Black holes and stars in the Horava-Lifshitz theory with the projectability condition

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We systematically study spherically symmetric static spacetimes filled with a fluid in the Horava-Lifshitz theory of gravity with the projectability condition, but without the detailed balance. We establish that when the spacetime is spatially Ricci flat the unique vacuum solution is the de Sitter Schwarzshcild solution, while when the spacetime has a nonzero constant curvature, there exist two different vacuum solutions; one is an (Einstein) static universe, and the other is a new spacetime. This latter spacetime is maximally symmetric and not flat. We find all the perfect fluid solutions for such spacetimes, in addition to a class of anisotropic fluid solutions of the spatially Ricci flat spacetimes. To construct spacetimes that represent stars, we investigate junction conditions across the surfaces of stars and obtain the general matching conditions with or without the presence of infinitely thin shells. It is remarkable that, in contrast to general relativity, the radial pressure of a star does not necessarily vanish on its surface even without the presence of a thin shell, due to the presence of high order derivative terms. Applying the junction conditions to our explicit solutions, we show that it is possible to match smoothly these solutions (all with nonzero radial pressures) to vacuum spacetimes without the presence of thin matter shells on the surfaces of stars.

I. INTRODUCTION

There has been considerable interest recently on a nonrelativistic theory of gravity proposed by Horava [1]. Inspired by the theory of a Lifshitz scalar [2] relevant in condensed matter systems, Horava postulated a nonrelativistic anisotropic scaling symmetry of space and time. This allows for the addition of higher spatial derivative terms in the action without their time derivative counterparts, rendering the theory power counting renormalizable. The theory is supposed to flow dynamically from a scale invariant theory in the ultraviolet (UV) to general relativity (GR) in the infrared (IR), thus restoring the diffeomorphism symmetry at low energies.

In the class of Horava-Lifshitz (HL) theories studied during the past year, there have been many variations regarding both the potential that is used and whether the lapse function is allowed to depend on the spatial coordinates or not. In the initial proposal by Horava [1], a simplifying assumption was made to reduce the number of terms in the potential. The potential was required to be derived by a "superpotential," thus giving is it a form dubbed as detailed balance. However, it was soon realized that the breaking of detailed balance is necessary in order to obtain Minkowski vacua at low energies [3,4]. This breaking can be done minimally by adding a soft breaking term in the action, or can be done in more generality by adding all possible terms. In the present paper, we will

adopt the potential of [3], where all operators that conserve parity are included in the potential. Since the detailed balance potential has operators that break parity, there will be no limit of the dynamics of the potential constructed in [3], unless the contributions of these operators vanish. It can be shown that when the spacetime is static and spherically symmetric, these operators indeed have zero contributions [4]. A second assumption that distinguishes HL theories, is whether one assumes that the laspe function N is a function only of time (projectable case) or a spacetime function (nonprojectable case). In the original proposal, projectability was assumed, but since the Hamiltonian constraint is not local (it becomes an integral constraint [5]), it was preferred in the literature to use the nonprojectable assumption. However, the latter seems to be inconsistent [6] as the Poisson brackets of the theory do not form a closed structure. For this reason, in the present paper we choose to work with the projectable assumption. As is expected, the breaking of diffeomorphism invariance results in the existence of an extra (scalar) mode in the spectrum, which is absent in GR. This mode introduces pathologies for almost all variations of the theory, giving rise to sometimes instabilities (classical or quantum), and always to strong coupling. In particular, in the projectable case this mode exists already in Minkowski backgrounds and is either classically or quantum mechanically unstable [3,7]. In the nonprojectable case, this mode exists for time-dependent and spatially inhomogeneous backgrounds and is classically unstable [8]. If one includes spatial gradients of the lapse function in the nonprojectable case [9], the mode can be rendered stable. The cases described above that are classically unstable can be rendered stable by

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higher order interactions with the price of introducing strong coupling in the theory [8,10]. Strong coupling also exists in the limit that the theory approaches GR [9], which brings the effective perturbative scale lower than the GR cutoff [11].

To understand the theory further, other aspects have also been studied [12]. In particular, solutions of this theory have been extensively investigated in the directions of cosmology as well as black holes. Isotropic cosmological solutions were studied in [13] and revealed that the new terms added in the theory typically modify the dynamics for nonzero spatial curvature. It is interesting to note that in [5] it was advocated that the projectable theory has a dark matter component built-in, due to the nonlocality of the Hamiltonian constraint.

The other set of solutions studied so far are the spherically symmetric ones [14]. In particular, Lu, Mei, and Pope found all the vacuum solutions of the HL theory with the detailed balance condition [15]. Cai *et al* generalized them to topological black holes with or without charges, and paid particular attention to their thermodynamics [16], while Colgain and Yavartanoo obtained a class of dyonic solutions with detailed balance and assuming that high order derivative terms in the potential of the massless vector field were absent [17]. (See also [18].) Adding a linear term of the three-spatial curvature into the action, so that the detailed balance condition was softly broken, Kehagias and Sfetsos constructed a class of vacuum solutions that is asymptotically flat [4]. Park generalized it to the case with a nonvanishing cosmological constant [19], while Lee, Kim, and Myung studied $AdS_2 \times S^2$ solutions in such a deformed generalization of the HL theory [20]. Capasso and Polychronakos considered the case with non-trivial lapse function and shift vector and found all the solutions with soft breaking of the detailed balance [21]. Kiritsis and Kofinas further generalized the HL theory to include quadratic terms of the three-dimensional spatial curvature, and found the most general solutions [22], while Kiritsis himself [23] considered static spherically symmetric solutions in a version proposed in [9]. Other kinds of solutions can be found in [12,14]. It should be noted that all the solutions mentioned above were constructed without assuming the projectability condition. As we shall show in this paper, all such solutions can be written in a canonical ADM form that exhibits explicitly the projectability condition. In addition, static spherically symmetric spacetimes with the projectability condition were studied recently by Tang and Chen, and some solutions with or without charge were found [24].

In this paper, we systematically study static spherically symmetric solutions in the projectable case, with the full potential of [3]. We first present a self-contained introduction of the theory in Sec. II, and then in Sec. III we write down the equations of motion for projectable gauge that we will work with. In Secs. IV and V, we present all the vacuum solutions as well as solutions in the presence of a perfect fluid for two cases: the case that the spatial curvature vanishes (Sec. IV) and the case where the spatial curvature is a nonzero constant (Sec. V). In the former case we also present a class of solutions that represent an anisotropic fluid whose radial pressure is proportional to its tangential one. In Sec. VI, we first develop the general junction conditions across the surface of a star with or without the presence of an infinitely thin shell. Then, we join our vacuum solutions with the ones of a fluid by requiring that no thin shell be present on the surface of the star. Finally, in Sec. VI we present our main results. Three appendices A, B, and C, are also included. In Appendix A, the general expressions of the $(F_s)_{ij}$ tensors appearing in the equations of motion for static spherically symmetric spacetimes are given, while in Appendix B static solutions in GR are studied in the canonical Arnowitt-Deser-Misner (ADM) form. In the latter, we show explicitly how one can bring any given static metric into an ADM form with the projectability condition. However, the kind of coordinate transformations needed to achieve this, are not allowed by the foliation-preserving diffeomorphisms of the HL theory, and the actions are generally not invariant. In Appendix C, we calculate the trace of the extrinsic curvature $K = K^i_{\ i}$ for all the solutions found in Secs. IV and V, and study its singular behavior.

Before proceeding, we would like to note that spherically symmetric spacetimes in the framework of the HL theory with the projectability condition are also studied recently by Izumi and Mukohyama [25], and found that, among other things, globally static and regular perfect fluid solutions do not exist. Our results presented in this paper do not contradict to it, since in this paper we have different assumptions. In particular, the perfect fluid to be considered here usually conducts heat along the radial direction [26], while the one considered in [25] does not. This is another peculiar feature of the HL theory: Static stars in GR do not conduct heat. In addition, black holes in the HL theory might not black, because of the different dispersion relations [22,25].

II. HORAVA-LIFSHITZ GRAVITY WITHOUT DETAILED BALANCE

We give a very brief introduction to HL gravity without detailed balance, but with the projectability condition. (For further details, see [3,7,27].) The dynamical variables are N, N^i and $g_{ij}(i, j = 1, 2, 3)$, in terms of which the metric takes the ADM form,

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \quad (2.1)$$

The theory is invariant under the scalings

$$t \to \ell^3 t, \qquad x^i \to \ell x^i, \qquad N \to \ell^{-2} N,$$
$$N^i \to \ell^{-2} N^i, \qquad g_{ij} \to g_{ij}. \qquad (2.2)$$

It should be noted that there was a constant term c^2 in front of the lapse function N in the ADM metric used in [3], so that N was rescaling as $N \to N$, where c has dimensions $[c] = [dx/dt]$. In [7,27] and this paper, we absorb it into N so that now N is scaling as that given above. The projectability condition requires a homogeneous lapse function:

$$N = N(t), \qquad N^i = N^i(t, x^k), \qquad g_{ij} = g_{ij}(t, x^k). \qquad (2.3)$$

The form of the metric is invariant under the foliation-preserving diffeomorphisms of the HL theory,

$$\tilde{t} = t + \chi^0(t), \qquad \tilde{x}^i = x^i + \chi^i(t, x^k). \qquad (2.4)$$

The total action consists of kinetic, potential and matter parts,

$$S = \zeta^2 \int dt d^3 x N \sqrt{g} (\mathcal{L}_K - \mathcal{L}_V + \zeta^{-2} \mathcal{L}_M), \qquad (2.5)$$

where $g = \det g_{ij}$, and

$$\mathcal{L}_K = K_{ij} K^{ij} - (1 - \xi) K^2,$$
$$\mathcal{L}_V = 2\Lambda - R + \frac{1}{\zeta^2} (g_2 R^2 + g_3 R_{ij} R^{ij})$$
$$+ \frac{1}{\zeta^4} (g_4 R^3 + g_5 R R_{ij} R^{ij} + g_6 R^i_j R^j_k R^k_i)$$
$$+ \frac{1}{\zeta^4} [g_7 R \nabla^2 R + g_8 (\nabla_i R_{jk})(\nabla^i R^{jk})]. \qquad (2.6)$$

Here $\zeta^2 = 1/16\pi G$, the covariant derivatives and Ricci and Riemann terms are all constructed from the three-metric g_{ij}, while K_{ij} is the extrinsic curvature,

$$K_{ij} = \frac{1}{2N} (-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i), \qquad (2.7)$$

where $N_i = g_{ij} N^j$. The constants ξ, $g_I (I = 2, \ldots 8)$ are coupling constants, and Λ is the cosmological constant. It should be noted that Horava included a cross term $C_{ij} R^{ij}$, where C_{ij} is the Cotton tensor. This term scales as ℓ^5 and explicitly violates parity. To restore parity, this term was excluded in [3].

In the IR limit, all the high order curvature terms (with coefficients g_I) drop out, and the total action reduces when $\xi = 0$ to the Einstein-Hilbert action.

Variation with respect to the lapse function, $N(t)$, yields the Hamiltonian constraint,

$$\int d^3 x \sqrt{g} (\mathcal{L}_K + \mathcal{L}_V) = 8\pi G \int d^3 x \sqrt{g} J^t, \qquad (2.8)$$

where

$$J^t = 2 \left(N \frac{\delta \mathcal{L}_M}{\delta N} + \mathcal{L}_M \right). \qquad (2.9)$$

Because of the projectability condition $N = N(t)$, the Hamiltonian constraint takes a nonlocal integral form. If one relaxes projectability and allows $N = N(t, x^i)$, then the corresponding variation with respect to N will yield a local super-Hamiltonian constraint $\mathcal{L}_K + \mathcal{L}_V = 8\pi G J^t$.

Variation with respect to the shift, N^i, yields the super-momentum constraint,

$$\nabla_j \pi^{ij} = 8\pi G J^i, \qquad (2.10)$$

where the supermomentum, π^{ij}, and matter current, J^i, are

$$\pi^{ij} \equiv \frac{\delta \mathcal{L}_K}{\delta \dot{g}_{ij}} = -K^{ij} + (1 - \xi) K g^{ij}, \qquad J^i \equiv -N \frac{\delta \mathcal{L}_M}{\delta N_i}. \qquad (2.11)$$

Varying with respect to g_{ij}, on the other hand, leads to the dynamical equations,

$$\frac{1}{N\sqrt{g}} (\sqrt{g} \pi^{ij})^{\cdot} = -2(K^2)^{ij} + 2(1 - \xi) K K^{ij}$$
$$+ \frac{1}{N} \nabla_k (N^k \pi^{ij} - N^i \pi^{jk} - N^j \pi^{ik})$$
$$+ \frac{1}{2} \mathcal{L}_K g^{ij} + F^{ij} + 8\pi G \tau^{ij}, \qquad (2.12)$$

where $(K^2)^{ij} \equiv K^{il} K^j_l$, and

$$F^{ij} \equiv \frac{1}{\sqrt{g}} \frac{\delta(-\sqrt{g} \mathcal{L}_V)}{\delta g_{ij}} = \sum_{s=0}^{8} g_s \zeta^{n_s} (F_s)^{ij}. \qquad (2.13)$$

The constants are given by $g_0 = 2\Lambda \zeta^{-2}$, $g_1 = -1$, and $n_s = (2, 0, -2, -2, -4, -4, -4, -4, -4)$. The stress 3-tensor is defined as

$$\tau^{ij} = \frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g} \mathcal{L}_M)}{\delta g_{ij}}, \qquad (2.14)$$

and the geometric 3-tensors $(F_s)_{ij}$ are defined as follows:

$$(F_0)_{ij} = -\frac{1}{2}g_{ij}, \qquad (F_1)_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}, \qquad (F_2)_{ij} = 2(R_{ij} - \nabla_i\nabla_j)R - \frac{1}{2}g_{ij}(R - 4\nabla^2)R,$$

$$(F_3)_{ij} = \nabla^2 R_{ij} - (\nabla_i\nabla_j - 3R_{ij})R - 4(R^2)_{ij} + \frac{1}{2}g_{ij}(3R_{kl}R^{kl} + \nabla^2 R - 2R^2),$$

$$(F_4)_{ij} = 3(R_{ij} - \nabla_i\nabla_j)R^2 - \frac{1}{2}g_{ij}(R - 6\nabla^2)R^2,$$

$$(F_5)_{ij} = (R_{ij} + \nabla_i\nabla_j)(R_{kl}R^{kl}) + 2R(R^2)_{ij} + \nabla^2(RR_{ij}) - \nabla^k[\nabla_i(RR_{jk}) + \nabla_j(RR_{ik})]$$
$$- \frac{1}{2}g_{ij}[(R - 2\nabla^2)(R_{kl}R^{kl}) - 2\nabla_k\nabla_l(RR^{kl})],$$

$$(F_6)_{ij} = 3(R^3)_{ij} + \frac{3}{2}[\nabla^2(R^2)_{ij} - \nabla^k(\nabla_i(R^2)_{jk} + \nabla_j(R^2)_{ik})] - \frac{1}{2}g_{ij}[R_l^k R_m^l R_k^m - 3\nabla_k\nabla_l(R^2)^{kl}],$$

$$(F_7)_{ij} = 2\nabla_i\nabla_j(\nabla^2 R) - 2(\nabla^2 R)R_{ij} + (\nabla_i R)(\nabla_j R) - \frac{1}{2}g_{ij}[(\nabla R)^2 + 4\nabla^4 R],$$

$$(F_8)_{ij} = \nabla^4 R_{ij} - \nabla_k(\nabla_i\nabla^2 R_j^k + \nabla_j\nabla^2 R_i^k) - (\nabla_i R_l^k)(\nabla_j R_k^l) - 2(\nabla^k R_i^l)(\nabla_k R_{jl}) - \frac{1}{2}g_{ij}[(\nabla_k R_{lm})^2 - 2(\nabla_k\nabla_l\nabla^2 R^{kl})]. \tag{2.15}$$

The matter quantities (J^t, J^i, τ^{ij}) satisfy the conservation laws [7,27,28],

$$\int d^3x\sqrt{g}\left[\dot{g}_{kl}\tau^{kl} - \frac{1}{\sqrt{g}}(\sqrt{g}J^t)^{\cdot} + \frac{2N_k}{N\sqrt{g}}(\sqrt{g}J^k)^{\cdot}\right] = 0, \tag{2.16}$$

$$\nabla^k\tau_{ik} - \frac{1}{N\sqrt{g}}(\sqrt{g}J_i)^{\cdot} - \frac{N_i}{N}\nabla_k J^k - \frac{J^k}{N}(\nabla_k N_i - \nabla_i N_k) = 0. \tag{2.17}$$

It should be noted that the energy-momentum tensor in GR is defined as,

$$T^{\mu\nu} = \frac{1}{\sqrt{-g^{(4)}}}\frac{\delta(\sqrt{-g^{(4)}}\mathcal{L}_M)}{\delta g^{(4)}_{\mu\nu}}, \tag{2.18}$$

where $\mu, \nu = 0, 1, 2, 3$, and $g^{(4)}_{00} = -N^2 + N^iN_i$, $g^{(4)}_{0i} = N_i$, and $g^{(4)}_{ij} = g_{ij}$. Introducing the normal vector n_ν to the hypersurface $t = $ const as,

$$n_\mu = N\delta^t_\mu, \qquad n^\mu = \frac{1}{N}(-1, +N^i), \tag{2.19}$$

one can decompose $T_{\mu\nu}$ as [29],

$$\rho_H \equiv T_{\mu\nu}n^\mu n^\nu, \qquad s_i \equiv -T_{\mu\nu}h_i^{(4)\mu}n^\nu,$$
$$s_{ij} \equiv T_{\mu\nu}h_i^{(4)\mu}h_j^{(4)\nu}, \tag{2.20}$$

where $h^{(4)}_{\mu\nu}$ is the projection operator, defined as $h^{(4)}_{\mu\nu} \equiv g^{(4)}_{\mu\nu} + n_\mu n_\nu$. In the GR limit, one may identify J^t, J_i, τ_{ij} with $-2\rho_H, s_i, s_{ij}$, respectively.

III. SPHERICALLY SYMMETRIC STATIC SPACETIMES

The general spherically symmetric spacetime that preserves the form of Eq. (2.1) with the projectability condi-

tion is described by the metric,

$$ds^2 = -N^2(t)dt^2 + e^{2\nu(t,r)}(dr + N^r(t,r)dt)^2 + R^2(t,r)d\Omega^2, \tag{3.1}$$

in the spherical coordinates $x^i = (r, \theta, \phi)$, where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$ and $N^i = \{N^r, 0, 0\}$. Clearly, it is invariant under the transformations,

$$t = f(t'), \qquad r = g(t', r'), \tag{3.2}$$

where f and g are arbitrary functions of their indicated arguments. With this gauge freedom, we see that, without loss of generality, we can set

$$N(t) = 1, \qquad R(t, r) = r, \tag{3.3}$$

a gauge we refer to as *the canonical ADM gauge.* From now on we shall work with this gauge.

To consider spherically symmetric static spacetimes in the HL theory with projectability, we assume that there exists a timelike Killing vector, ξ^μ, along t, namely $\xi = \partial_t$. It can then be shown that the Killing equations, $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$, lead to

$$\nu(t, r) = \nu(r), \qquad N^r(t, r) = N^r(r), \tag{3.4}$$

for which the metric can be finally written as

$$ds^2 = -dt^2 + (e^{\mu(r)}dt + e^{\nu(r)}dr)^2 + r^2 d\Omega^2, \tag{3.5}$$

where

$$\mu = \nu + \ln N^r, \qquad N^r = e^{\mu - \nu}. \tag{3.6}$$

For the metric (3.5), we find

$$K_{ij} = e^{\mu+\nu}(\mu'\delta_i^r\delta_j^r + re^{-2\nu}\Omega_{ij}),$$

$$R_{ij} = \frac{2\nu'}{r}\delta_i^r\delta_j^r + e^{-2\nu}[r\nu' - (1-e^{2\nu})]\Omega_{ij},$$

$$\mathcal{L}_K = e^{2(\mu-\nu)}\left[\xi\mu'^2 - \frac{4(1-\xi)}{r}\mu' - \frac{2(1-2\xi)}{r^2}\right], \quad (3.7)$$

$$\mathcal{L}_V = \sum_{s=0}^{3}\mathcal{L}_V^{(s)},$$

where a prime denotes the ordinary derivative with respect to its indicated argument, $\Omega_{ij} \equiv \delta_i^\theta\delta_j^\theta + \sin^2\theta\delta_i^\phi\delta_j^\phi$, and $\mathcal{L}_V^{(s)}$'s are given by Eq. (A1). Then, we find that the Hamiltonian constraint (2.8) reads,

$$\int(\mathcal{L}_K + \mathcal{L}_V - 8\pi GJ^t)e^\nu r^2 dr = 0, \quad (3.8)$$

while the momentum constraint (2.10) yields,

$$\xi[\mu'' + (\mu'-\nu')\mu'] + \frac{2}{r}[\xi\mu' + (1-\xi)\nu'] - \frac{2\xi}{r^2}$$

$$= -8\pi Ge^{2(\nu-\mu)}v, \quad (3.9)$$

where

$$J^i = e^{-(\mu+\nu)}(v, 0, 0). \quad (3.10)$$

The dynamical equations (2.12), on the other hand, yield,

$$\xi(2\mu'' - 2\nu'\mu' + \mu'^2) + \frac{4}{r}[\mu' + (1-\xi)\nu'] + \frac{2(1-4\xi)}{r^2}$$

$$= -2e^{2(\nu-\mu)}(e^{-2\nu}F_{rr} + 8\pi Gp_r), \quad (3.11)$$

$$2(1-\xi)(\mu'' - \mu'\nu') + (4-3\xi)\mu'^2$$

$$+ \frac{2(1-2\xi)}{r}(2\mu' - \nu')$$

$$= -2e^{2(\nu-\mu)}\left(\frac{1}{r^2}F_{\theta\theta} + 8\pi Gp_\theta\right), \quad (3.12)$$

where

$$\tau_{ij} = e^{2\nu}p_r\delta_i^r\delta_j^r + r^2p_\theta\Omega_{ij}, \quad (3.13)$$

and F_{ij} is given by Eqs. (2.13) and (A2). In this paper we define a fluid with $p_r = p_\theta$ as a perfect fluid, which in general conducts heat flow along the radial direction [26].

Since the spacetime is static, one can see that now the energy conservation law (2.16) is satisfied identically, while the momentum conservation (2.17) yields,

$$v\mu' - (v' - p_r') - \frac{2}{r}(v - p_r + p_\theta) = 0. \quad (3.14)$$

It should be noted that this equation is not independent of Eqs. (3.9), (3.11), and (3.12). As a result, one cannot use it as an additional condition to determinate the six unknown functions, μ, ν, J^t, v, p_r, p_θ. But, since it involves only first-order derivatives, it is often very useful to use it to replace one of the three equations (3.9), (3.11), and (3.12). In summary, in the present case we have one integral Hamiltonian constraint (3.8), one momentum constraint (3.9), and two dynamical equations (3.11) and (3.12), for the six unknowns. Thus, in order to determinate them uniquely, additional conditions are required.

In GR, for a perfect fluid [cf. Appendix B], one condition usually comes from the equation of state, $p = p(\rho)$, where ρ denotes the energy density of the fluid and is related to J^t (but not exactly equal to it [7]). However, in the present case, the Hamiltonian constraint is not local and cannot be used to close the system. Therefore, to have the system closed one may take the Hamiltonian constraint (3.8) as a constraint on $J^t(r)$ and then use the three remaining differential equations (3.9), (3.11), and (3.12), together with two additional conditions, to determinate uniquely the five unknowns, μ, ν, v, p_r, and p_θ. In this paper, we shall adopt this strategy and obtain one of the two conditions by specifying the spatial curvature. In particular, we will consider two cases: (i) the spacetime is spatially Ricci-flat, and (ii) the spatial curvature is a nonzero constant. Certainly, one can equally choose other physical conditions to close the system.

To relate the decomposition of the quantities J^t, J^i and τ_{ij} defined above to the ones introduced in Eq. (2.20), one may introduce another spacelike unit vector, χ_μ, which is orthogonal to n_μ, θ_μ and ϕ_μ, where

$$n_\mu = \delta_\mu^t, \qquad n^\mu = -\delta_t^\mu + e^{\mu-\nu}\delta_r^\mu,$$

$$\theta_\mu = r\delta_\mu^\theta, \qquad \phi_\mu = r\sin\theta\delta_\mu^\phi. \quad (3.15)$$

Such a χ_μ is uniquely determined as

$$\chi^\mu = e^{-\nu}\delta_r^\mu, \qquad \chi_\mu = e^\mu\delta_\mu^t + e^\nu\delta_\mu^r. \quad (3.16)$$

In terms of the four unit vectors, n_μ, χ_μ, θ_μ, and ϕ_μ, one can decompose the energy-momentum tensor for an anisotropic fluid with heat flow as

$$T_{\mu\nu} = \rho_H n_\mu n_\nu + q(n_\mu\chi_\nu + n_\nu\chi_\mu) + p_r\chi_\mu\chi_\nu$$

$$+ p_\theta(\theta_\mu\theta_\nu + \phi_\mu\phi_\nu), \quad (3.17)$$

where ρ_H, q, p_r and p_θ denote, respectively, the energy density, heat flow along radial direction, radial, and tangential pressures, measured by the observer with the four-velocity n_μ. Then, combining it with Eq. (2.20) one can see that such a decomposition is consistent with the quantities J^t, J^i and τ_{ij} defined above with $v = qe^\mu$. It should be noted that the definitions of the energy density ρ_H, the radial pressure p_r and the heat flow q are different from the ones (ρ_o, p_R, q_o) given by Eq. (B7), which are defined by assuming that the fluid is comoving with respect to the orthogonal frame (B5).

IV. SPATIALLY RICCI-FLAT SOLUTIONS

Requiring that the spacetime be spatially Ricci flat, $R_{ij} = 0$, we find that $\nu = 0$, and

$$\mathcal{L}_K = e^{2\mu}\left[\xi\mu'^2 - \frac{4(1-\xi)}{r}\mu' - \frac{2(1-2\xi)}{r^2}\right]. \quad (4.1)$$

Then, Eqs. (A1) and (A2) yield

$$\mathcal{L}_V = 2\Lambda, \qquad F_{ij} = -\Lambda g_{ij}. \quad (4.2)$$

Inserting the above into Eqs. (3.8), (3.9), (3.11), (3.12), and (3.14), we obtain, respectively,

$$\int (\mathcal{L}_K + 16\pi G\rho)r^2 dr = 0, \quad (4.3)$$

$$\xi(\mu'' + \mu'^2) + \frac{2\xi}{r}\mu' - \frac{2\xi}{r^2} = -8\pi G e^{-2\mu}\upsilon, \quad (4.4)$$

$$\xi(2\mu'' + \mu'^2) + \frac{4}{r}\mu' + \frac{2(1-4\xi)}{r^2}$$
$$= -16\pi G e^{-2\mu}(p_r + p_\Lambda), \quad (4.5)$$

$$2(1-\xi)\mu'' + (4-3\xi)\mu'^2 + \frac{4(1-2\xi)}{r}\mu'$$
$$= -16\pi G e^{-2\mu}(p_\theta + p_\Lambda), \quad (4.6)$$

$$\upsilon\mu' - (\upsilon' - p_r') - \frac{2}{r}(\upsilon - p_r + p_\theta) = 0, \quad (4.7)$$

where

$$\rho \equiv \rho_\Lambda - \frac{J^t}{2}, \qquad \rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi G}. \quad (4.8)$$

To study the above equations further, we consider the cases $\xi = 0$ and $\xi \neq 0$ separately.

A. $\xi = 0$

When $\xi = 0$, from Eq. (4.4) we find that $\upsilon = 0$. Then, from Eq. (4.1) we obtain

$$\mathcal{L}_K = -2(2r\mu' + 1)\frac{e^{2\mu}}{r^2}, \quad (4.9)$$

while Eqs. (4.5), (4.6), and (4.7) reduce, respectively, to

$$2r\mu' + 1 = -8\pi G r^2 e^{-2\mu}(p_r + p_\Lambda), \quad (4.10)$$

$$\mu'' + 2\mu'^2 + \frac{2}{r}\mu' = -8\pi G e^{-2\mu}(p_\theta + p_\Lambda), \quad (4.11)$$

$$p_r' + \frac{2}{r}(p_r - p_\theta) = 0. \quad (4.12)$$

1. de Sitter Schwarzschild solution

If we further require the fluid be perfect, $p_r = p_\theta = p$, Eq. (4.12) tells us that the pressure must be constant. Without loss of generality, we can absorb this constant into p_Λ, and then Eqs. (4.5) and (4.6) have the solution,

$$\mu = \frac{1}{2}\ln\left(\frac{M}{r} + \frac{\Lambda}{3}r^2\right). \quad (4.13)$$

Inserting it into Eq. (4.1) we find that $\mathcal{L}_K = -2\Lambda$. Then, Eq. (4.3) requires $J^t = 0$. This is exactly the de Sitter Schwarzschild solution written in the ADM form. It was first noticed in the framework of the HL theory in [15], and rederived later by several others. When $\Lambda < 0$, it represents the anti–de Sitter Schwarzschild solution.

2. Anisotropic fluid

When $p_\theta = \gamma p_r$, where γ is a constant but $\gamma \neq 1$, from Eq. (4.12) we find that

$$p_r = c_0 r^{2(\gamma-1)}, \quad (4.14)$$

where c_0 is an integration constant. Substituting it into Eqs. (4.10) and (4.11), for $\Lambda = 0$ we obtain

$$2r\mu' + 1 = -8c_0\pi G e^{-2\mu}r^{2\gamma}, \quad (4.15)$$

$$r^2\mu'' + 2r^2\mu'^2 + 2r\mu' = -8c_0\pi\gamma G e^{-2\mu}r^{2\gamma}, \quad (4.16)$$

from which we find that

$$\mu = \frac{1}{2}\ln\left[\frac{M}{r} + \left(\frac{r}{\ell}\right)^{2\gamma}\right] + \mu_0,$$
$$\mu_0 = -\frac{1}{2}\ln\left(-\frac{1+2\gamma}{8c_0\pi G\ell^{2\gamma}}\right). \quad (4.17)$$

Clearly, when $\gamma = 1$, choosing $c_0 = -\Lambda/(8\pi G)$ and $\ell = \sqrt{3/\Lambda}$, the above solutions reduce exactly to the (anti–) de Sitter Schwarzschild solution (4.13). When $\gamma \neq 1$, for the solutions to be real, we must have

$$\gamma = \begin{cases} < -\frac{1}{2}, & c_0 > 0, \\ > -\frac{1}{2}, & c_0 < 0, \end{cases} \quad (4.18)$$

for $\ell^{2\gamma} > 0$, and

$$\gamma = \begin{cases} > -\frac{1}{2}, & c_0 > 0, \\ < -\frac{1}{2}, & c_0 < 0, \end{cases} \quad (4.19)$$

for $\ell^{2\gamma} < 0$. On the other hand, inserting the above solution into Eq. (4.9) we obtain

$$\mathcal{L}_K = \frac{16\pi Gc_0}{r^{2(1-\gamma)}}. \quad (4.20)$$

Setting

$$\rho(r) = -c_0 r^{2(\gamma-1)} + \tilde{\rho}(r), \quad (4.21)$$

where

$$\tilde{\rho}(r) \simeq \begin{cases} \rho_c + a_0 r^{\beta_0}, & r \to 0, \\ a_\infty r^{\beta_\infty}, & r \to \infty, \end{cases} \quad (4.22)$$

with ρ_c, a_0, a_∞, β_0, and β_∞ being constants, we find that the Hamiltonian constraint (4.3) requires

$$\beta_0 > -3, \qquad \beta_\infty < -3. \quad (4.23)$$

However, to have the center, $r = 0$, free of spacetime singularity, from Eqs. (4.14), (4.21), and (4.22), we find that we must assume that

$$\gamma > 1, \qquad \beta_0 > 0. \quad (4.24)$$

Equation (4.18) then shows that this is possible only when $c_0 < 0$ if $\ell^{2\gamma} > 0$. Hence, the pressures become negative in this case. However, unlike GR [cf. Eq. (B25)], all the three energy conditions: weak, strong and dominant [30], can be satisfied by properly choosing $\tilde{\rho}(r)$, which now is only subject to the global Hamiltonian constraint (4.3). On the other hand, when $\ell^{2\gamma} < 0$, the condition (4.24) can be satisfied for $c_0 > 0$. Then, both the energy density and pressures can be positive. Once again, due to the integral form of the Hamiltonian constraint, one can always choose ρ properly, so that all the three energy conditions can be satisfied. It should be noted that the above conclusions do not contradict with the results obtained in [25], in which perfect fluid with some conditions between ρ and p was considered.

When $r \to \infty$, the pressures become infinitely large, and a spacetime singularity is indicated to exist there. One may cut the spacetime at a finite radius, and then join the solution to the de Sitter Schwarzschild solution, given by Eq. (4.13). We shall consider this issue in Sec. VI.

B. $\xi \neq 0$

When $\xi \neq 0$, to have the system (4.4), (4.5), and (4.6) closed, one additional condition is required. In this paper, we shall take it to be $\nu = 0$. Certainly, other physical conditions might be equally possible. Setting $\nu = 0$, we find that Eq. (4.4) has the general solution,

$$\mu = \ln\left(ar + \frac{b}{r^2}\right), \quad (4.25)$$

where a and b are integration constants. Then, for $\Lambda = 0$ Eqs. (4.5) and (4.6) yield,

$$\begin{aligned} p_r &= \frac{3}{16\pi G}\left[a^2(3\xi - 2) + \frac{2b^2}{r^6}\right], \\ p_\theta &= \frac{3}{16\pi G}\left[a^2(3\xi - 2) - \frac{4b^2}{r^6}\right], \end{aligned} \quad (4.26)$$

while Eq. (4.7) is satisfied identically, as one would expect. On the other hand, for the solution (4.25), we find that

$$\mathcal{L}_K = -6\left(a^2 - \frac{b^2}{r^6}\right). \quad (4.27)$$

Thus, setting

$$\rho = \frac{3a^2}{8\pi G} + \tilde{\rho}(r), \quad (4.28)$$

we find that the Hamiltonian constraint (4.3) reads,

$$\int \tilde{\rho}(r)r^2 dr = 0. \quad (4.29)$$

From Eqs. (4.26) and (4.27) we can see that the spacetime is usually singular at the center, $r = 0$, unless $b = 0$. In the latter case, we have

$$p_r = p_\theta = \frac{3(3\xi - 2)a^2}{16\pi G} = \begin{cases} \geq 0, & \xi \geq 2/3 \\ < 0, & \xi < 2/3. \end{cases} \quad (4.30)$$

That is, now the fluid is a perfect fluid with constant pressures, which are non-negative for $\xi \geq 2/3$. Setting $\tilde{\rho}(r) = 0$, for which the Hamiltonian constraint is satisfied identically, we find that the energy density also becomes a positive constant.

It is interesting to note that $p_r = p_\theta = 0$ when $\xi = 2/3$. In other words, the fluid becomes a dust. In GR, a dust cannot have a static configuration [26]. But, now the field equations involve second-order derivatives of μ (in GR only the first-order terms are involved.), which produce a repulsive force to hold the collapse [31]. If we further choose $\tilde{\rho} = -3a^2/(8\pi G)$ so that $\rho = 0$, the corresponding spacetime becomes vacuum.

V. SOLUTIONS WITH NONZERO CONSTANT CURVATURE

From Eq. (3.7), we find that

$$R = \frac{2e^{-2\nu}}{r^2}[2r\nu' - (1 - e^{2\nu})]. \quad (5.1)$$

When R is a constant, say, k, the above equation can be cast in the form,

$$2r\nu' + \left(1 - \frac{k}{2}r^2\right)e^{2\nu} - 1 = 0, \quad (5.2)$$

which has a particular solution $\nu = 0$, $k = 0$. These are the solutions studied in the last section. Therefore, in the following we shall consider only the case where $\nu' \neq 0$. Then, we find that

$$\nu = -\frac{1}{2}\ln\left(1 - \frac{k}{6}r^2\right), \quad (5.3)$$

for which we have

$$R_{ij} = \frac{k}{3} g_{ij}, \qquad F_{ij} = \mathcal{F}_0 g_{ij},$$

$$\mathcal{L}_V = 2\Lambda - k + \frac{3g_2 + g_3}{3\zeta^2} k^2 + \frac{9g_4 + 3g_5 + g_6}{9\zeta^4} k^3,$$

$$\mathcal{L}_K = \left(1 - \frac{k}{6}r^2\right)e^{2\mu}\left[\xi\mu'^2 - \frac{4(1-\xi)}{r}\mu' - \frac{2(1-2\xi)}{r^2}\right],$$
$$(5.4)$$

where

$$\mathcal{F}_0 \equiv -\Lambda + \frac{k}{6} + \frac{3g_2 + g_3}{18\zeta^2} k^2 + \frac{9g_4 + 3g_5 + g_6}{18\zeta^4} k^3.$$
$$(5.5)$$

To study this case further, we again consider solutions with $\xi = 0$ and $\xi \neq 0$ separately.

A. $\xi = 0$

When $\xi = 0$, we find that the corresponding dynamical equations, momentum constraint and the conservation law can be written, respectively, as

$$4\left(\frac{1}{r} - \frac{k}{6}r\right)\mu' + 2\left(\frac{1}{r^2} + \frac{k}{6}\right) = -2e^{-2\mu}(\mathcal{F}_0 + 8\pi G p_r),$$
$$(5.6)$$

$$\left(1 - \frac{k}{6}r^2\right)(\mu'' + 2\mu'^2) + \left(\frac{2}{r} - \frac{k}{2}r\right)\mu' - \frac{k}{6}$$
$$= -e^{-2\mu}(\mathcal{F}_0 + 8\pi G p_\theta),$$
$$(5.7)$$

$$v = -\frac{k}{24\pi G} e^{2\mu},$$
$$(5.8)$$

$$p_r' + \frac{2}{r}(p_r - p_\theta) = -\frac{ke^{2\mu}}{24\pi G r}(r\mu' + 2).$$
$$(5.9)$$

1. Vacuum solutions

When the spacetime is vacuum, $v = p_r = p_\theta = J^t = 0$, the above equations show that we must have

$$\mu = -\infty, \qquad \mathcal{F}_0 = 0,$$
$$(5.10)$$

for which we have

$$N^r \equiv e^{\mu - v} = 0, \qquad \mathcal{L}_K = 0,$$

$$\Lambda = \frac{k}{6} + \frac{3g_2 + g_3}{18\zeta^2} k^2 + \frac{9g_4 + 3g_5 + g_6}{18\zeta^4} k^3,$$

$$\mathcal{L}_V = -\frac{2k}{3} + \frac{4(3g_2 + g_3)}{9\zeta^2} k^2 + \frac{2(9g_4 + 3g_5 + g_6)}{9\zeta^4} k^3.$$
$$(5.11)$$

The Hamiltonian constraint (3.8) will be satisfied identically if the coupling constants are chosen so that $\mathcal{L}_V = 0$.

These solutions are (Einstein) static universe solutions for either sign of the spatial curvature k [24].

2. Perfect Fluid

For a perfect fluid, Eqs. (5.6) and (5.7) yield,

$$\left(1 - \frac{k}{6}r^2\right)(\mu'' + 2\mu'^2) - \frac{kr}{6}\mu' - \frac{1}{r^2} - \frac{k}{3} = 0. \quad (5.12)$$

Setting

$$\mu = \frac{1}{4}\ln\left(\sqrt{\frac{k}{6}}r\right) + \frac{1}{2}\ln w(z), \qquad z \equiv \sqrt{1 - \frac{k}{6}r^2}, \quad (5.13)$$

we find that Eq. (5.12) can be cast in the form,

$$(1 - z^2)w'' - 2zw' + \left[a(a+1) - \frac{b^2}{1-z^2}\right]w = 0,$$
$$(5.14)$$

with

$$a = -\frac{1}{2} + 2i, \qquad b = \frac{3}{2}. \quad (5.15)$$

The general solution of Eq. (5.14) is given by

$$w = c_1 P_a^b(z) + c_2 Q_a^b(z), \quad (5.16)$$

where c_1 and c_2 are the integration constants and should be chosen so that the solution is real. $P_a^b(z)$ and $Q_a^b(z)$ are, respectively, the associated Legendre functions of the first and second kinds [32]. Inserting Eq. (5.16) into Eq. (5.13), we find that

$$\mu = \frac{1}{4}\ln(r) + \frac{1}{2}\ln[c_1 P_a^b(z) + c_2 Q_a^b(z)] + \mu_0, \quad (5.17)$$

where $\mu_0 \equiv [\ln(k/6)]/8$. Then, Eqs. (3.7), (5.6), and (5.8) yield,

$$\mathcal{L}_K = -\frac{e^{2\mu_0}}{r^{3/2}}\left(1 - \frac{k}{6}r^2\right)\left[3(c_1 P_a^b(z) + c_2 Q_a^b(z))\right.$$
$$\left. - \frac{kr^2}{3\sqrt{1 - \frac{k}{6}r^2}}(c_1 P_a'^b(z) + c_2 Q_a'^b(z))\right],$$

$$p = -\frac{\mathcal{F}_0}{8\pi G} - \frac{\sqrt{r}e^{2\mu_0}}{16\pi G}\left[\left(\frac{3}{r^2} + \frac{k}{6}\right)(c_1 P_a^b(z) + c_2 Q_a^b(z))\right.$$
$$\left. - \frac{k}{3}\sqrt{1 - \frac{k}{6}r^2}(c_1 P_a'^b(z) + c_2 Q_a'^b(z))\right],$$

$$v = -\frac{k\sqrt{r}e^{2\mu_0}}{24\pi G}(c_1 P_a^b(z) + c_2 Q_a^b(z)).$$
$$(5.18)$$

Note that in writing the above expressions we did not impose any conditions obtained from the vacuum case, so that our solutions are as much applicable as possible. In particular, by properly choosing the parameters, we can have $\mathcal{F}_0 < 0$ so the pressure in the center of the star is positive and the fluid satisfies all the energy conditions [See the discussions below.]. On the other hand, setting

$$\rho(r) = \rho_k + \tilde{\rho}(r) - \frac{\mathcal{L}_K}{16\pi G}, \qquad (5.19)$$

we find that the Hamiltonian constraint (3.8) reads

$$\int \frac{\tilde{\rho}(r)r^2 dr}{\sqrt{1 - \frac{k}{6}r^2}} = 0, \qquad (5.20)$$

where

$$\rho_k \equiv \frac{k}{16\pi G}\left(1 - \frac{3g_2 + g_3}{3\zeta^2}k - \frac{9g_4 + 3g_5 + g_6}{9\zeta^4}k^2\right). \qquad (5.21)$$

To study the singular behavior of the solution near the center, we first note that [32]

$$P_a^b(z) = \frac{1}{\Gamma(1-b)}\left(\frac{z+1}{z-1}\right)^{b/2}F\left(-a, a+1; 1-b; \frac{1-z}{2}\right),$$

$$Q_a^b(z) = e^{ib\pi}\left(\frac{z+1}{z-1}\right)^{b/2}\left[\frac{1}{2}\Gamma(b)F\left(-a, a+1; 1-b; \frac{1-z}{2}\right)\right.$$

$$+ \frac{\Gamma(1+a+b)\Gamma(-b)}{2\Gamma(1+a-b)}\left(\frac{z-1}{z+1}\right)^b$$

$$\left. \times F\left(-a, a+1; 1+b; \frac{1-z}{2}\right)\right], \qquad (5.22)$$

for $|1 - z| < 2$, where $F(a, b; c; z)$ denotes the hypergeometric function. When $|z| \ll 1$, it is given by

$$F(a, b; c; z) \simeq 1 + \frac{ab}{c}z + \frac{ab(a+1)(b+1)}{c(c+1)}z^2 + \mathcal{O}(z^3). \qquad (5.23)$$

Thus, as $r \rightarrow 0^+$, we find

$$P_a^b(z) \simeq \frac{a_1}{r^{3/2}}\left\{1 - \frac{17k}{48}r^2 - \frac{425k^2}{4608}r^4 + \mathcal{O}(r^{9/2})\right\},$$

$$Q_a^b(z) \simeq \frac{a_2}{r^{3/2}}\left\{1 - \frac{17k}{48}r^2 + a_3 r^3 + \mathcal{O}(r^{5/2})\right\}, \qquad (5.24)$$

where

$$a_1 \equiv \frac{2^{3/2}}{\Gamma(-\frac{1}{2})(-\frac{k}{6})^{3/4}}, \qquad a_2 \equiv -i\frac{\sqrt{2}\Gamma(\frac{3}{2})}{(-\frac{k}{6})^{3/4}},$$

$$a_3 \equiv \frac{\Gamma(-\frac{3}{2})\Gamma(2+2i)}{\Gamma(\frac{3}{2})\Gamma(-1+2i)}\left(-\frac{k}{24}\right)^{3/2}. \qquad (5.25)$$

Then, from Eqs. (5.18) and (5.19) we find that when

$$a_1 c_1 + a_2 c_2 = 0, \qquad (5.26)$$

all the quantities, $\mathcal{L}_K$, ρ, v, and p are free of singularity at the center $r = 0$. In fact, for such a choice we have

$$\mathcal{L}_K \simeq -6c_2 a_2 a_3 e^{2\mu_0}, \qquad v \simeq 0,$$

$$\rho(r) \simeq \rho_k + \tilde{\rho}(0), \qquad p \simeq -\frac{\mathcal{F}_0}{8\pi G}, \qquad (5.27)$$

as $r \rightarrow 0^+$.

B. $\xi \neq 0$

When $\xi \neq 0$, the dynamical equations and the momentum constraint read, respectively,

$$\xi\left(1 - \frac{k}{6}r^2\right)(2\mu'' + \mu'^2) + \left(\frac{4}{r} - \frac{(2+\xi)k}{3}r\right)\mu'$$

$$+ \frac{2(1-4\xi)}{r^2} + \frac{(1+2\xi)k}{3}$$

$$= -2e^{-2\mu}(\mathcal{F}_0 + 8\pi G p_r), \qquad (5.28)$$

$$\left(1 - \frac{k}{6}r^2\right)[2(1-\xi)\mu'' + (4-3\xi)\mu'^2]$$

$$+ \left[\frac{4(1-2\xi)}{r} - \frac{(3-5\xi)k}{3}r\right]\mu' - \frac{(1-2\xi)k}{3}$$

$$= -2e^{-2\mu}(\mathcal{F}_0 + 8\pi G p_\theta), \qquad (5.29)$$

$$\xi\left(1 - \frac{k}{6}r^2\right)(\mu'' + \mu'^2) + \xi\left(\frac{2}{r} - \frac{k}{2}r\right)\mu'$$

$$- \frac{2\xi}{r^2} + \frac{k}{3} = -8\pi G e^{-2\mu}v. \qquad (5.30)$$

1. Vacuum solutions

In the vacuum case, one of the solutions is still given by

$$\mu = -\infty, \qquad \mathcal{F}_0 = 0, \qquad (5.31)$$

for which we have $N^r = 0 = \mathcal{L}_K$. When μ is finite, from Eqs. (5.28), (5.29), and (5.30) we find that the vacuum equations can be cast in the forms,

$$\xi\left(1 - \frac{k}{6}r^2\right)(2\mu'' + \mu'^2) + \left(\frac{4}{r} - \frac{(2+\xi)k}{3}r\right)\mu'$$

$$+ \frac{2(1-4\xi)}{r^2} + \frac{(1+2\xi)k}{3}$$

$$= -2e^{-2\mu}\mathcal{F}_0, \qquad (5.32)$$

$$\left(1 - \frac{k}{6}r^2\right)[2(1-\xi)\mu'' + (4-3\xi)\mu'^2]$$

$$+ \left[\frac{4(1-2\xi)}{r} - \frac{(3-5\xi)k}{3}r\right]\mu' - \frac{(1-2\xi)k}{3}$$

$$= -2e^{-2\mu}\mathcal{F}_0, \qquad (5.33)$$

and

$$(1 - x^2)\mu_{,xx} + \frac{1}{x}(4 - 5x^2)\mu_{,x} - \frac{3}{x^2} + \frac{2(2-\xi)}{\xi} = 0, \qquad (5.34)$$

where $x \equiv \sqrt{k/6}r$. Equation (5.34) has the general solution,

$$\mu = \mu_0 + \ln(r) + \frac{3(3\xi - 2)}{2\xi k r^2} + \frac{\sqrt{1 - \frac{k}{6}r^2}}{r^3}\left(1 + \frac{k}{3}r^2\right)$$

$$\times \left[\mu_1 + \frac{2 - 3\xi}{4\xi}\arcsin\left(\sqrt{\frac{k}{6}}r\right)\right], \qquad (5.35)$$

where μ_0 and μ_1 are integration constants. Inserting the above into Eq. (5.32), we find that it is satisfied only when

$$\mu_1 = 0, \qquad \xi = \frac{2}{3}, \qquad (5.36)$$

and for which Eq. (5.33) gives $\mathcal{F}_0 = 0$. That is

$$\Lambda - \frac{k}{6} - \frac{3g_2 + g_3}{18\zeta^2}k^2 - \frac{9g_4 + 3g_5 + g_6}{18\zeta^4}k^3 = 0. \quad (5.37)$$

When $\mu_1 = 0$ and $\xi = 2/3$ it can be also shown that $\mathcal{L}_K = 0$. The Hamiltonian constraint is then satisfied identically, when $\mathcal{L}_V = 0$, i.e.,

$$2\Lambda - k + \frac{3g_2 + g_3}{3\zeta^2}k^2 + \frac{9g_4 + 3g_5 + g_6}{9\zeta^4}k^3 = 0, \quad (5.38)$$

as one can see from Eq. (5.4). Therefore, provided that the coupling constants g_n, $(n = 2, 3, \ldots, 6)$ are chosen so that Eqs. (5.37) and (5.38) hold, the solution

$$\nu = -\frac{1}{2}\ln\left(1 - \frac{k}{6}r^2\right), \qquad \mu = \ln(r) + \mu_0, \qquad (\xi = 2/3),$$
$$(5.39)$$

represents *the unique vacuum solution of the HL theory with maximal symmetry for any given curvature k and nonzero ξ*. It is interesting to note that $\xi = 2/3$ is the case where an anisotropic Weyl symmetry exists in the UV limit [1]. Note also that the spacetime described by the solution (5.39) is not flat even in the sense of the 4-dimensional geometry. For example, the corresponding 4-dimensional Ricci scalar is given by,

$$R^{(4)} = 12e^{2\mu_0}\left(1 - \frac{k}{4}r^2\right) + k, \qquad (\xi = 2/3), \quad (5.40)$$

which shows that the spacetime is not flat even when $k = 0$.

2. Perfect Fluid

On the other hand, for a perfect fluid Eqs. (5.28) and (5.29) yield,

$$\left(1 - \frac{k}{6}r^2\right)[(1 - 2\xi)\mu'' + 2(1 - \xi)\mu'^2]$$

$$- \left(\frac{4\xi}{r} + \frac{(1 - 6\xi)k}{6}r\right)\mu' - \frac{1 - 4\xi}{r^2} - \frac{k}{3} = 0. \quad (5.41)$$

Setting

$$\mu = \frac{1 + 2\xi}{4(1 - \xi)}\ln(r) + \frac{1 - 2\xi}{2(1 - \xi)}\ln w(r), \qquad (5.42)$$

we find that Eq. (5.41) can be cast in the form of Eq. (5.14), but now with

$$a = \frac{1 - 2\xi + 4\sqrt{\xi(\xi + 1) - 1}}{2(2\xi - 1)}, \qquad b = \frac{3}{2}. \quad (5.43)$$

Therefore, in the present case the general solution of (5.41) is given by

$$\mu = \frac{1 + 2\xi}{4(1 - \xi)}\ln(r) + \frac{1 - 2\xi}{2(1 - \xi)}\ln[c_1 P_b^a(z) + c_2 Q_b^a(z)],$$
$$(5.44)$$

where, as previously, $z \equiv \sqrt{1 - \frac{k}{6}r^2}$. Once μ is given, from Eqs. (5.28) and (5.30) we can find p and v, which are too complicated to be written explicitly here.

To study the asymptotical behavior of the above solutions near the center, we first notice that $P_a^b(z)$ and $Q_a^b(z)$ take the same forms as those given by Eq. (5.24), as they do not depend explicitly on the parameter a as $r \to 0$. We find that

$$p \simeq -\frac{\mathcal{F}_0}{8\pi G} + \frac{3\xi(1 - 4\xi)}{64\pi G(1 - \xi)^2}\left(\frac{a_1c_1 + a_2c_2}{r^3}\right)^{1-2\xi/1-\xi},$$

$$v \simeq \frac{9\xi(1 - 2\xi)}{32\pi G(1 - \xi)^2}\left(\frac{a_1c_1 + a_2c_2}{r^3}\right)^{1-2\xi/1-\xi},$$

$$\mathcal{L}_K \simeq -\frac{3\xi(5 - 8\xi)}{4(1 - \xi)^2}\left(\frac{a_1c_1 + a_2c_2}{r^3}\right)^{1-2\xi/1-\xi}, \quad (5.45)$$

as $r \to 0$. Thus, when $-1/2 \le \xi < 1$, all these quantities are finite at the center for any given c_1 and c_2, provided that $a_1c_1 + a_2c_2 \ne 0$. When $\xi \ge 1$ or $\xi < 1/2$, they diverge there unless c_1 and c_2 are chosen such that $a_1c_1 + a_2c_2 = 0$. Therefore, in the present case c_1 and c_2 must be chosen so that

$$a_1c_1 + a_2c_2 = \begin{cases} \ne 0, & -1/2 \le \xi < 1, \\ = 0, & \text{otherwise.} \end{cases} \quad (5.46)$$

VI. JUNCTION CONDITIONS

Let us consider a surface Σ, defined by $r = r_0$, in the spacetime described by the metric (3.5), which divides the whole spacetime into two regions, the internal region $r < r_0$, and the external region $r > r_0$, denoted, respectively, by V^- and V^+. Note that once the metric is cast in the form (3.5), the coordinates t and r are all uniquely defined. As a result, the coordinates used in V^+ and V^- must be the same, i.e.,

$$\{x^{+\mu}\} = \{x^{-\mu}\} = (t, r, \theta, \phi). \tag{6.1}$$

Since the highest order of derivatives in the HL theory is six, one may require that the metric coefficients be at least C^6; that is, their derivatives up to six-order exist and are continuous across Σ. However, this requirement often is too strict, and, in particular, will exclude the existence of infinitely thin shells [33]. To relax this condition, from Eqs. (2.6) and (2.15) we can see that the quadratic terms of the highest derivatives are third-order, so we may require that the metric coefficients be at least C^3. In this way we can avoid terms that are powers of Dirac δ-functions, which mathematically are not well defined.

For the spherically static spacetime, this condition is still very strict, since the quadratic terms of the highest derivatives now are only terms involving ν''^2, $\nu''\nu'''$, and μ'^2, as we can see from Eqs. (3.7), (3.8), (3.9), (3.10), (3.11), (3.12), (A1), and (A2). Therefore, without loss of generality, we shall assume that $\nu(r)$ and $\mu(r)$ are at least C^1 and C^0, respectively, across the surface $r = r_0$, and at least C^4 and C^1 elsewhere. Denoting quantities defined in $V^+(V^-)$ by $F^+(F^-)$, we find that μ and ν can be written as

$$F(r) = F^+(r)H(x) + F^-(r)[1 - H(x)], \tag{6.2}$$

where $F = (\mu, \nu)$, $x \equiv r - r_0$ (It must noted that there is no confusion between x used in this section and the one used in Secs. IV and V.),

$$\text{limit}_{r \to r_0^+} \mu^+(r) = \text{limit}_{r \to r_0^-} \mu^-(r),$$

$$\text{limit}_{r \to r_0^+} \nu^+(r) = \text{limit}_{r \to r_0^-} \nu^-(r), \tag{6.3}$$

$$\text{limit}_{r \to r_0^+} \nu_{,r}^+(r) = \text{limit}_{r \to r_0^-} \nu_{,r}^-(r),$$

and $H(x)$ denotes the Heaviside function, defined as

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \tag{6.4}$$

which has the properties [34],

$$H^n(x) = H(x), \qquad [1 - H(x)]^n = [1 - H(x)],$$
$$H(x)[1 - H(x)] = 0, \qquad H'(x) = \delta(x), \tag{6.5}$$

in the sense of distributions, where $\delta(x)$ denotes the Dirac delta function. Although the high-order derivatives of μ and ν are not continuous across the hypersurface $r = r_0$, we assume that they all exist and are finite in the limits $r \to r_0^\pm$. Then, we find that

$$\mu' = \mu'^D, \qquad \mu'' = \mu''^D + [\mu']^-\delta(x), \qquad \nu' = \nu'^D,$$
$$\nu'' = \nu''^D, \qquad \nu''' = \nu'''^D + [\nu'']^-\delta(x),$$
$$\nu^{(4)} = \nu^{(4)D}H + [\nu''']^-\delta(x) + [\nu'']^-\delta'(x),$$
$$\nu^{(5)} = \nu^{(5)D} + [\nu^{(4)}]^-\delta(x) + [\nu''']^-\delta'(x) + [\nu'']^-\delta''(x), \tag{6.6}$$

where

$$[\nu^{(n)}]^- \equiv \text{limit}_{r \to r_0^+} \nu^{+(n)}(r) - \text{limit}_{r \to r_0^-} \nu^{-(n)}(r),$$
$$F^{(n)D} \equiv F^{+(n)}H + F^{-(n)}(1 - H). \tag{6.7}$$

Inserting the above expressions into Eqs. (3.7) and (A1) we find that

$$\mathcal{L}_K = \mathcal{L}_K^D, \qquad \mathcal{L}_V^{(0)} = \mathcal{L}_V^{(0)D}, \qquad \mathcal{L}_V^{(1)} = \mathcal{L}_V^{(1)D},$$
$$\mathcal{L}_V^{(2)} = \mathcal{L}_V^{(2)D},$$
$$\mathcal{L}_V^{(3)} = \mathcal{L}_V^{(3)D} + \mathcal{L}_V^{(3)\text{Im}}$$
$$\equiv \mathcal{L}_V^{(3)D} + \frac{8g_7 e^{-6\nu}}{\zeta^4 r^3}[2r\nu' - (1 - e^{2\nu})][\nu'']^-\delta(x), \tag{6.8}$$

while from Eq. (A2) we find that $(F_n)_{ij}$'s are given by Eq. (A3). The superindex "Im" represents the impulsive part of the quantity considered, which is usually proportional to $\delta(x)$ and its derivatives [cf. Eqs. (6.9) and (A5).]. Separating the nondistributional from the distributional parts of the matter content as

$$J^t = J^{t,D} + J^{t,\text{Im}}, \qquad v = v^D + v^{\text{Im}},$$
$$p_r = p_r^D + p_r^{\text{Im}}, \qquad p_\theta = p_\theta^D + p_\theta^{\text{Im}}, \tag{6.9}$$

we find that the Hamiltonian constraint (3.8) now reads

$$\int (\mathcal{L}_K^D + \mathcal{L}_V^D - 8\pi G J^{t,D})e^\nu r^2 dr$$
$$= \int (8\pi G J^{t,\text{Im}} - \mathcal{L}_V^{(3)\text{Im}})e^\nu r^2 dr. \tag{6.10}$$

While the momentum constraint (3.9) and the dynamical equations (3.11) and (3.12) remain the same in regions V^+ and V^-, but across the thin shell at $r = r_0$, they read

$$\xi[\mu']^-\delta(x) = -8\pi G e^{2(\nu-\mu)}v^{\text{Im}}, \tag{6.11}$$

$$\xi[\mu']^-\delta(x) + e^{-2\mu}F_{rr}^{\text{Im}} = -8\pi G e^{2(\nu-\mu)}p_r^{\text{Im}}, \tag{6.12}$$

$$(1 - \xi)[\mu']^-\delta(x) + \frac{1}{r^2}e^{2(\nu-\mu)}F_{\theta\theta}^{\text{Im}} = -8\pi G e^{2(\nu-\mu)}p_\theta^{\text{Im}}, \tag{6.13}$$

where F_{rr}^{Im} and $F_{\theta\theta}^{\text{Im}}$ are given by Eq. (A4). Assuming that the matter content has distributional contributions no more singular than a δ-function, we see from above that in order to cancel the δ-function derivative terms in F_{rr}^{Im} and $F_{\theta\theta}^{\text{Im}}$, it is sufficient that there is some tuning of the couplings as $g_8 = 8g_7/3$.

It is interesting to note that in the GR limits: $\xi = 0$ and $\zeta \rightarrow \infty$, we have $F_{rr}^{\mathrm{Im}} = F_{\theta\theta}^{\mathrm{Im}} = 0$, and Eqs. (6.11), (6.12), and (6.13) reduce to

$$p_\theta^{\mathrm{Im}} = -\frac{e^{2(\mu-\nu)}}{8\pi G}[\mu']^- \delta(x), \qquad (6.14)$$

$$\nu^{\mathrm{Im}} = p_r^{\mathrm{Im}} = 0, \qquad (\xi = 0, \zeta \rightarrow \infty). \qquad (6.15)$$

Thus, in this limit the radial pressure of the infinitely thin shell always vanishes. This is consistent with the conclusion obtained early by Santos [26].

However, this is no longer true when $\xi \neq 0$ even in the low energy limit where $F_{rr}^{\mathrm{Im}} = F_{\theta\theta}^{\mathrm{Im}} = 0$, as can be seen from Eqs. (6.11), (6.12), and (6.13). In particular, when $\nu = 0$, we find that

$$\xi[\mu']^- e^{2\mu} \delta(x) = -8\pi G \nu^{\mathrm{Im}}, \qquad (6.16)$$

$$\xi[\mu']^- e^{2\mu} \delta(x) = -8\pi G p_r^{\mathrm{Im}}, \qquad (6.17)$$

$$(1 - \xi)[\mu']^- e^{2\mu} \delta(x) = -8\pi G p_\theta^{\mathrm{Im}}. \qquad (6.18)$$

This completes the general description of the junctions of a spherically symmetric star in the HL theory of gravity. In the rest of this section, we shall apply the above general formulas to the solutions found in the last sections. We first notice that solutions with nonzero constant curvature k cannot be matched with the ones with zero constant curvature. This is because in the former the function ν cannot be zero for any given r_0. As a result, ν cannot be continuous across $r = r_0$. Therefore, only the solutions with the same curvature k can be matched to each other. However, since ξ is a running coupling constant, in principle ξ can have different values at different energies. In particular, the spacetime deep inside a very massive star is expected to have a very high temperature, and one would expect that ξ will have different values in the regions inside and outside of the star. Thus, in the following we shall consider the possibility of matching a fluid to a vacuum solution with different ξ. In addition, we shall consider only the match without an infinitely thin shell at $r = r_0$, that is, we shall set

$$\rho^{\mathrm{Im}} = \nu^{\mathrm{Im}} = p_r^{\mathrm{Im}} = p_\theta^{\mathrm{Im}} = 0, \qquad (6.19)$$

which implies that μ and ν must be at least C^1 and C^4, respectively.

A. Spatially Ricci flat solutions

When the spacetime is spatially Ricci flat, in Sec. IV we showed that the de Sitter Schwarzschild solution (4.13) is the unique vacuum solution. Therefore, in this case the spacetime outside the star is uniquely described by this solution,

$$\mu_+ = \frac{1}{2}\ln\left(\frac{M_+}{r} + \frac{\Lambda}{3}r^2\right), \qquad \nu_+ = 0. \qquad (6.20)$$

Inside the star, two solutions were found, one is for $\xi = 0$ given by Eq. (4.17) and the other is for $\xi \neq 0$ given by Eq. (4.25) with $b = 0$. In the following let us consider them separately.

1. $\xi = 0$

When $\xi = 0$, the spacetime inside the star is described by Eq. (4.17), which now can be written as

$$\mu_- = \frac{1}{2}\ln\left[\frac{M_-}{r} + \left(\frac{r}{\ell}\right)^{2\gamma}\right] + \mu_0, \qquad \nu_- = 0. \qquad (6.21)$$

From the above expressions we can see that ν is analytical across $r = r_0$, while the condition that μ being C^1 requires

$$\mu_+(r_0) = \mu_-(r_0), \qquad (6.22)$$

$$\mu_{+,r}(r_0) = \mu_{-,r}(r_0). \qquad (6.23)$$

Inserting Eqs. (6.20) and (6.21) into the above conditions, we find that

$$\frac{3M_+ + \Lambda r_0^3}{M_- + r_0(\frac{r_0}{\ell})^{2\gamma}} = 3e^{2\mu_0}, \qquad (6.24)$$

$$\frac{3M_+ - 2\Lambda r_0^3}{3M_+ + \Lambda r_0^3} = \frac{M_- - 2\gamma r_0(\frac{r_0}{\ell})^{2\gamma}}{M_- + r_0(\frac{r_0}{\ell})^{2\gamma}}, \qquad (6.25)$$

from which we obtain

$$M_+ = \frac{1}{3}e^{2\mu_0}\left[3M_- + 2(1 - \gamma)r_0\left(\frac{r_0}{\ell}\right)^{2\gamma}\right], \qquad (6.26)$$

$$\Lambda = -8\pi G c_0 r_0^{2(\gamma-1)}. \qquad (6.27)$$

Note that the condition of Eq. (6.27) guarantees that the radial pressure is continuous across the surface $r = r_0$, i.e., $p_r(r_0) = p_\Lambda$, as can be seen from Eq. (4.14).

2. $\xi \neq 0$

When $\xi \neq 0$, in Sec. IV, we found the perfect fluid solution given by Eq. (4.25) with $b = 0$, that is,

$$\mu_- = \ln(ar), \qquad \nu_- = 0. \qquad (6.28)$$

The corresponding pressure is given by

$$p = \frac{3(3\xi - 2)a^2}{16\pi G}. \qquad (6.29)$$

It is interesting to note that this solution is exactly the de

Sitter solution in GR. However, in the HL theory it corresponds to a perfect fluid with its pressure given by Eq. (6.29). As shown explicitly in Sec. IV, choosing $\tilde{\rho}(r) = 0$ the energy density becomes $\rho = 3a^2/(8\pi G)$ [cf. Eq. (4.28)], which satisfies all the three energy conditions for $4/9 \leq \xi \leq 4/3$. For such an internal solution, the conditions (6.22) and (6.23) read

$$M_+ = 0, \qquad a = \sqrt{\frac{\Lambda}{3}}. \qquad (6.30)$$

B. Stars with nonzero constant curvature

When the spatial three-curvature R is a nonzero constant, we found two vacuum solutions, one is given by Eq. (5.31) with $\mu = -\infty(N^r = 0)$, and the other is given by Eq. (5.39) with $\xi = 2/3$. The one with $\mu = -\infty$ cannot be matched to any solution with finite μ across $r = r_0$. As a result, the only possible solution that describes the spacetime outside of the star in the present case is the one given by Eq. (5.39),

$$\mu_+ = \ln(r) + \mu_0, \qquad \nu_+ = -\frac{1}{2} \ln\left(1 - \frac{k}{6} r^2\right). \quad (6.31)$$

On the other hand, two perfect fluid solutions were found, one is for $\xi = 0$ given by (5.17), and the other is for $\xi \neq 0$, given by Eq. (5.44). Redefining the integration constants c_1 and c_2 appearing in Eq. (5.17), we find that in both cases the solutions can be written as

$$\mu_- = \frac{1 + 2\xi}{4(1 - \xi)} \ln(r) + \frac{1 - 2\xi}{2(1 - \xi)} \ln[c_1 P_b^a(z) + c_2 Q_b^a(z)],$$

$$\nu_- = -\frac{1}{2} \ln\left(1 - \frac{k}{6} r^2\right). \qquad (6.32)$$

Clearly, in the present case ν is analytical across $r = r_0$, and we have $F_{rr}^{\text{Im}} = F_{\theta\theta}^{\text{Im}} = 0$. The conditions of Eq. (6.19) reduce, then, to those given by Eqs. (6.22) and (6.23). For the solutions given by Eqs. (6.31) and (6.32), those conditions read

$$\mu_0 = \frac{1 - 2\xi}{2(1 - \xi)} \ln\frac{c_1 P_a^b(z_0) + c_2 Q_a^b(z_0)}{r_0^{3/2}}, \qquad (6.33)$$

$$\frac{c_1 P_a'^b(z_0) + c_2 Q_a'^b(z_0)}{c_1 P_a^b(z_0) + c_2 Q_a^b(z_0)} = \frac{9z_0}{kr_0^2}, \qquad (6.34)$$

where $z_0 \equiv z(r_0)$. Clearly, by properly choosing the free parameters involved in this model, the above equations will be satisfied.

VII. CONCLUSIONS

In this paper, we have systematically studied spherically symmetric static spacetimes filled with a fluid in the HL theory of gravity with projectability, but without detailed balance conditions.

After writing down the relevant field equations coupled with a fluid (including the Hamiltonian, momentum constraints, dynamical equations, and conservation laws) in Sec. III, we systematically studied spatially Ricci flat spacetimes in Sec. IV, and spacetimes with nonzero constant curvature in Sec. V, for both cases where the spacetimes are vacuum and filled with a fluid. In particular, in Sec. IV we showed that the de Sitter Schwarzschild solution is the unique vacuum solution that is spatially flat. In this section, we found two classes of solutions coupled with a fluid. The first class, given by Eq. (4.17), represents spacetimes filled with an anisotropic fluid in which the tangential pressure is proportional to its radial pressure, given by Eq. (4.14). The second class is given by Eq. (4.25) with $b = 0$, which represents a perfect fluid with constant pressure. This class of solutions actually describes the de Sitter space, but corresponds to a perfect fluid with positive energy density and pressure. This is in contrast to GR, where de Sitter space does not satisfy the strong energy condition [30]. The main reason is that in the HL theory the Hamiltonian constraint becomes a global one, and has less constraint on the energy density. When $\xi = 2/3$, the pressure vanishes thus representing dust. In GR, dust cannot have a static configuration and it necessarily develops spacetime singularities [26]. In the HL theory, higher order derivatives are present, and it is exactly the existence of these terms that produce repulsive forces, which prevent the collapse of the dust. This provides another concrete example in which a would-be caustic is regularized by the repulsive gravitational forces, created from the gradients of high order derivatives of curvature [31].

In Sec. V, we found that there are two different vacuum solutions for spacetimes with nonzero constant curvature. One is an (Einstein) static universe, given by Eq. (5.10) or (5.31), and the other is given by Eq. (5.39), which has the maximal symmetry and is not flat. The general solutions for a perfect fluid was found explicitly, and are given, respectively, by Eq. (5.17) for $\xi = 0$, and Eq. (5.44) for $\xi \neq 0$.

To construct spacetimes that represent stars, we investigated the junction conditions across the surfaces of stars in Sec. VI, and obtained the general junction conditions with/without infinitely thin shells. It is remarkable that, in contrast to GR [26], the radial pressure of the star does not necessarily vanish on the surface of the star, neither does the radial pressure of the thin shell. This is due to the high order derivatives of the spacetime curvature. As a result, a star can be formed much easily than that in GR. Applying those general formulas to the solutions found in Secs. IV

and V, we showed that this is indeed the case. In particular, all the internal solutions have a nonzero radial pressure on the surface of the star but it is still possible for them to be matched smoothly to a vacuum spacetime without the presence of an infinitely thin shell on the surface.

In Appendix B, we studied anisotropic fluid with heat flow in general relativity with the metric written in an ADM form. Among other things, we showed explicitly that any given static solution written in an orthogonal form (B10) can be always brought into an ADM form (B12) with the projectability condition by the coordinate transformations Eq. (B11). However, such coordinate transformations are not allowed by the restricted diffeomorphisms, (2.4), of the HL theory. In particular, the extrinsic curvature tensor K_{ij} and the 3-dimensional Ricci tensor R_{ij} are no longer tensors under the transformations given by Eq. (B11). As a result, any actions constructed from K_{ij} and R_{ij}, such as the one given by (2.5), are in general not invariant and therefore the transformed metric is not a solution of the projectable theory. In particular, a vacuum solution is no longer vacuum in the new frame with the projectability condition [35].

Finally, let us comment on another delicate issue, namely, the singular behavior of the extrinsic curvature K for some of the solutions discussed in this paper. We have not used it to identify spacetime singularities in the present paper, because the singularities given by K seem not to be as serious as those given by other quantities, such as the Ricci curvature R, energy density ρ and pressure p.

For example, Cai and one of the present authors found in [35] that K is singular at $r = (3M/|\Lambda|)^{1/3}$ for the anti–de Sitter Schwarzschild solution. This singularity is absent in general relativity and the tidal forces and distortions felt by observers at these singularities are all finite. Therefore, it is not clear whether spacetimes beyond these points are extendable or not [36]. It is exactly due to these considerations that we did not use K to identify spacetime singularities, although the singularities of K are scalar ones and cannot be removed by the foliation-preserving diffeomorphisms [35]. In fact, in Appendix C we showed explicitly for which solutions found in this paper K is regular or singular at the center. Understanding the nature of singularities of K is an important issue in the HL theory, and we wish to return to this problem in the near future.

ACKNOWLEDGMENTS

APPENDIX A: FUNCTIONS $\mathcal{L}_V^{(n)}$ AND $(F_s)_{ij}$

The Lagrangians $\mathcal{L}_V^{(n)}$'s in Eq. (2.7) for the static spherically symmetric spacetime (3.5) are given by

$$\mathcal{L}_V^{(0)} = 2\Lambda - \frac{2e^{-2\nu}}{r^2}[2r\nu' - (1 - e^{2\nu})],$$

$$\mathcal{L}_V^{(1)} = \frac{2e^{-4\nu}}{\zeta^2 r^4}\{2g_2[2r\nu' - (1 - e^{2\nu})]^2 + g_3[3r^2\nu'^2 - 2r(1 - e^{2\nu})\nu' + (1 - e^{2\nu})^2]\},$$

$$\mathcal{L}_V^{(2)} = \frac{2e^{-6\nu}}{\zeta^4 r^6}\{4g_4[2r\nu' - (1 - e^{2\nu})]^3 + 2g_5[6r^3\nu'^3 - 7r^2(1 - e^{2\nu})\nu'^2 + 4r(1 - e^{2\nu})^2\nu' - (1 - e^{2\nu})^3]$$
$$+ g_6[5r^3\nu'^3 - 3r^2(1 - e^{2\nu})\nu'^2 + 3r(1 - e^{2\nu})^2\nu' - (1 - e^{2\nu})^3]\},$$

$$\mathcal{L}_V^{(3)} = \frac{2e^{-6\nu}}{\zeta^4 r^6}\{4g_7[2r^4\nu'(\nu''' - 7\nu'\nu'' + 6\nu'^3) - r^3[(1 - e^{2\nu})\nu''' - (9 - 7e^{2\nu})\nu'\nu'' + 2(5 - 3e^{2\nu})\nu'^3]$$
$$- r^2[(1 - e^{2\nu})\nu'' + 4\nu'^2] + r(1 - e^{2\nu})^2\nu' + (1 - e^{2\nu})^2] + g_8[3r^4[(\nu'' - 4\nu'^2)\nu'' + 4\nu'^4]$$
$$- 2r^3(\nu'' - 2\nu'^2)\nu' + r^2[4(1 - e^{2\nu})\nu'' - (3 - 8e^{2\nu})\nu'^2] + 8r(1 - e^{2\nu})\nu' + 6(1 - e^{2\nu})^2]\}. \tag{A1}$$

The functions $(F_s)_{ij}$ defined by Eq. (2.15) are given by

$$(F_0)_{ij} = -\frac{1}{2}e^{2\nu}\delta_i^r\delta_j^r - \frac{1}{2}r^2\Omega_{ij}, \qquad (F_1)_{ij} = \frac{1}{r^2}(1 - e^{2\nu})\delta_i^r\delta_j^r - re^{-2\nu}\nu'\Omega_{ij},$$

$$(F_2)_{ij} = \frac{e^{-2\nu}}{r^4}[4r^2(2\nu'' - 3\nu'^2) + (1 - e^{2\nu})(7 + e^{2\nu})]\delta_i^r\delta_j^r$$
$$+ \frac{2e^{-4\nu}}{r^2}[4r^3(2\nu''' - 7\nu'\nu'' + 6\nu'^3) - 2r\nu'(7 - 3e^{2\nu}) - (1 - e^{2\nu})(7 + e^{2\nu})]\Omega_{ij},$$

$$(F_3)_{ij} = \frac{e^{-2\nu}}{r^4}[3r^2(2\nu'' - 3\nu'^2) + (1 - e^{2\nu})(5 + e^{2\nu})]\delta_i^r\delta_j^r$$
$$+ \frac{e^{-4\nu}}{r^2}[3r^3(\nu''' - 7\nu'\nu'' + 6\nu'^3) - 2r\nu'(5 - 2e^{2\nu}) - (1 - e^{2\nu})(5 + e^{2\nu})]\Omega_{ij},$$

$$(F_4)_{ij} = \frac{4e^{-4\nu}}{r^6}[16r^3\nu'(3\nu'' - 5\nu'^2) - 12r(1 - e^{2\nu})(2r\nu'' - 3r\nu'^2 - 4\nu') - (1 - e^{2\nu})(23 - 22e^{2\nu} - e^{4\nu})]\delta_i^r\delta_j^r$$
$$+ \frac{4e^{-6\nu}}{r^4}[24r^4[\nu'\nu''' + (\nu'' - 11\nu'^2)\nu'' + 10\nu'^4] - 4r^3(17 - 18e^{2\nu})\nu'^3 - 12r^2(15 - 11e^{2\nu})\nu'^2$$
$$- (1 - e^{2\nu})[12r^3(\nu''' - 7\nu'\nu'') - 48r^2\nu'' + 3r(1 + 7e^{2\nu})\nu' - 2(1 - e^{2\nu})(23 + e^{2\nu})]]\Omega_{ij},$$

$$(F_5)_{ij} = \frac{2e^{-4\nu}}{r^6}\{12r^4[\nu'(\nu''' - 11\nu'\nu'' + 10\nu'^3) + \nu''^2] - 4r^3[(1 - e^{2\nu})(\nu''' - 7\nu'\nu'' + 6\nu'^3) - \nu'(3\nu'' - 2\nu'^2)]$$
$$+ r^2(1 - e^{2\nu})(2\nu'' - 15\nu'^2) + 4r(1 - e^{2\nu})(13 - 2e^{2\nu})\nu' + (1 - e^{2\nu})^2(23 - e^{2\nu})\}\delta_i^r\delta_j^r$$
$$+ \frac{2e^{-6\nu}}{r^4}\{18r^4[\nu'\nu''' + \nu''^2 - \nu'^2(11\nu'' - 10\nu'^2)] - r^3[7(1 - e^{2\nu})\nu''' - (53 - 49e^{2\nu})\nu'\nu'' + (45 - 42e^{2\nu})\nu'^3]$$
$$+ r^2[24(1 - e^{2\nu})\nu'' - (97 - 69e^{2\nu})\nu'^2] + r(1 - e^{2\nu})(13 - 15e^{2\nu})\nu' + 2(1 - e^{2\nu})^2(13 + e^{2\nu})\}\Omega_{ij},$$

$$(F_6)_{ij} = \frac{e^{-4\nu}}{r^6}\{10r^3(3\nu'' - 5\nu'^2)\nu' - 3r^2[2(1 - e^{2\nu})\nu'' + 3e^{2\nu}\nu'^2] + 12r(1 - e^{2\nu})\nu' - (1 - e^{2\nu})^2(14 + e^{2\nu})\}\delta_i^r\delta_j^r$$
$$+ \frac{e^{-6\nu}}{r^4}\{15r^4[\nu'\nu''' + (\nu'' - \nu'^2)(\nu'' - 10\nu'^2)] - r^3[3(1 - e^{2\nu})\nu''' - 3(1 - 7e^{2\nu})\nu'\nu'' - (25 + 18e^{2\nu})\nu'^3]$$
$$+ 3r^2[4(1 - e^{2\nu})\nu'' - (12 - 11e^{2\nu})\nu'^2] + 12r(1 - e^{2\nu})(2 - e^{2\nu})\nu' + 2(1 - e^{2\nu})^2(14 + e^{2\nu})\}\Omega_{ij},$$

$$(F_7)_{ij} = \frac{8e^{-4\nu}}{r^6}\{r^4[-2\nu^{(4)} + 20\nu'\nu''' + (15\nu'' - 82\nu'^2)\nu'' + 40\nu'^4] + 2r^2[2(3 - e^{2\nu})\nu'' - 3(5 - e^{2\nu})\nu'^2]$$
$$- 8r(3 - e^{2\nu})\nu' - (1 - e^{2\nu})(7 + e^{2\nu})\}\delta_i^r\delta_j^r + \frac{8e^{-6\nu}}{r^4}\{r^5[-\nu^{(5)} + 16\nu'\nu^{(4)} + (25\nu'' - 101\nu'^2)\nu'''$$
$$- (127\nu'' - 326\nu'^2)\nu'\nu'' - 120\nu'^5] + 2r^3[(3 - e^{2\nu})\nu''' - (33 - 7e^{2\nu})\nu'\nu'' + (45 - 6e^{2\nu})\nu'^3]$$
$$- 2r^2[4(3 - e^{2\nu})\nu'' - (51 - 11e^{2\nu})\nu'^2] + r(57 - 24e^{2\nu} - e^{4\nu})\nu' + 2(1 - e^{2\nu})(7 + e^{2\nu})\}\Omega_{ij},$$

$$(F_8)_{ij} = \frac{e^{-4\nu}}{r^6}\{r^4[6\nu^{(4)} - 68\nu'\nu''' - (59\nu'' - 358\nu'^2)\nu'' - 224\nu'^4] + 2r^3(13\nu'' - 29\nu'^2)\nu' - r^2[8(5 - 2e^{2\nu})\nu''$$
$$- 7(13 - 4e^{2\nu})\nu'^2] + 16r(4 - e^{2\nu})\nu' + 6(1 - e^{2\nu})(1 + 3e^{2\nu})\}\delta_i^r\delta_j^r + \frac{e^{-6\nu}}{r^4}\{3r^5[\nu^{(5)} - 16\nu'\nu^{(4)}$$
$$- (25\nu'' - 101\nu'^2)\nu''' + (127\nu'' - 326\nu'^2)\nu'\nu'' + 120\nu'^5] + r^4[\nu'\nu''' - (5\nu'' - 13\nu'^2)\nu'' - 14\nu'^4]$$
$$- r^3[2(7 - e^{2\nu})\nu''' - 2(78 - 7e^{2\nu})\nu'\nu'' + 2(107 - 6e^{2\nu})\nu'^3] + r^2[8(7 - e^{2\nu})\nu'' - (277 - 30e^{2\nu})\nu'^2]$$
$$- 16r(13 - 7e^{2\nu})\nu' - 6(1 - e^{2\nu})(11 - 3e^{2\nu})\}\Omega_{ij}, \tag{A2}$$

where we denoted $\nu^{(n)} \equiv d^n\nu/dr^n$. Inserting Eq. (6.6) into the above expressions, we find that

$$(F_0)_{ij} = (F_0)_{ij}^D, \qquad (F_1)_{ij} = (F_1)_{ij}^D, \qquad (F_2)_{ij} = (F_2)_{ij}^D + 16re^{-4\nu}[\nu'']^-\delta(x)\Omega_{ij},$$

$$(F_3)_{ij} = (F_3)_{ij}^D + 3re^{-4\nu}[\nu'']^-\delta(x)\Omega_{ij}, \qquad (F_4)_{ij} = (F_4)_{ij}^D + \frac{48e^{-6\nu}}{r}[2r\nu' - (1 - e^{2\nu})][\nu'']^-\delta(x)\Omega_{ij},$$

$$(F_5)_{ij} = (F_5)_{ij}^D + \frac{8e^{-4\nu}}{r^3}[3r\nu' - (1 - e^{2\nu})][\nu'']^-\delta(x)\delta_i^r\delta_j^r + \frac{2e^{-6\nu}}{r}[18r\nu' - 7(1 - e^{2\nu})][\nu'']^-\delta(x)\Omega_{ij},$$

$$(F_6)_{ij} = (F_6)_{ij}^D + \frac{3e^{-6\nu}}{r}[5r\nu' - (1 - e^{2\nu})][\nu'']^-\delta(x)\Omega_{ij}, (F_7)_{ij}$$

$$= (F_7)_{ij}^D - \frac{16e^{-4\nu}}{r^2}\{[[\nu^{(3)}]^- - 10\nu'[\nu'']^-]\delta(x) + [\nu'']^-\delta'(x)\}\delta_i^r\delta_j^r - \frac{8e^{-6\nu}}{r}\{[r^2[\nu^{(4)}]^- - 16r^2\nu'[\nu^{(3)}]^- \qquad (A3)$$

$$- r^2(25\{\nu''\}^+ - 101\nu'^2)[\nu'']^- - 2(3 - e^{2\nu})[\nu'']^-]\delta(x) + r^2[[\nu^{(3)}]^- - 16\nu'[\nu'']^-]\delta'(x)$$

$$+ r^2[\nu'']^-\delta''(x)\}\Omega_{ij}, (F_8)_{ij}$$

$$= (F_8)_{ij}^D + \frac{2e^{-4\nu}}{r^2}\{[3[\nu^{(3)}]^- - 34\nu'[\nu'']^-]\delta(x) + 3[\nu'']^-\delta'(x)\}\delta_i^r\delta_j^r + \frac{e^{-6\nu}}{r}\{[3r^2[\nu^{(4)}]^- - 48r^2\nu'[\nu^{(3)}]^-$$

$$- 3r^2(25\{\nu''\}^+ - 101\nu'^2)[\nu'']^- + (r\nu' - 14 + 2e^{2\nu})[\nu'']^-]\delta(x) + 3r^2[[\nu^{(3)}]^- - 16\nu'[\nu'']^-]\delta'(x)$$

$$+ 3r^2[\nu'']^-\delta''(x)\}\Omega_{ij},$$

where

$$\{\nu''\}^+ \equiv \frac{1}{2}[\text{limit}_{r\to r_0^+}\nu^{+''}(r) + \text{limit}_{r\to r_0^-}\nu^{-''}(r)], \qquad (F_n)_{ij}^D \equiv (F_n^+)_{ij}H(x) + (F_n^-)_{ij}[1 - H(x)], \qquad (A4)$$

with $x \equiv r - r_0$. Thus, we find that

$$F_{rr}^{\text{Im}} = \frac{2e^{-4\nu}}{\zeta^4 r^2}\left\{\left[4g_5\left(3\nu' - \frac{1}{r}(1 - e^{2\nu})\right)[\nu'']^- - 8g_7([\nu^{(3)}]^- - 10\nu'[\nu'']^-) + g_8(3[\nu^{(3)}]^- - 34\nu'[\nu'']^-)\right]\delta(x)\right.$$

$$\left. - (8g_7 - 3g_8)[\nu'']^-\delta'(x)\right\},$$

$$F_{\theta\theta}^{\text{Im}} = \frac{(16g_2 + 3g_3)re^{-4\nu}}{\zeta^2}[\nu'']^-\delta(x) + \frac{e^{-6\nu}}{r\zeta^4}\{48g_4[2r\nu' - (1 - e^{2\nu})][\nu'']^- + 2g_5[18r\nu' - 7(1 - e^{2\nu})][\nu'']^-$$

$$+ 3g_6[5r\nu' - (1 - e^{2\nu})][\nu'']^- - 8g_7[r^2[\nu^{(4)}]^- - 16r^2\nu'[\nu^{(3)}]^- - r^2(25\{\nu''\}^+ - 101\nu'^2)[\nu'']^-$$

$$- 2(3 - e^{2\nu})[\nu'']^-] + g_8[3r^2[\nu^{(4)}]^- - 48r^2\nu'[\nu^{(3)}]^- - 3r^2(25\{\nu''\}^+ - 101\nu'^2)[\nu'']^-$$

$$+ (r\nu' - 14 + 2e^{2\nu})[\nu'']^-]\}\delta(x) - \frac{(8g_7 - 3g_8)r}{\zeta^4 e^{6\nu}}([\nu^{(3)}]^- - 16\nu'[\nu'']^-)\delta'(x) - \frac{(8g_7 - 3g_8)r}{\zeta^4 e^{6\nu}}[\nu'']^-\delta''(x). \qquad (A5)$$

APPENDIX B: SPHERICALLY SYMMETRIC AND STATIC SPACETIMES IN GENERAL RELATIVITY

In this appendix we will see how static spherically symmetric metrics in theories with unbroken diffeomorphism invariance can always be brought to a projectable form. Furthermore, we specialize our discussion to general relativity and see how the equations of motion read in the new gauge.

The metric for spacetimes with spherical symmetry takes the general form,

$$ds^2 = g_{ab}dx^a dx^b + R^2 d\Omega^2, \qquad (B1)$$

where $a, b = 0, 1$, and g_{ab} and R are all functions of x^0 and x^1, and $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. The four-velocity of a fluid moving radially in such a spacetime usually has

only two nonvanishing components,

$$u_\mu = (u_0, u_1, 0, 0), \qquad (\mu = 0, 1, 2, 3) \qquad (B2)$$

subject to the condition,

$$u_\lambda u^\lambda = -1. \qquad (B3)$$

Clearly, the metric (B1) is invariant under the coordinate transformations,

$$x^0 = f(x'^0, x'^1), \qquad x^1 = g(x'^0, x'^1), \qquad (B4)$$

where f and g are arbitrary functions of their indicated arguments. Using one degree of the freedom, one usually sets $g_{01} = 0$. When one considers a fluid, one often uses the other degree of freedom to choose the coordinates to be comoving with the fluid, so that the four-velocity of the

fluid is given by $u_\mu \propto \delta_\mu^0$. Then, in this gauge we have

$$ds^2 = -e^{2\Psi(\tau,r)}d\tau^2 + e^{2\Phi(\tau,r)}dr^2 + R^2(\tau, r)d\Omega^2, \quad \text{(B5)}$$

with

$$u_\mu = e^\Psi \delta_\mu^\tau. \quad \text{(B6)}$$

An anisotropic fluid with heat moving along the radial direction takes the form,

$$T_{\mu\nu} = \rho_o u_\mu u_\nu + p_R r_\mu r_\nu + p_\theta(\theta_\mu \theta_\nu + \phi_\mu \phi_\nu)$$
$$+ q_o(u_\mu r_\nu + u_\nu r_\mu), \quad \text{(B7)}$$

where r_μ, θ_μ and ϕ_μ are unit vectors, defined by

$$r_\mu = e^\Phi \delta_\mu^r, \qquad \theta_\mu = R\delta_\mu^\theta, \qquad \phi_\mu = R\sin\theta \delta_\mu^\phi. \quad \text{(B8)}$$

ρ_o, p_R, p_θ and q_o are, respectively, the energy density, radial pressure, tangential pressure, and heat of the fluid comoving in the orthonormal frame. Note that the metric (B5) is still invariant under the rescaling,

$$\tau = \tilde{f}(\tau'), \qquad r = \tilde{g}(r'), \quad \text{(B9)}$$

where $\tilde{f}$ and $\tilde{g}$ are arbitrary functions of their indicated arguments.

When the spacetime is static, Ψ, Φ and R become functions of r only. Then, using the remaining gauge freedom (B9) we can always set $R(r) = r$, so that the metric finally reads

$$ds^2 = -e^{2\Psi(r)}d\tau^2 + e^{2\Phi(r)}dr^2 + r^2 d\Omega^2. \quad \text{(B10)}$$

Let us now make the coordinate transformations,

$$\tau = t - \int^r \sqrt{e^{-2\Psi} - 1}\, e^\Phi dr. \quad \text{(B11)}$$

Then, in terms of t, the above metric takes explicitly the canonical ADM form with the projectability condition,

$$ds^2 = -dt^2 + (e^{\mu(r)}dt + e^{\nu(r)}dr)^2 + r^2 d\Omega^2, \quad \text{(B12)}$$

with

$$\Phi(r) = \nu(r) - \frac{1}{2}\ln(1 - e^{2\mu}), \qquad \Psi(r) = \frac{1}{2}\ln(1 - e^{2\mu}),$$

$$u_\mu = \sqrt{1 - e^{2\mu}}\delta_\mu^t - \frac{e^{\mu+\nu}}{\sqrt{1 - e^{2\mu}}}\delta_\mu^r,$$

$$r_\mu = \frac{e^\nu}{\sqrt{1 - e^{2\mu}}}\delta_\mu^r, \qquad u^\mu = -\frac{1}{\sqrt{1 - e^{2\mu}}}\delta_t^\mu,$$

$$r^\mu = \frac{e^\mu}{\sqrt{1 - e^{2\mu}}}\delta_t^\mu + e^{-\nu}\sqrt{1 - e^{2\mu}}\delta_r^\mu. \quad \text{(B13)}$$

Clearly, to have the coordinate transformations be real, we must assume

$$e^{2\Psi} \leq 1. \quad \text{(B14)}$$

Ψ is often written as [37]

$$e^{2\Psi} = 1 - \frac{2m(r)}{r}, \quad \text{(B15)}$$

where $m(r)$ represents the gravitational mass within the shell r. When $m(r) \geq 0$, the condition (B14) is satisfied identically.

It should be noted that the coordinate transformations given by Eq. (B11) are not allowed by the foliation-preserving diffeomorphisms (2.4). In particular, the action of Eq. (2.5) is not invariant, because now the extrinsic curvature and Ricci tensors K_{ij} and R_{ij} no longer behave like tensors under these transformations.

Let us note here that the definitions of the energy density ρ_o, the radial pressure p_R and the heat flow q_o are different from the ones (ρ_H, p_r, q) given by Eq. (3.17), which are defined by assuming that the fluid is comoving with respect to the canonical ADM frame (B12). The relation between the two sets of quantities is the following

$$\rho_H = \frac{1}{1 - e^{2\mu}}(\rho_o + e^{2\mu}p_R - 2e^\mu q_o),$$

$$p_r = \frac{1}{1 - e^{2\mu}}(p_R + e^{2\mu}\rho_o - 2e^\mu q_o), \quad \text{(B16)}$$

$$q = \frac{1}{1 - e^{2\mu}}[(1 + e^{2\mu})q_o - e^\mu(\rho_o + p_R)].$$

The nonvanishing components of the Einstein tensor for the metric (B7) are given by

$$G_{00} = \frac{(1 - e^{2\mu})e^{2(\mu-\nu)}}{r^2}[2r\mu' + (1 - e^{-2\mu})(1 - 2r\nu')$$
$$+ e^{2(\nu-\mu)}],$$

$$G_{01} = -\frac{e^{3\mu-\nu}}{r^2}[2r\mu' + (1 - e^{-2\mu})(1 - 2r\nu') + e^{2(\nu-\mu)}],$$

$$G_{11} = -\frac{e^{2\mu}}{r^2}[2r(\mu' - \nu') + (1 - e^{-2\mu}) + e^{2(\nu-\mu)}],$$

$$G_{22} = -re^{2(\mu-\nu)}[r(\mu'' + 2\mu'^2 - \nu'\mu') + 2\mu'$$
$$- (1 - e^{-2\mu})\nu']. \quad \text{(B17)}$$

Then, for an anisotropic fluid (B7), the Einstein field equations, $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, yield,

$$2r\mu' - 2r(1 - e^{-2\mu})\nu' - (1 - e^{2\nu})e^{-2\mu} + 1$$
$$= 8\pi G r^2 e^{2(\nu-\mu)}\rho_o, [G_{00}], \quad \text{(B18)}$$

$$(1 - e^{-2\mu})\nu' = -4\pi G r e^{2(\nu-\mu)}(\rho_o + p_R - 2e^\mu q_o),$$
$$[e^{2\nu}G_{00} + (1 - e^{2\mu})G_{11}], \quad \text{(B19)}$$

$$r(\mu'' + 2\mu'^2 - \nu'\mu') + 2\mu' - (1 - e^{-2\mu})\nu'$$
$$= -8\pi G r e^{2(\nu-\mu)}p_\theta, [G_{22}], \quad \text{(B20)}$$

$$q_o = 0, [e^{\mu+\nu}G_{00} + (1 - e^{2\mu})G_{01}]. \quad \text{(B21)}$$

The last equation shows clearly that in GR heat flow along radial direction is not allowed in static spherically symmetric spacetimes.

The conservation laws $\nabla^\nu T_{\nu\mu} = 0$, on the other hand, give

$$2rq_o\mu' + (1 - e^{-2\mu})(rq_o' + 2q_o) = 0, \tag{B22}$$

$$\begin{aligned}
[(\rho_o + p_R) - 2e^\mu q_o]\mu' + (1 - e^{-2\mu})[(p_R' - e^\mu q_o') \\
+ \frac{2}{r}(p_R - p_\theta - e^\mu q_o)] = 0.
\end{aligned} \tag{B23}$$

For $q_o = 0$, Eq. (B22) is satisfied identically, while Eq. (B23) reduces to

$$(\rho_o + p_R)\mu' + (1 - e^{-2\mu})\left[p_R' + \frac{2}{r}(p_R - p_\theta)\right] = 0,$$

$$(q_o = 0). \tag{B24}$$

When $\nu = 0$, from Eq. (B19) we see that $p_R = -\rho_o$. If in addition we have a perfect fluid $p_R = p_\theta$, from Eq. (B24) we find that $p_R' = 0$, that is, the pressure is constant. Then, from Eq. (B18) we find that μ is exactly given by Eq. (4.13), which is identically the de Sitter Schwarzshcild solution written in the ADM form [38].

When $\nu = 0$ and $p_\theta = \gamma p_R$, from Eqs. (B18)–(B24) we find that

$$\mu = \frac{1}{2}\ln\left[\frac{M}{r} + \left(\frac{r}{\ell}\right)^{2\gamma}\right] + \mu_0,$$

$$\rho_o = -p_R = -\gamma^{-1}p_\theta = -c_0 r^{2(\gamma-1)}, \tag{B25}$$

where μ_0 is given by Eq. (4.17), and c_0 is a constant. To have ρ_o non-negative we must assume $c_0 < 0$, while finiteness at the center requires $\gamma > 1$. As a result, we find that $\rho_o + p_\theta = (1 - \gamma)\rho_o < 0$, that is, the fluid does not satisfies the weak energy condition [30]. In fact, it does not satisfies any of the three energy conditions.

APPENDIX C: SINGULAR BEHAVIOR OF EXTRINSIC CURVATURE K

For the static spacetimes described by metric (3.5), we have

$$K = e^{\mu-\nu}\left(\mu' + \frac{2}{r}\right). \tag{C1}$$

For the (anti–)de Sitter Schwarzschild solutions (4.13), K is given by [35]

$$K = \sqrt{\frac{3M + \Lambda r^3}{12r^3}}\left(4 - \frac{3M - 2\Lambda r^3}{3M + \Lambda r^3}\right). \tag{C2}$$

As noticed in [35], K is singular for the anti–de Sitter Schwarzschild solution ($\Lambda < 0$) not only at the center $r = 0$ but also at $r = (3M/|\Lambda|)^{1/3}$. The latter singularity is absent in GR.

For the Ricci-flat solutions given by Eq. (4.17), we have

$$K = \frac{e^{\mu_0}}{2r^{3/2}[M + r(\frac{r}{\ell})^{2\gamma}]^{1/2}}\left\{3M + 2r(2 + \gamma)\left(\frac{r}{\ell}\right)^{2\gamma}\right\}, \tag{C3}$$

which is singular at the center, unless $M = 0$ and $\gamma \geq 1$.

For the solutions of Eq. (4.25) with $b = 0$, we find that

$$K = 3a, \tag{C4}$$

and therefore everywhere finite.

For the solutions of Eq. (5.17), we find that

$$K = 3e^{\mu_0}\sqrt{c_2 a_2 a_3}(1 + \mathcal{O}(r)), \tag{C5}$$

and therefore finite at $r = 0$.

For the solutions of Eq. (5.39), we find that

$$K = 3e^{\mu_0}\sqrt{1 - \frac{k}{6}r^2}, \tag{C6}$$

and therefore everywhere finite.

For the solutions of Eq. (5.44), we find that as $r \to 0$

$$K \simeq \frac{3}{2(1 - \xi)}(a_1 c_1 + a_2 c_2)^{1-2\xi/2(1-\xi)} r^{3(2\xi-1)/2(1-\xi)}, \tag{C7}$$

with nonsingular corrections. Thus, when $a_1 c_1 + a_2 c_2 \neq 0$ and $1/2 \leq \xi < 1$, K is finite at the center. For $a_1 c_1 + a_2 c_2 = 0$ one obtains the same condition for ξ, but these solutions are forbidden from the regularity condition (5.46).

We have not provided here the singular behavior of the invariant $K_{ij}K^{ij}$, but it turns out to have similar behavior as the one of the trace of the extrinsic curvature K.

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