

Uniqueness theorem for charged dipole rings in five-dimensional minimal supergravityShinya Tomizawa,¹ Yukinori Yasui,² and Akihiro Ishibashi¹¹*KEK Theory Center, Institute of Particle and Nuclear Studies, KEK, Tsukuba, Ibaraki, 305-0801, Japan*²*Department of Mathematics and Physics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi, Osaka 558-8585, Japan*

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We show a uniqueness theorem for charged dipole rotating black rings in the bosonic sector of five-dimensional minimal supergravity, generalizing our previous work [arXiv:0901.4724] on the uniqueness of charged rotating black holes with topologically spherical horizon in the same theory. More precisely, assuming the existence of two commuting axial Killing vector fields and the same rod structure as the known solutions, we prove that an asymptotically flat, stationary charged rotating black hole with nondegenerate connected event horizon of cross-section topology $S^1 \times S^2$ in the five-dimensional Einstein-Maxwell-Chern-Simons theory—if exists—is characterized by the mass, charge, two independent angular momenta, dipole charge, and the ratio of the S^2 radius to the S^1 radius. As anticipated, the necessity of specifying dipole charge—which is not a conserved charge—is the new, distinguished ingredient that highlights difference between the present theorem and the corresponding theorem for vacuum case, as well as difference from the case of topologically spherical horizon within the same minimal supergravity. We also consider a similar boundary value problem for other topologically nontrivial black holes within the same theory, and in particular, discuss some nontrivial issues that arise when attempting to generalize the present uniqueness results to include black lenses—provided there exists such a solution in the theory.

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I. INTRODUCTION

Classifying higher dimensional black holes in supergravity theories is one of the key issues toward understanding the structure of string theory. In our previous paper [1], we addressed a classification problem of black holes in five-dimensional minimal supergravity, and showed that if an asymptotically flat, stationary charged rotating black hole solution of the theory possesses two rotational symmetries, then it can be uniquely specified by its asymptotic conserved charges [2]. In this version of uniqueness theorem, we restricted attention to the case of topologically spherical black holes since in that case, relevant boundary value analysis becomes simple and also there is a known exact solution [4] which appears to be most general as a spherical black hole in the five-dimensional minimal supergravity. However, topology theorem [5–7] itself does not stop us from considering topologically nonspherical black holes as far as horizon cross-section is of positive Yamabe type. In fact, a number of topologically nontrivial exact solutions, such as black rings and their multiple combinations, have been discovered in various theories [8–21]. It is therefore of considerable interest to consider a generalization of our uniqueness result [1] to include nonspherical black holes within the same supergravity theory. The main purpose of this paper is to show, on the basis of Ref. [1], a uniqueness theorem for black ring solutions—assuming their existence—in the bosonic sector of five-dimensional minimal supergravity theory, or equivalently five-dimensional

Einstein-Maxwell-Chern-Simons (EMCS) theory with the Chern-Simons coupling appropriately chosen.

Under the assumption that stationary black hole solutions admit additionally two independent rotational symmetries one can reduce the five-dimensional minimal supergravity theory to precisely the same type of nonlinear sigma model considered in our previous paper [1], irrespective to the horizon topology. One can then construct formally the same divergence identity for the sigma model fields on two-dimensional base space. Therefore, as briefly discussed in the summary section of Ref. [1], the only difference in uniqueness properties between spherical and nonspherical black holes should arise in the boundary value analysis on the nonlinear sigma model. The necessary boundary data are given at infinity and at a one-dimensional boundary component that corresponds to points of either the horizon or “axis” of rotational symmetries. The latter boundary component is further divided, in a certain manner, into a set of segments or intervals of invariant finite (or semi-infinite) length. Associated with each interval is an integer-valued vector that tells which (or what combination) of the two rotational Killing fields vanishes on the interval. The collection of such intervals and vectors are called the *rod structure* [22] (see also [23,24]), which, in particular, specifies the horizon topology. For example, as discussed in Ref. [1], the rod structure for a single black ring may be given by the following: (i) the semi-infinite interval $[c, \infty]$ with the vector $(0, 0, 1)$, (ii) the finite interval $[ck^2, c]$ with $(0, 1, 0)$, (iii) $[-ck^2, ck^2]$ with no vector, corresponding to the event horizon, and

(iv) $[-\infty, -ck^2]$ with $(0, 1, 0)$, where $c > 0$, $0 < k^2 < 1$. We will discuss this in more detail in the next section. Then, noting that information about horizon topology can be encoded in the rod structure, one might expect that the desired generalization of the uniqueness theorem to include nonspherical black holes would be straightforwardly achieved by merely specifying appropriate rod structure as well as all possible global conserved charges. This is indeed the case for the vacuum solutions [23,25]. However, one has to be more careful when gauge fields are involved: For example, when a rotating black ring couples to Maxwell field, it generates type of a dipole field. Accordingly, the *dipole charge*—which is not a conserved charge—comes to play a role as an additional parameter to characterize the solution, as stated already in the first example of dipole ring solutions found by Emparan [26], which are electrically coupled to a two form or a dual magnetic oneform field. Further examples of dipole rings have been constructed by Elvang *et al.* [15] in five-dimensional minimal supergravity, starting from a seven-parameter family of nonsupersymmetric black ring solutions. Their solution, however, does not have any limit to a supersymmetric solution, and moreover the dipole charge of their solution is not an independent parameter: it can be determined in terms of the other asymptotic conserved charges. Hence, as conjectured by the authors of [15] themselves, it is natural to anticipate that there exists a more general non-BPS black ring solution characterized by its mass, two independent angular momenta, electric charge, and a dipole charge that is independent of the other asymptotic conserved charges [27]. Although such a seemingly most general dipole ring solution has not been discovered yet, assuming its existence, we would like to show the following theorem:

Theorem. Consider the bosonic part of five-dimensional minimal supergravity, i.e., five-dimensional Einstein-Maxwell-Chern-Simons theory with certain value of the Chern-Simons coupling, and suppose there exists a regular stationary charged rotating black ring with finite temperature: that is, a stationary black hole solution that possesses a nondegenerate connected event horizon with cross-section topology $S^1 \times S^2$, and is regular on and outside the horizon and asymptotically flat in the standard sense with spherical spatial infinity. If such a black ring solution further admits (1) two mutually commuting axial Killing vector fields, in addition to the stationary Killing vector field, so that the isometry group is $\mathbb{R} \times U(1) \times U(1)$, and (2) the rod structure of the type (i)–(iv) above, then the solution is uniquely characterized by its mass, electric charge, two independent angular momenta, dipole charge, and the rod data (which corresponds to the ratio of the S^2 radius to the S^1 radius).

Some remarks are in order. As discussed in detail in [24], the rod structure (more precisely, the *interval structure* of [23,24]) can specify not only the horizon topology but also

topology of the black hole exterior region, as well as the action of the rotational symmetries. In the present case, restricting the rod structure as above (i)–(iv), the black hole exterior is topologically $\mathbb{R} \times \mathbb{R}^4 \setminus \{D^2 \times S^2\}$. In fact, all known black ring solutions with a single horizon component admit the rod structure above. However, it is not obvious whether any black ring solution must always have the rod structure of this simple type. Also when one wishes to generalize the present theorem to include other nontrivial black objects, one would need to address the case with more general rod structure, as in fact we will attempt to do so in Sec. III. In this regard, a similar uniqueness problem with general rod structure, treating both black rings and spherical holes in a unified manner, has been addressed in a rather restricted class of five-dimensional Einstein-Maxwell system [28]. There, the necessity of specifying a dipole charge and other extra charges (for the general rod structure case) has also been pointed out.

In the next section, we prove the above theorem, starting from a brief description of general strategy for the black hole uniqueness proof. In Sec. III, we study the boundary conditions for black holes with other horizon topologies, i.e., black lenses. In Sec. IV, we summarize our results and comment on some open issues on uniqueness for black lenses and multirings.

II. PROOF

Our proof consists of the following three steps (i)–(iii), employing the basic techniques of the classic uniqueness proof for four-dimensional black holes (see e.g., Ref. [29] and references therein), as well as imposing additional conditions upon topology and symmetries. In fact, essentially the same strategy has widely been used in proof of uniqueness theorems proposed for some restricted classes of higher dimensional black holes in various—but different from the present—context (See e.g., Refs. [1,23–25,28,30–38]). It goes roughly as follows: (i) First, using symmetry conditions, we reduce the theory of interest to a certain nonlinear sigma model on a two-dimensional base space, Σ . Thanks to the symmetry G of the sigma model, the set of sigma-model fields Φ^A on Σ can collectively be described in terms of a symmetric, unimodular matrix, M , on the coset space G/H , where H is an isotropy subgroup of G . Thus, in principle, the solutions of the system can compactly be expressed by the matrix M . Furthermore, the matrix M formally defines a conserved current, J , for the solution. (ii) Next, we introduce the *deviation matrix*, Ψ , which is essentially the difference between two coset matrices, say $M_{[0]}$ and $M_{[1]}$, so that when two solutions coincide with each other, the deviation matrix vanishes, and vice versa. What we wish to show is that Ψ vanishes over the entire Σ when two solutions satisfy the same boundary conditions that specify relevant physical parameters characterizing the black hole solution of interest. For

this purpose, we construct a global identity, called the *Mazur identity*, (the integral version of) which equates an integration along the boundary $\partial\Sigma$ of a derivative of the trace of Ψ to an integration over the whole base space Σ of the trace of “square” of the deviation, \mathcal{M} , of the two conserved currents, $J_{[0]}$ and $J_{[1]}$. The latter is therefore non-negative. The identity is essentially a generalization of the Green’s divergence identity for the standard Laplace equation. (iii) Then, we perform boundary value analysis of the matrix Ψ . We identify boundary conditions for M that define physical parameters characterizing black hole solutions and that guarantee the regularity of the solutions. Then we examine the behavior of Ψ near $\partial\Sigma$. For higher dimensional case, this is the point where the topology and symmetry properties, translated into the language of the rod structure, come to play a role as additional parameters to specify solutions. When the integral along the boundary $\partial\Sigma$, say the left side of the Mazur identity, vanishes under our boundary conditions, it then follows from the right side of the identity, i.e., the non-negative integration over Σ , that \mathcal{M} has to vanish, hence the two currents, $J_{[0]}$ and $J_{[1]}$, must coincide with each other over Σ , implying that the deviation matrix Ψ must be constant over Σ . Then, if Ψ is shown to be zero on some part of the boundary $\partial\Sigma$, it follows that Ψ must be identically zero over the entire Σ , thus proving the two solutions, $M_{[0]}$ and $M_{[1]}$, must be identical.

In our present case, the first two steps (i)-(ii) completely parallel those in Ref. [1], and Step (iii) is the new result of this paper. In order to highlight difference from the spherical horizon case and also to avoid unnecessary repetition, in the following we provide only some key formulas of Steps (i) and (ii), needed in Step (iii) later on, quoting from Ref. [1].

Our starting point is the following five-dimensional minimal supergravity action

$$S = \frac{1}{16\pi} \left[\int d^5x \sqrt{-g} \left(R - \frac{1}{4} F^2 \right) - \frac{1}{3\sqrt{3}} \int F \wedge F \wedge A \right], \quad (1)$$

where we set the Newton constant to be unity and $F = dA$ with A being the gauge potential. Varying this action (1), we derive the Einstein equations with the standard stress-energy tensor for five-dimensional Maxwell field, as well as Maxwell’s equations which have the extra term coming from the Chern-Simons term of (1). We are concerned with asymptotically flat, stationary, charged rotating black ring solutions of this theory. We additionally impose two independent axial symmetries, so that the total isometry group is $\mathbb{R} \times U(1) \times U(1)$ with \mathbb{R} being stationary symmetry, generated by mutually commuting three Killing vector fields $\xi_t = \partial/\partial t$ and $\xi_a = (\xi_\phi, \xi_\psi) = (\partial/\partial\phi, \partial/\partial\psi)$ [39]. Using the Einstein equations and the Maxwell equations, we can show that the generators ξ_t, ξ_a of the isometry group satisfy type of integrability conditions discussed

in Refs. [22,43]. As a result, we obtain the coordinate system, $\{t, \phi, \psi, \rho, z\}$, in which the metric takes the *Weyl-Papapetrou form*

$$\begin{aligned} ds^2 = & \lambda_{\phi\phi}(d\phi + a^\phi dt)^2 + \lambda_{\psi\psi}(d\psi + a^\psi dt)^2 \\ & + 2\lambda_{\phi\psi}(d\phi + a^\phi dt)(d\psi + a^\psi dt) \\ & + |\tau|^{-1}[e^{2\sigma}(d\rho^2 + dz^2) - \rho^2 dt^2], \end{aligned} \quad (2)$$

and the gauge potential is written,

$$A = \sqrt{3}\psi_a dx^a + A_t dt, \quad (3)$$

where the coordinates $x^a = (\phi, \psi)$ denote the Killing parameters, and thus all functions λ_{ab} , $\tau := -\det(\lambda_{ab})$, a^a , σ , and (ψ_a, A_t) are independent of t and x^a , and where the potentials ψ_a are related to Maxwell field by eq. (8) of Ref. [1] [see also Appendix A of Ref. [1] for the gauge choice employed in Eq. (3)]. Note that the coordinates (ρ, z) that span a two-dimensional *base space*, $\Sigma = \{(\rho, z) | \rho \geq 0, -\infty < z < \infty\}$, are globally well-defined, harmonic, and mutually conjugate on Σ . See e.g., [44]. Furthermore, by using the Maxwell’s equation and Einstein’s equations, we introduce the magnetic potential μ and twist potentials ω_a by

$$d\mu = \frac{1}{\sqrt{3}} * (\xi_\phi \wedge \xi_\psi \wedge F) - \epsilon^{ab} \psi_a d\psi_b, \quad (4)$$

$$d\omega_a = * (\xi_\phi \wedge \xi_\psi \wedge d\xi_a) + \psi_a (3d\mu + \epsilon^{bc} \psi_b d\psi_c), \quad (5)$$

where $\epsilon^{\phi\psi} = -\epsilon^{\psi\phi} = 1$. Then, the nonlinear sigma model reduced from the theory (1) with the symmetry assumptions consists of the target space with the isometry $G = G_{2(2)}$ and the eight scalar fields $\Phi^A = (\lambda_{ab}, \omega_a, \psi_a, \mu)$ on the base space Σ . All the other fields such as σ, a^a , etc. can be determined by Φ^A through the equations of motion. It turns out that the sigma model fields, Φ^A , can be expressed by a 7×7 symmetric unimodular coset $G_{2(2)}/SO(4)$ matrix M [see eq. (34) of Ref. [1]], as shown by [45–47]. Then we define the deviation matrix, Ψ , for two solutions, $M_{[0]}$ and $M_{[1]}$, as in eq. (42) of Ref. [1], and derive the Mazur identity,

$$\int_{\partial\Sigma} \rho \partial_a \text{tr} \Psi dS^a = \int_{\Sigma} \text{tr}(\mathcal{M}^T \cdot \mathcal{M}) \rho d\rho dz, \quad (6)$$

where *dot* denotes the inner product on Σ . As briefly mentioned above, \mathcal{M} , in the right side essentially describes the difference between two matrix currents $J_{[0]}, J_{[1]}$, given by eq. (47) of Ref. [1], of which detail is irrelevant to discussion below. Our task is to show that the left side of Eq. (6) vanishes on the boundary, $\partial\Sigma$, and then show Ψ itself vanishes on some part of the boundary.

Now we proceed step (iii): The boundary value analysis. In the Weyl-Papapetrou coordinate system, the boundaries for black rings can be described as follows:

- (i) ψ -invariant plane: $\partial\Sigma_\psi = \{(\rho, z)|\rho = 0, c < z < \infty\}$ with the rod vector $\mathbf{v} = (0, 0, 1)$,
- (ii) ϕ -invariant plane inside the black ring: $\partial\Sigma_{\text{in}} = \{(\rho, z)|\rho = 0, ck^2 < z < c\}$ with the rod vector $\mathbf{v} = (0, 1, 0)$,
- (iii) Horizon: $\partial\Sigma_{\mathcal{H}} = \{(\rho, z)|\rho = 0, -ck^2 < z < ck^2\}$,
- (iv) ϕ -invariant plane outside the black ring: $\partial\Sigma_\phi = \{(\rho, z)|\rho = 0, -\infty < z < -ck^2\}$ with the rod vector $\mathbf{v} = (0, 1, 0)$,
- (v) Infinity: $\partial\Sigma_\infty = \{(\rho, z)|\sqrt{\rho^2 + z^2} \rightarrow \infty$ with $z/\sqrt{\rho^2 + z^2}$ finite},

where the two constants c and k satisfy $c > 0$ and $0 < k^2 < 1$.

Therefore, the boundary integral in the left-hand side of the Mazur identity, Eq. (6), is decomposed into the integrals over the four rods (i)–(iv), and the integral at infinity (v), as

$$\begin{aligned} \int_{\partial\Sigma} \rho \partial_\rho \text{tr}\Psi dS^p &= \int_{-\infty}^{-ck^2} \rho \frac{\partial \text{tr}\Psi}{\partial z} dz + \int_{-ck^2}^{ck^2} \rho \frac{\partial \text{tr}\Psi}{\partial z} dz \\ &+ \int_{ck^2}^c \rho \frac{\partial \text{tr}\Psi}{\partial z} dz + \int_c^\infty \rho \frac{\partial \text{tr}\Psi}{\partial z} dz \\ &+ \int_{\partial\Sigma_\infty} \rho \partial_a \text{tr}\Psi dS^a. \end{aligned} \quad (7)$$

Note that the only difference between black holes and black rings appears at the third term in the right side of Eq. (7), which corresponds to the integral over the ϕ -invariant plane inside the black ring. As will be seen below, because of the existence of this third integral, a dipole charge comes to appear in our boundary conditions.

We examine the behavior of Ψ at each boundary, (i)–(v), separately, starting from analysis at Infinity (v).

(v) Infinity: $\partial\Sigma_\infty = \{(\rho, z)|\sqrt{\rho^2 + z^2} \rightarrow \infty$ with $z/\sqrt{\rho^2 + z^2}$ kept finite}. Since we are concerned with asymptotically flat solutions in the standard sense and the behavior of the scalar fields near infinity does not depend on what the topology of the horizon is, the discussion here is the same as the case of a spherical horizon topology [1]. So we have

$$\rho \partial_\rho \text{tr}\Psi dS^p \simeq \mathcal{O}\left(\frac{1}{\sqrt{\rho^2 + z^2}}\right). \quad (8)$$

(i) ψ -invariant plane: $\partial\Sigma_\psi = \{(\rho, z)|\rho = 0, c < z < \infty\}$. This part is essentially the same as the boundary analysis at ϕ -invariant plane of Ref. [1]. Note that in this paper, we are taking ψ as the Killing parameter along the S^1 sector of the ring solution, and for this reason the role of $\partial\Sigma_\psi$ is played by the “ ϕ -invariant plane” of Ref. [1]. Therefore the behavior of Ψ near $\partial\Sigma_\psi$ can be read off from the formulas of eqs. (63)–(70) of Ref. [1]. As a result, for two solutions, $M_{[0]}$ and $M_{[1]}$, with the same mass, the same angular momenta, and the same electric charge, we have,

$$\rho \partial_z \text{tr}\Psi \simeq \mathcal{O}(\rho). \quad (9)$$

(iv) ϕ -invariant plane outside the black ring: $\partial\Sigma_\phi = \{(\rho, z)|\rho = 0, -\infty < z < -ck^2\}$. Similarly, the behavior of Ψ around $\partial\Sigma_\phi$ can be read off from Eqs. (75)–(82) of Ref. [1], and we have $\rho \partial_z \text{tr}\Psi \simeq \mathcal{O}(\rho)$, for $\rho \rightarrow 0$.

(iii) Horizon: $\partial\Sigma_{\mathcal{H}} = \{(\rho, z)|\rho = 0, -ck^2 < z < ck^2\}$. The regularity on the horizon requires that for $\rho \rightarrow 0$,

$$\begin{aligned} \lambda_{ab} &\simeq \mathcal{O}(1), & \omega_a &\simeq \mathcal{O}(1), \\ \psi_a &\simeq \mathcal{O}(1), & \mu &\simeq \mathcal{O}(1). \end{aligned} \quad (10)$$

Thus, we have for $\rho \rightarrow 0$, $\rho \partial_z \text{tr}\Psi \simeq \mathcal{O}(\rho)$.

(ii) ϕ -invariant plane inside the black ring: $\partial\Sigma_{\text{in}} = \{(\rho, z)|\rho = 0, ck^2 < z < c\}$. This is the key part of the present boundary value analysis. As in the case (iv), the regularity requires that the potentials, λ_{ab} , must behave as

$$\lambda_{\phi\phi} \simeq \mathcal{O}(\rho^2), \quad (11)$$

$$\lambda_{\phi\psi} \simeq \mathcal{O}(\rho^2), \quad (12)$$

$$\lambda_{\psi\psi} \simeq \mathcal{O}(1). \quad (13)$$

The dipole charge for a black ring is defined by

$$\begin{aligned} q &:= \frac{1}{2\pi} \int_{S^2} F \\ &= \frac{1}{2\pi} \int_{S^2} A_{\phi,z} dz \wedge d\phi \\ &= \sqrt{3} [\psi_\phi(\rho = 0, z = ck^2) - \psi_\phi(\rho = 0, z = -ck^2)] \\ &= \sqrt{3} \psi_\phi(\rho = 0, z = ck^2), \end{aligned} \quad (14)$$

where S^2 denotes the two-sphere spatial cross section of the black ring horizon and where, at the fourth equality, we have used the fact that $\psi_\phi \simeq \mathcal{O}(\rho^2)$ on the ϕ -invariant plane $\partial\Sigma_\phi$ [eq. (81) of Ref. [1]]. Note that the derivative of the electric potential, $d\psi_\phi$, vanishes on $\partial\Sigma_{\text{in}}$ by definition, and hence ψ_ϕ is constant over $\partial\Sigma_{\text{in}}$. Therefore, from Eq. (14), we immediately find that the electric potential, ψ_ϕ , must behave as

$$\psi_\phi \simeq \frac{q}{\sqrt{3}} + \mathcal{O}(\rho^2) \quad (15)$$

in a neighborhood of $\partial\Sigma_{\text{in}}$. We cannot determine how the other magnetic potential, ψ_ψ , behaves on $\partial\Sigma_{\text{in}}$, and therefore we write

$$\psi_\psi \simeq f(z) + \mathcal{O}(\rho^2), \quad (16)$$

where $f(z)$ is some function depending only on z .

Next, we consider the magnetic potential μ on the ϕ -invariant plane inside the black ring $\partial\Sigma_{\text{in}}$. The magnetic potential, μ , satisfies, Eq. (4), i.e.,

$$d\mu = \frac{1}{\sqrt{3}} * (\xi_\phi \wedge \xi_\psi \wedge F) - (\psi_\phi d\psi_\psi - \psi_\psi d\psi_\phi). \quad (17)$$

The first term vanishes on $\partial\Sigma_{\text{in}}$ by definition. Substituting Eq. (15) into the above equation, we find that the derivative of μ can be written as

$$d\mu = -\frac{q}{\sqrt{3}} d\psi_\psi. \quad (18)$$

Hence, integrating (18), we obtain

$$\mu = -\frac{q}{\sqrt{3}} \psi_\psi + c_{\text{in}}, \quad (19)$$

where c_{in} is a constant. Here, we note from the analysis (i) on $\partial\Sigma_\psi$ that $\psi_\psi \rightarrow 0$ [eq. (68) of Ref. [1]] and $\mu \rightarrow 2Q/\sqrt{3}\pi$ [eq. (70) of Ref. [1]] at the center of the black ring, $\rho = 0$, $z = c$. Then, the constant is determined as

$$c_{\text{in}} = \frac{2Q}{\sqrt{3}\pi}. \quad (20)$$

Thus, in terms of the undetermined function, $f(z)$, of z , the electric charge, Q , and the dipole charge, q , we can obtain the behavior of the magnetic potential, μ , near $\partial\Sigma_{\text{in}}$ as follows.

$$\mu \simeq -\frac{q}{\sqrt{3}} f(z) + \frac{2Q}{\sqrt{3}\pi} + \mathcal{O}(\rho^2). \quad (21)$$

Furthermore, consider the boundary condition for the twist potentials, ω_a , on $\partial\Sigma_{\text{in}}$. Recall that they are given by

$$d\omega_a = * (\xi_\phi \wedge \xi_\psi \wedge d\xi_a) + \psi_a (3d\mu + \psi_\phi d\psi_\psi - \psi_\psi d\psi_\phi). \quad (22)$$

The first term vanishes on the ψ -invariant plane, $\partial\Sigma_{\text{in}}$, by definition. Substituting Eqs. (15) and (21) into the above equation, we find that the derivative of the twist potentials, ω_a , can be written as

$$d\omega_a = -\frac{2q}{\sqrt{3}} \psi_a d\psi_\psi. \quad (23)$$

Then, by using Eqs. (15) and (16), they can be rewritten as

$$d\omega_\phi = -\frac{2}{3} q^2 df(z), \quad d\omega_\psi = -\frac{2}{\sqrt{3}} q f(z) df(z). \quad (24)$$

Integrating Eq. (24) on $\partial\Sigma_{\text{in}}$, we obtain

$$\omega_\phi = -\frac{2}{3} q^2 f(z) + c_\phi, \quad \omega_\psi = -\frac{1}{\sqrt{3}} q f(z)^2 + c_\psi, \quad (25)$$

where c_ϕ and c_ψ are arbitrary constants. From the analysis (i) on $\partial\Sigma_\psi$ we have

$$\omega_a = -\frac{2J_a}{\pi}, \quad \psi_\psi = 0 \quad (26)$$

at the center of the black ring, $\rho = 0$, $z = c$ [eqs. (66)–(68) of Ref. [1]]. Hence, the constants, c_ϕ and c_ψ , can be determined as

$$c_\phi = -\frac{2J_\phi}{\pi}, \quad c_\psi = -\frac{2J_\psi}{\pi}. \quad (27)$$

As a result, we find that the two twist potentials ω_ϕ and ω_ψ must behave as

$$\omega_\phi \simeq -\frac{2}{3} q^2 f(z) - \frac{2J_\phi}{\pi} + \mathcal{O}(\rho^2), \quad (28)$$

$$\omega_\psi \simeq -\frac{1}{\sqrt{3}} q f(z)^2 - \frac{2J_\psi}{\pi} + \mathcal{O}(\rho^2), \quad (29)$$

on $\partial\Sigma_{\text{in}}$. Thus, from Eqs. (11)–(13), (15), (16), (21), (28), and (29), we find for $\rho \rightarrow 0$, $\rho \partial_z \text{tr}\Psi \simeq \mathcal{O}(\rho)$. We emphasize here that in order to obtain this result, we do not need to, in advance, specify the functions, $f(z)_{[0]}$, $f(z)_{[1]}$, in the two solutions.

We conclude from (i)–(v) that the boundary integral vanishes on each rod and the infinity. We can also find, by continuity, that the boundary integral is bounded, hence vanishing, at the points where two adjacent rods meet. The deviation matrix, Ψ , is constant and has the asymptotic behavior, $\Psi \rightarrow 0$. Therefore, Ψ vanishes over Σ , and the two configurations, $M_{[0]}$ and $M_{[1]}$, coincide with each other for the two black ring solutions with the same mass, angular momenta, electric charge, dipole charge and rod structure (i.e., k^2). This completes our proof for the uniqueness theorem.

III. BOUNDARY VALUE PROBLEM FOR BLACK LENS

As discussed in [23], under the existence of two commuting axial Killing vectors, the cross-section topology of each connected component of the event horizon of stationary vacuum black hole solutions must be either S^3 , $S^1 \times S^2$ or a lens space. In this section we would like to consider the boundary value problem for an asymptotically flat, black lens, though such a solution has not been found even in the vacuum case. The rod structure for a black lens was given by Evslin [48]. In the Weyl-Papapetrou coordinate system, the boundaries for a black lens with the $L(p; 1)$ horizon topology, if exists, can be given as follows:

- (i) ψ -invariant plane: $\partial\Sigma_\psi = \{(\rho, z) | \rho = 0, c < z < \infty\}$ with the rod vector $\mathbf{v} = (0, 0, 1)$,
- (ii) Inner axis: $\partial\Sigma_{\text{in}} = \{(\rho, z) | \rho = 0, ck^2 < z < c\}$ with the rod vector $\mathbf{v} = (0, 1, p)$,
- (iii) Horizon: $\partial\Sigma_{\mathcal{H}} = \{(\rho, z) | \rho = 0, -ck^2 < z < ck^2\}$,
- (iv) ϕ -invariant plane: $\partial\Sigma_\phi = \{(\rho, z) | \rho = 0, -\infty < z < -ck^2\}$ with the rod vector $\mathbf{v} = (0, 1, 0)$,

(v) Infinity: $\partial\Sigma_\infty = \{(\rho, z) | \sqrt{\rho^2 + z^2} \rightarrow \infty \text{ with } z/\sqrt{\rho^2 + z^2} \text{ finite}\}$,

where constants c and k satisfy $c > 0$ and $0 < k^2 < 1$.

For the boundaries (i), (iii), (iv), and (v), the boundary conditions of the scalar fields, Φ^A , are exactly the same as those of black rings. Therefore we consider only (ii). First, note that the rod vector, $v = \partial/\partial\phi + p\partial/\partial\psi$, has fixed points for $\rho = 0$, $z \in [ck^2, c]$, i.e.,

$$g(v, v) = 0 \iff \lambda_{\phi\phi} + 2p\lambda_{\phi\psi} + p^2\lambda_{\psi\psi} = 0. \quad (30)$$

On the inner axis, we have

$$\tau = \lambda_{\phi\psi}^2 - \lambda_{\phi\phi}\lambda_{\psi\psi} = 0. \quad (31)$$

Therefore, we find that near the inner axis, the potentials, λ_{ab} , must behave as

$$\lambda_{\phi\phi} \simeq p^2g(z) + \mathcal{O}(\rho^2), \quad (32)$$

$$\lambda_{\phi\psi} \simeq -pg(z) + \mathcal{O}(\rho^2), \quad (33)$$

$$\lambda_{\psi\psi} \simeq g(z) + \mathcal{O}(\rho^2), \quad (34)$$

where $g(z)$ is some function of z .

Next, consider the boundary conditions for the electric potentials ψ_a . It follows from Eq. (30) that for $\rho = 0$, $z \in [ck^2, c]$,

$$0 = -i_v F = \sqrt{3}(d\psi_\phi + p d\psi_\psi). \quad (35)$$

Integrating this, we obtain

$$\psi_\phi + p\psi_\psi = c_0, \quad (36)$$

where c_0 is a constant. Therefore, we can set the electric potentials to behave as

$$\psi_\phi \simeq c_0 - ph(z) + \mathcal{O}(\rho^2), \quad (37)$$

$$\psi_\psi \simeq h(z) + \mathcal{O}(\rho^2), \quad (38)$$

with $h(z)$ being some function of z .

We further consider the behavior of the magnetic potential μ defined by Eq. (4). Since the norm of the rod vector v vanishes over the inner axis, the first term in the right-hand side of Eq. (4) vanishes there. Then, it follows from Eqs. (37) and (38) that the derivative of the magnetic potential, μ , is given by

$$d\mu = -c_0 dh(z). \quad (39)$$

Integrating this, we obtain

$$\mu = -c_0 h(z) + c_1, \quad (40)$$

where c_1 is an integration constant. Here, note that $\mu = 2Q/(\sqrt{3}\pi)$, and $\psi_\psi = 0$ (i.e., $h(z=0) = 0$) hold at $\rho = 0$, $z = c$. Therefore, the constant c_1 is determined as

$$c_1 = \frac{2Q}{\sqrt{3}\pi}. \quad (41)$$

Thus, we find that near the inner axis, the magnetic potential, μ , must behave as

$$\mu \simeq -c_0 h(z) + \frac{2Q}{\sqrt{3}\pi} + \mathcal{O}(\rho^2). \quad (42)$$

Finally, let us consider the twist potentials ω_a on the inner axis. From Eqs. (37) and (38), the derivatives of the twist potentials on the inner axis are given by

$$d\omega_a = -2c_0 \psi_a dh(z). \quad (43)$$

Then, it follows that ω_a can be written in terms of the integration constants

$$\begin{aligned} \omega_\phi &= -2c_0^2 h(z) + pc_0 h(z)^2 + c_2, \\ \omega_\psi &= -c_0 h(z)^2 + c_3. \end{aligned} \quad (44)$$

We easily find that

$$\omega_\phi = -\frac{2J_a}{\pi}, \quad \psi_\psi = 0 \quad (45)$$

at $\rho = 0$, $z = c$. From continuity of the potentials, the constants c_2 and c_3 can be determined as

$$c_2 = -\frac{2J_\phi}{\pi}, \quad c_3 = -\frac{2J_\psi}{\pi}. \quad (46)$$

Therefore, we find the twist potentials behave as

$$\omega_\phi \simeq -2c_0^2 h(z) + pc_0 h(z)^2 - \frac{2J_\phi}{\pi} + \mathcal{O}(\rho^2), \quad (47)$$

$$\omega_\psi \simeq -c_0 h(z)^2 - \frac{2J_\psi}{\pi} + \mathcal{O}(\rho^2) \quad (48)$$

near the inner axis.

From the above behavior of the scalar fields, we find that the leading term of the boundary integral $\int \rho \partial_z \text{tr} \Psi dz$ is proportional to $(c_{0[0]} - c_{0[1]})\rho^{-3}$. Therefore, if the integration constants, $c_{0[0]}$ and $c_{0[1]}$, for two solutions with the same mass, two angular momenta and electric charge do not coincide with each other, the boundary integral does not vanish on the inner axis. Since in the vacuum case, the constant c_0 vanishes, our analysis above immediately implies that the boundary integral vanishes on the inner axis. This coincides with the results obtained in [28]. However, in the present case with Maxwell field being nonvanishing, there seems to be no obvious way to relate the constant c_0 to asymptotic charges and the rod data.

IV. SUMMARY AND DISCUSSION

In this paper, we have considered asymptotically flat, stationary charged rotating black rings, i.e., holes having $S^1 \times S^2$ horizon cross-section topology, in the bosonic sector of five-dimensional minimal supergravity, and

have proven a uniqueness theorem that states that under the assumptions of the existence of two commuting axial isometries, such a black ring with nondegenerate horizon is characterized by the rod data which corresponds to the ratio of the S^2 radius to the S^1 radius, and the following five charges: i.e., the mass, charge, two independent angular momenta, and dipole charge. As mentioned before, so far no such black ring solutions have been discovered. The solution obtained by Elvang *et al.* [15] admit no limit to a supersymmetric black ring solution because the solution does not have enough independent parameters, i.e., the dipole charge is not an independent parameter, except the case in which the net charge Q vanishes. One can expect that there should exist a nonsupersymmetric charged dipole ring solution with five independent parameters. If it is the case, our theorem states that such a solution must be uniquely determined by the five charges mentioned above and the rod data. Note that such a most general charged dipole ring solution may turn out to be generically unbalanced, having a naked conical singularity of the form of a disk inside the ring, as in the first example of a static black ring in vacuum [43]. Even in the case, our theorem would still apply (with removing the requirement that the space-time itself be regular on and outside the horizon in the statement of the theorem), since the existence of such a conical singularity does not affect the regularity of our target space scalar fields in a neighborhood of the boundary.

We have also considered a similar boundary value problem for asymptotically flat, black lens solutions—even

though no such a black lens solution has been found so far. We have not been able to relate the integration constant c_0 in Eq. (36) to any of the other charges, except for the vacuum case ($Q = q = 0$). This indicates that the constant c_0 arises as a result of interplay between the nonvanishing gauge field and nontrivial topology of the horizon, just like the dipole charge in the black ring case, and therefore may possibly play a role of an independent parameter to uniquely specify a black lens solution (if exists) in the minimal supergravity. In this paper, however, we have not been able to identify the physical interpretation of c_0 . We also expect that a similar problem just mentioned above may occur when considering uniqueness theorems for multirings, black Saturn, or more complicated black objects which couple to some nonvanishing gauge field and which admit the rod structure that contains a rod similar to $\partial\Sigma_{\text{in}}$ in the above black lens example. This issue deserves further study.

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