Black hole lasers, a mode analysis

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We show that the black hole laser effect discovered by Corley and Jacobson should be described in terms of frequency eigenmodes that are spatially bound. The spectrum contains a discrete and finite set of complex frequency modes, which appear in pairs and which encode the laser effect. In addition, it contains real frequency modes that form a continuous set when space is infinite, and which are only elastically scattered, i.e., not subject to any Bogoliubov transformation. The quantization is straightforward, but the calculation of the asymptotic fluxes is rather involved. When the number of complex frequency modes is small, our expressions differ from those given earlier. In particular, when the region between the horizons shrinks, there is a minimal distance under which no complex frequency mode exists, and no radiation is emitted. Finally, we relate this effect to other dynamical instabilities found for rotating black holes and in electric fields, and we give the conditions to get this type of instability.

I. INTRODUCTION

In 1998, Corley and Jacobson made the following interesting observation [1]: They noticed that the propagation of a superluminal dispersive field in a stationary geometry containing two horizons, as it is the case for nonextreme charged black holes, leads to a self-amplified Hawking radiation (for bosonic fields). The origin of this laser effect can be attributed to the closed trajectories followed by the negative Killing frequency partners of Hawking quanta. Because of the superluminal character of the dispersion, the partners indeed bounce from one horizon to the other.

Besides charged black holes, the acoustic geometries associated with a moving fluid that crosses twice the speed of sound also contain a pair of horizons. For one-dimensional flows these geometries are of the form

$$ds^2 = -dt^2 + (dx - v(x)dt)^2. \qquad (1)$$

One obtains this expression [2,3] when considering the propagation of low frequency phonons in a moving fluid of velocity $v(x)$. The sound velocity is assumed to be constant and has been set to 1. This metric possesses a black hole (BH) or a white hole (WH) horizon when the gradient $\partial_x v$ at the horizon $|v| = 1$ is positive or negative. When the dispersion is superluminal (anomalous) all such systems should experience a lasing effect. This should apply, in particular, to the BH-WH pair realized in a Bose Einstein condensate [4].

The original analysis and that of [5] were both carried out using wave packets. In the present work we show that there exists a more fundamental description based on frequency eigenmodes, which are asymptotically bound (in space). When $v(x)$ is constant and subsonic for both $x \to \pm\infty$, there is a discrete set of complex frequency modes and a continuous family of real frequency modes. (If instead the subsonic region is finite and periodic conditions imposed, the situation is more complicated [6,7] and will not be considered here.) In our case, the real frequency modes are asymptotically oscillating and normalized by a delta of Dirac. They are not subject to any Bogoliubov transformation. In fact, the scattering matrix at fixed ω only contains reflection and transmission coefficients mixing right and left moving (positive norm) modes. This was not *a priori* expected since the matrix associated with a single BH (or a WH) is 3×3 and mixes positive and negative norm modes in a nontrivial way [8,9].

The discrete set is composed of modes that vanish for $x \to \pm\infty$. They form two modes subsets of complex conjugated frequencies, each containing a growing and a decaying mode. The time dependence of the coefficients in each subset corresponds to that of a complex upside down and rotating harmonic oscillator. Both the real and the imaginary part of the frequency play important roles in determining the asymptotic properties of the fluxes. We notice such modes were encountered in several other situations [10–15]. We also mention that a stability analysis of BH-WH flows in Bose Einstein condensates was presented in [16]. We reach different conclusions because we use different boundary conditions.

In Sec. II, we present our settings. In Sec. III, we demonstrate that the set of spatially bound modes contains a continuous part and a discrete part composed of complex frequency modes. In Secs. IV and V, we study the properties of the modes, and show how the complex frequency modes determines the fluxes. We also relate our approach to that based on wave packets [1,5], and explain why the predictions differ in general, and, in particular, when the number of discrete modes is small. In Sec. VI, we give the conditions to get complex frequency modes in general terms.

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II. THE SETTINGS

We work in two space-time dimensions and consider the stationary metrics of Eq. (1), which contain a BH-WH pair. We restrict ourselves to flows that are asymptotically constant, i.e. we consider velocity profiles such as

$$v(x) = -1 + D \tanh\left[\frac{\kappa_B(x-L)}{D}\right]\tanh\left[\frac{\kappa_W(x+L)}{D}\right],$$
(2)

see Fig. 1. The BH (WH) horizon is situated at $x = L$ ($x = -L$). We suppose that the inequality $\kappa L/D \gg 1$ is satisfied for both values of κ, where $D \in]0, 1]$. In this case, the two near horizon regions of width $\Delta x \sim D/\kappa$ are well separated, and the surface gravities of the BH and the WH are, respectively, given by $\kappa_B = \partial_x v|_{x=L}$ and $\kappa_W = -\partial_x v|_{x=-L}$. The minimal speed $|v_-| < 1$ is reached for $x \to \pm\infty$, whereas the maximal speed $|v_+| > 1$ is found at $x = 0$ between the two horizons. When $\kappa L/D \gg 1$, their values are

$$v_\pm = -1 \mp D.$$
(3)

As emphasized in [8,9], when using nonlinear dispersion relations, D fixes the critical frequency ω_{max} above which no radiation is emitted by a single BH, or WH. Similarly here, there will be no unstable mode above ω_{max}.

As in [1], we work with a real field ϕ obeying a quartic superluminal dispersion relation

$$\Omega^2 = k^2 + k^4/\Lambda^2.$$
(4)

The UV scale characterizing the dispersion is given by Λ, and Ω is the frequency measured in the preferred frame, that comoving with the fluid. Most of the results we shall derive also apply to higher order superluminal dispersion relations. They also apply to the Bogoliubov theory of phonons in Bose condensates, which is a variant of the present case, as can be verified by comparing [9] to [8]. The
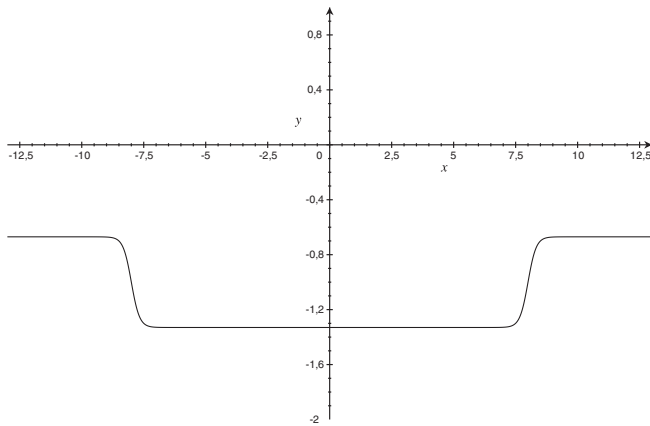

FIG. 1. Velocity profile $v(x)$ as a function of κx, for $\kappa_B = \kappa_W$, $D = 0.33$, and $\kappa L = 8$. The horizons are located at $\kappa x = \pm 8$, where $v(x) + 1 = 0$.

action of ϕ in the metric of Eq. (1) is

$$S = \frac{1}{2}\int dtdx\left[(\partial_t\phi + v\partial_x\phi)^2 - (\partial_x\phi)^2 - \frac{1}{\Lambda^2}(\partial_x^2\phi)^2\right],$$
(5)

and the wave equation is

$$\left[(\partial_t + \partial_x v)(\partial_t + v\partial_x) - \partial_x^2 + \frac{1}{\Lambda^2}\partial_x^4\right]\phi = 0.$$
(6)

When the flow is stationary, one can look for solutions with a fixed frequency $\lambda = i\partial_t$. Inserting $\phi = e^{-i\lambda t}\phi_\lambda(x)$ in Eq. (6) yields

$$\left[(-i\lambda + \partial_x v)(-i\lambda + v\partial_x) - \partial_x^2 + \frac{1}{\Lambda^2}\partial_x^4\right]\phi_\lambda = 0.$$
(7)

Because of the quartic dispersion, the number of linearly independent solutions is four. It would have been n if the dispersion relation would have been $\Omega^2 = p^2 + p^n/\Lambda^{n-2}$ rather than Eq. (4). However, when imposing that the modes ϕ_λ be asymptotically bound for $x \to \pm\infty$, the dimensionality is reduced to two or one, depending on whether λ is real or complex, but irrespective of the value of the power n. (To avoid confusion about the real/complex character of λ we shall write it as $\lambda = \omega + i\Gamma$, with ω and Γ both real and positive. The other cases can be reached by complex conjugation and by multiplication by -1.)

The necessity of considering only asymptotically bound modes (ABM) comes from the requirement that the observables, such as the energy of Eq. (15), be well defined. Returning to Eq. (5), the conjugate momentum is

$$\pi = \partial_t\phi + v\partial_x\phi,$$
(8)

the scalar product is

$$(\phi_1|\phi_2) = i\int_{-\infty}^{\infty}dx[\phi_1^*\pi_2 - \phi_2\pi_1^*],$$
(9)

and the Hamiltonian is given by

$$H = \frac{1}{2}\int dx\left[(\partial_t\phi)^2 + (1-v^2)(\partial_x\phi)^2 + \frac{1}{\Lambda^2}(\partial_x^2\phi)^2\right].$$
(10)

In the subspace of ABM, the Hamiltonian is Hermitian, i.e. $(\phi_1|H\phi_2) = (H\phi_1|\phi_2)$.

We conclude with some remarks. First, when written in the form Eq. (9), the scalar product is conserved in virtue of Hamilton's equations, and the Hermiticity of H. Second, from Eq. (10), one sees that the Hamiltonian density is negative where the flow is supersonic: $v^2 > 1$. We shall later see that the supersonic region should be "deep" enough so that it can sustain at least a bound mode, thereby engendering a laser effect. Third, when considering ABM, eigenmodes characterized by different frequencies are orthogonal in virtue of the identity [10,13]

$$(\lambda' - \lambda^*)(\phi_\lambda|\phi_{\lambda'}) = 0,$$
(11)

which follows from the Hermiticity of H. Finally, we notice that complex frequency ABM can exist in the present case because neither the scalar product Eq. (9), nor the Hamiltonian Eq. (10) is positive definite, see [10], p. 228. On the contrary, since fermionic fields are endowed with a positive scalar product [17], no complex frequency ABM could possibly be found in their spectrum [18].

III. THE SET OF ASYMPTOTICALLY BOUND MODES

A. Main results

In the BH-WH flows of Eq. (2), the set of ABM contains a continuous spectrum of dimensionality 2 labeled by a positive real frequency ω, and a discrete spectrum of $N < \infty$ pairs of complex frequencies eigenmodes. We shall suppose that this set is complete.[1] That is, any solution of Eq. (6) with $(\phi|\phi) < \infty$ can be decomposed as

$$\phi = \int_0^\infty d\omega (e^{-i\omega t}[a_{\omega,u}\phi_\omega^u(x) + a_{\omega,v}\phi_\omega^v(x)] + \text{H.c.})$$
$$+ \Sigma_{a=1,N}(e^{-i\lambda_a t}b_a\varphi_a(x) + e^{-i\lambda_a^* t}c_a\psi_a(x) + \text{H.c.}).$$
$$(12)$$

For flows that are asymptotically constant on both sides of the BH-WH pair, we shall show that the real frequency modes can be normalized according to

$$(\phi_{\omega'}^i|\phi_\omega^j) = \delta^{ij}\delta(\omega - \omega'), \qquad (\phi_{\omega'}^{i*}|\phi_\omega^j) = 0, \quad (13)$$

where the discrete index i takes two values u, v, and where ω, $\omega' > 0$. The index u, v characterizes modes that are asymptotically left (v) or right moving modes (u) with respect to a stationary frame.

When λ is complex, the situation is unusual. Yet it closely corresponds to that described in the Appendix of [10]. In fact, whenever a Hermitian Hamiltonian possesses complex frequency ABM, one obtains a discrete set of two-modes (φ_a, ψ_a) of complex conjugated frequency λ_a, λ_a^*. Their "normalization" can be chosen to be

$$(\varphi_{a'}|\varphi_a) = 0, \qquad (\psi_{a'}|\varphi_a) = i\delta_{a,a'}, \quad (14)$$

with all the other (independent) products vanishing in virtue of Eq. (11). Since the overlap between modes belonging to the continuous and discrete sectors, such as $(\phi_\omega|\varphi_a)$, also vanish, eigenmodes of different frequency never mix. Moreover, since the positive norm modes ϕ_ω^i all have $\omega > 0$, one cannot obtain Bogoliubov transformations as those characterizing the Hawking radiation associated with a single BH (or WH). This implies that the (late time) radiation emitted by BH-WH pairs *entirely* comes from the discrete set of modes.

Using the above equations, the energy carried by ϕ of Eq. (12) is given by

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[1]We are currently trying to demonstrate this assumption.

$$E = (\phi|H\phi)$$
$$= \int_0^\infty d\omega\,\omega(|a_{\omega,u}|^2 + |a_{\omega,u}|^2)$$
$$+ \Sigma_{a=1,N}(-i\lambda_a b_a c_a^* + \text{H.c.}). \quad (15)$$

Because of the complex frequency modes, it is unbound from below. Notice also that the absence of terms such as $|b_a|^2$ is necessary to have at the same time complex frequency eigenmodes and real energies.

B. Asymptotic behavior and roots k_λ

The material presented below closely follows that of [8]. In fact, the lengthy presentation of that work was written having in mind its applicability to the present case. The novelties are related to the fact that, for the metrics of Eq. (2), the supersonic region is bound (from $-L$ to L), and the velocity is subsonic for $x \to \pm\infty$.

Since the velocity v is asymptotically constant for $|x/L| \gg 1$, in both asymptotic regions, the solutions of Eq. (7) are superpositions of four exponentials $e^{ik_\lambda x}$ weighted by constant amplitudes. To characterize a solution, one thus needs to know (on one side, say on the left) the four amplitudes A_k associated to the corresponding asymptotic wave vectors $k(\lambda)$. These are the roots of

$$(\lambda - v(x)k)^2 = k^2 + \frac{k^4}{\Lambda^2} = \Omega^2(k), \quad (16)$$

evaluated for $v(x \to -\infty) = v_-$. We shall not assume *a priori* that λ is real. Rather we shall look for all ABM. Notice that when considering complex frequencies $\lambda = \omega + i\Gamma$, the roots of Eq. (16) are continuous functions of Γ. In addition, when the scales are well separated, i.e. when $\kappa/\Lambda \ll 1$, the relevant values of Γ will obey $\Gamma/\Lambda \ll 1$. It is therefore appropriate to start the analysis with $\lambda = \omega$ real, and then to study how the roots migrate when Γ increases.

When $\omega > 0$, since the flow is subsonic for $x \to \pm\infty$, there exist two real asymptotic roots: $k_\omega^u > 0$ and $k_\omega^v < 0$, which correspond to a right and a left mover, respectively. There also exist a pair of complex conjugated roots, since Eq. (16) is real. Thus, on each side of the BH-WH pair there is a growing and a decaying mode. As in [8] we define them according to the behavior of the mode when moving away from the BH-WH horizons.

In preparation for what follows, we study the roots in the supersonic region between the horizons. For ω smaller than a critical frequency ω_{max}, whose value is discussed below, the four roots are real. For flows to the left, $v < 0$, the two new real roots correspond to two right movers (with respect to the fluid). Indeed, they live on the u branch of the dispersion relation Eq. (16), that with $\partial_k\Omega > 0$, see Fig. 2. When ω increases at fixed $|v| > 1$, these roots approach to each other. Thus, for flows characterized by a maximal velocity v_+, there is a frequency ω_{max} above which they no longer exist as real roots. It is given by the
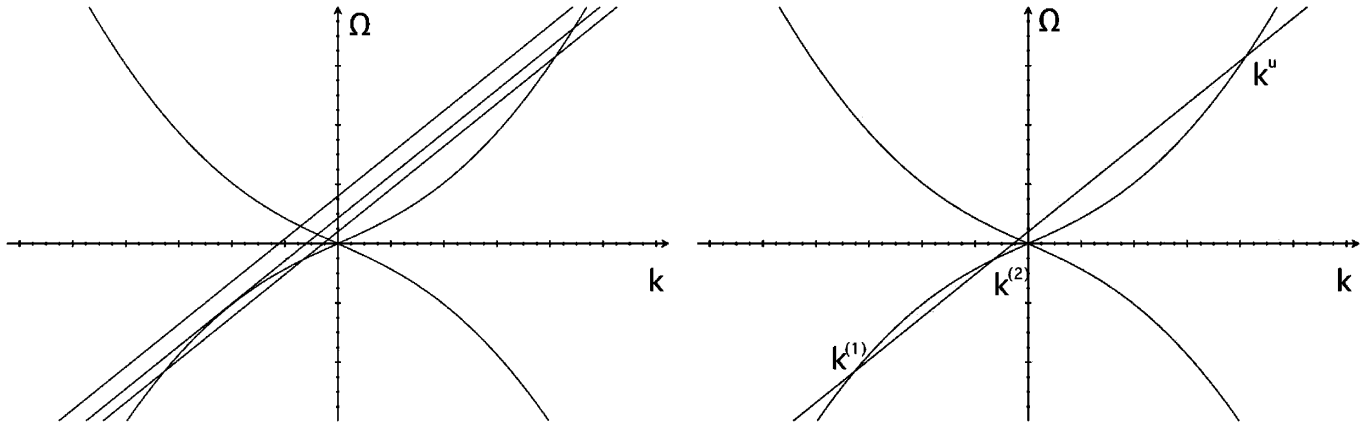
FIG. 2.   Graphical resolution of Eq. (16). On the left, the straight lines represent $\omega - k v_+$ in the supersonic regime $v_+ < -1$ for three different values of ω. The middle one, which is tangent to $\Omega(k)$, determines the critical frequency $\omega_{\max}$. For $\omega < \omega_{\max}$, the two extra negative roots correspond to right moving modes with respect to the fluid since $\partial_k \Omega > 0$. On the right, k^u is the positive root describing the right moving mode, see Eq. (18), $k^{(1)}$ the most negative root, $k^{(2)}$ that with the smallest norm, see Eq. (22).

value of ω where they merge for $v = v_+$. When the two horizons are well separated, v_+ is given in Eq. (3), up to exponentially small terms. As shown in [8], $\omega_{\max}$ is of the form $\omega_{\max} = \Lambda f(D)$, where D is defined in Eq. (2). For $D \ll 1$, one finds $f(D) \propto D^{3/2}$. Thus, for a given dispersion scale Λ, $\omega_{\max}$ can be arbitrarily small. This will be important when considering the appearance of the laser effect in parameter space.

C. The continuous spectrum

We now have all the elements to show that the continuous part of the spectrum of ABM is labeled by positive real frequencies ω, and that, for a fixed ω, its dimensionality is two.

The general solution of Eq. (7) with ω real can be characterized by the four amplitudes, which multiply the four exponentials evaluated in the asymptotic left region. When imposing that the growing mode is absent on that side, only three independent solutions remain. However, when propagating these solutions in the asymptotic region on the right side of the BH-WH pair, the growing mode will be generally present on that side. Thus, when requiring that it be also absent imposes to take particular combinations, and this reduces the dimensionality of ABM to two.

One can then take appropriate linear combinations to construct the *in* (*out*) modes describing the left and right movers propagating toward (escaping from) the BH-WH pair. The *in* right moving solution $\phi_\omega^{u,in}$ is the combination which on the left is asymptotically proportional to $e^{ik_\omega^u x}$, where k_ω^u is the asymptotic real positive root of Eq. (16). Similarly, one can identify the *in* left moving mode ϕ_ω^{vin}, and the two *out* modes ϕ_ω^{uout} and ϕ_ω^{vout}.

More precisely, because of the infinite and flat character of the space on either side of the BH-WH pair, the two *in* and the two *out* modes can be normalized as in a constant velocity flow (by considering a series of broad wave pack-

ets localized in one asymptotic region and whose spread in frequency progressively vanishes). Considering ϕ_ω^{uin} for $x \to -\infty$, Eq. (13) and (9) imply that it asymptotes to

$$\phi_\omega^{uin}(x) \to \sqrt{\frac{dk_\omega^u}{d\omega}} \frac{\exp i k_\omega^u x}{\sqrt{4\pi \Omega(k_\omega^u)}}. \tag{17}$$

A similar expression valid for $x \to \infty$ gives ϕ_ω^{vin}. These two *in* modes are orthogonal to each other, establishing the Kronecker δ^{ij} in Eq. (13). For $\omega > 0$, these modes have a positive Klein-Gordon norm. For $\omega < 0$, they have a negative norm. Therefore, negative norm modes can all be described as superpositions of complex conjugated positive norm modes.

When propagated across the BH-WH geometry, the *in* modes are scattered by the gradients of $v(x)$. When the variation of v is slow, i.e. $\partial_x \ln v \ll \partial_x \ln k_\omega$, the exact solutions are globally well approximated by the WKB solutions of Eq. (7). For the u mode, one finds, see Eq. (17),

$$\varphi_\omega^u(x) = \sqrt{\frac{\partial k_\omega^u(x)}{\partial \omega}} \frac{\exp(i \int_{-L}^x dx' k_\omega^u(x'))}{\sqrt{4\pi \Omega(k_\omega^u(x))}}. \tag{18}$$

We use the symbol φ_ω^u (φ_ω^v) to differentiate the WKB solution from the exact one ϕ_ω^u (ϕ_ω^v). At high frequency, $\omega/\kappa \gg 1$, the inequality $\partial_x \ln v \ll \partial_x \ln k_\omega$ is satisfied, and u and v modes do not mix. At lower frequency, they do. Nevertheless, far away from the BH-WH pair, since $v(x)$ is asymptotically constant, exact solutions decompose into superpositions of φ_ω^u and φ_ω^v with constant amplitudes. This applies for both the real and the complex frequency modes of Eq. (12). We shall use this fact several times to characterize the properties of the exact solutions.

Introducing the *out* modes as in Eq. (17), this scattering is described by

$$\phi_\omega^{u,in} = T_\omega \phi_\omega^{u,out} + R_\omega \phi_\omega^{v,out},$$
$$\phi_\omega^{v,in} = \tilde{T}_\omega \phi_\omega^{v,out} + \tilde{R}_\omega \phi_\omega^{u,out}. \qquad (19)$$

Unitarity imposes $|T_\omega|^2 + |R_\omega|^2 = 1 = |\tilde{T}_\omega|^2 + |\tilde{R}_\omega|^2$, and $R_\omega \tilde{T}_\omega^* + T_\omega \tilde{R}_\omega^* = 0$. For all values of ω, one thus has an elastic scattering, without spontaneous pair creation. For frequencies $\omega > \omega_{max}$, Eq. (19) coincides with what is found in single horizon scatterings [8]. Instead, for frequencies $0 < \omega < \omega_{max}$, this radically differs from the scattering on a single because, in that case, the matrix was 3×3 and mixed ϕ_ω^u, ϕ_ω^v with the negative frequency u mode $(\phi_{-\omega}^u)^*$. The presence of the second horizon therefore "removes" these modes. As we shall later see, they shall be "replaced" by a finite and discrete set of complex frequency modes. This is not so surprising since the classical trajectories associated with the negative frequency modes are closed (hence the discretization), and since these trapped modes mix with the continuous spectrum through each horizon (hence the imaginary part of the frequency). In fact, this is reminiscent, but not identical, to quasinormal modes [19] or resonances. The main difference is that quasinormal modes are not asymptotically bound. Thus, they should not be used in the mode expansion of Eq. (12).

It should be also mentioned that both *in* and *out* modes contain a trapped component in the supersonic region,[2] which plays no role as far as their normalization is concerned since the modes are everywhere regular, and the supersonic domain is finite. The case where the subsonic domain is also finite should be analyzed separately. If periodic conditions are imposed at the edges of the condensate, one obtains discrete frequencies and resonances effects [6,7]. If instead absorptive conditions are used, the frequencies are continuous and the situation is closer to the case we are studying.

D. The discrete spectrum

On general grounds we explain why, when considering Eq. (7) in BH-WH flows of Eq. (2), there exists a discrete and finite set of complex frequency modes. To this end, we first show that for a generic complex frequency $\lambda = \omega + i\Gamma$, there exists no ABM. When $\Gamma \ll \Lambda$, the four roots have not crossed each other with respect to the case where $\Gamma = 0$ since the imaginary part of the two complex roots $k_\omega^\pm$ with ω real is proportional to Λ. Thus, in this regime, one can still meaningfully talk about the two "propagating" roots k_λ^u, k_λ^v, and the growing and decaying roots $k_\lambda^\pm$. Then, as in the former subsection, when imposing that the growing

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[2]Because of this trapped wave, the *in* (and *out*) modes are *not* asymptotic modes in a strong sense since a wave packet made with $\phi_\omega^{u,in}$ will have a double spatial support for $t \to -\infty$: the standard incoming packet coming from $x = -\infty$, and the unusual trapped piece. This additional component, see Eq. (22), ensures that $\phi_\omega^{u,in}$ is orthogonal to the complex frequency modes of Eq. (12).

mode be asymptotically absent on both sides of the BH-WH pair, the space of solutions is still two. However when $\Gamma > 0$, the u root k^u acquires a positive imaginary contribution, which means that the u mode of Eq. (17) diverges for $x \to -\infty$. To get a bound mode, its amplitude should be set to 0. For similar reasons, on the right, the incoming v mode diverges for $x \to \infty$. Hence, for a general value of λ, the set of ABM is empty. The above reasoning applies to flat backgrounds with v constant, and establishes that in that case, complex frequency modes should not be considered in Eq. (12).

We should now explain why when v is supersonic in a finite region, some complex frequency ABM exist. The basic reason is the same as that which gives rise to a discrete set of bound modes when considering the Schroedinger equation in a potential well (or the propagation of light through a cavity). In the supersonic region, (in the well), there exist two additional real roots k_ω when $\lambda = \omega$ real. The classical trajectories associated with them are closed, and, as in a Bohr-Sommerfeld treatment, the discrete set of modes is related to the requirement that the mode be single valued and bound. The complex character of the frequency λ is due to the finite tunneling amplitude across the horizons. Indeed, as we shall later see, the imaginary part of λ_a is proportional to some quadratic expression in the β Bogoliubov coefficients characterizing the scattering through the horizons. Were these coefficients equal to zero, λ_a would have been real. When "turning on" these coefficients, the frequency λ_a migrates in the complex plane, and the bound modes are continuously deformed.

These ABM appear in pairs with complex conjugated frequencies. This stems from the Hermiticity of H, which guarantees that there exists an ABM of frequency λ_a^* whenever there is one of frequency λ_a. At this point it should be re-emphasized that the existence of these complex frequency ABM is due to the fact that the scalar product is not definite positive. Indeed, these modes all have a vanishing norm in virtue of Eq. (11).

We can also conclude that the discrete set of complex frequencies ABM is finite. In fact, there are no closed orbits for $\omega < -\omega_{max}$, since the extra real roots k_ω no longer exist and since there is a gap between the eigenfrequencies.

E. The quantization

The canonical quantization of the field ϕ is straightforward since each eigenfrequency sector evolves independently from the others. Indeed, when decomposing the field as in Eq. (12), with the coefficients a, b, c promoted operators, the equal time commutation $[\phi(t, x), \pi(t, x')] = i\delta(x - x')$ and the orthonormality conditions Eq. (13) and (14) entirely fix their commutation relations. For real frequency modes, the operators $a_{\omega,i} = (\phi_\omega^i | \phi)$ obey the standard commutation relations

$$[a_{\omega,i}, a_{\omega,j}^{\dagger}] = \delta_{ij}\delta(\omega - \omega'). \qquad (20)$$

Instead, for complex frequency modes, one gets

$$[b_a, c_{a'}^{\dagger}] = i\delta_{aa'}. \qquad (21)$$

All the other commutators vanish.

Because of this disconnection, the ground state of the real frequency modes is stable and in fact subject to no evolution. Hence the number of quanta of these modes is constant. The evolution of the states associated with the complex frequency modes is also rather simple and described in the Appendix. What remains to be done is to determine the properties of the asymptotic fluxes. To this end we need a better understanding of the modes.

IV. THE PROPERTIES OF THE MODES

From Eq. (7), it is not easy to determine the complex frequencies λ_a and the properties of the modes. Several routes can be envisaged. One can adopt numerical techniques. We are presently modifying [20] the code used in [8,9] to address this problem. One can also bypass the calculation of the eigenmodes and directly compute the propagation of coherent states, or the density-density correlation function [21], using the techniques of [22]. This is currently under study [23]. One can also envisage to use analytical methods by choosing the flow $v(x)$ as in [16]. This method is also currently under study [24].

In what follows, we use an approximative treatment which is valid when the two horizons are well separated. Doing so, we make contact with the original treatment [1,5] based on wave packets. More importantly, we determine algebraic relations which do not rely on the validity of our approximations. In particular, we establish that the real frequency modes ϕ_{ω}^u are intimately related to the complex frequency modes even though their overlap vanish.

A. The limit of thin near horizon regions

To simplify the mode propagation, we assume that the near horizon regions are thin and well separated, i.e. $L\kappa \gg D$ in Eq. (2). In this case, the propagation through the BH-WH geometry resembles very much to that through a cavity. Indeed, the following apply. First, the nontrivial propagation across the two thin horizon regions can be described by matrices that connect a solution evaluated on one side to that on the other side. Second, the modes can be analyzed separately in three regions: in L, the external left region, for $-(x + L) \gg D/\kappa_W$; in R, the external right region, for $(x - L) \gg D/\kappa_B$; and in the inside region I, for $L - |x| \gg D/\kappa$. Within each region, the gradient of v is small. Hence, any solution is well approximated by a superposition of WKB waves Eq. (18) with constant amplitudes.

To further simplify the analysis, we use the fact that the u-v mode mixing coefficients are generally much smaller than those mixing the negative frequency modes to the positive u ones [8,9]. Hence, it is a reliable (and consistent) approximation to assume that the v modes completely decouple. After having analyzed this case, we shall briefly present the modifications introduced by relaxing this hypothesis. Adopting the hypothesis that the u-v mixing can be neglected, for each ω real, one has the following situation: In the left region L, one only has the WKB mode φ_{ω}^u of Eq. (18). Thus, the only solution is $\phi_{\omega}^{u,in}$ (up to an overall irrelevant phase we take to be 1). In the inside region I, one has three modes:

$$\phi_{\omega}^{u,in} = \mathcal{A}_{\omega}\varphi_{\omega}^u + \mathcal{B}_{\omega}^{(1)}(\varphi_{-\omega}^{(1)})^* + \mathcal{B}_{\omega}^{(2)}(\varphi_{-\omega}^{(2)})^*, \qquad (22)$$

since in supersonic flows, there exist two extra real roots in Eq. (16). The superscripts u, (1), (2) characterize the coefficients and the WKB modes associated with the three roots shown in Fig. 2. Since we are considering a solution with $i\partial_t = \omega > 0$, the (positive norm) negative frequency modes $\varphi_{-\omega}^{(i)}$ appear complex conjugated in Eq. (22).

In the external R region the solution must be again proportional to the WKB mode φ_{ω}^u. By unitarity, the solution must be of the form $\phi_{\omega}^{u,in} = e^{i\theta_{\omega}}\varphi_{\omega}^u$. Thus, a full characterization of $\phi_{\omega}^{u,in}$ requires to compute the phase θ_{ω} and the above three coefficients. At this point, it should be noticed that $\lambda = \omega$ is *a priori* real. However, the S matrix, and therefore the three coefficients, are holomorphic functions in λ. Hence, nothing prevents to leave the real axis. In fact, we shall show that the complex frequencies correspond to poles associated with a coefficient of the S matrix.

B. An S-matrix approach

1. The S matrix

The simplest way we found to compute the above coefficients is to follow the approach of [5], up to a certain point. In this treatment, a solution of frequency $\omega > 0$ is represented by a two component vector $(\varphi_{\omega}^u, \varphi_{-\omega}^*)$. The time evolution of a wave packet of such solutions is then considered in the thin horizon limit. Since the frequency content of the wave packet plays no role, we do not need to introduce a new notation to differentiate it from an eigenmode. In this language, the S matrix characterizing a bounce of the trapped mode $\varphi_{-\omega}^*$ can be decomposed as

$$S = U_4 U_3 U_2 U_1. \qquad (23)$$

The first matrix describes the scattering across the WH horizon. In full generality we parameterize it by

$$U_1 = \begin{pmatrix} \alpha_{\omega} & \alpha_{\omega}z_{\omega} \\ \tilde{\alpha}_{\omega}z_{\omega}^* & \tilde{\alpha}_{\omega} \end{pmatrix}. \qquad (24)$$

Unitarity imposes that $|\alpha_{\omega}|^2 = |\tilde{\alpha}_{\omega}|^2$ and $|\alpha_{\omega}|^2(1 - |z_{\omega}|^2) = 1$. The second matrix describes the free propagation (i.e. without backscattering) from the WH to the BH

horizon

$$U_2 = \begin{pmatrix} e^{iS^u_\omega} & 0 \\ 0 & e^{-iS^{(1)}_{-\omega}} \end{pmatrix}. \qquad (25)$$

In a WKB approximation, the two phases are, respectively, given by the actions

$$S^u_\omega = \int_{-L}^{L} dx k^u_\omega(x), \qquad S^{(1)}_{-\omega} = \int_{L_\omega}^{R_\omega} dx [-k^{(1)}_\omega(x)]. \quad (26)$$

In U_2, $S^{(1)}_{-\omega}$ is multiplied by $-i$ since it governs the evolution of $\varphi^*_{-\omega}$. Its momentum is $k^u_{-\omega} = -k^{(1)}_\omega > 0$, where $k^{(1)}_\omega$ is the most negative root of Eq. (16) found in supersonic flows. The ends of integration L_ω, R_ω are, respectively, the left and right locations of the turning points of the trajectories with $-\omega < 0$. These obey Hamilton's equations [25,26]

$$dx/dt = (\partial_\omega k^u_\omega)^{-1}, \qquad dk^u/dt = -\partial_x \omega = -k^u \partial_x v, \quad (27)$$

for negative frequency. In the thin horizon approximation, the turning points hardly differ from $-L$ and L. Unitarity brings no conditions on these phases.

The third matrix describes the scattering across the WH horizon, and we write it as

$$U_3 = \begin{pmatrix} \gamma_\omega & \gamma_\omega w_\omega \\ \tilde{\gamma}_\omega w^*_\omega & \tilde{\gamma}_\omega \end{pmatrix}. \qquad (28)$$

Unitarity imposes that $|\gamma_\omega|^2 = |\tilde{\gamma}_\omega|^2$ and $|\gamma_\omega|^2(1 - |w_\omega|^2) = 1$. The fourth matrix describes the return of the negative frequency partner towards the WH horizon, whereas the positive frequency mode propagates away in the R region. This is described by

$$U_4 = \begin{pmatrix} 1 & 0 \\ 0 & e^{iS^{(2)}_{-\omega}} \end{pmatrix}. \qquad (29)$$

In the WKB approximation, this backwards movement (hence the $+i$ in the front of $S^{(2)}$) is governed by

$$S^{(2)}_{-\omega} = \int_{L_\omega}^{R_\omega} dx [-k^{(2)}_\omega(x)], \qquad (30)$$

where the momentum $k^{(2)}_\omega$ is the least negative u root of Eq. (16). Since the positive frequency mode further propagates to the right, there is no meaning to attribute it a phase in U_4. In any case this phase would drop from all physical quantities.

The matrix S of Eq. (23) is unitary since its four constituents are. Hence, $|S_{22}|^2 = |S_{11}|^2 = 1 + |S_{12}|^2$. The components S_{22} and S_{21} we shall later use are given by

$$S_{22} = \tilde{\gamma}_\omega \tilde{\alpha}_\omega e^{-i(S^{(1)}_{-\omega} - S^{(2)}_{-\omega})} \left(1 + z_\omega w^*_\omega \frac{\alpha_\omega}{\tilde{\alpha}_\omega} e^{i(S^u_\omega + S^{(1)}_{-\omega})} \right),$$

$$S_{21} = \tilde{\gamma}_\omega \tilde{\alpha}_\omega e^{-i(S^{(1)}_{-\omega} - S^{(2)}_{-\omega})} \left(z^*_\omega + w^*_\omega \frac{\alpha_\omega}{\tilde{\alpha}_\omega} e^{i(S^u_\omega + S^{(1)}_{-\omega})} \right).$$
$$(31)$$

Hitherto we followed the method of [5]. Henceforth, we proceed differently by adding a key element. We require that the mode propagated by S be single valued. For real ω, this unequivocally defines ϕ^u_ω of Eq. (22). Moreover, when looking for complex frequency bound modes, this will give us the modes φ_a, ψ_a we are seeking.

2. The real frequency modes

Imposing that the trapped mode of negative frequency is single valued translates into

$$\begin{pmatrix} e^{i\theta_\omega} \\ b_\omega \end{pmatrix} = S \begin{pmatrix} 1 \\ b_\omega \end{pmatrix}. \qquad (32)$$

The phase θ_ω is that mentioned after Eq. (22). It should not be constrained since the positive frequency component keeps propagating to the right. The matricial equation gives

$$b_\omega = \frac{S_{21}}{1 - S_{22}}, \qquad e^{i\theta_\omega} = -\frac{S_{11}}{S^*_{22}} \frac{1 - S^*_{22}}{1 - S_{22}}. \qquad (33)$$

These equations constitute the first important result of this section. They do not rest on the WKB approximation. Of course, this approximation can be used to estimate the four elements of S. But once these are known, e.g. using a numerical treatment, these equations apply. What is needed to get these equations is the neglect of the u-v mixing, and no significant frequency mixing in the inside region in order to obtain well-defined amplitudes in Eq. (22).

Using $S^\dagger = S$ one verifies that the norm of the right-hand side of the second equation is unity. This is as it is must be, since in the absence of u-v mixing, the u component only acquires a phase, here measured with respect to the WKB wave. Adopting the convention that WKB modes φ_ω have a vanishing phase at $x = -L$, the coefficients of Eq. (22) are

$$\mathcal{A}_\omega = \alpha_\omega(1 + z_\omega b_\omega),$$
$$\mathcal{B}^{(1)}_\omega = \tilde{\alpha}_\omega(z^*_\omega + b_\omega), \qquad \mathcal{B}^{(2)}_\omega = b_\omega. \qquad (34)$$

These amplitudes are governed by b_ω, which can *a priori* be larger or smaller than unity. In particular it diverges if $S_{22} \to 1$ for some ω, thereby approaching a resonance, see Fig. 3. We now show that the complex frequency modes correspond to these resonances.

3. The pairs of complex frequency modes

Following the discussion of Sec. III D, we impose that the amplitude of the incoming u branch be zero, and, as above, that the trapped mode of negative frequency is single valued. This gives

$$\begin{pmatrix} \beta_a \\ 1 \end{pmatrix} = S \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (35)$$

which implies

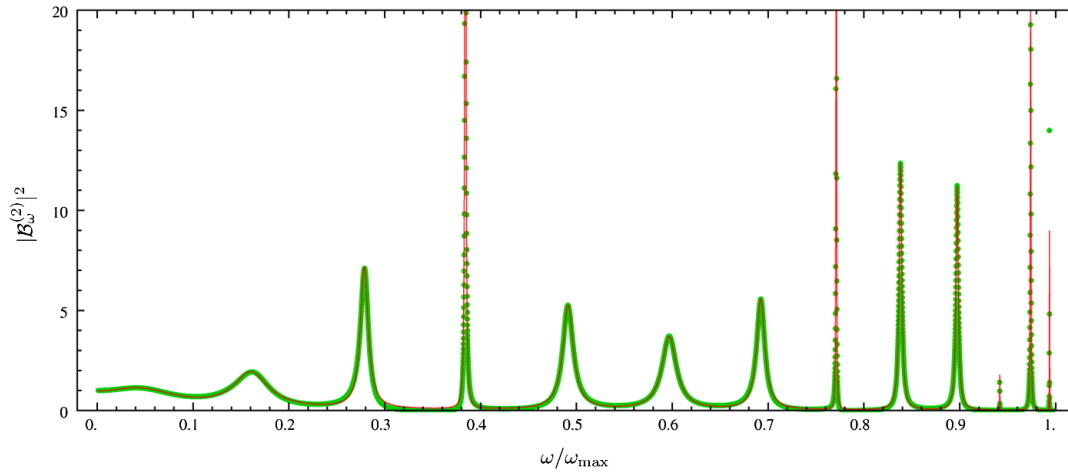$$\beta_a = S_{12}, \qquad 1 = S_{22}. \qquad (36)$$

FIG. 3 (color online). By making use of numerical results borrowed from [20], we have represented the square norm of $\mathcal{B}_\omega^{(2)}$, the amplitude of the negative frequency trapped mode of Eq. (22), as a function of ω real. Near a complex frequency $\lambda_a = \omega_a - i\Gamma_a$, which solves $S_{22} = 1$, see Eq. (36), $|\mathcal{B}_\omega^{(2)}|^2$ behaves as a Lorentzian: $\sim|\omega - \omega_a - i\Gamma_a|^{-2}$, see Eqs. (33) and (34). The dots are the numerical values, whereas the continuous line is a fitted sum of Lorentzians. The remarkable agreement establishes that the complex frequencies λ_a can be deduced from the analysis of the real frequency modes. The frequency ω has been expressed in the units of ω_{max}, see Fig. 2, so that there is no resonance above $\omega/\omega_{max} = 1$. In the present case there are 13 resonances. The narrow peaks, $\Gamma_a \sim 0$, are due to the fact that the surface gravities are equal, $\kappa_B = \kappa_W$, which leads to $z_\omega = w_\omega$ in Eq. (41).

Two important lessons are obtained. First, to get the complex frequencies λ_a with a positive imaginary part, it suffices to solve the roots of $S_{22} = 1$. Second, as mentioned, these correspond to the poles characterizing the propagating modes, see Eq. (33).

Before computing these frequencies, we explain how to get the decaying modes ψ_a, the "partners" of the φ_a in Eq. (12). Since these bound modes have a negative imaginary frequency, the amplitude of the escaping mode must be zero, i.e.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = S \begin{pmatrix} \tilde{\beta}_a \\ 1 \end{pmatrix}. \qquad (37)$$

Using the Hermitian conjugated $S^\dagger$, and the unitarity relation $S^\dagger S = 1$, this condition gives

$$\begin{pmatrix} \tilde{\beta}_a \\ 1 \end{pmatrix} = S^\dagger \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (38)$$

from which we get $\tilde{\beta}_a = (S^\dagger)_{12}$ and $1 = (S^\dagger)_{22}$. As explained in the Appendix, when expressed in terms of the elements of S, these equations give

$$\tilde{\beta}_a = -[S_{21}(\lambda^*)]^*, \qquad 1 = [S_{22}(\lambda^*)]^*. \qquad (39)$$

Thus, the solutions of $1 = S_{22}^*$ are the complex conjugated of those that solve Eq. (36), thereby establishing the partnership between φ_a and ψ_a. In addition, one has $\tilde{\beta}_a = -(\det S)^* \beta_a$.

C. The set of complex frequencies λ_a

To compute the roots of Eq. (36) we need to know S_{22} as a function of $\lambda = \omega + i\Gamma$. In what follows we shall relate

them to the quantities which enter in Eq. (31). To this end, we suppose that the "tunneling" across the horizons is small, i.e. the β-Bogoliubov coefficients associated with each horizon are small. This is true for ω/κ sufficiently large, see [8] where it was shown that z_ω and w_ω, which are related to the Bogoliubov coefficients by $z_\omega = \beta_\omega/\alpha_\omega$, behave as $\sim e^{-\pi\omega/\kappa}(1 - \omega/\omega_{max})^{1/4}$ where ω_{max} is defined in Fig. 2. To proceed we suppose that the norms of z_ω^2, w_ω^2 and $z_\omega w_\omega$ are much smaller than one. In this case, one can expand Eq. (31) in these three products, and in Γ, since the roots of Eq. (36) are real when $z_\omega = w_\omega = 0$.

Indeed, to zeroth order in these quantities, one gets $\tilde{\alpha}_\omega \tilde{\gamma}_\omega = e^{-i\pi}$. The norm is trivially constrained by unitarity, whereas the phase takes its value from the contributions stemming from the prefactor of the trapped mode at the two turning points, see Eq. (18). Taking this into account, $S_{22} = 1$ gives

$$S_{-\omega}^{(1)} - S_{-\omega}^{(2)} + \pi = \int_{L_\omega}^{R_\omega} dx[-k_\omega^{(1)}(x) + k_\omega^{(2)}(x)] + \pi$$
$$= 2\pi n, \qquad (40)$$

with $n \in \mathbb{N}_1$. This is the Bohr-Sommerfeld condition applied to the negative frequency mode $\varphi_{-\omega}$. (In fact, subtracting from both $k^{(1)}$ and $k^{(2)}$ the value of k at the turning points, k_ω^{tp}, the differences $k_\omega^{(1)} - k_\omega^{tp}$, $k_\omega^{(2)} - k_\omega^{tp}$ have opposite sign. Hence, Eq. (40) contains a sum of two positive contributions, as in the Bohr-Sommerfeld condition.) We call ω_a, $a = 1, 2, \ldots, N$ the discrete set of frequencies, which is finite because no solution exists above ω_{max}.

To first order, for each ω_a, one gets a complex phase shift $\delta\lambda_a = \delta\omega_a + i\Gamma_a$. The imaginary shift is

$$2\Gamma_a T^b_{\omega_a} = |z_{\omega_a}|^2 + |w_{\omega_a}|^2 + 2|z_{\omega_a} w_{\omega_a}| \cos\psi_a$$
$$= |S_{12}(\omega_a)|^2. \tag{41}$$

The phase in the cosine is

$$\psi_a = S^u_{\omega_a} + S^{(1)}_{-\omega_a} + \arg(z_{\omega_a} w^*_{\omega_a} \alpha_{\omega_a}/\tilde\alpha_{\omega_a}), \tag{42}$$

and $T^b_{\omega_a} > 0$ is the time for the negative frequency partner to make a bounce. It is given by

$$T^b_\omega = \frac{\partial}{\partial\omega}(S^{(2)}_{-\omega} - S^{(1)}_{-\omega} + \arg\tilde\alpha_\omega \tilde\gamma_\omega), \tag{43}$$

evaluated for $\omega = \omega_a$. The first two terms give the classical (Hamilton-Jacobi) time, whereas the last one gives the contribution from the scattering coefficients (which is a small correction when $L\Lambda D^{1/2} \gg 1$).[3] The last equality of Eq. (41) tells us that Γ_a is linearly related to the norm of the effective β-Bogoliubov coefficient of the pair, which obeys $|S_{22}|^2 = 1 + |S_{12}|^2$. It is also worth noticing that S_{12} fixes the amplitude β_a of the leaking mode in Eq. (36).

Because $|S_{22}|^2 = 1 + |S_{12}|^2$, Γ_a defined in Eq. (41) is positive, thereby implying that φ_a, the solution of Eq. (35), is a growing mode in time, and an ABM in space. What distinguishes the present case from usual resonances characterized by a decay rate ($\Gamma < 0$) is the fact that the norm of the trapped mode $\varphi^*_{-\omega}$ is opposite to that of the leaking wave φ^u_ω. Even though unitarity in both cases implies a decrease of the norm of the trapped mode, in the present case it becomes more negative whereas in the standard case it tends to zero since it has the same sign as that of φ^u_ω. As a corollary of this, the fact that resonances have an opposite sign of Γ while satisfying an outgoing condition as in Eq. (35) implies that they are not ABM, and therefore not included in the set of modes of Eq. (12). This remark applies to fermionic fields and implies that the set of ABM for the fermionic dispersive field considered in [1] and propagating in the BH-WH metric of Eq. (2), is restricted to the continous set of Eq. (12), i.e. positive real frequency modes elastically scattered as given in Eq. (19). To complete these remarks, one should notice that ψ_a, our decaying modes, are also ABM because they obey the incoming condition Eq. (38).

Our treatment is similar to the interesting analysis presented in [11]. In that work, the general solution is constructed in terms of WKB waves. Then the Bohr-Sommerfeld and the outgoing conditions are separately imposed in the small tunneling approximation, thereby fixing both the real part of the frequency ω_a and its imaginary part Γ_a. We followed another logic which leads

---

[3]Using the techniques of [26,27], we verified that it gives a small correction to the classical actions. We also obtained a refined expression for Eq. (40), which goes beyond the WKB approximation and which has been validated by the numerical analysis of [20].

to the same result, namely, the requirement that the mode be an ABM gives Eq. (35), which in turn gives the complex equation $S_{22} = 1$ that encodes both conditions. We note that $S_{22} = 1$ does not require that the tunneling amplitudes be small to be well defined. We also note that our quantization scheme applies to the cases studied in [11,12] and allows to remove the "formal trick" used in the second paper.

D. The density of unstable modes

To further characterize the instability, it is of interest to inquire about the density of unstable modes, about the end of the set, its beginning, and about the most unstable mode.

The number of unstable modes will be either large, or small, depending on the value of $L/\omega_{\max}$. When $L/\omega_{\max} \gg 1$, the number is large because, for $\omega < \omega_{\max}$, the gap between two neighboring modes roughly given by D/L. The end of the set is controlled by $\omega_{\max}$ as explained after Eq. (40). It does not significantly contribute to the instability when $\kappa/\omega_{\max} \ll 1$, since z_ω and w_ω tend to zero as $e^{-\pi\omega/\kappa}(1 - \omega/\omega_{\max})^{1/4}$ for $\omega \to \omega_{\max}$. Thus, the growth rate $\Gamma_a \to 0$ as $\omega \to \omega_{\max}$.

The beginning of the set is governed by the first Bohr-Sommerfeld modes. Their frequency is of the order of D/L. A proper evaluation of Γ_a is harder since, generically, the coefficients α_ω and γ_ω both diverge as $\sim\omega^{-1/2}$. We thus conjecture that the most unstable mode is the first mode, or one of the first ones, because the cosine of Eq. (41) could in some cases lower the value of Γ_1 below that of the next ones. We can also characterize the migration of the roots as L increases. The (total) variation of Eq. (40) with respect to L and Eq. (43) tells us that $\partial\omega_a/\partial L > 0$. From this we conjecture that new unstable modes (obtained by increasing L) appear with $\omega_a = 0$. This is corroborated by the first runs of [20]. The determination of Γ_a for $\omega_a \to 0$ is difficult, and further study is needed to establish if Γ_a follows the abrupt behavior present in Fig. 1 of [14].

E. Taking into account the u-v mixing

A nonvanishing u-v mixing hardly modifies the above results. However, the algebra becomes more complicated because the two nontrivial matrices describing the propagation across the WH and the BH horizon (U_1 and U_3) are now the 3×3. Let us only sketch this enlarged case.

When considering the in mode $\phi^{u,in}_\omega$, in the left region L, one has $\phi^{u,in}_\omega = \varphi^u_\omega + R_\omega\varphi^v_\omega$, see Eq. (19). In the inside region I, one has

$$\phi^{u,in}_\omega = \mathcal{A}^u_\omega\varphi^u_\omega + \mathcal{A}^v_\omega\varphi^v_\omega + \mathcal{B}^{(1)}_\omega(\varphi^{(1)}_{-\omega})^* + \mathcal{B}^{(2)}_\omega(\varphi^{(2)}_{-\omega})^*, \tag{44}$$

and in the external R region, the mode asymptotes to $T_\omega\varphi^u_\omega$. As before, the coefficients are determined by the matching conditions at the horizons. Expressing $\phi^{u,in}_\omega$ in

terms of three coefficients associated with the three WKB waves, $(\varphi_\omega^u, \varphi_\omega^v, \varphi_{-\omega}^*)$, using the Bogoliubov matrices [8] S_{WH}, S_{BH} which give, for the WH and the BH, respectively, the coefficients after the scattering given the incident ones, one has

$$
\begin{pmatrix} \mathcal{A}_\omega^u \\ R_\omega \\ \mathcal{B}_\omega^{(1)} \end{pmatrix} = S_{\mathrm{WH}} \begin{pmatrix} 1 \\ \mathcal{A}_\omega^v \\ \mathcal{B}_\omega^{(2)} \end{pmatrix},
\tag{45}
$$

and

$$
\begin{pmatrix} T_\omega e^{iS_\omega^u} \\ \mathcal{A}_\omega^v e^{iS_\omega^v} \\ \mathcal{B}_\omega^{(2)} e^{-iS_{-\omega}^{(2)}} \end{pmatrix} = S_{\mathrm{BH}} \begin{pmatrix} \mathcal{A}_\omega^u e^{iS_\omega^u} \\ 0 \\ \mathcal{B}_\omega^{(1)} e^{-iS_{-\omega}^{(1)}} \end{pmatrix}.
\tag{46}
$$

In the second equation, the exponentials arise from the propagation from the WH to the BH horizon, and from our choice that the phase of WKB waves vanishes at the WH horizon. The six relations fix the six coefficients. T_ω and R_ω characterize the asymptotic scattering, see Eq. (19), whereas the four others determine $\phi_\omega^{u,in}$ in the inside region.

To get the complex frequencies λ_a, since the weight of the incoming v mode is already set to 0, one simply needs to replace the 1 in the right-hand side of Eq. (45) by a 0, as done in Eq. (35).

V. PHYSICAL PREDICTIONS

A. Classical settings

From an abstract point of view, any (bounded) initial condition, i.e. the data of the field amplitude and its derivative with respect to time, can be translated into the coefficients a_ω^i, b_a, c_a of Eq. (12). This is guaranteed by the completeness of the mode basis. Yet, it is instructive to study more closely how initial conditions describing a wave packet initially moving towards the WH-BH pair translate into these coefficients. This will allow us to relate our mode basis to the wave packet analysis of [1,5]. Moreover it is a warming up exercise for the determination of the fluxes in quantum settings. For simplicity, we work in the regime in which the number of unstable modes is large, i.e. $L\omega_{\max} \gg 1$, and with a wave packet of mean frequency obeying $\bar{\omega} \ll \omega_{\max}$ and $L\bar{\omega} \gg 1$, e.g. with $\bar{\omega} = \sqrt{\omega_{\max}/L}$.

We consider a unit norm wave packet which is initially in the left region and propagating to the right. At early times, before it approaches the WH horizon, it can thus be expressed in terms of the WKB modes of Eq. (18) as

$$
\bar{\phi}_{\bar{\omega}}^u(t, x) = \int_0^\infty d\omega (f_{\bar{\omega}, x_0}(\omega) e^{-i\omega t} \varphi_\omega^u(x) + c.c.).
\tag{47}
$$

As an example, one can consider the following Fourier components:

$$
f_{\bar{\omega}, x_0}(\omega) = \frac{e^{-(\omega - \bar{\omega})^2/2\sigma^2}}{\pi^{1/4} \sigma^{1/2}} e^{i\omega t_0} e^{iS_\omega^u(x_0, -L)}.
\tag{48}
$$

The first factor fixes the mean frequency $\bar{\omega}$ and the spread σ taken to satisfy $\sigma/\bar{\omega} \ll 1$ and $\sigma L \gg 1$. The last two exponentials ensure that at $t = t_0$ the incoming wave packet is centered around some $x_0 \ll -L$. $S_\omega^u(x, x')$ is the classical action of the u mode evaluated from x to x', see the phase of φ_ω^u in Eq. (18).

When decomposing $\bar{\phi}_{\bar{\omega}}^u$ as in Eq. (12), the coefficients are given by the overlaps

$$
\bar{a}_\omega^u = (\varphi_\omega^{u,in} | \bar{\phi}_{\bar{\omega}}^u) = f_{\bar{\omega}, x_0}(\omega), \qquad \bar{a}_\omega^v = (\varphi_\omega^{v,in} | \bar{\phi}_{\bar{\omega}}^u) = 0,
$$
$$
\bar{b}_a = (-i)(\psi_a | \bar{\phi}_{\bar{\omega}}^u) \neq 0, \qquad \bar{c}_a = (-i)(\varphi_a | \bar{\phi}_{\bar{\omega}}^u) = 0.
\tag{49}
$$

Hence, at all times, the wave packet can be expressed as

$$
\bar{\phi}_{\bar{\omega}}^u(t, x) = \int_0^\infty d\omega (f_{\bar{\omega}, x_0}(\omega) e^{-i\omega t} \phi_\omega^{u,in}(x) + c.c.)
$$
$$
+ \Sigma_a (\bar{b}_a e^{-i\lambda_a t} \varphi_a(x) + c.c.).
\tag{50}
$$

The first two coefficients in Eq. (49) are easily computed and interpreted. They fix the real frequency contribution of the wave packet. The last two are very interesting and encode the instability. We first notice that, because of Eq. (14), they are given by the overlap with the partner wave. Hence, the coefficients $\bar{c}_a$ identically vanish. Indeed, the overlap of our wave packet with the growing modes φ_a vanishes since these, by construction, have no incoming branch. On the contrary $\bar{b}_a$, the amplitudes of the growing modes, do not vanish since the *decaying* modes ψ_a contain an incoming branch, see Eq. (38). It is thus through the nondiagonal character of Eq. (14) that the instability enters in the game. In this respect, our analysis differs from Ref. [16]. We do not understand the rules adopted in that work.

To compute $\bar{b}_a$ we use the fact that at time t_0 the wave packet is localized around $x_0 \ll -L$. Thus we need the behavior of ψ_a in this region. Using the results of Sec. IV C and Eq. (18) extended to complex frequencies, for $\Gamma_a T_{\omega_a}^b \ll 1$, one gets

$$
\psi_a(x) = \tilde{\beta}_a \varphi_{\lambda_a}^u(x) = \tilde{\beta}_a \varphi_{\omega_a}^u(x) \times \exp[-\Gamma_a t_{\omega_a}(x, -L)],
\tag{51}
$$

where $t_{\omega_a}(x, -L) = \partial_\omega S^u(x, -L) > 0$ is the time taken by a u mode of frequency ω_a to travel from x to $-L$.[4] Inserting this result in $(\psi_a | \bar{\phi}_{\bar{\omega}}^u)$, using the fact that the wave packet has a narrow spread in $x - x_0$ given by $1/(\sigma \partial_\omega k^u)$, the overlap is approximately

---

[4]It is interesting to notice how ψ_a and ϕ_a acquire a vanishing norm. Because the leaking wave has an amplitude $S_{21} \propto \Gamma_a^{1/2}$ and decreases in x in Γ_a^{-1}, its positive contribution to the norm is independent of Γ_a and cancels out that negative of the trapped mode.

$$\bar{b}_a = \frac{\tilde{\beta}_a^*}{\pi^{1/4}(2\sigma)^{1/2}} e^{-i(\bar{\omega}-\omega_a)t_0} e^{iS_{\omega a}^u(x_0,-L)} \times e^{-\Gamma_a t_{\omega a}(x_0,-L)},$$

(52)

The meaning of this result is clear. The spatial decay of ψ_a governed by Γ_a has the role to delay the growth of the amplitude of the wave packet until it reaches the WH-BH pair. The phase ensures the spatial coherence of the propagation. That is, the sum of a in Eq. (47) will give constructive interferences along the classical trajectory emerging from x_0 at t_0. Let us also mention that, because of Eq. (14), the normalization of $\bar{b}_a$ is arbitrary, but the product $\bar{b}_a \varphi_a$ is well defined and physically meaningful.

From this analysis, we see that at early times the wave packet $\bar{\phi}_{\bar{\omega}}^u$ behaves as described in [1,5]. It propagates freely without any growth until it reaches the horizon of the WH. Then one piece is reflected and becomes a v mode with amplitude $R_{\bar{\omega}}$. The other piece enters in the inside region. A later times, a component stays trapped and bounces back and forth while its amplitude increases in as $e^{\bar{\Gamma}t}$. Every time it bounces there is leakage of a u wave packet at the BH horizon, and a v mode at the WH one.

This back and forth semiclassical movement goes on until the most unstable mode, that with the largest Γ_a, progressively dominates and therefore progressively destroys the coherence of the successive emissions. In Sec. IV D we saw that the most unstable mode is likely to be that characterized by the smallest real frequency, $n = 1$ in Eq. (40). This implies that at late time, our wave packet will be completely governed by the corresponding wave $\varphi_1(x)$ (unless of course $\bar{b}_1 = 0$)

$$\bar{\phi}_{\bar{\omega}}^u \rightarrow e^{\Gamma_1(t-t_0)} \times \text{Re}[e^{-i\omega_1 t}\bar{b}_1\varphi_1(x)].$$ (53)

In the late time limit, its behavior differs from the above semiclassical one. Indeed, in the inside region, one essentially has a standing wave whose amplitude exponentially grows (it would have been one if the tunneling amplitudes were zero). Outside, for $x \gg L$, using $\varphi_1(x) \sim \beta_1 \varphi_{\lambda_1}^u(x)$ and the same approximation as in Eq. (51), one has a modulated oscillatory pattern given by

$$\bar{\phi}_{\bar{\omega}}^u \rightarrow e^{\Gamma_1[(t-t_0)-t_{\omega_1}(L,x)]} \times \text{Re}[e^{-i\omega_1 t}\bar{b}_1\beta_1\varphi_{\omega_1}^u(x)].$$ (54)

It moves with a speed equal to ~ 1 in the dispersionless regime, when $\omega_1/\Lambda \ll 1$. The energy flow is now given by a sine squared of period $2\pi/\omega_1$ rather than being composed of localized packets separated by the bounce time T^b.

B. Quantum settings

1. The initial state

Because of the instability, there is no clear definition of what the vacuum state should be. Indeed, as can be seen from Eq. (15), the energy is unbound from below. Therefore, to identify the physically relevant states, one

should inquire what would be the state, or better the subset of states, which would obtain when the BH-WH pair is formed at some time t_0. If this formation is adiabatic, the initial state would be close to a vacuum state at that time. That is, the expectation values of the square of the various field amplitudes would be close to their values in minimal uncertainty states, with no squeezing, i.e. no anisotropy in $\phi_\omega - \pi_\omega$ plane, where π_ω is the conjugated momentum of ϕ_ω. Because of the orthogonality of the eigenmodes, this adiabatic state would be, and stay, a tensor product of states associated with each mode separately.

There is no difficulty to apply these considerations to the real (positive) frequency oscillators which are described by standard destruction (creation) operators $\hat{a}_\omega^u$, $\hat{a}_\omega^v$ ($\hat{a}_\omega^{u\dagger}$, $\hat{a}_\omega^{v\dagger}$). The adiabaticity guarantees that one obtains a state close to the ground state annihilated by the destruction operators. Because of the elastic character of Eq. (19), for all real frequency oscillators, one gets stationary vacuum expectation values with no sign of instability.

There is no difficulty either for the complex frequency oscillators described by $\hat{b}_a$ and $\hat{c}_a$. Indeed, as shown in the Appendix, one can define two destruction operators $\hat{d}_{a+}$ and $\hat{d}_{a-}$, and use them to define the state as that annihilated by them at t_0. In this state, we get the following nonstationary vacuum expectation values:

$$\langle \hat{b}_{a'}(t)\hat{b}_a^\dagger(t')\rangle = \frac{\delta_{a',a}}{2} e^{-i\omega_a(t-t')} e^{\Gamma_a(t+t'-2t_0)},$$

$$\langle \hat{c}_{a'}(t)\hat{c}_a^\dagger(t')\rangle = \frac{\delta_{a',a}}{2} e^{-i\omega_a(t-t')} e^{-\Gamma_a(t+t'-2t_0)},$$ (55)

$$\langle \hat{b}_{a'}(t)\hat{c}_a^\dagger(t')\rangle = i\frac{\delta_{a',a}}{2} e^{-i\lambda_a(t-t')}.$$

As expected, the expression in $bb^\dagger$ ($cc^\dagger$) leads to an exponentially growing (decaying) contribution, whereas the cross term is constant at equal time. This behavior is really peculiar to unstable systems. Even though the metric is stationary, there is no normalizable state in which the expectation values of the b, c operators are constant. Stationary states do exist though, but they all have an infinite norm, see the Appendix.

Using Eq. (12), and putting $t_0 = 0$ for simplicity, the two-point function in this vacuum state is

$$\langle \hat{\phi}(t,x)\hat{\phi}(t',x')\rangle = \int_0^\infty d\omega e^{-i\omega(t-t')}[\phi_\omega^u(x)(\phi_\omega^u(x'))^*$$
$$+ \phi_\omega^v(x)(\phi_\omega^v(x'))^*]$$
$$+ i\Sigma_{a=1,N} \text{Re}(e^{-i\lambda_a(t-t')}\varphi_a(x)\psi_a^*(x')$$
$$- e^{-i\lambda_a^*(t-t')}\psi_a(x)\varphi_a^*(x'))$$
$$+ \Sigma_{a=1,N} \text{Re}(e^{-i(\lambda_a t-\lambda_a^* t')}\varphi_a(x)\varphi_a^*(x')$$
$$+ e^{-i(\lambda_a^* t-\lambda_a t')}\psi_a(x)\psi_a^*(x')).$$ (56)

We notice that the last term, the second contribution of complex frequency modes is real, as a classical term (a

stochastic noise) would be. We shall return to this point below. We also notice that the second term is purely imaginary and, when evaluated at the same point $x = x'$, it is confined inside the horizons since φ_a vanishes for $x \ll -L$ whereas ψ_a does it for $x \gg L$. Thus, it will give no asymptotic contribution to local observables.

2. The asymptotic fluxes

Our aim is to characterize the asymptotic particle content encoded in the growing modes φ_a. To this end it is useful to introduce a particle detector localized far away from the BH-WH pair. We take it to be sitting at $x \gg L$, in the R region on the right of the BH horizon. We assume that it oscillates with a constant frequency $\omega_0 > 0$, and that its coupling to ϕ is switched on at $t = -\infty$, and switched off suddenly at $t = T \gg t_0$ in order to see how the response function is affected by the laser effect a finite time after the formation of the BH-WH pair at $t_0 = 0$.

When the detector is initially in its ground state, to second order in the coupling g with the field, the probability to find it excited at time T is given by [28]

$$P_e(T) = g^2 \int_{-\infty}^{T} dt' \int_{-\infty}^{T} dt e^{-i\omega_0(t-t')} \langle \hat{\phi}(t, x) \hat{\phi}(t', x) \rangle,$$

$$= g^2 \Sigma_a |\beta_a|^2 |\varphi_{\lambda_a}^u(x)|^2 \left| \int_{t_{\omega_a}(L,x)}^{T} dt e^{-i(\omega_0 - \lambda_a)t} \right|^2,$$

$$= g^2 \Sigma_a \frac{|\varphi_{\omega_a}^u(x)|^2}{(\omega_0 - \omega_a)^2 + (\Gamma_a)^2} \bar{n}_{\omega_a}(T, x), \qquad (57)$$

where

$$\bar{n}_{\omega_a}(T, x) = |\beta_a|^2 e^{2\Gamma_a(T - t_{\omega_a}(L,x))}$$
$$\times |1 - \exp\{-[\Gamma_a + i(\omega_0 - \omega_a)](T - t_{\omega_a})\}|^2. \qquad (58)$$

To get the second line of Eq. (57), we used $\varphi_a(x) \sim \beta_a \varphi_{\lambda_a}^u(x)$, the fact that the BH-WH pair is formed at $t = 0$, and that it takes a time $t_{\omega_a}(L, x)$ for the mode $\varphi_{\lambda_a}^u$ to reach the detector at x. To get the third line, we used the inequality $\Gamma_a T_{\omega_a}^b \ll 1$ as in Eq. (51). The meaning of the various factors appearing in P_e is the following. The sum over a means that all unstable modes contribute, but the Lorentz functions restrict the significant contributions to frequencies ω_a near ω_0, that of the detector. The prefactor $|\varphi_{\omega_a}^u(x)|^2$ depends on the norm of the corresponding mode evaluated at the detector location, as in the usual case. The function $\bar{n}_{\omega_a}(T, x)$ acts as the number of particles of frequency ω_a received by the detector at time T, and at a distance $x - L \gg \kappa_B^{-1}$ from the BH horizon. It depends on the number initially emitted ($= |\beta_a|^2$) multiplied by the exponential governed by $T - t_{\omega_a}$, the lapse of time since the onset of the BH-WH pair minus the time needed to reach the detector at x.

From the response function of a localized detector, it is clear that one cannot distinguish between the noise due to quanta of the real frequency modes ϕ_ω^u and that carried by the growing modes φ_a, because both modes asymptote to the WKB waves φ_ω^u which are asymptotically complete. In this respect it is particularly interesting to compute the de-excitation probability P_d which governs the (spontaneous + induced) decay of the detector. It is obtained by replacing ω_0 by $-\omega_0$ in the first line of Eq. (57) [28]. In that case, one finds that the spontaneous decay only comes from the ϕ_ω^u whereas the induced part only comes from the φ_a. The induced part equals that of P_e since the asymptotic contribution of the φ_a to Eq. (56) is real. We are not aware of other circumstances where orthogonal modes with different eigenfrequency (here ϕ_ω^u and φ_a) are combined in this way in the spontaneous + induced de-excitation probability P_d, or equivalently, contribute in this way to the commutator and the anticommutator of the field, i.e. with the φ_a only contributing to the latter. (For damped modes, the commutator and the anticommutator are related differently, see, e.g. Appendices A and B in [29].) The lesson we can draw is the following: even though the modes ϕ_ω^u are orthogonal to the growing modes φ_a, their respective contribution to the asymptotic particle content cannot be distinguished by external devices coupled to the field.

It is also interesting to compute the asymptotic outgoing energy flux $\langle \hat{T}_{uu}(t, x) \rangle$, where $T_{uu} = [(\partial_t - \partial_x)\phi]^2$. At large distances in the R region, using $\varphi_a \sim \beta_a \varphi_{\lambda_a}^u$ and Eq. (56), the renormalized value of the flux is

$$\langle \hat{T}_{uu}(t, x) \rangle = \Sigma_a |(\partial_t - \partial_x) e^{-i\lambda_a t} \varphi_a(x)|^2$$
$$= \Sigma_a |\beta_a|^2 |(\partial_t - \partial_x) e^{-i\lambda_a t} \varphi_{\lambda_a}^u(x)|^2. \qquad (59)$$

It only depends on the discrete set of complex frequency modes. Yet, because of the imaginary part of λ_a defines a width $= \Gamma_a$, the spectrum of real frequencies ω is continuous, as can be see in Eq. (57). In fact the observables can either be written in terms of a discrete sum over complex frequencies, or as a continuous integral of a sum of Lorentz functions centered at ω_a and of width Γ_a. However, this second writing is only approximative and requires that the inequality $\Gamma_a T_a^b \ll 1$ of Sec. IV C be satisfied to provide a reliable approximation.

3. The correlation pattern

As noticed in [1], because of the bounces of the trapped modes, the asymptotic fluxes possess nontrivial correlations on the *same* side of the horizon, and not across the horizon as in the case of Hawking radiation without dispersion [21,28,30], or in a dispersive medium [25]. These new correlations are easily described in the wave packet language of that reference, or that of Sec. V A. When using frequency eigenmodes, they can be recovered through constructive interferences, as in Sec IV F. of [9]. Indeed,

when the complex frequencies modes form a dense set so that the dispersion of the waves can be neglected, the sum over a in Eq. (56) constructively interferes at equal time for two different positions x and x' separated by a propagation time

$$\partial_\omega S^u(x, x') = \int_x^{x'} dx \partial_\omega k^u_\omega(x) = T^b_\omega, \qquad (60)$$

where T^b_ω is the bounce time of Eq. (43). This is because the differences $\omega_a - \omega_{a+n}$ are equal $n \times 2\pi/T^b_\omega$ since the ω_a are solutions of Eq. (40). With more precision, the conditions for having these multimodes interferences are, on one hand, $\omega_{\max}/\kappa \gg 1$ so that dispersion hardly affects the modes and, on the other hand, $\kappa L \gg 1$, so as to have many modes for ω below the Hawking temperature $\sim \kappa$. (For higher frequencies $|\beta_a|^2$, which governs the intensity of the correlations, is exponentially damped.)

However, since dispersive effects grow and since the most unstable mode progressively dominates the two-point function of Eq. (56), at sufficiently large time the above multimode coherence will be destroyed and replaced by the single mode coherence of the most unstable one. This is unlike what is obtained when dealing with a single BH or WH horizon because in that case [30] the pattern is stationary, and all frequencies steadily contribute (significantly for $\omega \lesssim \kappa$). In the present case, at late time, if ϕ_1 is the most unstable one, the correlation pattern is given by

$$\langle \hat{\phi}(t, x)\hat{\phi}(t', x')\rangle = e^{\Gamma_1(t+t')} \times \mathrm{Re}(e^{-i\omega_1(t-t')}\varphi_1(x)\varphi_1^*(x')). \qquad (61)$$

The asymptotic pattern, for $x \gg L$, is obtained using $\varphi_1(x) \sim \beta_1 e^{-\Gamma_1 t_{\omega_1}(L,x)}\varphi^u_{\omega_1}(x)$. It is very similar to that of Eq. (54) found by sending a classical wave packet, see Appendix C of [9] for a discussion of the correspondence between statistical correlations encoded in the two-point function and deterministic correlations encoded in the mean value when dealing with a wave packet described by a coherent state.

So far we worked under the assumption that the u-v mixing coefficients are negligible. When taking them into account, one obtains a richer pattern which is determined by the complex frequency modes solutions of Sec. IV E.

4. The small supersonic region limit

When L of Eq. (2) [or $D = |v_+|/c - 1$] decreases, the number of solutions of Eq. (40) diminishes. Therefore, in the narrow supersonic limit $\kappa L \to 0$, there is a threshold value for κL given D, below which there is no solution. In that case, there are no unstable mode, and no laser effect. In fact no flux is emitted, and this even though the surface gravity of the BH (and that of the WH) is not zero. The reason is that there is no room for the negative frequency modes $\phi_{-\omega}$ to exist. In agreement with the absence of radiation, the entanglement entropy of the BH [31] would vanish, because it accounts for the number of entangled

modes across the horizon and thus of opposite frequency, see [32] for the effects of dispersion on the entanglement entropy.

5. Comparison with former works

It is instructive to compare our expressions to those obtained in [1] and in [5]. Our expressions differ from theirs because the discrete character of the set of complex frequency modes was ignored in these works. As a result, a continuous spectrum was obtained. Yet this spectrum possesses rapid superimposed oscillations stemming from the interferences that are present in $S_{21}(\omega)$ of Eq. (31). *A priori* one might think that they could coincide with our frequencies $\omega_a = \mathrm{Re}\lambda_a$. However, as noticed in [5], their value are insensitive the phase governed by $S^{(2)}_\omega$, whereas it plays a crucial role in Eq. (40). We found no regime in which the two sets could approximately agree. Therefore, as far as the fine properties of the spectrum are concerned, the predictions of [1,5] are not trustworthy.

Nevertheless, when the density of complex frequency modes is high, and when ignoring these fine properties, the average properties derived using [1,5] coincide with ours. Indeed, when considering the mean flux in frequency intervals $\Delta\omega \gg 1/L$, the rapid oscillations found in [1,5] are averaged out. As a result the mean agrees with that over the contributions of complex frequency modes. This can be explicitly verified by comparing the norm of our discrete modes φ_a, ψ_a with the continuous norm of the negative frequency modes used in [1,5]. In the limit of Sec. IV C the relevant contribution to the overlap $(\psi_a|\varphi_a)$ comes from the negative frequency mode $\varphi_{-\omega}$. Moreover, φ_a and ψ_a are given by a sum of (normalized) WKB waves $\varphi^{(1)}_{-\omega_a}$ and $\varphi^{(2)}_{-\omega_a}$ times a prefactor $= \sqrt{2\pi/T^b_{\omega_a}}$, where $T^b_{\omega_a}$ is the bounce time given in Eq. (43). This is just what is needed for approximating the discrete sum in Eq. (56) by a continuous integral $\int d\omega$ with a measure equal to one.

In conclusion, when the number of bound modes becomes small, the difference between our description and the continuous one increases. This difference is maximal when there are no solution of Eq. (40). In this case, no radiation is emitted, something which cannot be derived by the continuous approach of [1,5].

VI. CONDITIONS FOR HAVING A LASER EFFECT

Having understood the black hole laser effect, it is worth identify in more general terms the conditions under which a laser effect would develop. We define a "laser effect" by the fact that a free field possesses complex frequency ABM while being governed by a quadratic Hermitian Hamiltonian, as in Eq. (10), and a conserved scalar product, as in Eq. (9). The field thus obeys an equation which is stationary, homogeneous, and second order in time. Let us note that this type of instability is often referred to as a *dynamical instability* [6,16,33], a denomination which in-

dicates that quantum mechanics is not needed to describe/obtain it. Let us also note that we do not consider the case where the frequency of the ABM is purely imaginary. Such dynamical instability seems to belong to another class than that we are considering, see the end of this subsection for more discussion on this.

Using semiclassical concepts, the conditions for obtaining complex frequency ABM are the following :

(i) For a finite range of the real part of frequency ω, WKB solutions with both signs of norm should exist, or equivalently, positive norm WKB solutions should exist for both sign of ω. This is a rather strong condition which requires that the external field (gravitational or electric) must be strong enough for this level crossing to take place, i.e. for the general solution be a superposition as in Eq. (22).

(ii) These WKB solutions of opposite norm must mix when considering the exact solutions of the mode equation. In other words, there should be connected by a nonzero tunneling amplitude. This is a very weak condition as different WKB branches are generally connected to each other. In our case it means that z_ω and w_ω appearing in Eq. (31) should not vanish.

(iii) One of these WKB solutions must be trapped so that the associated wave packets will bounce back and forth. This is also a rather strong condition.

(iv) The depth of the potential trapping these modes should be deep enough so that at least one pair of bound modes can exist, see Eq. (40). This condition is rather mild once the first three are satisfied.

When these conditions are met for a sufficiently wide domain of frequency ω, they are sufficient to get a laser effect, and they apply both when the external region is finite [6] or infinite. Being based on semiclassical concepts, *stricto sensu*, they cannot be considered as necessary. But we are not here after mathematical rigor, rather we wish to identify the relevant conditions in physically interesting situations.

In this respect, it should be noticed that when only the first two conditions are satisfied, one obtains a *vacuum instability* [28], also called a *superradiance* in the context of rotating bodies [33,34]. Hence, whenever there is a vacuum instability, one can engender a laser effect by introducing a reflecting condition, as was done in [13,14], or by modifying the potential, so that the last two conditions are also satisfied. It should be clear that when the laser effect takes place, it replaces the vacuum instability rather than occurs together with it. Indeed, as proven in Sec. IV C, the frequency of the trapped modes are generically complex. The possibility of having a trapped mode (subjected to a vacuum instability prior introducing the reflecting condition) with a real frequency is of measure zero, as two conditions must be simultaneously satisfied. To give an example of the replacement of a vacuum instability by a laser effect, let us consider the archetypal case of pair production in a static electric field studied by Heisenberg [35] and Schwinger [36]. In that case, one obtains a laser effect by replacing the Coulomb potential $A_0 = Ex$ by $A_0 = E|x|$ which traps particles of charge q for $qE < 0$, for frequencies $\omega < -m$ where m is their mass. We hope to return to such pedagogical examples in the near future.

In conclusion, we make several remarks. Even though the complex frequency ABM are orthogonal to the real frequency modes, as is it guaranteed by Eq. (11), the *asymptotic* quanta associated with these modes are not of a new type but are, as we saw, superpositions of the standard ones associated with real frequency modes. If laser instabilities can be studied in classical terms, the quantum aspects are not washed out. For instance, when considering a charged field, the charge received as infinity is still quantized, albeit its mean value is described by a complex frequency ABM. Moreover, in all cases, when the instability ceases, the number of emitted quanta is a well-defined observable governed by standard destruction/creation operators as those appearing in Eq. (A13). From this it appears that dynamical instabilities governed by an ABM with a purely imaginary frequency, as e.g. the Gregory-Laflamme instability [37], belong to another class since this asymptotic decomposition in terms of quanta does not seem available. Whether it could nevertheless makes sense to quantize such instability is a moot point.

VII. CONCLUSIONS

We showed that the black hole laser effect should be described in terms of a finite and discrete set of complex frequency modes, which asymptotically vanish. We also showed that these modes are orthogonal to the continuous set of real frequency modes which are only elastically scattered, and which therefore play no role in the laser effect. In Sec. IV, using the simplifying assumption that the near horizon regions are thin, we determined the set of complex frequencies and the properties of the modes using the approach that combines and generalizes [5,11].

We described how an initial wave packet is amplified as it propagates in the BH-WH geometry. When the density of complex frequency modes is high we recovered the picture of [1] at early times. Instead, at late time, or when the density is low, the successive emissions of distinct wave packets associated with the bouncing trajectories are replaced by an oscillating flux governed by the most unstable mode.

We then computed in quantum settings how the growth of the complex frequency modes determine the asymptotic fluxes when the initial state at the formation of the BH-WH pair is vacuum. Because of the width associated with the instability, the spectral properties of the fluxes are continuous albeit they arise from a discrete set of modes. The properties we obtained significantly differ from those

found in [1,5]. We also briefly described the properties of the correlation pattern at early times when the number of complex frequency modes is large, and at late time when only the most unstable mode contributes. When the supersonic region between the two horizons is too small so that there is no solution to Eq. (40), we concluded that there is no instability, that no flux is emitted, and that the entanglement entropy vanishes. Finally, in Sec. VI we gave the general WKB conditions under which a laser effect would obtain starting from the standard concepts that govern a vacuum instability in quantum field theory.

This work poses several questions which deserve further study. In Bose condensates, thebackreaction due to the instability and the suppression of the instability could be computed using the Gross-Pitaevskii equation. Fermionic fields [1,12] should be further studied to reveal the roles of the N pairs of complex frequency modes, which are not ABM, but which indicate that the naive vacuum will decay in the lowest energy state vacuum plus N asymptotic quanta by spontaneously emptying the N Dirac holes which are trapped inside the horizons.

APPENDIX A: UPSIDE DOWN HARMONIC OSCILLATORS

1. Real upside down oscillator

We review the quantization of upside down oscillators. To begin with, we start with a single real upside down harmonic oscillator. Its Hamiltonian is

$$H = \tfrac{1}{2}(p^2 - \Gamma^2 q^2), \qquad (A1)$$

when written in terms of position q and conjugated momentum p, obeying the standard equal time commutator (ETC) $[q, p] = i$. Introducing the "null" combinations

$$b = \frac{1}{\sqrt{2\Gamma}}(p + \Gamma q), \qquad c = \frac{1}{\sqrt{2\Gamma}}(p - \Gamma q), \qquad (A2)$$

one gets

$$H = \frac{\Gamma}{2}(bc + cb). \qquad (A3)$$

One verifies that they obey the ETC $[b, c] = i$. The ordering of b and c in H follows from that of Eq. (A1). The equations of motions are

$$\dot{b} = (-i)[b, H] = \Gamma b, \qquad \dot{c} = (-i)[c, H] = -\Gamma c, \qquad (A4)$$

thereby establishing that b (c) is the growing (decaying) mode $b = b_0 e^{\Gamma t}$, ($c = c_0 e^{-\Gamma t}$).

It is now relevant to look for stationary states. In the b representation ($c = -i\partial_b$), the stationary Schroedinger equation $H\Psi_E = E\Psi_E$ reads

$$-ib\partial_b \Psi_E(b) = \left(\frac{E}{\Gamma} - \frac{i}{2}\right)\Psi_E(b). \qquad (A5)$$

Solutions exist for all real values of E, and the general solution is

$$\Psi_E(b) = A_E \theta(b)(b)^{iE/\Gamma - 1/2} + B_E \theta(-b)(-b)^{iE/\Gamma - 1/2}. \qquad (A6)$$

Since the spectrum is continuum, one should adopt a Dirac delta normalization $\langle \Psi_{E'} | \Psi_E \rangle = \delta(E - E')$. This gives

$$|A_E|^2 + |B_E|^2 = \frac{1}{2\pi\Gamma}. \qquad (A7)$$

Imposing that the solution be even in q (p, b, or c) imposes $A_E = B_E$. The important lesson one should retain is that there is no square integrable stationary states. Therefore, in all physically acceptable states (i.e. square integrable) the expectation values of $q^2 + p^2$ will exponentially grow $\sim e^{2\Gamma t}$ at late time.

2. Complex oscillators

More relevant for us is the complex upside down harmonic oscillator. It can be described by the complex variables $q = q_1 + iq_2$, $p = p_1 + ip_2$ where q_1, p_1 and q_2, p_2 are Hermitian and obey the standard ETC given above. We then introduce the complex b and c variables

$$b = \frac{1}{\sqrt{4\Gamma}}(p + \Gamma q), \qquad c = \frac{1}{\sqrt{4\Gamma}}(p - \Gamma q), \qquad (A8)$$

which are normalized so that they obey the ETC

$$[b, c^\dagger] = i. \qquad (A9)$$

We then look for the (Hermitian) Hamiltonian, which gives the following equations:

$$\dot{b} = (-i)[b, H] = -i\lambda b, \qquad \dot{c} = (-i)[c, H] = -i\lambda^* c, \qquad (A10)$$

where $\lambda = \omega + i\Gamma$. It is given by

$$H = -i\lambda c^\dagger b + i\lambda^* b^\dagger c. \qquad (A11)$$

It is instructive to reexpress this system is term of a couple of destruction and creation operators d_-, $d_-^\dagger$, d_+, $d_+^\dagger$, which are given, at a given time t_0, by

$$b = \frac{1}{\sqrt{2}}(d_+ - id_-^\dagger), \qquad c = \frac{1}{\sqrt{2}}(-id_+ + d_-^\dagger). \quad (A12)$$

They obey the standard commutation relations $[d_i, d_j^\dagger] = \delta_{ij}$, and Eq. (A11) reads

$$H = \omega(d_+^\dagger d_+ - d_-^\dagger d_-) + \Gamma(d_- d_+ + d_+^\dagger d_-^\dagger),$$
$$= H_0 + H_{sq}. \tag{A13}$$

In the first term one recovers the standard form of an Hamiltonian is the presence of stationary modes with opposite frequencies [9]. The second term induces a squeezing of the state of the d_-, d_+ oscillators, which grows linearly with time.

To set initial conditions, and to be able to read the result of the instability in terms of quanta, it is appropriate to use this decomposition of H and to work in the "interacting" picture where the operators b, c only evolve according to H_0, and where the squeezing operator acts on the state of the field. Indeed, in this picture the states can be expressed at any time as a superposition of states with a definite occupation numbers n_- and n_+.

3. The Hermitian conjugated $S^\dagger$

To make contact with the treatment of Sec. IV B, it is appropriate to express the time evolution of linear operators in a S-matrix language. We introduce the operator

$$A[Z(t)] = x(t)b_0 + y(t)c_0 = \begin{pmatrix} b_0 & c_0 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{A14}$$

where b_0 and c_0 are the operators b, c evaluated at t_0. Using Eq. (A10), the time evolution of $Z = (x, y)$ is

$$i\partial_t \begin{pmatrix} x \\ y \end{pmatrix} = H\begin{pmatrix} x \\ y \end{pmatrix}, \tag{A15}$$

where the 2×2 matrix H is diag(λ, λ^*). By definition, the S matrix brings Z from t_0 to $t_0 + t$. It is given by

$$S = \begin{pmatrix} e^{-i\lambda t} & 0 \\ 0 & e^{-i\lambda^* t} \end{pmatrix}. \tag{A16}$$

To define $S^\dagger$ we need to refer to the matrix encoding the ETC of b and c, see Eq. (A9):

$$Q = \begin{pmatrix} [b, b^\dagger] & [b, c^\dagger] \\ [c, b^\dagger] & [c, c^\dagger] \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{A17}$$

This matrix defines a scalar product in the space of the vectors Z through

$$[A(Z'), A(Z)^\dagger] = (Z|Z') = \begin{pmatrix} x^* & y^* \end{pmatrix}Q\begin{pmatrix} x' \\ y' \end{pmatrix}. \tag{A18}$$

The Hermitian conjugate defined by $(SZ|Z') = (Z|S^\dagger Z')$ obeys

$$QS^\dagger = S^{*T}Q. \tag{A19}$$

In the present case, using Eq. (A16), we find

$$S^\dagger(\lambda) = \begin{pmatrix} e^{i\lambda t} & 0 \\ 0 & e^{i\lambda^* t} \end{pmatrix} = [S(\lambda^*)]^*. \tag{A20}$$

In the case where S would have been nondiagonal, we should also transpose the matrix

$$S^\dagger(\lambda) = [S(\lambda^*)]^{*T}. \tag{A21}$$

We see that the expression of the components of $S^\dagger$ in terms of those of S is unusual because the norm of Z, instead of being $|x|^2 + |y|^2$, it is of the form Im(xy^*). However, when λ is real, Eq. (A20) gives the usual relation. Hence, for complex λ, it can be viewed as an analytical continuation of the real frequency case.

The Hermicity of the Hamiltonian guarantees that the S matrix is unitary in the sense that the scalar product $(Z'|Z)$, or the ETC relations Eq. (A9), are conserved. One verifies that Eq. (A20) implies the usual unitarity condition

$$S^\dagger(\lambda)S(\lambda) = I, \tag{A22}$$

for λ real and complex.

The above analysis applies to the modes φ_a, ψ_a when considering the field operator restricted to the a sector, i.e. $A(Z_a) = xb_a\varphi_a + yc_a\psi_a$, because φ_a, ψ_a have complex conjugated frequencies and because b_a, c_a obey Eq. (21). Hence, the Hermitian conjugated matrix appearing in Eq. (38) should be understood as in Eq. (A21).

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