PHYSICAL REVIEW D **81**, 084058 (2010)

Kaluza-Klein two-brane-worlds cosmology at low energy

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(Received 5 January 2010; published 30 April 2010)

We study two $(4 + n)$-dimensional branes embedded in $(5 + n)$-dimensional spacetime. Using the gradient expansion approximation, we find that the effective theory is described by $(4 + n)$-dimensional scalar-tensor gravity with a specific coupling function. Based on this theory we investigate the Kaluza-Klein two-brane-worlds cosmology at low energy, in both the static and the nonstatic internal dimensions. In the case of the static internal dimensions, the effective gravitational constant in the induced Friedmann equation depends on the equations of state of the brane matter, and the dark radiation term naturally appears. In the nonstatic case we take a relation between the external and internal scale factors of the form $b(t) = a^{\gamma}(t)$ in which the brane world evolves with two scale factors. In this case, the induced Friedmann equation on the brane is modified in the effective gravitational constant and the term proportional to $a^{-4\beta}$. For dark radiation, we find $\gamma = -2/(1 + n)$. Finally, we discuss the issue of conformal frames which naturally arises with scalar-tensor theories. We find that the static internal dimensions in the Jordan frame may become nonstatic in the Einstein frame.

DOI: 10.1103/PhysRevD.81.084058 PACS numbers: 04.50.−h, 98.80.Cq, 98.80.Jk, 98.80.Qc

I. INTRODUCTION

One of the most interesting and surprising aspects of string theory or M theory is the fact that it can only be correctly formulated in a higher-dimensional spacetime. On the other hand, our observed Universe is a four-dimensional spacetime. Therefore we need a mechanism of compactification of the extra dimensions, so that they become invisible at least at low energy scales. Moreover, investigations of nonperturbative string theory have led to the discovery that string theory must contain higher-dimensional extended objects called branes. The existence of these branes has inspired a new method of compactification of extra dimensions, so that they become invisible at least at low energy scales. Previously, the preferred method was Kaluza-Klein compactification, in which the extra dimensions are compact and extremely small. This method of compactification has further inspired a group of classical models of the Universe, in which extra dimensions can be included in general relativity, and their possible implications for classical cosmology can be investigated phenomenologically without any dependence on a particular model of string theory. This is known as the brane-world scenario, in which the standard particles or fields are confined to a brane, while the graviton can propagate into the bulk as well as into the brane. Much effort has been put into revealing the cosmology on the brane in the context of five-dimensional spacetime, especially after the stimulating proposals by Randall and Sundrum (RS) [1,2]. The RS model is a five-dimensional realization of the Horava-Witten solution [3], in which the hierarchy problem can be solved by introducing an appropriated exponential warp factor in the metric. The various properties and characteristics of the RS model have been extensively analyzed: the cosmology framework [4–8], the low energy effective theory [9–20], black hole physics [21–26], the Lorentz violation [27–37], etc. However, the RS model with a codimension-one brane world is insufficient to reconcile a higher-dimensional theory with the observed four-dimensional spacetime as suggested by string theory.

Recently, the hybrid construction of the Kaluza-Klein and brane-world compactifications, i.e., a Kaluza-Klein compactification on the brane, has been investigated [38–45]. Such a construction is called a Kaluza-Klein brane world. A basic equation was derived by Yamauchi and Sasaki [43] for the study of Kaluza-Klein brane worlds in which some dimensions on the brane are compactified or for a regularization scheme for a higher codimension brane world. In order to analyze the Kaluza-Klein cosmology some authors have used the Shiromizu-Maeda-Sasaki equation [4] or tried to solve the bulk geometry. However, it difficult to solve the bulk geometry in most cases.

In this paper, our main purpose is to study low energy two-brane cosmological models in higher-dimensional spacetime. We generalize the case of four-dimensional two-brane models to $(4 + n)$-dimensional two-brane models, where n represents internal dimensions of the brane.

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We derive the effective equations of motion for higher-dimensional two-brane models using a low energy expansion method [13]. This perturbative method solves the full $(5+n)$-dimensional equations of motion using an approximation, and after imposing the junction conditions, one obtains the $(4+n)$-dimensional effective equations of motion. The effective equations can be solved without knowing the bulk geometry. Based on this theory we discuss the cosmological two-brane models at low energy, which we study in both static and nonstatic internal dimensions.

This paper is organized as follows. In Sec. II, we study a higher brane-world model in a $(5+n)$-dimensional spacetime bulk with a cosmological constant. We solve the $(5+n)$-dimensional Einstein equations at low energy using the gradient expansion approximation. We see that the effective theory is described by the $(4+n)$-dimensional quasi-scalar-tensor gravity with a specific coupling function. In Sec. III, the Kaluza-Klein two-brane-worlds cosmology is presented. We derive the effective Friedmann equations in both static and nonstatic internal dimensions. Section IV is devoted to the conclusions. In the Appendix, we present detailed calculations.

II. LOW ENERGY EFFECTIVE THEORY FOR HIGHER-DIMENSIONAL TWO-BRANE WORLDS

In this section, we derive the low energy effective theory for higher-dimensional two-branes systems, formally solving the bulk geometry in the gradient expansion approximation developed by Kanno and Soda [13] (see also [12]). We consider that the two branes represent a $(4+n)$-dimensional spacetime embedded in a $(5+n)$-dimensional spacetime. We assume that there is no matter in the bulk and the energy-momentum tensor of the bulk is proportional to the $(5+n)$-dimensional cosmological constant, $-2\Lambda_{5+n} = (4+n)(3+n)/l^2$. Then the higher-dimensional brane-world model is described by the action

$$S = \frac{1}{2\kappa^2}\int d^{5+n}x\sqrt{-g}\left[\mathcal{R} + \frac{(4+n)(3+n)}{l^2}\right] - \sum_{i=A,B}\int d^{4+n}x\sqrt{-g^{i\mathrm{brane}}}(\sigma_i - \mathcal{L}^i_{\mathrm{matter}}), \tag{1}$$

where $\mathcal{R}$, $g^{i\mathrm{brane}}_{\mu\nu}$, l, and κ^2 are the $(5+n)$-dimensional scalar curvature, the induced metric on branes, the scale of the bulk curvature radius, and the gravitational constant in $(5+n)$ dimensions, respectively. Because we will consider the matter terms in (1), the branes will not, in general, be flat. Consequently, we cannot put both branes at $y=0$ and $y=l$ and use Gaussian normal coordinates. Therefore, we use the following coordinate system to describe the geometry of the brane model,

$$ds^2 = e^{2\phi(y,x^\mu)}dy^2 + g_{\mu\nu}(y,x^\mu)dx^\mu dx^\nu. \tag{2}$$

The proper distance between the A-brane and B-brane with fixed x coordinates can be written as

$$d(x) = \int_0^l e^{\phi(y,x)}dy. \tag{3}$$

The extrinsic curvature is defined as

$$K_{\mu\nu} = -\frac{1}{2}\frac{\partial}{\partial y}g_{\mu\nu} \equiv -\frac{1}{2}g_{\mu\nu,y}. \tag{4}$$

In the coordinate system (2) and using the extrinsic curvature (4), we can write down the components of the Einstein equations in $(5+n)$ dimensions as

$$\begin{aligned} {}^{(5+n)}G^\mu{}_\nu &= G^\mu{}_\nu + e^{-\phi}(e^{-\phi}K^\mu{}_\nu - \delta^\mu{}_\nu e^{-\phi}K)_{,y} - (e^{-\phi}K)(e^{-\phi}K^\mu{}_\nu) + \frac{1}{2}\delta^\mu{}_\nu[(e^{-\phi}K)(e^{-\phi}K) + (e^{-\phi}K^{\alpha\beta})(e^{-\phi}K_{\alpha\beta})] \\ &\quad - \nabla^\alpha\nabla_\alpha\phi - \nabla^\alpha\phi\nabla_\alpha\phi + \delta^\mu{}_\nu(\nabla^\alpha\nabla_\alpha\phi + \nabla^\alpha\phi\nabla_\alpha\phi) \\ &= \frac{(4+n)(3+n)}{2l^2}\delta^\mu{}_\nu + \kappa^2(-\sigma^A\delta^\mu{}_\nu + T^{A\mu}{}_\nu)e^{-\phi}\delta(y) + \kappa^2(-\sigma^B\delta^\mu{}_\nu + \tilde{T}^{B\mu}{}_\nu)e^{-\phi}\delta(y-l), \end{aligned} \tag{5}$$

$$ {}^{(5+n)}G^y{}_y = -\frac{1}{2}R + \frac{1}{2}(e^{-\phi}K)(e^{-\phi}K) - \frac{1}{2}(e^{-\phi}K^{\alpha\beta})(e^{-\phi}K_{\alpha\beta}) = \frac{(4+n)(3+n)}{2l^2}, \tag{6}$$

$$ {}^{(5+n)}G^y{}_\mu = -\nabla_\nu(e^{-\phi}K_\mu{}^\nu) + \nabla_\mu(e^{-\phi}K) = 0, \tag{7}$$

where $G^\mu{}_\nu = R^\mu{}_\nu - \delta^\mu{}_\nu R/2$ is the $(4+n)$-dimensional Einstein tensor and ∇_μ denotes the covariant derivative with respect to the metric $g_{\mu\nu}$. $T^\mu{}_\nu$ is the energy-momentum tensor of the brane matter other than the tension. The junction conditions are obtained by collecting together the terms in the field equations which contain a δ function; then we obtain

$$e^{-\phi}[K^\mu_\nu - \delta^\mu_\nu K]|_{y=0} = \frac{\kappa^2}{2}(-\sigma_A\delta^\mu_\nu + T^{A\mu}{}_\nu), \tag{8}$$

$$e^{-\phi}[K^\mu_\nu - \delta^\mu_\nu K]|_{y=l} = -\frac{\kappa^2}{2}(-\sigma_B\delta^\mu_\nu + \tilde{T}^{B\mu}{}_\nu), \tag{9}$$

where $K^\mu_\nu = g^{\mu\alpha}K_{\alpha\nu}$. Note that the junction conditions constrain the induced metrics on both branes; they naturally give rise to the effective equations of motion for the gravity on the branes. In order to solve the bulk field equations, we use the gradient expansion scheme. The basic idea of the approximation is the assumption that the energy density of matter ρ on the brane is smaller than the brane tension σ. Equivalently, the bulk curvature scale l is much smaller than the characteristic length scale of the curvature L on the brane. Then, the small expansion parameter is given by $\epsilon = (l/L)^2 \ll 1$. This allows us to expand the metric in perturbative series starting from the induced metric on the A-brane $h_{\mu\nu}$ as the first term,

$$g_{\mu\nu}(y, x^\mu) = a^2(y)[h_{\mu\nu}(x^\mu) + {}^{(1)}g_{\mu\nu}(y, x^\mu) + \cdots], \tag{10}$$

where the boundary conditions on the A-brane are given by

$${}^{(i)}g_{\mu\nu}(y = 0, x^\mu) = \begin{cases} h_{\mu\nu}(x^\mu) & i = 0 \\ 0 & i = 1, 2, 3, \ldots. \end{cases} \tag{11}$$

For the extrinsic curvature tensor we expand this as

$$K^\mu{}_\nu = {}^{(0)}K^\mu{}_\nu + {}^{(1)}K^\mu{}_\nu + {}^{(2)}K^\mu{}_\nu + \cdots, \tag{12}$$

where ${}^{(i)}K^\mu{}_\nu = \mathcal{O}(\epsilon^i)$.

Applying the above scheme (see the Appendix for more details), we write down the $(4 + n)$-dimensional effective Einstein equations on the branes in closed form, subject to the low energy expansion, as follows:

$$G^\mu{}_\nu(h) = \frac{(2 + n)\kappa^2}{2l} T^{A\mu}{}_\nu - \frac{(2 + n)}{l} \chi^\mu{}_\nu, \tag{13}$$

$$G^\mu{}_\nu(f) = -\frac{(2 + n)\kappa^2}{2l} T^{B\mu}{}_\nu - \frac{(2 + n)}{l} \frac{\chi^\mu{}_\nu}{\Omega^{4+n}}, \tag{14}$$

where the A-brane metric is defined as $h_{\mu\nu} \equiv g^{A\text{-brane}}_{\mu\nu}$, while the B-brane metric is $f_{\mu\nu} \equiv g^{B\text{-brane}}_{\mu\nu}$. A conformal factor Ω relates the metric on the A-brane to that on the B-brane, $g^{B\text{-brane}}_{\mu\nu} = \Omega^2 g^{A\text{-brane}}_{\mu\nu}$. The terms proportional to χ^μ_ν are $(5 + n)$-dimensional Weyl tensor contributions, which describe the nonlocal $(5 + n)$-dimensional effect.

A. Effective theory on the A-brane

Eliminating χ^μ_ν from Eqs. (13) and (14), the $(4 + n)$-dimensional field equations on the A-brane can be written as

$$\begin{aligned} G^\mu{}_\nu(h) = {} & \frac{(2 + n)\kappa^2}{2l} \frac{1}{\Psi} [T^{A\mu}{}_\nu + (1 - \Psi) T^{B\mu}{}_\nu] \\ & + \frac{1}{\Psi} (\Psi^{|\mu}{}_{|\nu} - \delta^\mu_\nu \Psi^{|\alpha}{}_{|\alpha}) \\ & + \frac{\omega_A}{\Psi^2} \left(\Psi^{|\mu} \Psi_{|\nu} - \frac{1}{2} \delta^\mu_\nu \Psi^{|\alpha} \Psi_{|\alpha} \right), \end{aligned} \tag{15}$$

where $|$ denotes the covariant derivative with respect to the A-brane metric $h_{\mu\nu}$ and the new scalar field $\Psi = 1 - \Omega^{2+n}$. The coupling function ω_A is defined as

$$\omega_A(\Psi) \equiv \frac{3 + n}{2 + n} \frac{\Psi}{1 - \Psi}. \tag{16}$$

We can also determine χ^μ_ν by eliminating G^μ_ν from Eqs. (13) and (14). Then, we obtain

$$\begin{aligned} \frac{(2 + n)}{l} \chi^\mu{}_\nu = {} & -\frac{(2 + n)\kappa^2}{2l} \frac{(1 - \Psi)}{\Psi} (T^{A\mu}{}_\nu + T^{B\mu}{}_\nu) \\ & - \frac{1}{\Psi} (\Psi^{|\mu}{}_{|\nu} - \delta^\mu_\nu \Psi^{|\alpha}{}_{|\alpha}) \\ & + \frac{\omega_A}{\Psi^2} \left(\Psi^{|\mu} \Psi_{|\nu} - \frac{1}{2} \delta^\mu_\nu \Psi^{|\alpha} \Psi_{|\alpha} \right). \end{aligned} \tag{17}$$

Note that $\chi^\mu{}_\nu$ is expressed through the quantities on the branes, $\chi^\mu{}_\nu = \chi^\mu{}_\nu(x^\mu)$. Since $\chi^\mu{}_\nu$ is traceless, Eq. (17) leads to an equation of motion for the scalar field Ψ,

$$\begin{aligned} \Psi^{|\mu}{}_{|\mu} = {} & \frac{1}{(3 + n) + (2 + n)\omega_A} \bigg[\frac{(2 + n)\kappa^2}{2l} (T^A + T^B) \\ & - \frac{d\omega_A}{d\Psi} \Psi^{|\mu} \Psi_{|\mu} \bigg], \end{aligned} \tag{18}$$

where we have taken Eq. (16) into account. The conservation laws for the A-brane and B-brane matter with respect to the A-brane metric $h_{\mu\nu}$ are given by

$$\begin{aligned} & T^{A\mu}{}_{\nu|\mu} = 0, \\ & T^{B\mu}{}_{\nu|\mu} = \frac{\Psi_{|\mu}}{1 - \Psi} T^{B\mu}{}_\nu - \frac{1}{(2 + n)} \frac{\Psi_{|\nu}}{1 - \Psi} T^B. \end{aligned} \tag{19}$$

One can see that Eqs. (15) and (19) do not include the term $\chi^\mu{}_\nu$, but they include the energy-momentum tensor of the B-brane. For this reason Kanno and Soda called this theory "quasi-scalar-tensor" gravity.

The effective action on the A-brane can be derived from the original $(5 + n)$-dimensional action by substituting the solution of the equations of motion in the bulk and integrating out over the bulk coordinate. Up to the first order, we obtain the effective action for the A-brane as

$$\begin{aligned} S_A = {} & \frac{l}{(2 + n)\kappa^2} \int d^{4+n}x \sqrt{-h} \left[\Psi R(h) - \frac{\omega_A}{\Psi} \Psi^{|\alpha} \Psi_{|\alpha} \right] \\ & + \int d^{4+n}x \sqrt{-h} \mathcal{L}^A \\ & + \int d^{4+n}x \sqrt{-h} (1 - \Psi)^{(4+n)/(2+n)} \mathcal{L}^B. \end{aligned} \tag{20}$$

Notice that the action (20) represents the action of the general $(4 + n)$-dimensional scalar-tensor theory with a specific form of the coupling function (16) and an extra matter term from the B-brane.

B. Effective theory on the B-brane

In order to obtain the effective equations of motion on the B-brane, we simply reverse the role of the A-brane and that of the B-brane. Solving Eq. (14) for $G^\mu_\nu(f)$, the $(4 + n)$-dimensional field equations on the B-brane can be written as

$$G^{\mu}{}_{\nu}(f) = \frac{(2+n)\kappa^2}{2l}\frac{1}{\Phi}[T^{B\mu}{}_{\nu} + (1+\Phi)T^{A\mu}{}_{\nu}] + \frac{1}{\Phi}(\Phi^{;\mu}{}_{;\nu} - \delta^{\mu}_{\nu}\Phi^{;\alpha}_{\alpha}) + \frac{\omega_B}{\Phi^2}\left(\Phi^{;\mu}\Phi_{;\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\Phi^{;\alpha}\Phi_{;\alpha}\right), \tag{21}$$

where ";" denotes the covariant derivative with respect to the B-brane metric $f_{\mu\nu}$ and $\Phi = \Omega^{-(2+n)} - 1$. Here, the coupling function ω_B is defined as

$$\omega_B(\Phi) = -\frac{3+n}{2+n}\frac{\Phi}{1+\Phi}. \tag{22}$$

The equation of motion for the scalar field Φ becomes

$$\Phi^{;\mu}{}_{;\mu} = \frac{1}{(3+n)+(2+n)\omega_B}\left[\frac{(2+n)\kappa^2}{2l}(T^A + T^B) - \frac{d\omega_B}{d\Phi}\Phi^{;\mu}\Phi_{;\mu}\right]. \tag{23}$$

The conservation laws of the A-brane and B-brane matter with respect to the B-brane metric $f_{\mu\nu}$ are as follows:

$$T^{A\mu}{}_{\nu;\mu} = \frac{\Phi_{;\mu}}{1+\Phi}T^{A\mu}{}_{\nu} - \frac{1}{(2+n)}\frac{\Phi_{;\nu}}{1+\Phi}T^A, \qquad T^{B\mu}{}_{\nu;\mu} = 0. \tag{24}$$

Finally, the corresponding effective action for the B-brane is

$$S_B = \frac{l}{(2+n)\kappa^2}\int d^{4+n}x\sqrt{-f}\left[\Phi R(f) - \frac{\omega_B}{\Phi}\Phi^{;\alpha}\Phi_{;\alpha}\right] + \int d^{4+n}x\sqrt{-f}\mathcal{L}^B + \int d^{4+n}x\sqrt{-f}(1+\Phi)^{(4+n)/(2+n)}\mathcal{L}^A. \tag{25}$$

In the derivation of the equations of motion above, we first need to know the dynamics on one brane. Then we know the gravity on the other branes. Therefore, the dynamics on both branes are not independent. The transformation rules for scalar radion and the metric in $(4+n)$ dimensions are given by

$$\Phi = \frac{\Psi}{1-\Psi}, \tag{26}$$

$$g^{B\text{-brane}}_{\mu\nu} = (1-\Psi)^{2/(2+n)} \times [h_{\mu\nu} + g^{(1)}_{\mu\nu}(h_{\mu\nu}, \Psi, T^A_{\mu\nu}, T^B_{\mu\nu}, y = l)]. \tag{27}$$

The bulk metric can be determined if we know the energy-momentum tensors on both branes, the induced metric on the A-brane, and the scalar field Ψ. Since $(4+n)$-dimensional fields allow us to construct the $(5+n)$-dimensional bulk geometry, the quasi-scalar-tensor theory works as a holographic at low energy.

In the following section, for the realization at the first order expansion, we study the cosmological consequences of the model. We solve the effective equations without knowing the bulk geometry. Then, we can determine the Friedmann equation on the brane. Here we focus on the positive tension brane, the A-brane.

III. KALUZA-KLEIN TWO-BRANE-WORLDS COSMOLOGY AT LOW ENERGY

A. Effective Friedmann equation

In this section, we discuss the cosmological consequences of the higher-dimensional brane worlds. We take the induced metric on the A-brane of the form

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j + b^2(t)\delta_{\alpha\beta}dz^\alpha dz^\beta, \tag{28}$$

where δ_{ij} represents the metric of three-dimensional ordinary spaces with the spatial coordinates x^i ($i = 1, 2, 3$), while $\delta_{\alpha\beta}$ represents the metric of n-dimensional compact spaces with the coordinates z^α ($\alpha = 1, \ldots, n$). The scale factor b denotes the size of the internal dimensions, while the scale factor a is the usual scale factor for the external space. We choose the energy-momentum tensors of the A-brane and B-brane of the following form:

$$T^A_{\mu\nu} = (\rho_A, P_A a^2\delta_{ij}, Q_A b^2\delta_{\alpha\beta}), \tag{29}$$

$$T^B_{\mu\nu} = \Omega^2(\rho_B, P_B a^2\delta_{ij}, Q_B b^2\delta_{\alpha\beta}), \tag{30}$$

where ρ_i is the energy density, P_i the external pressure, and Q_i the internal pressure, where $i = A, B$. The Ω^2 factor results from the fact that the B-brane metric is $f_{\mu\nu} = \Omega^2 h_{\mu\nu}$. The symmetries imply that Ψ only depends on time.

Using the metric (28) and the energy-momentum tensors (29) and (30) in the effective Einstein equations (15), one finds

$$3H_a^2 + 3nH_aH_b + \frac{n(n-1)}{2}H_b^2 = \frac{8\pi G}{\Psi}[\rho_A + \rho_B(1-\Psi)^{(4+n)/(2+n)}] + \frac{1}{\Psi}\left[\frac{(n+3)}{2(n+2)}\frac{\dot{\Psi}^2}{(1-\Psi)} - 3H_a\dot{\Psi} - nH_b\dot{\Psi}\right], \tag{31}$$

$$-2\dot{H}_a - 3H_a^2 - 2nH_aH_b - n\dot{H}_b - \frac{n(n+1)}{2}H_b^2 = \frac{8\pi G}{\Psi}[P_A + P_B(1-\Psi)^{(4+n)/(2+n)}] + \frac{1}{\Psi}\left[\ddot{\Psi} + \frac{(n+3)}{2(n+2)}\frac{\dot{\Psi}^2}{(1-\Psi)} + 2H_a\dot{\Psi} + nH_b\dot{\Psi}\right], \tag{32}$$

$$-3\dot{H}_a - 6H_a^2 - 3(n-1)H_aH_b - (n-1)\dot{H}_b - \frac{n(n-1)}{2}H_b^2 = \frac{8\pi G}{\Psi}[Q_A + Q_B(1-\Psi)^{(4+n)/(2+n)}] + \frac{1}{\Psi}\left[\ddot{\Psi} + \frac{(n+3)}{2(n+2)}\frac{\dot{\Psi}^2}{(1-\Psi)} + 3H_a\dot{\Psi} + (n-1)H_b\dot{\Psi}\right], \tag{33}$$

where we have defined the Hubble parameters $H_a = \dot{a}/a$ and $H_b = \dot{b}/b$ and

$$8\pi G = \frac{(2+n)\kappa^2}{2l}. \tag{34}$$

In the case $n = 0$, the above equations reduce to a five-dimensional brane world. For $n = 0$, $\Psi = 1$, $\dot{\Psi} = 0$, the above equations reduce to the general relativistic Friedmann-Lemaitre-Robertson-Walker equations with barotropic perfect fluid.

The equation of motion for the scalar field Ψ is

$$\ddot{\Psi} = \frac{8\pi G}{(3+n)}[(\rho_A - 3P_A - nQ_A)(1-\Psi) + (\rho_B - 3P_B - nQ_B)(1-\Psi)^{(4+n)/(2+n)}] - \frac{1}{2}\frac{\dot{\Psi}^2}{(1-\Psi)} - 3H_a\dot{\Psi} - nH_b\dot{\Psi}. \tag{35}$$

In addition, the conservation laws for the matter with respect to the A-brane metric (19) are given by

$$\dot{\rho}_A + 3H_a(\rho_A + P_A) + nH_b(\rho_A + Q_A) = 0, \tag{36}$$

$$\dot{\rho}_B + 3H_a(\rho_B + P_B) + nH_b(\rho_B + Q_B) = \frac{3(\rho_B + P_B) + n(\rho_B + Q_B)}{2+n}\frac{\dot{\Psi}}{1-\Psi}. \tag{37}$$

Equation (35) can be used to eliminate $\ddot{\Psi}$ in Eqs. (32) and (33), and then we have

$$-2\dot{H}_a - 3H_a^2 - 2nH_aH_b - n\dot{H}_b - \frac{n(n+1)}{2}H_b^2 + H_a\frac{\dot{\Psi}}{\Psi} = \frac{8\pi G}{\Psi}\left[w_A\rho_A + \frac{(1-3w_A - nv_A)}{(3+n)}\rho_A(1-\Psi) + \frac{(1+nw_B - nv_B)}{(3+n)}\rho_B(1-\Psi)^{(4+n)/(2+n)}\right] + \frac{1}{2(n+2)}\frac{\dot{\Psi}^2}{\Psi(1-\Psi)}, \tag{38}$$

$$-3\dot{H}_a - 6H_a^2 - 3(n-1)H_aH_b - (n-1)\dot{H}_b - \frac{n(n-1)}{2}H_b^2 + H_b\frac{\dot{\Psi}}{\Psi} = \frac{8\pi G}{\Psi}\left[v_A\rho_A + \frac{(1-3w_A - nv_A)}{(3+n)}\rho_A(1-\Psi) + \frac{(1-3w_B + 3v_B)}{(3+n)}\rho_B(1-\Psi)^{(4+n)/(2+n)}\right] + \frac{1}{2(n+2)}\frac{\dot{\Psi}^2}{\Psi(1-\Psi)}. \tag{39}$$

Here, we have assumed that the matter distribution on the branes is given by the equations of state $P_i = w_i\rho_i$ and $Q_i = v_i\rho_i$ ($i = A, B$). From Eqs. (31), (38), and (39), we eliminate the $\dot{\Psi}^2$ term to obtain

$$2\dot{H}_a + \frac{3(4+n)}{(3+n)}H_a^2 + \frac{n(9+2n)}{(3+n)}H_aH_b + n\dot{H}_b + \frac{n(n^2+5n+2)}{2(3+n)}H_b^2 - \frac{n}{(3+n)}(H_a - H_b)\frac{\dot{\Psi}}{\Psi} = \frac{8\pi G}{\Psi}\left[\frac{(1-(3+n)w_A)\rho_A}{(3+n)} - \frac{(1-3w_A - nv_A)\rho_A(1-\Psi)}{(3+n)} - \frac{n(w_B - v_B)}{(3+n)} \times \rho_B(1-\Psi)^{(4+n)/(2+n)}\right], \tag{40}$$

$$\dot{H}_a + 3H_a^2 + (n-3)H_aH_b - \dot{H}_b - nH_b^2 + (H_a - H_b)\frac{\dot{\Psi}}{\Psi} = \frac{8\pi G}{\Psi}[(w_A - v_A)\rho_A + (w_B - v_B)\rho_B(1-\Psi)^{(4+n)/(2+n)}]. \tag{41}$$

Combining Eqs. (40) and (41) we get the dynamical equation for the Hubble parameters in $(4+n)$ dimensions,

$$\dot{H}_a + 2H_a^2 + nH_aH_b + \frac{n(1+n)}{6}H_b^2 + \frac{n}{3}\dot{H}_b = \frac{8\pi G}{3}\frac{(1-3w_A - nv_A)}{(2+n)}\rho_A. \tag{42}$$

The conservation laws reduce to

$$\dot{\rho}_A + 3H_a(1+w_A)\rho_A + nH_b(1+v_A)\rho_A = 0, \tag{43}$$

$$\dot{\rho}_B + 3H_a(1+w_B)\rho_B + nH_b(1+v_B)\rho_B = \frac{[3(1+w_B) + n(1+v_B)]\rho_B}{2+n}\frac{\dot{\Psi}}{1-\Psi}. \tag{44}$$

In general, Eq. (42) is a second order differential equation for scale factors $a(t)$ and $b(t)$. In the case of a four-dimensional brane world ($n = 0$), Eq. (42) can be solved analytically, and this results in the Friedmann equation on the brane with the dark radiation term as an integration constant. In our case Eq. (42) cannot be integrated analytically, and therefore, the usual form of the Friedmann equation on the brane cannot be extracted. In the following two subsections we consider two cases: static and nonstatic internal dimensions.

B. Friedmann equation with static internal dimensions

In the case of static internal extra dimensions, the dynamical A-brane is described by the following equations:

$$H_a^2 + H_a \frac{\dot{\Psi}}{\Psi} - \frac{(n+3)}{6(n+2)} \frac{\dot{\Psi}^2}{\Psi(1-\Psi)} = \frac{8\pi G}{3\Psi}[\rho_A + \rho_B(1-\Psi)^{(4+n)/(2+n)}], \tag{45}$$

$$\dot{H}_a + 2H_a^2 = \frac{8\pi G}{3} \frac{(1-3w_A - n\upsilon_A)}{(2+n)} \rho_A, \tag{46}$$

$$\ddot{\Psi} + 3H_a\dot{\Psi} + \frac{1}{2}\frac{\dot{\Psi}^2}{(1-\Psi)} = \frac{8\pi G}{(3+n)}[(1-3w_A - n\upsilon_A) \times \rho_A(1-\Psi) + (1-3w_B - n\upsilon_B)\rho_B(1-\Psi)^{(4+n)/(2+n)}]. \tag{47}$$

Here we have assumed that the compact dimensions are stabilized, $b(t) = 1$ [44]. We see that the above equations do not contain any additional term compared with five-dimensional brane-world cosmology. However, the differences from the usual two-brane models are concealed in the gravitational constant and also in the form of the constraint equation (45).

The conservation laws for the matter with respect to the A-brane metric reduce to

$$\dot{\rho}_A + 3H_a(1+w_A)\rho_A = 0, \tag{48}$$

$$\dot{\rho}_B + 3H_a(1+w_B)\rho_B = \frac{3(1+w_B)\rho_B + n(1+\upsilon_B)\rho_B}{2+n} \times \frac{\dot{\Psi}}{1-\Psi}, \tag{49}$$

and we obtain

$$\rho_A \propto a^{-3(1+w_A)}, \tag{50}$$

$$\rho_B \propto a^{-3(1+w_B)}(1-\Psi)^{(3(1+w_B)+n(1+\upsilon_B))/(2+n)}. \tag{51}$$

A relation between the energy densities on both branes can be obtained by eliminating a,

$$\rho_B \propto \rho_A^{(1+w_B)/(1+w_A)}(1-\Psi)^{(3(1+w_B)+n(1+\upsilon_B))/(2+n)}. \tag{52}$$

In the case $w_A \neq 1/3$, leaving υ_A as a free parameter and using the matter conservation equation (48), we can write (46) as

$$\frac{d}{dt}\left(a^4H_a^2 - \frac{8\pi G}{3}\frac{2(1-3w_A - n\upsilon_A)}{(2+n)(1-3w_A)}a^4\rho_A\right) = 0. \tag{53}$$

Then, we obtain an expression for the effective Hubble parameter on the A-brane as

$$H_a^2 = \frac{8\pi G_{\rm eff}}{3}\rho_A + \frac{\mathcal{C}}{a^4}, \tag{54}$$

where $\mathcal{C}$ is an integration constant which can be interpreted as dark radiation. We have defined the effective gravitational constant as

$$G_{\rm eff} = \frac{2(1-3w_A - n\upsilon_A)}{(2+n)(1-3w_A)}G. \tag{55}$$

For $w_A < 1/3$, $n\upsilon_A < 1 - 3w_A$ and $w_A > 1/3$, $n\upsilon_A > 1 - 3w_A$, the effective gravitational constant becomes positive.

In the case of a radiation dominated universe, $w_A = 1/3$, we have

$$\dot{H}_a + 2H_a^2 = -\frac{8\pi G n\upsilon_A}{3(2+n)}\rho_A. \tag{56}$$

Using the matter conservation equation, we can write Eq. (56) as

$$\frac{d}{dt}\left(a^4H_a^2 + \frac{8\pi G}{3}\frac{2n\upsilon_A}{(2+n)}\log a\right) = 0, \tag{57}$$

with

$$H_a^2 = -\frac{8\pi G}{3}\frac{2n\upsilon_A \log a}{(2+n)}\rho_A + \frac{K}{a^4}, \tag{58}$$

where K is an integration constant which can be redefined as a sum of the initial value of radiative matter density and the initial value of the dark radiation density $\mathcal{C}$. Then Eq. (58) becomes

$$H_a^2 = \frac{8\pi G}{3}\left(1 - \frac{2n}{(2+n)}\upsilon_A \log\frac{a}{a_*}\right)\rho_A + \frac{\mathcal{C}}{a^4}, \tag{59}$$

where a_* is a constant corresponding to the dark radiation component $\mathcal{C}$. Defining the effective gravitational constant

$$G_{\rm eff} = \left[1 - \frac{2n}{(2+n)}\upsilon_A \log\frac{a}{a_*}\right]G, \tag{60}$$

then we have the effective Friedmann equation (54). As expected, the expression for the effective Friedmann equation on the A-brane coincides with the Kaluza-Klein brane-world cosmology with one-brane model in the low energy approximation, where the term of quadratic energy density is neglected [44]. In contrast to the usual four-dimensional two-brane model, the effective gravitational constant depends on the equation of state and the external scale factor explicitly, and may become positive or negative.

C. Friedmann equation with nonstatic internal dimensions

Let us now consider the case of nonstatic internal dimensions, in which the brane world evolves with two scale factors. We take a simple relation between the scale factors on the A-brane of the form

$$b(t) = a^{\gamma}(t), \tag{61}$$

where γ is a constant. For the internal scale factor $b(t)$ to be small compared to the external scale factor $a(t)$, the constant γ should be negative.

For nonstatic internal dimensions, the dynamical A-brane is described by the following equations:

$$\left[\frac{6(1+n\gamma)+n(n-1)\gamma^2}{2}\right]H_a^2 + (3+n\gamma)H_a\frac{\dot{\Psi}}{\Psi} = \frac{8\pi G}{\Psi}[\rho_A + \rho_B(1-\Psi)^{(4+n)/(2+n)}] + \frac{(n+3)}{2(n+2)} \times \frac{\dot{\Psi}^2}{\Psi(1-\Psi)}, \tag{62}$$

$$\dot{H}_a + \frac{6(2+n\gamma)+n(1+n)\gamma^2}{2(3+n\gamma)}H_a^2 = \frac{8\pi G(1-3w_A - n\upsilon_A)}{(2+n)(3+n\gamma)}\rho_A, \tag{63}$$

$$\ddot{\Psi} + (3+n\gamma)H_a\dot{\Psi} = \frac{8\pi G}{(3+n)}[(1-3w_A-n\upsilon_A) \times \rho_A(1-\Psi) + (1-3w_B-n\upsilon_B) \times \rho_B(1-\Psi)^{(4+n)/(2+n)}] - \frac{1}{2}\frac{\dot{\Psi}^2}{(1-\Psi)}. \tag{64}$$

The conservation laws become

$$\dot{\rho}_A + [3(1+w_A) + n\gamma(1+\upsilon_A)]H_a\rho_A = 0, \tag{65}$$

$$\dot{\rho}_B + [3(1+w_B) + n\gamma(1+\upsilon_B)]H_a\rho_B = \frac{[3(1+w_B)+n(1+\upsilon_B)]\rho_B}{2+n}\frac{\dot{\Psi}}{1-\Psi}. \tag{66}$$

Using the matter conservation equation (65),

$$[4\beta - 3(1+w_A) - n\gamma(1+\upsilon_A)]H_a\rho_A = \dot{\rho} + 4\beta H_a\rho_A = \frac{1}{a^{4\beta}}\frac{d}{dt}(a^{4\beta}\rho_A), \tag{67}$$

and so we can write Eq. (63) as

$$\frac{d}{dt}\left(a^{4\beta}H_a^2 - \frac{8\pi G_{\rm eff}}{3}a^{4\beta}\rho_A\right) = 0, \tag{68}$$

where

$$\beta = \frac{6(2+n\gamma)+n(1+n)\gamma^2}{4(3+n\gamma)}. \tag{69}$$

Then the effective Friedmann equation for nonstatic internal dimensions on the A-brane is given by

$$H_a^2 = \frac{8\pi G_{\rm eff}}{3}\rho_A + \frac{C}{a^{4\beta}}, \tag{70}$$

where C is a constant of integration and we have defined the effective gravitational constant as follows:

$$G_{\rm eff} = \frac{6(1-3w_A-n\upsilon_A)G}{(2+n)(3+n\gamma)[4\beta - 3(1+w_A) - n\gamma(1+\upsilon_A)]}. \tag{71}$$

Notice that for $n=3$ and nonstatic internal dimensions, the setup is symmetric under the exchange of internal and external pressures ($w_i \leftrightarrow \upsilon_i$), and $a(t) \leftrightarrow b(t)$.

The above results also include the well-known five-dimensional brane world, corresponding to $n=0$ and for which $\beta = 1$, $G_{\rm eff} = G$. For $\gamma = 0$ the above results reduce to the static internal dimensions. If $\gamma = 1$, the internal scale factor $b(t)$ is related to $a(t)$ as $b(t) = a(t)$, and we obtain the Friedmann equation of the generalized Randall-Sundrum model in $(5+n)$ dimensions describing a $(4+n)$-dimensional universe.

$$H_a^2 = \frac{8\pi G_{\rm eff}}{3}\rho_A + \frac{C}{a^{4+n}}, \tag{72}$$

where the effective gravitational constant is now given by

$$G_{\rm eff} = \frac{6}{(2+n)(3+n)}G. \tag{73}$$

In the case $n=0$, the above Friedmann equation reduces to the usual Friedmann equation on a four-dimensional brane.

Leaving β as a free parameter, we can solve Eq. (69) for γ. We obtain

$$\gamma = -\frac{3 - 2\beta \pm \sqrt{\frac{4\beta(3+n\beta)-3(4+n)}{n}}}{1+n}. \tag{74}$$

The negative values of γ indicate that the internal scale factor $b(t)$ is small compared to the external scale factor $a(t)$. Taking $\beta = 1$ such that the second term of the Friedmann equation (70) contributes the "dark" radiation, we have

$$\gamma = -\frac{2}{1+n} \quad \text{or} \quad \gamma = 0, \tag{75}$$

where $\gamma = 0$ corresponds to the static internal dimensions. Therefore, the dark radiation component in the Friedmann equation can also be realized in the Kaluza-Klein brane worlds with nonstatic internal dimensions.

D. Hubble parameters in conformal frames

The action on the A-brane is written in the Jordan frame, for which the gravitational sector has a noncanonical form. We can, however, perform a conformal transformation in the Einstein frame: $\tilde{h}_{\mu\nu} = \Psi^{2/(2+n)} h_{\mu\nu}$. In the Einstein frame, the metric (28) is

$$\begin{aligned} d\tilde{s}^2 &= \tilde{h}_{\mu\nu} dx^\mu dx^\nu \\ &= \Psi^{2/(2+n)}[-dt^2 + a^2(t)\delta_{ij}dx^i dx^j \\ &\quad + b^2(t)\delta_{\alpha\beta}dz^\alpha dz^\beta] \\ &= -d\tilde{t}^2 + \tilde{a}^2(\tilde{t})\delta_{ij}dx^i dx^j + \tilde{b}^2(\tilde{t})\delta_{\alpha\beta}dz^\alpha dz^\beta, \end{aligned} \tag{76}$$

and the Hubble parameters satisfy

$$\tilde{H}_a - \tilde{H}_b = \Psi^{-(1/(2+n))}(H_a - H_b), \tag{77}$$

where $\tilde{H}_a = \tilde{a}^{-1}(d\tilde{a}/d\tilde{t})$ and $\tilde{H}_b = \tilde{b}^{-1}(d\tilde{b}/d\tilde{t})$. One can see that the static internal dimensions (in the Jordan frame) may become dynamical in the Einstein frame. In this case we have

$$\tilde{H}_a - \tilde{H}_b = \Psi^{-(1/(2+n))}H_a, \qquad \tilde{H}_b = \frac{1}{(2+n)\Psi}\frac{d\Psi}{d\tilde{t}}. \tag{78}$$

In the case $b(t) = a^\gamma(t)$, we have

$$\tilde{H}_a - \tilde{H}_b = (1-\gamma)\Psi^{-(1/(2+n))}H_a. \tag{79}$$

Dynamics of the Hubble parameters H_a and H_b in the Jordan frame are also dynamics in the Einstein frame.

IV. CONCLUSION

In this paper we have derived the low energy effective equations for higher-dimensional two-brane models by using the gradient expansion approximation. As expected, the effective theory is described by the $(4+n)$-dimensional quasi-scalar-tensor gravity with a specific coupling function. The effective equations presented can be used as the basic equations for higher-dimensional two-brane-worlds cosmology, in which some spatial dimensions on the brane are Kaluza-Klein compactified.

We can see already from the Friedmann equations that the Kaluza-Klein brane world can be realized at low energies. Because of their complicated structure, the field equations appearing in the theories are very difficult to solve analytically; we have restricted our discussions to the following special cases: static internal dimensions and nonstatic internal dimensions, where a relation between the external and internal scale factors is given by $b(t) = a^\gamma(t)$. In the case of static internal dimensions, $\gamma = 0$, our results coincide with the Kaluza-Klein brane-world cosmology with one brane model in the low energy approximation, where the term of quadratic energy density is neglected [44]. In the case of nonstatic internal dimensions, the induced Friedmann equation on the brane is modified in the effective gravitational constant and the term proportional to $a^{-4\beta}$.

Another important result of this work is the dynamics of the internal Hubble parameter in conformal frames. Both the static and nonstatic internal dimensions in the Jordan frame are always dynamics in the Einstein frame. However, the physical interpretation and equivalence of these two frames are a problem in the case of static internal dimensions in the Jordan frame. We plan to investigate the correspondence between the Jordan and the Einstein frame descriptions, including the dynamical scalar field.

ACKNOWLEDGMENTS

We thank the referees for many helpful suggestions which improved the presentation of the paper.

APPENDIX: DETAILED CALCULATIONS

Let us decompose the extrinsic curvature into the traceless part and the trace part,

$$\begin{aligned} e^{-\phi}K_{\mu\nu} &= \Sigma_{\mu\nu} + \frac{1}{4+n} g_{\mu\nu} Q, \\ Q &= -e^{-\phi}\frac{\partial}{\partial y}\log\sqrt{-g}, \end{aligned} \tag{A1}$$

which allows us to write the field equations (5)–(7) in the bulk as follows:

$$\begin{aligned} e^{-\phi}\Sigma^\mu{}_{\nu,y} - Q\Sigma^\mu{}_\nu = -\Big[& R^\mu{}_\nu - \frac{1}{4+n}\delta^\mu{}_\nu R - \nabla^\mu\nabla_\nu\phi \\ & - \nabla^\mu\phi\nabla_\nu\phi + \frac{1}{4+n}\delta^\mu_\nu(\nabla^\alpha\nabla_\alpha\phi \\ & + \nabla^\alpha\phi\nabla_\alpha\phi)\Big], \end{aligned} \tag{A2}$$

$$\frac{3+n}{4+n}Q^2 - \Sigma^\alpha{}_\beta\Sigma^\beta{}_\alpha = [R] + \frac{(4+n)(3+n)}{l^2}, \tag{A3}$$

$$\begin{aligned} & e^{-\phi}Q_{,y} - \frac{1}{4+n}Q^2 - \Sigma^{\alpha\beta}\Sigma_{\alpha\beta} \\ & = \nabla^\alpha\nabla_\alpha\phi + \nabla^\alpha\phi\nabla_\alpha\phi - \frac{4+n}{l^2}, \end{aligned} \tag{A4}$$

$$\nabla_\lambda\Sigma_\mu{}^\lambda - \frac{3+n}{4+n}\nabla_\mu Q = 0. \tag{A5}$$

The junction conditions determine the dynamics of the induced metric and provide the effective theory of gravity on the brane, reduced to

$$\left[\Sigma^\mu_\nu - \frac{3}{4}\delta^\mu_\nu Q\right]\Big|_{y=0} = \frac{\kappa^2}{2}(-\sigma_A\delta^\mu_\nu + T^{A\mu}{}_\nu), \tag{A6}$$

$$\left[\Sigma^\mu_\nu - \frac{3}{4}\delta^\mu_\nu Q\right]\Big|_{y=l} = -\frac{\kappa^2}{2}(-\sigma_B\delta^\mu_\nu + \tilde{T}^{B\mu}{}_\nu). \tag{A7}$$

1. Zeroth order

At zeroth order, the gradient terms and matter on the brane can be ignored. We find

$$ {}^{(0)}\Sigma^{\mu}_{\nu} = 0, \qquad {}^{(0)}Q = \frac{4+n}{l}. \tag{A8}$$

The junction conditions (A6) and (A7) yield

$$\sigma_A = \frac{2(3+n)}{\kappa^2 l}, \qquad \sigma_B = -\frac{2(3+n)}{\kappa^2 l}. \tag{A9}$$

Using the definition of the extrinsic curvature, we get the zeroth order metric as

$$ds^2 = e^{2\phi(y,x)}dy^2 + a^2(y,x)h_{\mu\nu}dx^\mu dx^\nu, \tag{A10}$$

$$a(y,x) = \exp\left[-\frac{1}{l}\int_0^y dy e^{\phi(y,x)}\right], \tag{A11}$$

where the tensor $h_{\mu\nu}$ is the induced metric on the A-brane. To proceed, we will assume $\phi(y,x) \equiv \phi(x)$, and thus $a(y,x) = \exp[-ye^{\phi(x)}/l]$.

2. First order

In the first order, the curvature term that has been ignored in the zeroth order calculation comes into play. Substituting the solutions at zeroth order, the field equations (A2)–(A5) can be written as follows:

$$e^{-\phi(1)}\Sigma^{\mu}{}_{\nu,y} - \frac{4+n}{l}{}^{(1)}\Sigma^{\mu}{}_{\nu} = -\Big[R^{\mu}{}_{\nu} - \frac{1}{4+n}\delta^{\mu}{}_{\nu}R - (\nabla^\mu\nabla_\nu\phi + \nabla^\mu\phi\nabla_\nu\phi) + \frac{1}{4+n}\delta^{\mu}_{\nu}(\nabla^\alpha\nabla_\alpha\phi + \nabla^\alpha\phi\nabla_\alpha\phi)\Big]^{(1)}, \tag{A12}$$

$$\frac{2(3+n)}{l}{}^{(1)}Q = [R]^{(1)}, \tag{A13}$$

$$e^{-\phi(1)}Q_{,y} - \frac{2}{l}{}^{(1)}Q = [\nabla^\alpha\nabla_\alpha\phi + \nabla^\alpha\phi\nabla_\alpha\phi]^{(1)}, \tag{A14}$$

$$\nabla_\lambda{}^{(1)}\Sigma_{\mu}{}^{\lambda} - \frac{3+n}{4+n}\nabla_\mu{}^{(1)}Q = 0. \tag{A15}$$

And the junction conditions are given by

$$\left[{}^{(1)}\Sigma^{\mu}_{\nu} - \frac{3}{4}\delta^{\mu}_{\nu}{}^{(1)}Q\right]\Big|_{y=0} = \frac{\kappa^2}{2}T^{A\mu}{}_{\nu}, \tag{A16}$$

$$\left[{}^{(1)}\Sigma^{\mu}_{\nu} - \frac{3}{4}\delta^{\mu}_{\nu}{}^{(1)}Q\right]\Big|_{y=l} = -\frac{\kappa^2}{2}\tilde{T}^{B\mu}{}_{\nu}, \tag{A17}$$

where the superscript (1) represents the order of the gradient expansion. Now one can express the Ricci tensor $[R^{\mu}{}_{\nu}(g)]^{(1)}$ in terms of the Ricci tensor of the A-brane metric $h_{\mu\nu} \equiv g^{A\text{-brane}}_{\mu\nu}$ [denoted by $R^{\mu}{}_{\nu}(h)$] and ϕ:

$$[R^{\mu}{}_{\nu}(g)]^{(1)} = \frac{1}{a^2}\Big[R^{\mu}{}_{\nu}(h) + \frac{(2+n)ye^{\phi}}{l}(\phi^{|\mu}{}_{|\nu} + \phi^{|\mu}\phi_{|\nu}) + \frac{ye^{\phi}}{l}\delta^{\mu}_{\nu}(\phi^{|\alpha}{}_{|\alpha} + \phi^{|\alpha}\phi_{|\alpha}) + \frac{(2+n)y^2e^{2\phi}}{l^2}\phi^{|\mu}\phi_{|\nu} - \frac{(2+n)y^2e^{2\phi}}{l^2}\delta^{\mu}_{\nu}\phi^{|\alpha}\phi_{|\alpha}\Big], \tag{A18}$$

where $|$ denotes the covariant derivative with respect to the A-brane metric $h_{\mu\nu}$. Taking the trace of Eq. (A18) and using Eq. (A13), the trace part of the extrinsic curvature can be obtained without solving the bulk geometry,

$${}^{(1)}Q(y,x) = \frac{l}{2(3+n)a^2}[R(g)] = \frac{l}{a^2}\Big[\frac{1}{2(3+n)}R(h) + \frac{ye^{\phi}}{l}(\phi^{|\alpha}{}_{|\alpha} + \phi^{|\alpha}\phi_{|\alpha}) - \frac{(2+n)y^2e^{2\phi}}{2l^2}\phi^{|\alpha}\phi_{|\alpha}\Big]. \tag{A19}$$

The second derivatives of ϕ are given by

$$[\nabla^\mu\nabla_\nu\phi]^{(1)} = \frac{1}{a^2}\Big[\phi^{|\mu}{}_{|\nu} + 2\frac{ye^{\phi}}{l}\phi^{|\mu}\phi_{|\nu} - \frac{ye^{\phi}}{l}\delta^{\mu}_{\nu}\phi^{|\alpha}\phi_{|\alpha}\Big]. \tag{A20}$$

It is easy to see that the Hamiltonian constraint equation (A14) is trivially satisfied now. Then, Eq. (A12) can be integrated to give

$${}^{(1)}\Sigma^{\mu}{}_{\nu}(y,x) = \frac{l}{a^2}\Big[\frac{1}{(2+n)}\Big(R^{\mu}{}_{\nu} - \frac{1}{4+n}\delta^{\mu}_{\nu}R\Big) + \frac{ye^{\phi}}{l}\Big(\phi^{|\mu}{}_{|\nu} - \frac{1}{4+n}\delta^{\mu}_{\nu}\phi^{|\alpha}{}_{|\alpha}\Big) + \Big(\frac{y^2e^{2\phi}}{l^2} + \frac{ye^{\phi}}{l}\Big) \times \Big(\phi^{|\mu}\phi_{|\nu} - \frac{1}{4+n}\delta^{\mu}_{\nu}\phi^{|\alpha}\phi_{|\alpha}\Big)\Big] + \frac{\chi^{\mu}_{\nu}(x)}{a^{4+n}}, \tag{A21}$$

where $\chi^{\mu}{}_{\nu}(x)$ is an integration constant whose trace vanishes: $\chi^{\mu}_{\mu} = 0$, and Eq. (A15) requires that $\chi^{\mu}{}_{\nu|\mu} = 0$.

Substituting Eqs. (A19) and (A21) into the junction condition at the A-brane (A16), we obtain

$$\frac{l}{(2+n)}G^{\mu}{}_{\nu}(h) + \chi^{\mu}{}_{\nu} = \frac{\kappa^2}{2}T^{A\mu}{}_{\nu}, \tag{A22}$$

and the junction condition at the B-brane (A17) yields

$$\frac{l}{(2+n)\Omega^2}G^{\mu}{}_{\nu} + \frac{le^{\phi}}{\Omega^2}(\phi^{|\mu}{}_{|\nu} - \delta^{\mu}_{\nu}\phi^{|\alpha}{}_{|\alpha} + \phi^{|\mu}\phi_{|\nu} - \delta^{\mu}_{\nu}\phi^{|\alpha}\phi_{|\alpha}) + \frac{le^{2\phi}}{\Omega^2}\left(\phi^{|\mu}\phi_{|\nu} + \frac{(1+n)}{2}\delta^{\mu}_{\nu}\phi^{|\alpha}\phi_{|\alpha}\right) + \frac{\chi^{\mu}_{\nu}}{\Omega^{4+n}} = -\frac{\kappa^2}{2\Omega^2}T^{B\mu}{}_{\nu}, \tag{A23}$$

where $\Omega(x) = a(y = l, x) = \exp[-e^{\phi}]$ and the index of $T^{B\mu}{}_{\nu}$ is the energy-momentum tensor with the index raised by the induced A-brane metric $h_{\mu\nu}$, while $\tilde{T}^{B\mu}{}_{\nu}$ is the one raised by the induced metric on the B-brane, $f_{\mu\nu} \equiv g^{B\text{-brane}}_{\mu\nu}$. Using $f_{\mu\nu} = \Omega^2 h_{\mu\nu} = \exp[-2e^{\phi}]h_{\mu\nu}$, Eq. (A23) can be rewritten as

$$\frac{l}{(2+n)}G^{\mu}{}_{\nu}(f) + \frac{\chi^{\mu}{}_{\nu}}{\Omega^{4+n}} = -\frac{\kappa^2}{2}\tilde{T}^{B\mu}{}_{\nu}. \tag{A24}$$

We now solve the metric in the bulk. The definition (A1) gives

$$-\frac{e^{-\phi}}{2a^2}h^{\alpha\mu}\frac{\partial}{\partial y}{}^{(1)}g_{\alpha\nu} = {}^{(1)}\Sigma^{\mu}{}_{\nu} + \frac{1}{4+n}\delta^{\mu(1)}_{\nu}Q. \tag{A25}$$

Integrating Eq. (A25), we obtain the metric in the bulk:

$$\begin{aligned}{}^{(1)}g_{\mu\nu}(y,x) = &-\frac{l^2}{(2+n)}\left(\frac{1}{a^2}-1\right)\left[R_{\mu\nu} - \frac{1}{2(3+n)}h_{\mu\nu}R\right] \\ &+ \frac{l^2}{2}\left(\frac{1}{a^2} - 1 - \frac{2ye^{\phi}}{l}\frac{1}{a^2}\right) \times \left(\phi_{|\mu\nu} + \frac{1}{2}h_{\mu\nu}\phi^{|\alpha}\phi_{|\alpha}\right) \\ &- \frac{y^2e^{2\phi}}{a^2}\left(\phi_{|\mu}\phi_{|\nu} - \frac{1}{2}h_{\mu\nu}\phi^{|\alpha}\phi_{|\alpha}\right) \\ &- \frac{2l}{4+n}\left(\frac{1}{a^{4+n}} - 1\right)\chi_{\mu\nu},\end{aligned} \tag{A26}$$

where we have imposed the boundary condition ${}^{(1)}g_{\mu\nu}(y = 0, x^{\mu}) = 0$. We can use a schematic iteration [13] for the solutions at higher orders.

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