Weyl transverse gravity and the cosmological constant

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Scale invariant (transverse) gravitational theories are introduced. They are invariant under pure metric rescalings (i.e. the matter fields are inert under those). This symmetry forbids the presence of a cosmological constant. Those theories are not invariant under the full set of diffeomorphisms, but only with respect to those locally characterized by the fact that their generator is transverse $\partial_\alpha \xi^\alpha = 0$.

I. TRANSVERSE GRAVITY AND SCALE INVARIANCE

The hope that scale invariance could shed some light on the fact that the observed value of the cosmological constant scale is much lower than expected from the Wilsonian viewpoint is certainly an old and cherished one. Let us mention just a couple of recent works [1,2] where some entries into the bibliography can be found.

The aim of the present work is to present a new twist of this idea in the framework of transverse gravity, where the full diffeomorphism invariance (Diff) is broken to those (TDiff) that preserve the Lebesgue measure. Transverse gravity has been studied in previous papers [3–6] where references to the earlier literature are included.

Those transverse gravitational models that enjoy scale invariance (that is, rigid Weyl invariance in the sense of [7] which is dubbed WTDiff in [4]) are, naively as we shall see in a moment, characterized by tracefree field equations. This means that the actions must be scale invariant, at least on shell, that is,

$$g^{\alpha\beta} \frac{\delta}{\delta g^{\alpha\beta}} S = 0.$$

The big difference with Einstein's diffeomorphism invariant gravity is that now we can sprinkle powers of g here and there. Under a global (i.e. constant) Weyl rescaling

$$g_{\alpha\beta} \rightarrow \Omega^2 g_{\alpha\beta} \qquad g \equiv \det g_{\alpha\beta} \rightarrow \Omega^{2n} g.$$

At the linear level with $\Omega \sim 1 + \omega$, $\delta g_{\alpha\beta} = 2\omega g_{\alpha\beta}$.

Christoffel symbols are invariant, and so is the Riemann tensor, so that

$$R \rightarrow \Omega^{-2} R.$$

This means that there is a purely gravitational (without

scalar fields) scale invariant[1] action, i.e.,

$$S_W \equiv -\frac{1}{2\kappa^2} \int d^n x |g|^{1/n} R.$$

The scaling behavior of matter is determined by the kinetic term (including the power of $|g|$ in front).

For example, in Einstein's gravity, a scalar field with kinetic part

$$\sqrt{|g|} \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi$$

implies that

$$\Phi \rightarrow \Omega^{1-n/2} \Phi$$

which coincides with the naive dimension of the field.

For Dirac fermions instead

$$\sqrt{|g|} i \bar{\psi} e_a^\mu \gamma^a \partial_\mu \psi$$

yields the naive dimension again

$$\psi \rightarrow \Omega^{(1-n)/2} \psi.$$

Changing the power of $|g|$, for example, as in

$$|g|^a \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi,$$

implies that

$$\Phi \rightarrow \Omega^{1-na} \Phi.$$

It is plain that when $a = 1/n$ then the theory enjoys rigid Weyl invariance with inert matter fields.

This means that with the measure

$$|g|^{1/n} d^n x$$

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[1]This action can be made Weyl gauge invariant, along the lines of [7] by means of a gauge field W_μ that transforms as

$$\delta W_\mu = \Omega^{-1} \partial_\mu \Omega$$

and adding a term proportional to

$$\int d^n x |g|^{1/n} \left(\nabla_\alpha W^\alpha + \frac{n-2}{2} W_\mu W^\mu \right).$$

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rigid Weyl invariance implies that no potential is allowed, not even a mass term.

Interactions are, however, allowed, but must either be dressed with some gravitational scalar of weight -2, for example,

$$\frac{c_p}{M^{(p(n-2)+4-2n)/2}} R \Phi^p$$

(where c_p are dimensionless constants, and M is a mass scale). When perturbing around a nontrivial constant curvature background (such as de Sitter space), this gives rise to masses

$$m^2 \sim c_2 \bar{R}$$

which are naturally tiny if the radius of curvature is very large.

Interactions are also allowed when they are totally decoupled from gravitation [5], as in

$$d^n x V(\Phi_i).$$

Similarly for Dirac fermions,

$$|g|^a i \bar{\psi} e_\alpha^\mu \gamma^\alpha \partial_\mu \psi$$

yields

$$\psi \to \Omega^{(1-2an)/2} \psi.$$

The new condition for invariance with inert Dirac fermions is

$$a = \frac{1}{2n}.$$

It is remarkable that this measure does not coincide with the bosonic one.

II. LOW ENERGY EFFECTIVE LAGRANGIANS

It is expected that lowest dimension operators compatible with the assumed symmetry (WTDiff) are bound to dominate the physics at low energies. Let us classify transverse scalars according to their dimension, writing also the corresponding scale invariant combination.

(i) *Dimension zero*
   Transverse dimension zero operators are

$$L_0 \equiv F(|g|)$$

   so that the WTDiff cosmological constant is decoupled from gravity

$$\delta S_0 \equiv \lambda \delta \int d^n x L_0 = 0,$$

   where λ is be a dimension n constant.

(ii) *Dimension 2*
   Generic transverse dimension 2 operators are

$$L_2^{(1)} = F(|g|) g^{\alpha\beta} \partial_\alpha g \partial_\beta g \qquad L_2^{(2)} = F(|g|) R. \tag{2.1}$$

   The WTDiff operator corresponding to the first one is

$$S_2^{(1)} \equiv -\frac{1}{2\kappa_1^2} |g|^{(1-2n)/n} g^{\alpha\beta} \partial_\alpha |g| \partial_\beta |g|,$$

   where κ_1^2 is a new *gravitational constant* of dimension $2-n$ *a priori* unrelated to Newton's constant,

$$\delta S_2^{(1)} = -\frac{1}{2\kappa_1^2} \int d^n x \left(-\frac{1-2n}{n} |g|^{(1-2n)/n} g^{\mu\nu} \partial_\mu |g| \partial_\nu |g| g_{\alpha\beta} \delta g^{\alpha\beta} + |g|^{(1-2n)/n} \partial_\alpha |g| \partial_\beta |g| \delta g^{\alpha\beta} \right.$$

$$\left. - 2|g|^{(1-2n)/n} g^{\mu\nu} \partial_\mu |g| \partial_\nu (|g| g_{\alpha\beta} \delta g^{\alpha\beta}) \right)$$

$$= -\frac{1}{2\kappa_1^2} \int d^n x \left(-\frac{1-2n}{n} |g|^{(1-2n)/n} g^{\mu\nu} \partial_\mu |g| \partial_\nu |g| g_{\alpha\beta} \delta g^{\alpha\beta} + |g|^{(1-2n)/n} \partial_\alpha |g| \partial_\beta |g| \delta g^{\alpha\beta} \right.$$

$$\left. + 2|g| \partial_\nu (|g|^{(1-2n)/n} g^{\mu\nu} \partial_\mu |g|) g_{\alpha\beta} \delta g^{\alpha\beta} \right). \tag{2.2}$$

The gravitational equations of motion are now

$$\frac{\delta S_2^{(1)}}{\delta g^{\alpha\beta}} = |g|^{(1-2n)/n} \partial_\alpha |g| \partial_\beta |g|$$

$$- \left(\frac{1-2n}{n} |g|^{(1-2n)/n} g^{\mu\nu} \partial_\mu |g| \partial_\nu |g| \right.$$

$$\left. - 2|g| \partial_\nu (|g|^{(1-2n)/n} g^{\mu\nu} \partial_\mu |g|) \right) g_{\alpha\beta},$$

where the gravitational constant has been deleted because it is not important in the absence of matter.

These equations are traceless *up to a total derivative*

$$g^{\alpha\beta} \frac{\delta S_2^{(1)}}{\delta g^{\alpha\beta}} = +2n \partial_\nu (|g|^{(1-n)/n} g^{\mu\nu} \partial_\mu |g|).$$

This means that the Noether current associated to WTDiff is

$$W^\mu \equiv |g|^{(1-n)/n} g^{\mu\nu} \partial_\nu |g|.$$

(iii) *Dimension two (continued).*—The second transverse dimension 2 operator is just a generalization of the usual Einstein-Hilbert Lagrangian

$$L_2^{(2)} = F(|g|)R.$$

In order to compute the variation of the corresponding WTDiff operator

$$\delta S_2^{(2)} = \delta\left(-\frac{1}{2\kappa^2}\int d^n x |g|^{1/n} R\right).$$

The variation of the curvature scalar is needed

$$\delta R = \delta g^{\nu\sigma} R_{\nu\sigma} + (g_{\alpha\beta}\Delta - \nabla_{(\alpha}\nabla_{\beta)})\delta g^{\alpha\beta}.$$

It follows:

$$\delta S_2^{(2)} = \int d^n x |g|^{1/n}\delta g^{\alpha\beta}\left(\frac{1}{2\kappa^2 n}g_{\alpha\beta}R - \frac{1}{2\kappa^2}R_{\alpha\beta}\right)$$
$$- \int d^n x |g|^{1/n}\frac{1}{2\kappa^2}(g_{\alpha\beta}\Delta - \nabla_\alpha\nabla_\beta)\delta g^{\alpha\beta}.$$
$$(2.3)$$

When

$$\delta g^{\alpha\beta} = -\Omega^2 g^{\alpha\beta}$$

the action remains invariant, just because $\nabla_\alpha g_{\mu\nu} = 0$. We must be careful with the integration by parts. A good place to start is the formula valid for any contravariant vector [8]

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}V^\mu).$$

Let us integrate by parts the slightly more general integral

$$\int d^n x f(g)\nabla_\mu(\nabla^\mu g_{\alpha\beta}\delta g^{\alpha\beta} - \nabla_\beta\delta g^{\mu\beta}) \equiv I_1 - I_2$$

$$I_1 \equiv \int \partial_\mu\left(\frac{f}{\sqrt{|g|}}\right)\sqrt{|g|}\nabla_\beta\delta g^{\mu\beta} = \int \partial_\mu\left(\frac{f}{\sqrt{|g|}}\right)\sqrt{|g|}(\partial_\beta\delta g^{\mu\beta} + \Gamma^\mu_{\beta\sigma}\delta g^{\sigma\beta} + \Gamma^\beta_{\beta\sigma}\delta g^{\sigma\mu})$$

$$= \int \partial_\mu\left(\frac{f}{\sqrt{|g|}}\right)\sqrt{|g|}(\Gamma^\mu_{\beta\sigma}\delta g^{\sigma\beta} + \Gamma^\beta_{\beta\sigma}\delta g^{\sigma\mu}) - \partial_\beta\left(\partial_\mu\left(\frac{f}{\sqrt{|g|}}\right)\sqrt{|g|}\right)\delta g^{\mu\beta}$$

$$I_2 \equiv \int \partial_\mu\left(\frac{f}{\sqrt{|g|}}\right)\sqrt{|g|}\nabla^\mu g_{\alpha\beta}\delta g^{\alpha\beta} = \int \partial_\mu\left(\frac{f}{\sqrt{|g|}}\right)\sqrt{|g|}g^{\mu\lambda}\partial_\lambda(g_{\alpha\beta}\delta g^{\alpha\beta}) = -\int \partial_\lambda\left(\partial_\mu\left(\frac{f}{\sqrt{|g|}}\right)\sqrt{|g|}g^{\mu\lambda}\right)g_{\alpha\beta}\delta g^{\alpha\beta}.$$
$$(2.4)$$

In conclusion,[2] calling $\Phi_\mu = \partial_\mu(\frac{f}{\sqrt{|g|}})$

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[2]It is worth checking that this still gives zero for a metric rescaling. This means that both integrals must vanish separately when

$$\delta g^{\alpha\beta} = -\epsilon g^{\alpha\beta}.$$

This is obvious for I_2, which in this case reduces to the integral of a total derivative. With respect to the first integral, we shall employ the well-known formulas [8]

$$\Gamma^\beta_{\beta\sigma} = \frac{1}{\sqrt{|g|}}\partial_\sigma\sqrt{|g|} \qquad g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = -\frac{1}{\sqrt{|g|}}\partial_\lambda(\sqrt{|g|}g^{\mu\lambda}) \qquad (2.5)$$

relating the Christoffels and the determinant:

$$I_1 = \int \Phi_\mu\sqrt{|g|}\left(-\frac{1}{\sqrt{|g|}}\partial_\lambda(\sqrt{|g|}g^{\mu\lambda}) + \frac{1}{\sqrt{|g|}}g^{\sigma\mu}\partial_\sigma\sqrt{|g|}\right) - g^{\mu\beta}\partial_\beta(\sqrt{|g|}\phi_\mu)$$

$$= \int \sqrt{|g|}\Phi_\mu\partial_\beta g^{\mu\beta} - \Phi_\mu\sqrt{|g|}\partial_\lambda g^{\mu\lambda} - \Phi_\mu g^{\mu\lambda}\partial_\lambda\sqrt{|g|} + \Phi_\mu g^{\sigma\mu}\partial_\sigma\sqrt{|g|} = 0. \qquad (2.6)$$

That is, the integrand itself vanishes. Under an arbitrary variation

$$\frac{\delta I_1}{\delta g^{\alpha\beta}} = \Phi_\mu\sqrt{|g|}\Gamma^\mu_{\alpha\beta} + \Gamma^\lambda_{\lambda\alpha}\phi_\beta\sqrt{|g|} - \partial_\beta(\sqrt{|g|}\Phi_\alpha) = -\sqrt{|g|}\partial_\beta\Phi_\alpha + \Phi_\mu\sqrt{|g|}\Gamma^\mu_{\alpha\beta} \equiv -\sqrt{|g|}\nabla_\beta\Phi_\alpha$$
$$(2.7)$$
$$\frac{\delta I_2}{\delta g^{\alpha\beta}} = -\partial_\lambda(\Phi_\mu\sqrt{|g|}g^{\mu\lambda})g_{\alpha\beta} \equiv -\sqrt{|g|}\nabla_\lambda(\Phi_\mu g^{\mu\lambda})g_{\alpha\beta},$$

where the covariant derivatives are defined as if Φ_μ were a tensor; which it is not, so that those constructions do not enjoy all properties of covariant derivatives of tensors. Still, it is sometimes a useful abbreviation.

$$\delta S_2^{(2)} = \int d^n x |g|^{1/n} \delta g^{\alpha\beta} \left(\frac{1}{2\kappa^2 n} g_{\alpha\beta} R - \frac{1}{2\kappa^2} R_{\alpha\beta} \right) + \int d^n x \frac{1}{2\kappa^2} \sqrt{|g|} (\nabla_\beta \Phi_\alpha - \nabla_\lambda (\Phi_\mu g^{\mu\lambda}) g_{\alpha\beta}) \delta g^{\alpha\beta}$$

$$= \int |g|^{1/n} \frac{1}{2\kappa^2} \left[\left(\frac{1}{n} g_{\alpha\beta} R - R_{\alpha\beta} \right) + \frac{2-n}{2n} |g|^{-1} \left(\frac{2-3n}{2n} g^{-1} g_\alpha g_\beta - \nabla_\beta g_\alpha \right. \right.$$

$$\left. \left. - \left(\frac{1-n}{n} g^{-1} g_\mu g_\nu g^{\mu\nu} + \partial_\mu (g_\nu g^{\mu\nu}) \right) g_{\alpha\beta} \right) \right] \delta g^{\alpha\beta} d^n x \tag{2.8}$$

(where $g_\alpha \equiv \partial_\alpha g$). It is remarkable that Einstein's 1919 equations

$$R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} = \kappa^2 \left(T_{\mu\nu} - \frac{1}{n} T g_{\mu\nu} \right)$$

which are truly traceless [9] (cf. also [3]) do not seem to be obtainable from a variational principle of the sort we are studying which always yield equations of motion which are traceless only up to a total derivative.

III. CONCLUSIONS

We have studied in the body of the paper a gravitational symmetry that forbids the presence of a cosmological constant. We believe that this is some progress insofar as we were not aware of any such symmetry previously known.

It would be interesting to present our results in the Einstein frame. In the case of the second dimension 2 operator, which is the only one resembling the Einstein-Hilbert Lagrangian, this would stem from the redefinition of a new spacetime metric such that

$$\sqrt{|g_e|} R[g_e] = |g|^{1/n} R.$$

It is quite simple to realize that

$$g_{\mu\nu}^e \equiv g^{-(1/n)} g_{\mu\nu}$$

such that $g_e \equiv 1$. The restricted variational principle would then give true traceless equations of motion of the Einstein's 1919 sort [9], except that in Einstein's mind the metric was not restricted by any unimodularity condition.

We can understand our results from a different viewpoint. It is well known that transverse theories are equivalent, in a given reference system, to scalar-tensor theories [4,10]. A way of implementing this mapping is as follows: our second dimension 2 Lagrangian is equivalent to

$$L = -\frac{1}{2\kappa^2} \sqrt{|g|} \phi R + \sqrt{|g|} \chi (\phi - |g|^{(2-n)/2n}),$$

where ϕ and χ are two auxiliary scalar densities. It is now possible to find an unconstrained Einstein metric such that

$$\sqrt{|g_E|} R[g_E] = \sqrt{|g|} \phi R.$$

The answer is clearly

$$g_{\mu\nu}^E = \phi^{2/(n-2)} g_{\mu\nu}$$

(so that $g_E = g \phi^{2n/(n-2)}$) and the full scalar-tensor Lagrangian reads

$$L = -\frac{1}{2\kappa^2} \phi \sqrt{|g|} R + \sqrt{|g|} \chi (\phi - |g|^{2n/(2-n)})$$

$$= -\frac{1}{2\kappa^2} \sqrt{|g_E|} R_E + \sqrt{|g_E|} \phi^{-(2/(n-2))} \chi (1 - |g_E|^{2n/(2-n)})$$

$$+ \frac{n-1}{2\kappa^2(n-2)} \left(2 \partial_\mu \left(\sqrt{g_E} g_E^{\mu\nu} \frac{\partial_\nu \phi}{\phi} \right) \right.$$

$$\left. - \sqrt{g_E} g_E^{\mu\nu} \frac{\partial_\mu \phi \partial_\nu \phi}{\phi^2} \right), \tag{3.1}$$

i.e., it is of the unimodular type. The Lagrangian multiplier χ enforces the constraint $|g_E| = 1$ on shell only, so that the variations are *not* restricted by $g_E^{\mu\nu} \delta g_{\mu\nu}^E = 0$, as in Einstein's 1919 theory.

Quantum effects would generically give a kinetic energy to this field, which then becomes a propagating field.

It has however been stressed in the literature [3] that this is subtly not equivalent to choosing the unimodular gauge in general relativity, which is always allowed (and used many times by Einstein himself). The coupling to matter is independent of the scalar density ϕ. For example, for a scalar field Φ (not to be confused with the scalar density ϕ of gravitational origin),

$$L_I = |g_E|^{1/n} g_E^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi.$$

Under conformal transformations in the old frame

$$\phi \to \Omega^{2-n} \phi$$

and for consistency,

$$\chi \to \Omega^{-2} \chi,$$

whereas the unimodular Einstein metric is inert. What looks like a purely gravitational symmetry in one frame, looks like a *matter* symmetry in another. Potential energy coupled to gravity is again forbidden, because they appear in the new frame as

$$\phi^{-(2/(n-2))} V(\Phi).$$

It is also interesting to follow the first dimension 2 term under this change of frame. It is easy to check that, if the equations of motion are used, it reduces to

$$L_2^{(1)} = \frac{4n^2}{(n-2)^2} \phi^{-2} g_E^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi$$

(if the equations of motion are not used, there are other terms proportional to $\partial_\mu |g_E|$).

Nevertheless, transverse theories are most likely severely constrained by experiment [6] and besides scale invariance has to be broken, at least by the Weyl anomaly [11,12] (which has yet to be computed for transverse theories).

Actually WTDiff makes an overkill, in the sense that it not only forbids a cosmological constant, but also any potential energy whatsoever which is coupled to gravitation. There is experimental evidence[3] that potential energy

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[3]Although experiments tend to bound *differences* between properties of different objects, so that if those differences are universal there are not so constrained.

does couple to gravitation [13], which is again an indication that scale symmetry must be badly broken in nature.

The proper setting of the problem is most likely a cosmological one, in which the universe goes through different epochs characterized by different amounts of symmetry in the gravitational sector. Work on concrete models of this sort is in progress and we hope to report on that in the future.

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