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Note on gravity, entropy, and BF topological field theory

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In this article I argue that the expression for entropic force, used as a starting point in Verlinde's derivation of Newton's law [E. P. Verlinde, arXiv:1001.0785], can be deduced from the first principles if one assumes that the microscopic theory behind his construction is the topological SO(4, 1) BF theory coupled to particles.

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I. INTRODUCTION

There is a number of evidences suggesting a deep relation between gravity and thermodynamics. In the early 1970s four laws of black hole dynamics were formulated [1], whose form closely resembled the four laws of thermodynamics. It was then realized that this similarity between gravity and thermodynamics reaches far beyond formal analogy: the bold conjecture of Bekenstein [2] that an area of the black hole horizon is proportional to thermodynamical entropy has been strengthened by the Hawking discovery of black hole radiation [3]. It turned out that indeed, as suggested by the four laws of black hole dynamics, black holes behave as thermal systems, with entropy and temperature proportional to the area and surface gravity, respectively.

About 20 years later, in a remarkable paper Jacobson [4] showed that from the proportionality between area and entropy [2] taken as a fundamental principle one can derive the full Einstein equations of gravity. This idea was then discussed in depth by Padmanabhan and others; see [5] for a recent review and references.

Building on these developments, in a recent paper [6], Verlinde argued that the force of the second law of dynamics and that of Newton's law of gravity can both have their origin in thermodynamics, and can be understood in terms of the entropic force (a similar idea, based on equipartition of energy, appeared earlier in [7]). Within weeks several follow-up works appeared, testing this idea in various contexts (see, for example [8] for the discussion on the context of cosmology and [9] for derivation of the Coulomb law from thermodynamics.) In particular, in [10] Smolin argued that Verlinde's proposal could be naturally realized in the context of loop quantum gravity, and suggested its relations to constrained topological field theories. This idea is the starting point of the present work.

Let us recall the major points of Verlinde's reasoning. The basic postulate of his work (see [6] for a detailed discussion) is the following assumption:

(i) Consider a holographic screen $\mathcal{S}$. If a particle of mass m crosses the screen, then the change of entropy of the screen is proportional to the mass and displacement Δx

$$\Delta S \sim m\Delta x. \tag{1.1}$$

(ii) It then follows from the first law of thermodynamics that if there is the temperature T that can be associated with the screen, then the entropic force F exists, satisfying

$$F\Delta x = T\Delta S, \tag{1.2}$$

so that

$$F \sim mT. \tag{1.3}$$

As shown by Verlinde, Newton's law of gravity can be derived assuming just this postulate, energy equipartition, and the holographic principle. The reasoning goes as follows. Consider a spherical screen $\mathcal{S}$ at the center of which a localized, static chunk of matter of mass M is placed. Assume that the radius of the screen is much larger than the size of the chunk, so that we can assume spherical symmetry of the problem. The holographic principle says that the number of bits N on the screen $\mathcal{S}$ is proportional to its area[1]

$$N = \frac{A}{G}, \tag{1.4}$$

which is essentially the statement that the screen $\mathcal{S}$ is made of pixels of Planck size (this is the point where, as pointed out in [10], loop quantum gravity with its quantization of area operator [11] naturally fits). To complete the derivation of Newton's law one has to assume the equipartition of energy on the screen, from which it follows the relation between energy $E = M$ and the temperature

$$M = \tfrac{1}{2}NT. \tag{1.5}$$

Finally, assume that the area of the screen $\mathcal{S}$ is given by the Euclidean formula

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[1]In what follows I will use the units in which the velocity of light c, the Planck constant $\hbar$, and the Boltzmann constant k_B are all equal to 1.

$$A = 4\pi R^2. \tag{1.6}$$

It follows from Eqs. (1.4), (1.5), and (1.6) that the temperature satisfies

$$T = \frac{2GM}{4\pi R^2}, \tag{1.7}$$

which, together with the postulate (1.3) reproduces Newton's law, $F = GMm/R^2$. It is worth noticing that for Schwarzschild black hole horizon $R = 2GM$, Eq. (1.7) reproduces the correct expression for Bekenstein-Hawking temperature $T_{\rm BH} = (8\pi GM)^{-1}$.

As argued by Verlinde the above reasoning is robust and general, the only weak point being the origin of the entropic force (1.2) and (1.3). Certainly, there must be some microscopic degrees of freedom responsible for its emergence, and below I will argue that they, and the corresponding force, arise quite naturally in the formulation of gravity as a constrained topological BF theory.

The plan of this paper is as follows. In the next section, I will recall the formulation of gravity as a constrained $\mathsf{SO}(4,1)$ BF theory and its coupling to particles. These technical results will be needed for the derivation of Verlinde's entropic force. The reader might decide to skip these technicalities and jump directly to Sec. III, where the main argument of the paper will be presented. The last section is devoted to discussion and conclusions.

II. GRAVITY AS A CONSTRAINED TOPOLOGICAL FIELD THEORY

It has been known for quite some time that gravity can be formulated as a constrained topological field theory. The most popular model of this kind is given by the Plebanski action [12], being an action of the constrained BF theory of the Lorentz $\mathsf{SO}(3,1)$ group. This model is a starting point for four-dimensional spin foam models building (see e.g., [13]). In the present context, however, it is convenient to consider a different model, based on the de Sitter gauge group $\mathsf{SO}(4,1)$ (the anti–de Sitter model can be constructed analogously). The main reason for this choice is that the $\mathsf{SO}(4,1)$ model allows for natural particles coupling.

The action of the $\mathsf{SO}(4,1)$ constrained BF theory has the following form [14,15]:

$$S = \int \mathsf{B}^{IJ}\wedge \mathsf{F}_{IJ} - \frac{\beta}{2}\mathsf{B}^{IJ}\wedge \mathsf{B}_{IJ} - \frac{\alpha}{4}\mathsf{B}^{ab}\wedge \mathsf{B}^{cd}\epsilon_{abcd}, \tag{2.1}$$

where F_{IJ} is the curvature of the $\mathsf{SO}(4,1)$ connection one-form A_{IJ} and B^{IJ} is a two-form valued in the algebra of the $\mathsf{SO}(4,1)$ group. Here the algebra indices $I, J, \ldots$ take values $0, \ldots, 4$, wile the indices $a, b, \ldots, = 0, \ldots, 3$ label Lorentz subalgebra $\mathsf{SO}(3,1)$ of $\mathsf{SO}(4,1)$. If one decomposes the connection A_{IJ} into Lorentz and translational parts

$$\mathsf{A}^{ab} = \omega^{ab}, \qquad \mathsf{A}^{a4} = \frac{1}{\ell}e^a, \tag{2.2}$$

solves the equations of motion for B^{IJ} resulting from (2.1), and plugs the result back into this action, one gets as a result the first order action of general relativity

$$S = \frac{1}{2G}\int R^{ij}(\omega)\wedge e^k \wedge e^l \epsilon_{ijkl} - \frac{\Lambda}{12G}\int e^i \wedge e^j \wedge e^k \wedge e^l \epsilon_{ijkl}$$

accompanied by the Holst term and a number of topological terms (see [15] for details). To get the action of general relativity the coupling constants α, β of the action (2.1) and the length scale ℓ necessary for making the tetrad e^a_μ dimensionless are to be related to Newton's constant G, cosmological constant Λ, and Immirzi parameter γ as follows:

$$\gamma = \frac{\beta}{\alpha}, \qquad \frac{1}{\ell^2} = \frac{\Lambda}{3}, \qquad G = \frac{3\alpha(1-\gamma^2)}{\Lambda}. \tag{2.3}$$

Let us pause for a moment to comment on the structure of the action (2.1). If α vanishes, the resulting action is just that of a topological field theory with no dynamical degrees of freedom. The local degrees of freedom of gravity (like gravitational waves or the presence of Newton's potential) appear only if the gauge breaking term, controlled by the coupling constant α, is nonzero. Therefore the action (2.1) clearly exhibits the split between topological and local degrees of freedom. In other words it is only the last term of (2.1) that knows about the dynamics of gravity. In the context of the present paper an obvious question arises: is it possible that the topological action describes the primary degrees of freedom of theory, while the gauge breaking term (i.e., gravity) arises as an emergent phenomenon from entropic force? As it is argued below at least Newton's force between massive bodies can be understood in this way.

It is worth noticing also that the form of the gauge breaking term in the action (2.1) is justified only by the fact that the theory described by this action turns out, at the end of the day, to be equivalent on shell to general relativity. Therefore, it would be very interesting to find a principle, which would explain the presence of this term (see [14] for an interesting proposal in this context). Deducing Newton's law is the first step in this direction.

As explained in [16], one can straightforwardly add point particles to the theory described by (2.1) by identifying them with Wilson lines. To do that one includes the localized breaking of the gauge symmetry along the one-dimensional worldline. The gauge degrees of freedom are then promoted to dynamical degrees of freedom, which, in the case $\alpha \neq 0$ reproduce the dynamics of a relativistic particle coupled to gravity. For a single particle this idea is realized by choosing a worldline $\mathcal{P}$ and a fixed element K

in the Cartan subalgebra of the $\mathsf{so}(4,1)$ Lie algebra generated by two generators, the translational T^{04} and rotational T^{23} ones, depending on the particle rest mass and spin[2]

$$\mathsf{K} \equiv m\ell T^{04} + sT^{23}. \tag{2.4}$$

Note that the particle mass arises quite naturally in this picture in a purely algebraic way and is related to the one of the Casimirs of the gauge algebra. Then the action for the particle at rest takes the form

$$S_P(\mathsf{A}) = -\int d\tau \,\mathrm{Tr}(\mathsf{K}\mathsf{A}_\tau(\tau)), \tag{2.5}$$

where τ parametrizes the worldline $z^\mu(\tau)$ and $\mathsf{A}_\tau(\tau) \equiv \mathsf{A}_\mu(z(\tau))\dot{z}^\mu$.

The action of the particle moving in an arbitrary way is obtained by realizing that the moving particle is related to the one at rest by an appropriate $\mathsf{SO}(4,1)$ transformation acting on the worldline. In this way the gauge degrees of freedom at the location of the particle become its physical degrees of freedom. Thus the Lagrangian of the dynamical particle has the form

$$L(z,\mathsf{h};\mathsf{A}) = -\mathrm{Tr}(\mathsf{K}\mathsf{A}^{\mathsf{h}}_\tau(\tau)) \qquad S = \int d\tau L(z,\mathsf{h};\mathsf{A}), \tag{2.6}$$

with

$$\mathsf{A}^{\mathsf{h}} = \mathsf{h}^{-1}\mathsf{A}\mathsf{h} + \mathsf{h}^{-1}d\mathsf{h},$$

which can be rewritten as

$$L(z,\mathsf{h};\mathsf{A}) = L_1(z,\mathsf{h}) - \mathrm{Tr}(\mathsf{J}\mathsf{A}_\tau), \tag{2.7}$$

with the first term being the particle kinetic Lagrangian

$$L_1(z,\mathsf{h}) = -\mathrm{Tr}(\mathsf{h}^{-1}\dot{\mathsf{h}}\mathsf{K}), \tag{2.8}$$

while the second describes its coupling to the connection A, with J being the dynamical particle momentum/spin and is given by

$$\mathsf{J} \equiv \mathsf{h}\mathsf{K}\mathsf{h}^{-1}. \tag{2.9}$$

It can be shown that from (2.7) the correct particle equation of motion (Mathisson-Papapetrou equation) follows; the theory described by (2.1) and (2.7) leads to Einstein-Cartan equations with point sources carrying mass and spin (see [16] for a detailed discussion).

This completes our description of the theory. Let us now turn to the discussion of solutions of topological BF theory coupled to such defined particle.

[2]Here we consider massive particles only. An extension to the case of massless particles is straightforward.

Take the topological limit $\alpha \to 0$ in (2.1) and (2.7) and consider the resulting field equations [17] for the particle at rest at the origin of an appropriate coordinate system.[3] One finds

$$\mathsf{F}^{IJ} = \beta\mathsf{B}^{IJ}, \tag{2.10}$$

$$\mathsf{D}_{\mathsf{A}}\mathsf{B}^{IJ} = \mathsf{J}^{IJ}\delta_P, \qquad \delta_P = \delta^3(x)\varepsilon, \tag{2.11}$$

where D_{A} is the covariant derivative of connection A and ε is the volume three-form on a constant time surface.

If one then solves (2.10) for B and substitutes the result to (2.11) one finds that the left-hand side of the resulting equation is zero by virtue of Bianchi identity. It is clear therefore that there does not exist a nonsingular connection A satisfying these equations for a nonzero source. However, if one allows connections with stringlike singularity (Misner string [18], which is the gravitational counterpart of Dirac string), these equations can be solved.

In fact it turns out that a pointlike source must be accompanied by a string extending from the source to infinity. As argued in [17] the spacetime corresponding to the solution of these equations[4] is the (linearized) Taub-NUT spacetime

$$g = -(dt + n(1-\cos\theta)d\phi)^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.12}$$

with Taub-NUT charge

$$n = \bar{G}m, \qquad \bar{G} = G\frac{\gamma}{1+\gamma^2}. \tag{2.13}$$

This completes our brief description of the constrained $\mathsf{SO}(4,1)$ BF theory, its relation to gravity, and coupling to point sources.

III. ENTROPY AND GRAVITY FROM TOPOLOGICAL FIELD THEORY

In the previous section I argued that if one couples the $\mathsf{SO}(4,1)$ topological BF theory [which after gauge breaking down to $\mathsf{SO}(3,1)$ is equivalent to general relativity] to point particles, then the theory forces the particles to be accompanied by semi-infinite Misner strings. Moreover, the space-time corresponding to such solution is the Taub-NUT solution linearized in the charge n, which turns out to be proportional to Gm, where the particle mass m is the value of one of the two Casimirs of $\mathsf{SO}(4,1)$ (the second

[3]The reader may wonder that by using the coordinates, and the geometry, we let gravity sneak through the back door. Of course, we need geometry to formulate the model, but the relation between the local degrees of freedom of gravity and the geometrical quantities is not present at this level yet, because dynamical gravity is not there.

[4]In the limit $\ell \to \infty$, which corresponds to the vanishing cosmological constant. Since the Taub-NUT charge n does not depend on ℓ, taking the limit does not influence it.

one describes the spin, but here we discuss the spinless case only).

Knowing this, let us turn to deducing the form of entropic force acting on the particle. Suppose the test particle of mass m is at the distance R from the mass M, which we can assume to be also pointlike. Consider now, as in Verlinde's argument, a spherical screen $\mathcal{S}$ of radius R. Let the test particle move radially toward the central mass piercing the screen, and let its displacement be Δx. As a result we have now a segment of the Misner string of the test particle of the length Δx connecting it with the screen. Therefore the screen that previously was just a sphere[5] now becomes a sphere with a piece of Misner string, the line segment of length Δx attached, $\mathcal{S}'$.

Let me now turn to the main argument of this paper. It is well known that there is entropy associated with Misner string; see [19–22] where it is argued that the entropy of Misner string is intrinsically defined. In particular, using methods of conformal field theory Carlip [21] shows that the segment of the Misner string of the length Δx carries the entropy

$$\Delta S = \frac{1}{8\pi G} n\Delta x = \frac{1}{8\pi} m\Delta x. \tag{3.1}$$

Although this result has not been rigorously established in the present context of BF theory, it is unlikely that a formula analogous to (3.1) would not hold in this case as well. It seems clear that Misner string carries entropy, no matter what is the theory describing local and/or topological degrees of freedom. If one accepts this argument, it follows from simple dimensional analysis that the entropy of the segment of Misner string of the length Δx has to have the form

$$\Delta S = \zeta m\Delta x, \tag{3.2}$$

where ζ is the coefficient depending on the structure (and coupling constants) of the underlying theory.

The entropy (3.2) adds to the original entropy of the screen, and since it is proportional to the test particle displacement it leads to the emergence of the entropic force. Notice that since entropy increases when the test particle moves toward the mass M this entropic force is attractive. Also when the test particle which was initially inside the screen moves outside, the entropy decreases, since the contribution from the Misner string is no longer present.

Having (3.2) it is possible now to run the remaining part of Verlinde's argument essentially without modifications. The only point that is worth discussing is the equation relating the number of the screen pixels with area. Why is G the measure of area of a pixel? In loop quantum gravity this question finds its natural answer thanks to the fact that quantization of area in Planck scale units is the main result of this theory. It is not excluded that even in the context of BF theory one can define an area operator with discrete spectrum. Until this idea is supported (or disproved) by concrete calculations we can rely only on general intuitions. The theory at hand provides us with the dimensionful scale ℓ and the dimensionless coupling constant β. From the two it is possible to construct another constant of dimension of area

$$\bar{G} = \frac{3\beta}{\ell^2}, \tag{3.3}$$

which in the full theory (including nontrivial gauge breaking term) becomes proportional to Newton's constant of general relativity [cf. (2.3)]. Since β has some final value, and since ℓ is an infrared scale of the theory, it is quite natural to treat $\bar{G}$ as an intrinsic ultraviolet scale of the theory, and thus to replace (1.4) with

$$N = \frac{A}{\bar{G}}, \qquad A = 4\pi R^2, \tag{3.4}$$

which directly, by virtue of Verlinde's argument recalled in the Introduction, leads to Newton's law

$$F = \frac{GmM}{R^2}, \tag{3.5}$$

where $G = 4\pi\zeta\bar{G}$ is Newton's constant, whose value can be directly measured, e.g., in Cavendish experiment. This concludes the presentation of the main argument of this paper.

IV. CONCLUSIONS AND OUTLOOK

In this paper I argued that the form of entropic force being the starting point of the recent proposal of Verlinde [6] to seek the origin of gravity in thermodynamics can be understood if one assumes that the fundamental degrees of freedom behind it are described by the topological BF theory coupled to particle(s). The reason for this is that, as shown in [16] and discussed in [17], a particle carrying the charge of (anti) de Sitter $\mathsf{SO}(4,1)$ [$\mathsf{SO}(3,2)$] group coupled to the topological BF theory with the same gauge group must have Misner string attached. This string, in turn, carries entropy, which adds to the entropy of the holographic screen $\mathcal{S}$ when the particle crosses it, which results in emergence of the entropic force.

It should be stressed that the thought experiment of lowering the mass m described in the preceding section has been devised so as to ensure that the particle is indeed a test one, i.e., that one can safely neglect the backreaction in the form of the entanglement of the configurations of the test particle and the central mass. It can be therefore assumed that to the leading order the only result of this process will be the increase of the entropy of the screen as in Eq. (3.2). Notice that in order to run this argument one

[5]More precisely $\mathcal{S}$ consists of the sphere along with the attached string (or strings) emanating from the central mass M. But since we are only interested in the (infinitesimal) change of entropy, we do not have to consider them.

does not need to know what is the total entropy of the holographic screen; what matters is only its infinitesimal change.

There are however some more complicated configurations which should be analyzed and understood.[6] For example, consider again the test mass m hovering just outside the spherical holographic screen of radius R with mass M residing in its center. This time let the Misner string go from the test particle to the center of the screen and then radially to infinity. (It should be noted that this configuration cannot be obtained from the original one, considered above, by the action of diffeomorphism; the diffeomorphism acts on both the string and the screen and the number of points in which the string pierces the screen is diffeomorphism invariant.) Then pulling the string by an infinitesimal distance Δx should again lead to the increase of the entropy, and therefore the attractive force. However this configuration is much harder to analyze than the one considered above, because now the finite segment of test particle's Misner string is placed inside the screen and the entanglement of its degrees of freedom with the one originally present inside the screen cannot be, presumably, neglected any longer.

There are several problems of a more general nature that have to be solved before the idea described above turns to a solid proposal. First, one has to calculate the entropy of Misner string directly in the framework of BF theory, to fix the constant ζ in (3.2). This can be presumably done with the help of the method similar to that described in [21]. Second, it would be interesting to see if it is possible to improve on the part of the original Verlinde's argument that makes use of equipartition of energy to get the expression for temperature (1.7). One has therefore to analyze in depth the thermodynamics of the BF theory coupled to particles. It would be also desirable to investigate if in the case of BF theory one can define an area operator with discrete spectrum, resembling that of loop quantum gravity [11]. It is interesting to note in this context that in order to do this one would have to split the connection A into the translational and Lorentz parts, with the former related to area (and volume, and length) measurements. Thus it might be that the area measurement, necessary for Verlinde's construction to work, is the fundamental reason of the deep relation between the dynamics of gravity and breaking of the $\mathsf{SO}(4,1)$ gauge group down to its Lorentz subgroup, reflected in the form of the action (2.1). Last but not least one has to find out if it is possible to formulate the holographic principle for BF theory. Work on these problems is in progress.

[6] I thank the anonymous referee for bringing this example to my attention.

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