[21] K. Krasnov, arXiv:hep-th/0611182.
[22] I. Bengtsson, Phys. Lett. B **254**, 55 (1991).
[23] S. Alexandrov and K. Krasnov, Classical Quantum Gravity **26**, 055005 (2009).
[24] K. Krasnov, Europhys. Lett. **89**, 30002 (2010).
[25] R. Capovilla, T. Jacobson, and J. Dell, Phys. Rev. Lett. **63**, 2325 (1989).
[26] K. Krasnov, Classical Quantum Gravity **26**, 055002 (2009).
[27] K. Krasnov, Phys. Rev. Lett. **100**, 081102 (2008).
[28] L. Freidel, arXiv:0812.3200.
[29] H. Urbantke, J. Math. Phys. (N.Y.) **25**, 2321 (1984).
[30] H. Georgi, *Lie Algebras in Particle Physics* (Westview Press, Boulder, CO, 1991), 2nd ed.
[31] R. Cahn, *Semi-Simple Lie Algebras and Their Representation* (The Benjamin/Cummings Publishing Company, San Francisco, CA, 1984).
[32] S. Weinberg, Physica (Amsterdam) **96A**, 327 (1979).
[33] S. Weinberg, arXiv:0908.1964.