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Horizon thermodynamics and gravitational field equations in Hořava-Lifshitz gravity

Rong-Gen Cai[1,2,*] and Nobuyoshi Ohta[2,†]

[1] *Key Laboratory of Frontiers in Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, P.O. Box 2735, Beijing 100190, China*

[2] *Department of Physics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan*

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We explore the relationship between the first law of thermodynamics and gravitational field equation at a static, spherically symmetric black hole horizon in Hořava-Lifshitz theory with/without detailed balance. It turns out that as in the cases of Einstein gravity and Lovelock gravity, the gravitational field equation can be cast to a form of the first law of thermodynamics at the black hole horizon. This way we obtain the expressions for entropy and mass in terms of black hole horizon, consistent with those from other approaches. We also define a generalized Misner-Sharp energy for static, spherically symmetric spacetimes in Hořava-Lifshitz theory. The generalized Misner-Sharp energy is conserved in the case without matter field, and its variation gives the first law of black hole thermodynamics at the black hole horizon.

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I. INTRODUCTION

The holographic principle might be one of the principles of nature, which states that a theory with gravity could be equivalent to a theory without gravity in one less dimension. The well-known AdS/CFT correspondence [1] is a realization of the holographic principle, while the latter is motivated by black hole thermodynamics. The black hole thermodynamics says that a black hole behaves as an ordinary thermodynamic system with temperature and entropy. The temperature of a black hole is proportional to surface gravity at its horizon, while the entropy of the black hole is measured by its horizon area. Black hole mass, temperature, and entropy satisfy the first law of thermodynamics. These results come from a combination of quantum mechanics, black hole geometry, and general relativity. This implies that there might exist a deep connection between thermodynamics and gravity theory.

Indeed some pieces of evidence have been accumulated for the connection between thermodynamics and gravity theory in the literature. Assuming there is a proportionality between entropy and horizon area, Jacobson [2] derived the Einstein field equation by using the fundamental Clausius relation, $\delta Q = TdS$, connecting heat, temperature, and entropy. The key idea is to demand that this relation holds for all the local Rindler causal horizon through each spacetime point, with δQ and T interpreted as the energy flux and Unruh temperature seen by an accelerated observer just inside the horizon. In this way, the Einstein field equation is nothing but an equation of state of spacetime. More recently, Jacobson's argument has been generalized to all diffeomorphism-invariant theories of gravity [3] (however, see also [4]). For $f(R)$ theory and scalar-tensor theory, see also [5,6]. In fact, investigating the thermodynamics of spacetime for $f(R)$ theory [7–9] and scalar-tensor theory [8,10], it is found that a nonequilibrium thermodynamic setup has to be employed. Further, it is argued that if shear of spacetime is not assumed to vanish, the nonequilibrium thermodynamic setting is required even for the Einstein general relativity [11,12]. There an internal entropy production term has to be introduced to balance energy conservation. The internal entropy production term dS_i is proportional to the squared shear of the horizon and the ratio of the proportionality constant to the area entropy density is $1/4\pi$. The latter is a universal value for many kinds of conformal field theories with AdS duals [13].

There exists another route in exploring the relationship between thermodynamics and gravity theory. Padmanabhan [14] first noticed that the gravitational field equation in a static, spherically symmetric spacetime can be rewritten as a form of the ordinary first law of thermodynamics at a black hole horizon. This indicates that Einstein's equation is nothing but a thermodynamic identity. For a recent review on this, see [15]. This observation was then extended to the cases of stationary axisymmetric horizons and evolving spherically symmetric horizons in the Einstein gravity [16], static spherically symmetric horizons [17], and dynamical apparent horizons [18] in Lovelock gravity, and three-dimensional Banados-Teitelboim-Zanelli black hole horizons [19]. On the other hand, the relationship between the first law of thermodynamics and dynamical equation of spacetime has been intensively investigated in a Friedmann-Robertson-Walker (FRW) cosmological setup in various gravity theories [8–10,20–25]; it is shown that (modified) Friedmann equations can be cast to a form of the first law of thermodynamics, and there exists a Hawking radiation associated with apparent horizon in a FRW universe [26].

Recently, a field theory model for a UV complete theory of gravity was proposed by Hořava [27], which is a nonrelativistic renormalizable theory of gravity and is ex-

*cairg@itp.ac.cn
†ohtan@phys.kindai.ac.jp

pected to recover Einstein's general relativity at large scales. This theory is named Hořava-Lifshitz theory in the literature since at the UV fixed point of the theory space and time have different scalings. Since then a lot of work has been done in exploring various aspects of the theory; for a more or less complete list of references, see, for example, [28].

In this paper we discuss the relationship between the first law of thermodynamics and gravitational field equation in Hořava-Lifshitz theory. In static spherically symmetric black hole spacetimes, we show that the gravitational field equation can be rewritten as $dE - TdS = PdV$ at the black hole horizon. Note that in Hořava-Lifshitz theory the full diffeomorphism invariance is broken to the "foliation-preserving" diffeomorphism. Therefore our result is a nontrivial generalization of Padmanabhan's observation. In addition, we discuss the question of whether one can define a generalized Misner-Sharp quasilocal energy in Hořava-Lifshitz theory. The answer is positive. We define a generalized Misner-Sharp energy. It is a conserved charge when the matter field is absent, while its variation at a black hole horizon gives the first law of black hole thermodynamics.

This paper is organized as follows. In the next section we review the Padmanabhan's observation by extending his discussion to a more general spherically symmetric spacetime. In Sec. III we consider black hole spacetimes in Hořava-Lifshitz theory. In Sec. IV the case of IR modified Hořava-Lifshitz theory is discussed. In Sec. V we define a generalized Misner-Sharp quasilocal energy for static, spherically symmetric spacetimes in Hořava-Lifshitz theory and discuss its properties. The conclusion is given in Sec. VI.

II. BLACK HOLES IN EINSTEIN GRAVITY

As a warm-up exercise, in this section, we will briefly review the observation made by Padmanabhan [14] by generalizing his discussion to a more general spherically symmetric case. In Einstein's general relativity, the gravitational field equations are

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.1}$$

where $G_{\mu\nu}$ is Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor of matter field. On the other hand, for a general static, spherically symmetric spacetime, its metric can be written down as

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + b^2(r)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2.2}$$

where $f(r)$ and $b(r)$ are two functions of the radius coordinate r. [Note that in the Padmanabhan's discussion [14], the metric is assumed in a form (2.2) with $b(r) = r$; when matter is present, however, such a metric form is not always satisfied. See also [16].] Suppose the metric (2.2) describes a nonextremal black hole with horizon at r_+, then the function $f(r)$ has a simple zero at $r = r_+$. Namely, $f'(r)|_{r=r_+} = 0$, but $f''(r)|_{r=r_+} \neq 0$. It is easy to show that the Hawking temperature of the black hole associated with the horizon r_+ is

$$T = \frac{1}{4\pi} f'(r)|_{r=r_+} \equiv \frac{1}{4\pi} f'(r_+), \tag{2.3}$$

where a prime stands for the derivative with respective to r. Einstein's equations in the metric (2.2) have the components

$$\begin{aligned} G^t_t &= \frac{1}{b^2}(-1 + fb'^2 + b(f'b' + 2fb'')), \\ G^r_r &= \frac{1}{b^2}(-1 + bf'b' + fb'^2). \end{aligned} \tag{2.4}$$

Note that at the horizon, one has $f(r) = 0$, and then

$$G^t_t|_{r=r_+} = G^r_r|_{r=r_+} = \frac{1}{b^2}(-1 + bf'b')|_{r=r_+}. \tag{2.5}$$

Therefore at the horizon, the t-t component of Einstein's equations can be expressed as

$$-1 + bf'b' = 8\pi G b^2 P, \tag{2.6}$$

where $P = T^r_r|_{r=r_+}$ is the radial pressure of matter at the horizon. Note that here (2.5) guarantees $T^t_t = T^r_r$ at the horizon. Now we multiply dr_+ on both sides of (2.6) and rewrite this equation as

$$\frac{1}{2G} bf'b'dr_+ - \frac{1}{2G} dr_+ = 4\pi b^2 P dr_+. \tag{2.7}$$

Note that b is a function of r only and f' has a relation to the Hawking temperature as (2.3). One then can rewrite the above equation as

$$Td\left(\frac{4\pi b^2}{4G}\right) - d\left(\frac{r_+}{2G}\right) = PdV, \tag{2.8}$$

where $dV = 4\pi b^2 dr_+$. Therefore V is just the volume of the black hole with horizon radius r_+ in the coordinate (2.2). The equation (2.8) can be further rewritten as

$$TdS - dE = PdV, \tag{2.9}$$

with identifications

$$S = \frac{4\pi b^2}{4G} = \frac{A}{4G}, \qquad E = \frac{r_+}{2G}. \tag{2.10}$$

Clearly here S is precisely the entropy of the black hole, while E is the Misner-Sharp energy [29] at the horizon. Thus we have shown in general that at black hole horizon, Einstein's equations can be cast into the form of the first law of thermodynamics.

III. BLACK HOLES IN HOŘAVA-LIFSHITZ GRAVITY

In the (3 + 1)-dimensional Arnowitt-Deser-Misner formalism, where the metric can be written as

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (3.1)$$

and for a spacelike hypersurface with a fixed time, its extrinsic curvature K_{ij} is

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i), \quad (3.2)$$

where a dot denotes a derivative with respect to t and covariant derivatives defined with respect to the spatial metric g_{ij}.

The action of Hořava-Lifshitz theory is [27,30]

$$\begin{aligned} I &= \int dt d^3x(\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_m), \\ \mathcal{L}_0 &= \sqrt{g}N\left\{\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2\mu^2(\Lambda R - 3\Lambda^2)}{8(1-3\lambda)}\right\}, \\ \mathcal{L}_1 &= \sqrt{g}N\left\{\frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}R^2 - \frac{\kappa^2}{2\omega^4}\left(C_{ij} - \frac{\mu\omega^2}{2}R_{ij}\right)\right. \\ &\quad \left.\times\left(C^{ij} - \frac{\mu\omega^2}{2}R^{ij}\right)\right\}, \end{aligned} \quad (3.3)$$

where κ^2, λ, μ, ω, and Λ are constant parameters and the Cotten tensor, C_{ij}, is defined by

$$C^{ij} = \epsilon^{ikl}\nabla_k(R^j_l - \tfrac{1}{4}R\delta^j_l) = \epsilon^{ikl}\nabla_k R^j_l - \tfrac{1}{4}\epsilon^{ikj}\partial_k R. \quad (3.4)$$

The first two terms in $\mathcal{L}_0$ are the kinetic terms, others in $(\mathcal{L}_0 + \mathcal{L}_1)$ give the potential of the theory in the so-called "detailed-balance" form, and $\mathcal{L}_m$ stands for the Lagrangian of other matter field.

Comparing the action to that of general relativity, one can see that the speed of light, Newton's constant, and the cosmological constant are

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\Lambda}{1-3\lambda}}, \qquad G = \frac{\kappa^2 c}{32\pi}, \qquad \tilde{\Lambda} = \frac{3}{2}\Lambda, \quad (3.5)$$

respectively. Let us notice that when $\lambda = 1$, $\mathcal{L}_0$ could be reduced to the usual Lagrangian of Einstein's general relativity. Therefore it is expected that general relativity could be approximately recovered at large distances when $\lambda = 1$. Here we will mainly consider the case of $\lambda = 1$, but will also discuss the $\lambda \neq 1$ case briefly at the end of this paper.

Now we consider black hole spacetime with metric ansatz [30,31]

$$ds^2 = -\tilde{N}^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_k^2, \quad (3.6)$$

where $d\Omega_k^2$ denotes the line element for a two-dimensional Einstein space with constant scalar curvature $2k$. Without loss of generality, one may take $k = 0, \pm 1$, respectively. Substituting the metric (3.6) into (3.3), we have

$$\begin{aligned} I = \frac{\kappa^2\mu^2\Lambda\Omega_k}{8(1-3\lambda)}\int dt dr\tilde{N}\Big\{&-3\Lambda r^2 - 2(f-k) - 2r(f-k)' \\ &+ \frac{(\lambda-1)f'^2}{2\Lambda} + \frac{(2\lambda-1)(f-k)^2}{\Lambda r^2} - \frac{2\lambda(f-k)}{\Lambda r}f' \\ &+ \alpha r^2\mathcal{L}_m\Big\}, \end{aligned} \quad (3.7)$$

where a prime denotes the derivative with respect to r, Ω_k is the volume of the two-dimensional Einstein space, and the constant $\alpha = 8(1-3\lambda)/\kappa^2\mu^2\Lambda$. In the case of $\lambda = 1$ we can rewrite the action as

$$\begin{aligned} I = \frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\int dt dx\tilde{N}\Big\{&\left(x^3 - 2x(f-k) + \frac{(f-k)^2}{x}\right)' \\ &+ x^2\left(\frac{\alpha}{-\Lambda}\right)\mathcal{L}_m\Big\}. \end{aligned} \quad (3.8)$$

Note that here $x = \sqrt{-\Lambda}r$, a prime becomes the derivative with respect to x. Varying the action with $\tilde{N}$, we obtain the equations of motion

$$\begin{aligned} &-\frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\left(3x^2 - 2(f-k) - \frac{(f-k)^2}{x^2} - 2xf'\right. \\ &\left.+ \frac{2(f-k)f'}{x}\right) = x^2\frac{\Omega_k}{(-\Lambda)^{3/2}}\frac{\delta(\tilde{N}\mathcal{L}_m)}{\delta\tilde{N}}. \end{aligned} \quad (3.9)$$

Suppose the nonextremal black hole (3.6) has a horizon radius r_+, namely $x_+ = \sqrt{-\Lambda}r_+$. Then the Hawking temperature of the black hole is

$$T = \frac{1}{4\pi}\tilde{N}(r)\frac{df}{dr}\bigg|_{r=r_+} = \frac{\sqrt{-\Lambda}}{4\pi}\tilde{N}(x)f'|_{x=x_+}. \quad (3.10)$$

Now we consider a class of solutions with $\tilde{N}(r) = \text{const}$. For example, the charged black hole solution discussed in [31] belongs to this class of solutions. In this case, one can set $\tilde{N} = 1$ by rescaling the time coordinate t. Note that here not all solutions with matter field have the form $\tilde{N} = 1$. At the horizon x_+, Eq. (3.9) is reduced to

$$\begin{aligned} &-\frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\left(3x_+^2 + 2k - \frac{k^2}{x_+^2} - 2x_+f' - \frac{2kf'}{x_+}\right) \\ &= x_+^2\frac{\Omega_k}{(-\Lambda)^{3/2}}\frac{\delta(\tilde{N}\mathcal{L}_m)}{\delta\tilde{N}}\bigg|_{x=x_+}. \end{aligned} \quad (3.11)$$

Multiplying both sides with dx_+, a variation of the horizon radius, we have

$$\begin{aligned} &\frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\left(2\left(x_+ + \frac{k}{x_+}\right)f'dx_+\right. \\ &\left.-\left(3x_+^2 + 2k - \frac{k^2}{x_+^2}\right)dx_+\right) = x_+^2\frac{\Omega_k}{(-\Lambda)^{3/2}}Pdx_+, \end{aligned} \quad (3.12)$$

where $P = \{[\delta(\tilde{N}\mathcal{L}_m)]/(\delta\tilde{N})\}|_{x=x_+}$. Note that the Hawking temperature turns to be $T = f'\sqrt{-\Lambda}/4\pi$ when $\tilde{N} = 1$. The above equation then can be rewritten as

$$TdS - dE = PdV, \tag{3.13}$$

where

$$\begin{aligned} S &= \frac{\pi\kappa^2\mu^2\Omega_k}{4}(x_+^2 + 2k\ln x_+) + S_0, \\ E &= \frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16x_+}(x_+^2 + k)^2, \end{aligned} \tag{3.14}$$

$V = \frac{\Omega_k}{3}r_+^3$, and S_0 is an undetermined constant. Clearly V is the volume of black hole with radius r_+. Comparing (3.14) with black hole entropy and mass defined through a Hamiltonian approach in our previous papers [31], we see that S and E are just black hole entropy and mass in terms of horizon radius x_+, and the gravitational field equation at the black hole horizon can be cast to the form of the first law of thermodynamics. Note that here we have obtained expressions for black hole entropy and mass, but have not used any explicit black hole solutions. In other words, the above way provides a universal method to derive black hole entropy and mass.

Now we turn to the case without the detailed-balance condition by considering the action as [30,31]

$$I = \int dtd^3x(\mathcal{L}_0 + (1-\epsilon^2)\mathcal{L}_1 + \mathcal{L}_m), \tag{3.15}$$

where the parameter $\epsilon^2 \neq 0$. In this case, instead of (3.8), we have

$$\begin{aligned} I = \frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\int dtdx\tilde{N}\bigg\{\bigg(x^3 - 2x(f-k) \\ + (1-\epsilon^2)\frac{(f-k)^2}{x}\bigg)' + x^2\bigg(\frac{\alpha}{-\Lambda}\bigg)\mathcal{L}_m\bigg\}. \end{aligned} \tag{3.16}$$

Varying the action with respect to $\tilde{N}$ yields

$$\begin{aligned} -\frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\bigg(3x^2 - 2(f-k) - (1-\epsilon^2)\frac{(f-k)^2}{x^2} \\ - 2xf' + (1-\epsilon^2)\frac{2(f-k)f'}{x}\bigg) = x^2\frac{\Omega_k}{(-\Lambda)^{3/2}}\frac{\delta(\tilde{N}\mathcal{L}_m)}{\delta\tilde{N}}. \end{aligned} \tag{3.17}$$

At the black hole horizon where $f = 0$, the equation reduces to

$$\begin{aligned} -\frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\bigg(3x_+^2 + 2k - (1-\epsilon^2)\frac{k^2}{x_+^2} - 2x_+f' \\ - (1-\epsilon^2)\frac{2kf'}{x_+}\bigg) = x_+^2\frac{\Omega_k}{(-\Lambda)^{3/2}}\frac{\delta(\tilde{N}\mathcal{L}_m)}{\delta\tilde{N}}\bigg|_{x=x_+}. \end{aligned} \tag{3.18}$$

Multiplying dx_+ on both sides, one can express this equation as the form (3.13) with the condition $\tilde{N} = 1$, again. But this time we have

$$\begin{aligned} S &= \frac{\pi\kappa^2\mu^2\Omega_k}{4}(x_+^2 + 2k(1-\epsilon^2)\ln x_+) + S_0, \\ E &= \frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16x_+}(x_+^4 + 2kx_+ + (1-\epsilon^2)k^2). \end{aligned} \tag{3.19}$$

These are nothing but the entropy and mass, expressed in terms of horizon radius x_+, of black hole solutions found in [31].

Now we turn to the case with $z = 4$ terms, where z is the dynamical critical exponent. Such terms are super-renormalizable ones. The vacuum black hole solution for this case has been discussed in [32]. Including $z = 4$ terms changes $\mathcal{L}_1$ in (3.3) to

$$\begin{aligned} \mathcal{L}_1 = -\sqrt{g}N\frac{\kappa^2}{8}\bigg\{\frac{4}{\omega^4}C^{ij}C_{ij} - \frac{4\mu}{\omega^2}C^{ij}R_{ij} - \frac{4}{\omega^2 M}C^{ij}L_{ij} \\ + \mu^2G_{ij}G^{ij} + \frac{2\mu}{M}G^{ij}L_{ij} + \frac{2\mu}{M}\Lambda L + \frac{1}{M^2}L^{ij}L_{ij} \\ - \tilde{\lambda}\bigg(\frac{L^2}{M^2} - \frac{\mu L}{M}(R - 6\Lambda) + \frac{\mu^2}{4}R^2\bigg)\bigg\}, \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} G^{ij} &= R^{ij} - \frac{1}{2}g^{ij}R \\ L^{ij} &= (1+2\beta)(g^{ij}\nabla^2 - \nabla^i\nabla^j)R + \nabla^2G^{ij} \\ &\quad + 2\beta R\bigg(R^{ij} - \frac{1}{4}g^{ij}R\bigg) + 2\bigg(R^{imjn} - \frac{1}{4}g^{ij}R^{mn}\bigg)R_{mn}, \\ L &\equiv g^{ij}L_{ij} = \bigg(\frac{3}{2} + 4\beta\bigg)\nabla^2R + \frac{\beta}{2}R^2 + \frac{1}{2}R_{ij}R^{ij}, \end{aligned} \tag{3.21}$$

and $\tilde{\lambda} = \lambda/(3\lambda - 1)$, β and M are two new parameters. When $\beta = -3/8$ and $\lambda = 1$, the action in the metric (3.6) reduces to

$$\begin{aligned} I = \frac{\kappa^2\Omega_k}{16\sqrt{-\Lambda^3}}\int dtdx\tilde{N}\bigg\{\bigg[\tilde{\mu}^2\bigg(x^3 - 2x(f-k) \\ + \frac{(f-k)^2}{x}\bigg) - 2\tilde{\beta}\,\tilde{\mu}\bigg(\frac{(f-k)^3}{x^3} - \frac{(f-k)^2}{x}\bigg) \\ + \tilde{\beta}^2\frac{(f-k)^4}{x^5}\bigg]' + x^2\tilde{\alpha}\mathcal{L}_m\bigg\}, \end{aligned} \tag{3.22}$$

where we define $\tilde{\mu} = -\mu\Lambda$, $\tilde{\beta} = \frac{\Lambda^2}{4M}$, $\tilde{\alpha} = 16/\kappa^2$, and the prime is still the derivative with respect to x. Varying the action with respect to $\tilde{N}$ yields

$$\begin{aligned} -\frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\bigg(x^3 - 2x(f-k) + \frac{(f-k)^2}{x} \\ - 2\frac{\tilde{\beta}}{\tilde{\mu}}\bigg(\frac{(f-k)^3}{x^3} - \frac{(f-k)^2}{x}\bigg) + \frac{\tilde{\beta}^2}{\tilde{\mu}^2}\frac{(f-k)^4}{x^5}\bigg)' \\ = x^2\frac{\Omega_k}{(-\Lambda)^{3/2}}\frac{\delta(\tilde{N}\mathcal{L}_m)}{\delta\tilde{N}}. \end{aligned} \tag{3.23}$$

Again, we take the values of all quantities at the black hole horizon and then multiply dx_+ on both sides; the above equation turns to be

$$TdS - dE = PdV, \tag{3.24}$$

where $P = \{[\delta(\tilde{N}\mathcal{L}_m)]/(\delta\tilde{N})\}|_{x=x_+}$, $V = \Omega_k r_+^3/3$ is the volume of the black hole, and

$$\begin{aligned} E &= \frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\bigg(x_+^3 + 2kx_+ + \frac{k^2}{x_+} + 2\frac{\tilde{\beta}}{\tilde{\mu}}\bigg(\frac{k^3}{x_+^3} + \frac{k^2}{x}\bigg) \\ &\quad + \frac{\tilde{\beta}^2}{\tilde{\mu}^2}\frac{k^4}{x_+^5}\bigg) \\ S &= \frac{\pi\kappa^2\mu^2\Omega_k}{4}\bigg(x_+^2 + 2k\ln x_+ - 3\frac{\tilde{\beta}k^2}{\tilde{\mu}x_+^2} - \frac{\tilde{\beta}^2k^3}{\tilde{\mu}^2x_+^4} \\ &\quad + 4\frac{\tilde{\beta}k}{\tilde{\mu}}\ln x_+\bigg) + S_0. \end{aligned} \tag{3.25}$$

This way we have obtained entropy and mass of black hole solutions [32], again, and shown that at the black hole horizon, the gravitational field equation can be cast into the form of the first law of thermodynamics.

IV. BLACK HOLES IN IR MODIFIED HOŘAVA-LIFSHITZ GRAVITY

In this section we consider the case with broken detailed balance by introducing a term $\mu^4 R$ to the action [33]. Such theory is called IR modified Hořava-Lifshitz theory. In this way, it is found that one can get asymptotically flat solutions. In fact, introducing the parameter ϵ^2 to the original action of Hořava-Lifshitz theory with detailed balance can also lead to asymptotically flat solutions [30,31]. Here we show that, for the IR modified Hořava-Lifshitz theory, gravitational field equations at the black hole horizon can also be cast into a form of the first law of thermodynamics.

Now we add a new term,

$$\mathcal{L}_3 = \sqrt{g}N\frac{\kappa^2\mu^2\nu}{8(3\lambda - 1)}R, \tag{4.1}$$

to the action (3.3). Here ν is a new parameter. The term (4.1) "softly" violates the so-called detailed balance. The action in the metric (3.6) changes to [34]

$$\begin{aligned} I &= \frac{\kappa^2\mu^2\Omega}{8(1-3\lambda)}\int dtdr\tilde{N}\bigg((2\lambda - 1)\frac{(f-1)^2}{r^2} - 2\lambda\frac{f-1}{r}f' \\ &\quad + \frac{\lambda - 1}{2}f'^2 - 2(\nu - \Lambda)(1 - f - rf') - 3\Lambda^2r^2 \\ &\quad + \alpha\Lambda r^2\mathcal{L}_m\bigg), \end{aligned} \tag{4.2}$$

where we have restricted to the case with $k = 1$ and a prime stands for the derivative with respect to r. Considering the case with $\lambda = 1$, and varying the action with respect to $\tilde{N}$, one has the equation of motion:

$$\begin{aligned} &\frac{\kappa^2\mu^2\Omega}{16}\bigg(\frac{(f-1)^2}{r^2} - 2\frac{f-1}{r}f' - 2(\nu - \Lambda)(1 - f - rf') \\ &\quad - 3\Lambda^2r^2\bigg) = r^2\Omega\frac{\delta(\tilde{N}\mathcal{L}_m)}{\delta\tilde{N}}. \end{aligned} \tag{4.3}$$

At a black hole horizon where $f = 0$ and $f'|_{r=r_+} = 4\pi T$, by the same approach, we can rewrite the equation as

$$TdS - dE = PdV, \tag{4.4}$$

where

$$\begin{aligned} S &= \frac{\pi\kappa^2\mu^2\Omega}{4}((\nu - \Lambda)r_+^2 + 2\ln r_+) + S_0, \\ E &= \frac{\kappa^2\mu^2\Omega}{16}\bigg(\Lambda^2r_+^3 + 2(\nu - \Lambda)r_+ + \frac{1}{r_+}\bigg). \end{aligned} \tag{4.5}$$

This energy is the same as that given in [34], up to a factor, which is not figured out there. The entropy is given for the first time, although related results on thermodynamics of black holes in the modified Hořava-Lifshitz theory have been discussed in [35].

V. GENERALIZED MISNER-SHARP ENERGY AND THE FIRST LAW OF BLACK HOLE THERMODYNAMICS

Quasilocal energy is an important concept in general relativity. In particular, the so-called Misner-Sharp energy is intensively discussed in the literature. In a spherically symmetric spacetime with metric

$$ds^2 = h_{ab}dx^adx^b + r^2(x)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{5.1}$$

where $a = 0$ and 1, the Misner-Sharp energy is defined as [29]

$$M(r) = \frac{r}{2G}(1 - h^{ab}\partial_a r\partial_b r), \tag{5.2}$$

which is valid for general relativity in four dimensions. For Schwarzschild solution, (5.2) just gives the Schwarzschild mass, while it gives the effective Schwarzschild mass $m(r)$ at r for a static spherically symmetric spacetime (3.6) with $f(r) = 1 - \frac{2Gm(r)}{r}$. Therefore at a black hole horizon r_+, the Misner-Sharp energy (5.2) gives us the energy of gravitational field at the horizon r_+: $M = r_+/2G$.

In the previous sections, we have shown that the gravitational field equations at a black hole horizon can be cast to a form of the first law of thermodynamics in Hořava-Lifshitz theory. In this section, we show that the form of action for the Hořava-Lifshitz theory allows us to give a generalized Misner-Sharp quasilocal energy in the case of static, spherically symmetric spacetime (3.6).

Let us start with the action (3.3) with the detailed balance. In this case, the gravitational part of the action can be rewritten in a derivative form (3.8), which enables us to define a generalized Misner-Sharp energy as

$$M(r) = \frac{\kappa^2\mu^2\Omega_k}{16r}(\Lambda^2 r^4 - 2\Lambda r^2(k-f) + (k-f)^2). \tag{5.3}$$

It is easy to see that, at a black hole horizon r_+, this quasilocal energy $E(r)$ gives the mass (3.14) of the black hole solution. The variation of the generalized Misner-Sharp energy with respect to r gives

$$dM(r) = -r^2\Omega_k \frac{\delta(\tilde{N}\mathcal{L}_m)}{\delta\tilde{N}}dr, \tag{5.4}$$

from which one can see clearly that $-\frac{\delta(\tilde{N}\mathcal{L}_m)}{\delta\tilde{N}}$ is the energy density of matter field. In the case without matter field, the generalized Misner-Sharp energy is conserved, $dM(r) = 0$. At the horizon we have

$$dM(r)|_{r=r_+} = dE - TdS, \tag{5.5}$$

where E and S are just mass and entropy of the black hole, given in (3.14). When the matter field is absent, it gives us $dE = TdS$, which is the first law of black hole thermodynamics. In [31], we have used the first law to derive the black entropy.

In the case including the $z = 4$ term, the generalized Misner-Sharp energy can be read down from the action (3.22)

$$M(r) = \frac{\kappa^2\mu^2\Omega_k}{16r}\bigg(\Lambda^2 r^4 - 2\Lambda r^2(k-f) + (k-f)^2 - 2\frac{\tilde{\beta}}{\Lambda\mu}\bigg((k-f)^2 - \frac{(k-f)^3}{\Lambda r^2}\bigg) + \frac{\tilde{\beta}^2}{\Lambda^2\mu^2}\frac{(k-f)^4}{\Lambda^2 r^4}\bigg), \tag{5.6}$$

while for the IR modified Hořava-Lifshitz theory, we have, from (4.2),

$$M(r) = \frac{\kappa^2\mu^2\Omega}{16r}(\Lambda^2 r^4 + 2(\nu - \Lambda)r^2(1-f) + (1-f)^2). \tag{5.7}$$

It is easy to show that, when the matter field is absent, these two generalized Misner-Sharp energies are conserved, while when the matter field appears, their variation satisfies (5.4). At the black hole horizon, the variation of the generalized Misner-Sharp energy obeys (5.5), which is closely related to the first law of black hole thermodynamics.

VI. CONCLUSIONS AND DISCUSSIONS

The black hole thermodynamics implies that there might exist a deep connection between thermodynamics and gravity theory, although they are quite different subjects. Such a connection must be closely related to the holographic properties of gravity. The holography is an essential feature of gravity. In this paper we investigated the relationship between the first law of thermodynamics and gravitational field equation at a static, spherically symmetric black hole horizon in Hořava-Lifshitz theory with/without detailed balance. It turns out that, as in the cases of Einstein gravity and Lovelock gravity, the gravitational field equation can be cast to a form of the first law of thermodynamics at the black hole horizon. This way we obtained entropy and mass expressions in terms of black hole horizon, and they are exactly the same as those resulting from the integration method for black hole entropy and the Hamiltonian approach for black hole mass [31].

Note that Hořava-Lifshitz theory, different from general relativity, is not fully diffeomorphism invariant and only keeps the ''foliation-preserving'' diffeomorphism. Our results on the relation between the first law of thermodynamics and gravity field equation in the Hořava-Lifshitz theory indicate that this relation is a robust one, and is of some universality. In addition, unlike the case in general relativity, the first law of black hole mechanics has not yet been established so far in Hořava-Lifshitz theory. Our result is a first step towards that goal. Furthermore, let us stress that in the process to derive the entropy and mass of black holes in Hořava-Lifshitz theory, we have not employed an explicit solution of the theory. This is quite different from the previous works in the literature. This manifests that the relation between the first law and gravity field equation has a deep implication.

We also defined generalized Misner-Sharp energy for static, spherically symmetric spacetimes in Hořava-Lifshitz theory. The generalized Misner-Sharp energy is conserved in the case without matter field, and its variation gives the first law of black hole thermodynamics at the black hole horizon.

Note that we have restricted ourselves to the case with $\lambda = 1$ in Sec. III. Here let us make a simple discussion of the case with $\lambda \neq 1$. In this case, the reduced action can be expressed as [31]

$$I = \frac{\kappa^2\mu^2\Omega_k}{8(1-3\lambda)}\int dtdr\tilde{N}\bigg\{\frac{(\lambda-1)}{2}F'^2 - \frac{2\lambda}{r}FF' + \frac{(2\lambda-1)}{r^2}F^2\bigg\}, \tag{6.1}$$

where $F(r) = k - \Lambda r^2 - f(r)$. Varying the action with respect to F and $\tilde{N}$ yields the equations of motion:

$$0 = \bigg(\frac{2\lambda}{r}F - (\lambda-1)F'\bigg)\tilde{N}' + (\lambda-1)\bigg(\frac{2}{r^2}F - F''\bigg)\tilde{N}, \tag{6.2}$$

$$0 = \frac{(\lambda-1)}{2}F'^2 - \frac{2\lambda}{r}FF' + \frac{(2\lambda-1)}{r^2}F^2. \tag{6.3}$$

These equations have the solution with [30,31]

$$F(r) = \alpha r^s, \qquad \tilde{N}(r) = \gamma r^{1-2s}, \tag{6.4}$$

where α and γ are both integration constants and

$$s = \frac{2\lambda \pm \sqrt{2(3\lambda - 1)}}{\lambda - 1}.$$

As discussed in the second reference of [31], to have a well-defined physical quantities and well-behaved asymptotical behavior for the solution, we have to take the negative branch in s and s is in the range $s \in [-1, 2)$. The temperature of the black hole in this case is

$$T = \frac{1}{4\pi}\tilde{N}(r)f'(r)|_{r=r_+} = \frac{\gamma}{4\pi r_+^{2s}}(-\Lambda r_+^2(2 - s) - sk). \tag{6.5}$$

We can rewrite Eq. (6.3) as

$$\frac{2\lambda}{r}(k - \Lambda r^2 - f)f' + 4\lambda\Lambda(k - \Lambda r^2 - f) + \frac{(\lambda - 1)}{2}F'^2 + \frac{(2\lambda - 1)}{r^2}F^2 = 0. \tag{6.6}$$

On the black hole horizon where $f(r_+) = 0$, the above equation reduces to

$$\frac{2\lambda}{r_+}(k - \Lambda r_+^2)f' + 4\lambda\Lambda(k - \Lambda r_+^2) + \frac{(\lambda - 1)}{2}F'^2(r_+) + \frac{(2\lambda - 1)}{r^2}F^2(r_+) = 0. \tag{6.7}$$

Multiplying Eq. (6.7) by

$$\frac{\sqrt{2}\kappa^2\mu^2\Omega_k\tilde{N}(r_+)}{16\lambda\sqrt{3\lambda - 1}}dr_+,$$

and considering the expression of the temperature (6.5), we find that the first term in (6.7) can be expressed as TdS, where

$$S = \frac{\pi\kappa^2\mu^2\Omega_k}{\sqrt{2(3\lambda - 1)}}\left(k\ln(\sqrt{-\Lambda}r_+) + \frac{1}{2}(\sqrt{-\Lambda}r_+)^2\right) + S_0, \tag{6.8}$$

where S_0 is an integration constant. On the other hand, with the solution (6.4), the other three terms in (6.7) can be expressed as $-dM$, where M is

$$M = \frac{\sqrt{2}\kappa^2\mu^2\gamma\Omega_k}{16\sqrt{3\lambda - 1}}\frac{(k - \Lambda r_+^2)^2}{r_+^{2s}}. \tag{6.9}$$

Thus we have shown that, on the black hole horizon, the equation of motion (6.3) can be expressed as $TdS - dM = 0$, where S and M are just the black hole entropy and mass, as found in the second reference of [31]. Here we would like to mention that unlike the case of $\lambda = 1$, due to the presence of F^2 and F'^2 in the equation of motion, we have to use the black hole solution (6.4) in order to express the equation of motion in the form of the first law of thermodynamics. In addition, we here have discussed the case with the vacuum solution.

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