PHYSICAL REVIEW D **81,** 084028 (2010)

New framework for studying spherically symmetric static solutions in $f(R)$ gravity

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(Received 9 September 2009; published 15 April 2010)

We develop a new covariant formalism to treat spherically symmetric spacetimes in *metric* $f(R)$ theories of gravity. Using this formalism we derive the general equations for a static and spherically symmetric metric in a general $f(R)$ gravity. These equations are used to determine the conditions for which the Schwarzschild metric is the only vacuum solution with vanishing Ricci scalar. We also show that our general framework provides a clear way of showing that the Schwarzschild solution is not a unique static spherically symmetric solution, providing some insight into how the current form of Birkhoff's theorem breaks down for these theories.

DOI: 10.1103/PhysRevD.81.084028 PACS numbers: 04.50.Kd

I. INTRODUCTION

Ever since the publication of the Schwarzschild solution almost 100 years ago, the study of spherically symmetric solutions has played a fundamental role in determining the understanding of the nature of gravity and underlies many of the key tests of Einstein's theory of general relativity (GR). Until recently, general relativity has been unquestionably the only theory able to explain gravity on both astrophysical and cosmological scales. With the advent of new high precision cosmological tests, capable of probing physics at very large redshifts, this situation has completely changed and recently the large-scale validity of general relativity has begun to be questioned. This is largely due to the fact that in order to fit the standard model of cosmology, which is based on general relativity coupled to standard matter (baryons and radiation), the introduction of two dark components are needed to achieve a consistent picture. Specifically, dark matter is needed to fit the astrophysical dynamics at galactic and cluster scales, while a new ingredient, dubbed dark energy, is required in order to explain the observed accelerated behavior of the Hubble flow. Combining the luminosity distance data of supernovae type Ia [1], the large-scale structure [2], the anisotropy of cosmic microwave background [3], and baryon acoustic oscillations [4] suggest that if we retain general relativity as the theory of the gravitational interaction, the best fit model is a spatially flat Universe, dominated by cold dark matter (CDM) and dark energy (DE) in the form of an effective cosmological constant. Although CDM candidates have not yet been directly detected, there are strong arguments that suggest that CDM has a nongravitational origin [5]. The same is not true for DE. The cosmological constant and coincidence problems together with the fact that there are no convincing DE candidates, seems to suggest that the concordance model is incomplete, and despite enormous effort over the past few years, this problem remains one of the greatest puzzles in contemporary physics. One of the theoretical proposals that has received a considerable amount of attention recently, is that dark energy has a geometrical origin. This idea has been driven by the fact that modifications to general relativity appear in the low energy limit of many fundamental schemes [6,7] and that these modifications lead naturally to cosmologies that admit a dark energy -like era [8–12] without the introduction of any additional cosmological fields. Most of the work on this idea has focused on *fourth-order gravity*, in which the standard Hilbert-Einstein action is modified with terms that are at most of order four in the metric tensor.

The features of fourth-order gravity have been analyzed with different techniques [13], and all of these studies suggest that these cosmologies can give rise to a phase of accelerated expansion, which is considered to be an important footprint of dark energy. In particular, dynamical systems analysis shows that for Friedmann-Lemaître-Robertson-Walker models there exist classes of fourth-order theories, which admit a transient decelerated expansion phase, followed by one with an accelerated expansion rate. The first (Friedmann-like) phase provides a setting during which structure formation can take place, followed by a smooth transition to a DE-like era, which drives the cosmological acceleration.

More recently, the theory of linear perturbations for these models has been developed [14] using the $1 + 3$ covariant approach [15]. A number of important features were found which allows one to differentiate the structure growth scenario from what occurs in general relativity.

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First, it was found that the evolution of density perturbations is determined by a *fourth-order* differential equation rather than a second-order one. This implies that the evolution of the density fluctuations contains, in general, four modes rather that two and can give rise to a more complex evolution than the one of GR. Second, the perturbations are found to depend on the scale for any equation of state for standard matter (while in general relativity the evolution of the dust perturbations are scale invariant). This means that even for dust, the evolution of superhorizon and subhorizon perturbations are different. Third, it was found that the growth of large-scale density fluctuations can occur also in backgrounds in which the expansion rate is increasing in time. This is in striking contrast with what one finds in general relativity and would lead to a time-varying gravitational potential, putting tight constraints on the integrated Sachs-Wolfe effect for these models.

Although these results are very encouraging, there are still some important open problems to be addressed. Of particular interest is the degree to which the physics of fourth-order gravity is consistent with *both* cosmological and Solar System scales, indeed, there has been considerable debate about the short-scale behavior of higher order over the past few years, leading to much work on the Newtonian and post-Newtonian limits of these theories [16]. Consequently, measurements coming from weak field limit tests like the bending of light, the perihelion shift of planets, and frame dragging experiments represent critical tests for any theory of gravity. Fundamental to confronting such tests with fourth-order gravity is the existence of physically viable spherically symmetric solutions in these theories. The aim of this paper is this therefore twofold. First to determine a set of general results for spherically symmetric spacetimes in $f(R)$ gravity, and second to obtain a general procedure for generating solutions of this type.

The present analysis is based on a powerful extension of the $1 + 3$ covariant approach in which the three spatial degrees of freedom are further decomposed relative to a spatial vector [17]. In the case of spherical symmetry, this is chosen to be the radial direction. This leads to a larger set of covariant variables with their corresponding equations (evolution, propagation, and constraint). Furthermore, all the equations are developed in the Jordan frame without resorting to any conformal transformations.

Unless otherwise specified, natural units ($\hbar = c = k_B = 8\pi G = 1$) will be used throughout this paper, Latin indices run from 0 to 3. The symbol ∇ represents the usual covariant derivative, and ∂ corresponds to partial differentiation. We use the $-, +, +, +$ signature and the Riemann tensor is defined by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de}, \quad (1)$$

where the $\Gamma^a{}_{bd}$ are the Christoffel symbols (i.e. symmetric in the lower indices), defined by

$$\Gamma^a_{bd} = \tfrac{1}{2}g^{ae}(g_{be,d} + g_{ed,b} - g_{bd,e}). \quad (2)$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices

$$R_{ab} = g^{cd}R_{acbd}. \quad (3)$$

The symmetrization and the antisymmetrization over the indexes of a tensor are defined as

$$T_{(ab)} = \tfrac{1}{2}(T_{ab} + T_{ba}), \qquad T_{[ab]} = \tfrac{1}{2}(T_{ab} - T_{ba}). \quad (4)$$

Finally the Hilbert–Einstein action in the presence of matter is given by

$$\mathcal{A} = \frac{1}{2}\int d^4x\sqrt{-g}[R + 2\mathcal{L}_m]. \quad (5)$$

II. GENERAL EQUATIONS FOR FOURTH-ORDER GRAVITY

In a completely general context, a fourth-order theory of gravity is obtained by adding terms involving $f(R, R_{ab}R^{ab}, R_{abcd}R^{abcd})$ to the standard Einstein Hilbert action. However, we know the Gauss-Bonnet term ($\mathcal{G} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$) is a total differential in four dimensions and hence do not affect the field equations. Hence, we can replace all linear terms of $R_{abcd}R^{abcd}$ with the other two. Furthermore, if the spacetime is highly symmetric, then the variation of the term $R_{ab}R^{ab}$ can always be rewritten in terms of the variation of R^2 [18,19]. It follows that a *sufficiently general* fourth-order Lagrangian for a highly symmetric spacetime only contains powers of R, and we can write the action as

$$\mathcal{A} = \frac{1}{2}\int d^4x\sqrt{-g}[f(R) + 2\mathcal{L}_m], \quad (6)$$

where $\mathcal{L}_m$ represents the matter contribution.

Varying the action with respect to the metric gives the following field equations:

$$f'G_{ab} = T^m_{ab} + \tfrac{1}{2}(f - Rf')g_{ab} + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f', \quad (7)$$

where f' denotes the derivative of the function 'f' with respect to the Ricci scalar, and T^m_{ab} is the matter stress energy tensor defined as

$$T^m_{ab} = \mu^m u_a u_b + p^m h_{ab} + q^m_a u_b + q^m_b u_a + \pi^m_{ab}. \quad (8)$$

Here, u^a is the direction of a timelike observer, h_{ab} is the projected metric on the 3-space perpendicular to u^a. Also μ^m, p^m, q^m, and π^m_{ab} denote the standard matter density, pressure, heat flux, and anisotropic stress, respectively. Equations (7) reduce to the standard Einstein field equations when $f(R) = R$.

III. 1 + 1 + 2 COVARIANT APPROACH

We know that the $1+3$ covariant approach, initially developed by Ehlers and Ellis [15] has proven to be a very useful technique in many aspects of relativistic cosmology. The approach has been particularly useful in obtaining a deep understanding of many aspects of relativistic fluid flows, whether it is applied in terms of fully nonlinear GR effects or the gauge invariant, covariant perturbation formalism. In cosmology these methods have been applied, for example, to the formalism and evolution of density perturbations [20] in the Universe and to the physics of cosmic microwave background [21]. This approach is based on a $1+3$ threading decomposition of the spacetime manifold with respect to a timelike congruence as a splitting of spacetime onto a timelike and a orthogonal three-dimensional spacelike hypersurface. All the essential information in the system is captured in a set of *kinematic and dynamic* $1+3$ variables that have a well defined physical and geometrical significance. These variables satisfy a set of evolution and constraint equations derived from the Bianchi and Ricci identities, forming a closed system of equations for a chosen equation of state describing matter.

A natural extension to the $1+3$ approach, optimized for problems that have spherical symmetry, is the $1+1+2$ formalism developed recently by Clarkson and Barrett [17]. In this formalism one first proceeds to the same split of the $1+3$ approach and then a further one that isolates a specific spatial direction. This allows us to derive a set of variables that are more advantageous to treat systems with one preferred direction. For example, in the spherically symmetric system the equation for the $1+1+2$ variables are scalar equations and are much simpler than the ones of the $1+3$ formalism, which are in general tensorial. The $1+1+2$ formalism was applied to the study of linear perturbations of a Schwarzschild spacetime [17] and to the generation of electromagnetic radiation by gravitational waves interacting with a strong magnetic field around a vibrating Schwarzschild black hole [22].

In the following we give a brief review of these formalisms, before applying it to the specific case of $f(R)$ gravity.

A. Kinematics

In $(1+3)$ approach first we define a timelike congruence by a timelike unit vector u^a. Then the spacetime is split in the form $R\otimes V$, where R denotes the timeline along u^a and V is the 3-space perpendicular to u^a. Then any vector X^a can be projected on the 3-space by the projection tensor $h^a_b = g^a_b + u^a u_b$.

At this point, two derivatives are defined: the vector u^a is used to define the *covariant time derivative* (denoted by a dot) for any tensor $T^{a\ldots b}{}_{c\ldots d}$ along the observers' worldlines defined by

$$\dot{T}^{a\ldots b}{}_{c\ldots d} = u^e \nabla_e T^{a\ldots b}{}_{c\ldots d}, \tag{9}$$

and the tensor h_{ab} is used to define the fully orthogonally *projected covariant derivative* D for any tensor $T^{a\ldots b}{}_{c\ldots d}$,

$$D_e T^{a\ldots b}{}_{c\ldots d} = h^a{}_f h^p{}_c \ldots h^b{}_g h^q{}_d h^r{}_e \nabla_r T^{f\ldots g}{}_{p\ldots q}, \tag{10}$$

with total projection on all the free indices. Angle brackets to denote orthogonal projections of vectors and the orthogonally *projected symmetric trace-free* PSTF part of tensors [15]:

$$V^{\langle a\rangle} = h^a{}_b V^b, \qquad T^{\langle ab\rangle} = \left[h^{(a}{}_c h^{b)}{}_d - \frac{1}{3}h^{ab}h_{cd}\right]T^{cd}. \tag{11}$$

In the $(1+1+2)$ approach we further split the 3-space V, by introducing the unit vector e^a orthogonal to u^a so that

$$e_a u^a = 0, \qquad e_a e^a = 1. \tag{12}$$

Then the *projection tensor*

$$N_a{}^b \equiv h_a{}^b - e_a e^b = g_a{}^b + u_a u^b - e_a e^b, \qquad N^a{}_a = 2 \tag{13}$$

projects vectors onto the 2-surfaces orthogonal to e^a *and* u^a, which, following [17], we will refer to as "*sheets*." Hence, it is obvious that $e^a N_{ab} = 0 = u^a N_{ab}$. As we know in $(1+3)$ approach any second rank symmetric 4-tensor can be split into a scalar along u^a, a 3-vector and a PSTF 3-tensor. In $(1+1+2)$ slicing, we can take this split further by splitting the 3-vector and PSTF 3-tensor with respect to e^a. Any 3-vector, ψ^a, can be irreducibly split into a component along e^a and a sheet component Ψ^a, orthogonal to e^a, i.e.

$$\psi^a = \Psi e^a + \Psi^a, \qquad \Psi \equiv \psi^a e_a, \qquad \Psi^a \equiv N^{ab}\psi_b. \tag{14}$$

A similar decomposition can be done for PSTF 3-tensor, ψ_{ab}, which can be split into scalar (along e^a), 2-vector and 2-tensor part as follows:

$$\psi_{ab} = \psi_{\langle ab\rangle} = \Psi\left(e_a e_b - \tfrac{1}{2}N_{ab}\right) + 2\Psi_{(a}e_{b)} + \Psi_{ab}, \tag{15}$$

where

$$\Psi \equiv e^a e^b \psi_{ab} = -N^{ab}\psi_{ab}, \qquad \Psi_a \equiv N_a{}^b e^c \psi_{bc},$$
$$\Psi_{ab} \equiv \psi_{\{ab\}} \equiv \left(N^c{}_{(a}N_{b)}{}^d - \frac{1}{2}N_{ab}N^{cd}\right)\psi_{cd}, \tag{16}$$

and the curly brackets denote the PSTF part of a tensor with respect to e^a. We also have

$$h_{\{ab\}} = 0, \qquad N_{\langle ab\rangle} = -e_{\langle a}e_{b\rangle} = N_{ab} - \tfrac{2}{3}h_{ab}. \tag{17}$$

The sheet carries a natural 2-volume element, the alternating Levi-Civita 2-tensor

$$\varepsilon_{ab} \equiv \varepsilon_{abc}e^c = \eta_{dabc}e^c u^d, \tag{18}$$

where ε_{abc} is the 3-space permutation symbol the volume element of the 3-space and η_{abcd} is the spacetime permutator or the 4-volume.

With these definitions it follows that any 1 + 3 quantity can be locally split in the 1 + 1 + 2 setting into only three types of objects: scalars, 2-vectors in the sheet, and PSTF 2-tensors (also defined on the sheet).

B. Derivatives and the kinematical variables

Apart from the "*time*" (dot) derivative, of an object (scalar, vector or tensor), which is the derivative along the timelike congruence u^a, we now introduce two new derivatives, which e^a defines, for any object $\psi_{a\ldots b}{}^{c\ldots d}$:

$$\hat{\psi}_{a\ldots b}{}^{c\ldots d} \equiv e^f D_f \psi_{a\ldots b}{}^{c\ldots d}, \tag{19}$$

$$\delta_f \psi_{a\ldots b}{}^{c\ldots d} \equiv N_a{}^f \ldots N_b{}^g N_h{}^c \ldots N_i{}^d N_f{}^j D_j \psi_{f\ldots g}{}^{i\ldots j}. \tag{20}$$

The hat derivative is the derivative along the e^a vector field in the surfaces orthogonal to u^a. The δ derivative is the projected derivative onto the sheet, with the projection on every free index. We can now decompose the covariant derivative of e^a in the direction orthogonal to u^a into its irreducible parts giving

$$D_a e_b = e_a a_b + \tfrac{1}{2}\phi N_{ab} + \xi \varepsilon_{ab} + \zeta_{ab}, \tag{21}$$

where

$$a_a \equiv e^c D_c e_a = \hat{e}_a, \tag{22}$$

$$\phi \equiv \delta_a e^a, \tag{23}$$

$$\xi \equiv \tfrac{1}{2}\varepsilon^{ab}\delta_a e_b, \tag{24}$$

$$\zeta_{ab} \equiv \delta_{\{a}e_{b\}}. \tag{25}$$

We see that for an observer that chooses e^a as special direction in the spacetime, ϕ represents the *expansion of the sheet*, ζ_{ab} is the *shear of* e^a (i.e. the distortion of the sheet) and a^a its *acceleration*. We can also interpret ξ as the *vorticity* associated with e^a so that it is a representation of the "twisting" or rotation of the sheet.

Using Eqs. (14) and (15) one can split the (1 + 3) kinematical variables and Weyl tensors as

$$\dot{u}^a = \mathcal{A} e^a + \mathcal{A}^a, \tag{26}$$

$$\omega^a = \Omega e^a + \Omega^a, \tag{27}$$

$$\sigma_{ab} = \Sigma(e_a e_b - \tfrac{1}{2}N_{ab}) + 2\Sigma_{(a}e_{b)} + \Sigma_{ab}, \tag{28}$$

$$E_{ab} = \mathcal{E}(e_a e_b - \tfrac{1}{2}N_{ab}) + 2\mathcal{E}_{(a}e_{b)} + \mathcal{E}_{ab}, \tag{29}$$

$$H_{ab} = \mathcal{H}(e_a e_b - \tfrac{1}{2}N_{ab}) + 2\mathcal{H}_{(a}e_{b)} + \mathcal{H}_{ab}, \tag{30}$$

where E_{ab} and H_{ab} are the electric and magnetic part of the Weyl tensor, respectively. Therefore, the key variables of the 1 + 1 + 2 formalism are

$$\{\Theta, \mathcal{A}, \Omega, \Sigma, \mathcal{E}, \mathcal{H}, \mathcal{A}^a, \Omega^a, \Sigma^a, \mathcal{E}^a, \mathcal{H}^a, \Sigma_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}\}. \tag{31}$$

Similarly, we may split the anisotropic fluid variables q^a and π_{ab}:

$$q^a = Q e^a + Q^a, \tag{32}$$

$$\pi_{ab} = \Pi[e_a e_b - \tfrac{1}{2}N_{ab}] + 2\Pi_{(a}e_{b)} + \Pi_{ab}. \tag{33}$$

The full covariant derivatives of e^a and u^a in terms of these variables are given in the Appendix.

IV. 1 + 1 + 2 EQUATIONS FOR LRS-II SPACETIMES

Locally rotationally symmetric (LRS) spacetimes posses a continuous isotropy group at each point and hence a multitransitive isometry group acting on the spacetime manifold [23]. These spacetimes exhibit locally (at each point) a unique preferred spatial direction, covariantly defined, for example, by either vorticity vector field or a nonvanishing nongravitational acceleration of the matter fluids. The 1 + 1 + 2 formalism is therefore ideally suited for covariant description of these spacetimes, yielding a complete deviation in terms of invariant scalar quantities that have physical or direct geometrical meaning [24]. The preferred spatial direction in the LRS spacetimes constitutes a local axis of symmetry and in this case e^a is just a vector pointing along the axis of symmetry and is thus called a "radial" vector. Since LRS spacetimes are defined to be isotropic, this allows for the vanishing of all 1 + 1 + 2 vectors and tensors, such that there are no preferred directions in the sheet. Thus, all the nonzero 1 + 1 + 2 variables are covariantly defined scalars. The variables, $\{\mathcal{A}, \Theta, \phi, \xi, \Sigma, \Omega, \mathcal{E}, \mathcal{H}, \mu, p, \Pi, Q\}$ fully describe LRS spacetimes and are what is solved for in the 1 + 1 + 2 approach. A detailed discussion of the covariant approach to LRS perfect fluid spacetimes can be seen in [23].

A subclass of the LRS spacetimes, called LRS-II, contains all the LRS spacetimes that are rotation free. As consequence, in LRS-II spacetimes the variables Ω, ξ and $\mathcal{H}$ are identically zero, and the variables

$$\{\mathcal{A}, \Theta, \phi, \Sigma, \mathcal{E}, \mu, p, \Pi, Q\}$$

fully characterize the kinematics. The propagation and constraint equations for these variables are given in the following subsections:

A. Propagation equations

$$\hat{\phi} = -\tfrac{1}{2}\phi^2 + (\tfrac{1}{3}\Theta + \Sigma)(\tfrac{2}{3}\Theta - \Sigma) - \tfrac{2}{3}\mu - \tfrac{1}{2}\Pi - \mathcal{E}, \tag{34}$$

$$\hat{\Sigma} - \tfrac{2}{3}\hat{\Theta} = -\tfrac{3}{2}\phi\Sigma - Q, \tag{35}$$

$$\hat{\mathcal{E}} - \tfrac{1}{3}\hat{\mu} + \tfrac{1}{2}\hat{\Pi} = -\tfrac{3}{2}\phi(\mathcal{E} + \tfrac{1}{2}\Pi) + (\tfrac{1}{2}\Sigma - \tfrac{1}{3}\Theta)Q. \tag{36}$$

B. Evolution equations

$$\dot{\phi} = -(\Sigma - \tfrac{2}{3}\Theta)(\mathcal{A} - \tfrac{1}{2}\phi) + Q, \tag{37}$$

$$\dot{\Sigma} - \tfrac{2}{3}\dot{\Theta} = -\mathcal{A}\phi + 2(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma)^2 + \tfrac{1}{3}(\mu + 3p) - \mathcal{E} + \tfrac{1}{2}\Pi, \tag{38}$$

$$\dot{\mathcal{E}} - \tfrac{1}{3}\dot{\mu} + \tfrac{1}{2}\dot{\Pi} = (\tfrac{3}{2}\Sigma - \Theta)\mathcal{E} + \tfrac{1}{4}(\Sigma - \tfrac{2}{3}\Theta)\Pi + \tfrac{1}{2}\phi Q - \tfrac{1}{2}(\mu + p)(\Sigma - \tfrac{2}{3}\Theta). \tag{39}$$

C. Propagation/evolution equations

$$\dot{\mu} + \hat{Q} = -\Theta(\mu + p) - (\phi + 2\mathcal{A})Q - \tfrac{3}{2}\Sigma\Pi, \tag{40}$$

$$\dot{Q} + \hat{p} + \hat{\Pi} = -(\tfrac{3}{2}\phi + \mathcal{A})\Pi - (\tfrac{4}{3}\Theta + \Sigma)Q - (\mu + p)\mathcal{A}, \tag{41}$$

$$\hat{\mathcal{A}} - \dot{\Theta} = -(\mathcal{A} + \phi)\mathcal{A} + \tfrac{1}{3}\Theta^2 + \tfrac{3}{2}\Sigma^2 + \tfrac{1}{2}(\mu + 3p). \tag{42}$$

D. Commutation relation

$$\hat{\dot{\psi}} - \dot{\hat{\psi}} = -\mathcal{A}\dot{\psi} + (\tfrac{1}{3}\Theta + \Sigma)\hat{\psi}. \tag{43}$$

Here, the quantities μ and p are the total *effective* energy density and pressure. In context of fourth-order gravity we would define these quantities later. Since the vorticity vanishes, the unit vector field u^a is hypersurface orthogonal to the spacelike 3-surfaces whose intrinsic curvature can be calculated from the *Gauss equation* for u^a that is generally given as [24]

$$^{(3)}R_{abcd} = (R_{abcd})_{\perp} - K_{ac}K_{bd} + K_{bc}K_{ad}\,, \tag{44}$$

where $^{(3)}R_{abcd}$ is the *3-curvature tensor*, $\perp$ means projection with h_{ab} on all indices, and K_{ab} is the *extrinsic curvature*. With the additional constraint of the vanishing of the sheet distortion ξ, i.e. the sheet is a genuine 2-surface, The Gauss equation for e^a together with the 3-Ricci identities determine the 3-Ricci curvature tensor of the spacelike 3-surfaces orthogonal to u^a to be

$$^3R_{ab} = -[\hat{\phi} + \tfrac{1}{2}\phi^2]e_a e_b - [\tfrac{1}{2}\hat{\phi} + \tfrac{1}{2}\phi^2 - K]N_{ab}\,, \tag{45}$$

This gives the 3-Ricci-scalar as

$$^3R = -2[\tfrac{1}{2}\hat{\phi} + \tfrac{3}{4}\phi^2 - K], \tag{46}$$

where K is the *Gaussian curvature* of the sheet, $^2R_{ab} = KN_{ab}$. From this equation and (34) an expression for K is obtained in the form [24]

$$K = \tfrac{1}{3}\mu - \mathcal{E} - \tfrac{1}{2}\Pi + \tfrac{1}{4}\phi^2 - (\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma)^2. \tag{47}$$

From (34)–(39), the evolution and propagation equations of K can be determined as

$$\dot{K} = -\tfrac{2}{3}(\tfrac{2}{3}\Theta - \Sigma)K\,, \tag{48}$$

$$\hat{K} = -\phi K. \tag{49}$$

From Eq. (48), it follows that whenever the Gaussian curvature of the sheet is nonzero and constant in time, then the shear is always proportional to the expansion as $\Sigma = \frac{2}{3}\Theta$.

Let us now turn to the case of spherically symmetric static spacetimes, which belong naturally to LRS class II. The condition of staticity implies that the dot derivatives of all the quantities vanish. Furthermore, the expansion also vanishes, as a nonvanishing expansion would imply that the timelike congruence would contract or expand in time, which is not possible in a static spacetime. Hence, we have $\Theta = 0$, and as discussed in the previous section this implies $\Sigma = 0$. From Eq. (37) we then have the heat flux Q to vanish identically in these spacetimes. Hence, the set of $(1 + 1 + 2)$ equations, which describe the spacetime become

$$\hat{\phi} = -\tfrac{1}{2}\phi^2 - \tfrac{2}{3}\mu - \tfrac{1}{2}\Pi - \mathcal{E}, \tag{50}$$

$$\hat{\mathcal{E}} - \tfrac{1}{3}\hat{\mu} + \tfrac{1}{2}\hat{\Pi} = -\tfrac{3}{2}\phi(\mathcal{E} + \tfrac{1}{2}\Pi), \tag{51}$$

$$0 = -\mathcal{A}\phi + \tfrac{1}{3}(\mu + 3p) - \mathcal{E} + \tfrac{1}{2}\Pi, \tag{52}$$

$$\hat{p} + \hat{\Pi} = -(\tfrac{3}{2}\phi + \mathcal{A})\Pi - (\mu + p)\mathcal{A}\,, \tag{53}$$

$$\hat{\mathcal{A}} = -(A + \phi)\mathcal{A} + \tfrac{1}{2}(\mu + 3p). \tag{54}$$

V. SPHERICALLY SYMMETRIC STATIC SPACETIMES IN HIGHER ORDER GRAVITY

At this point one can rederive these equations in the case of $f(R)$ gravity. The quantities μ, p and Π are defined, in this case, as

$$\mu = \frac{1}{f'}\left(\mu^m + \frac{1}{2}(Rf' - f) + f''\hat{X} + f''X\phi + f'''X^2\right), \tag{55}$$

$$p = \frac{1}{f'}\left(p^m + \frac{1}{2}(f - Rf') - \frac{2}{3}f''\hat{X} - \frac{2}{3}f''X\phi - \frac{2}{3}f'''X^2 - \mathcal{A}f''X\right), \tag{56}$$

$$\Pi = \frac{1}{f'}\left(\frac{2}{3}f'''X^2 + \frac{2}{3}f''\hat{X} - \frac{1}{3}f''X\phi\right), \tag{57}$$

where we have defined $\hat{R} = X$. We will consider here the "external" field generated by a pointlike source so that $\mu^m = 0$ and $p^m = 0$. Because of the additional degrees of freedom the Eqs. (50)–(54) are not closed and we have to add an additional equation, the *trace equation*

$$Rf' - 2f = -3f''\hat{X} - 3f''X\phi - 3f'''X^2 - 3\mathcal{A}f''X. \tag{58}$$

Using the above equations in (50)–(54) and eliminating $\mathcal{E}$, we get the set of four coupled first order equations governing the spacetime in the fourth-order gravity as

$$f'[\hat{\phi} + \phi(\tfrac{1}{2}\phi - \mathcal{A})] = \tfrac{1}{3}Rf' - \tfrac{2}{3}f + f''X(\phi + 2\mathcal{A}), \tag{59}$$

$$f'[\hat{\mathcal{A}} + \mathcal{A}(\mathcal{A} + \phi)] = \tfrac{1}{6}f - \tfrac{1}{3}Rf' - f''X\mathcal{A}, \tag{60}$$

$$\hat{R} = X, \tag{61}$$

$$f''\hat{X} = -\tfrac{1}{3}Rf' + \tfrac{2}{3}f - f'''X^2 - X(\phi + \mathcal{A})f''. \tag{62}$$

We emphasize here that the above system of equations are written in terms of the covariant quantities in the 1 + 1 + 2 splitting and absolutely coordinate independent. Note that the system reduces to the second-order system of GR in vacuum [17], if we put $f(R) = R$, $R = 0$ and $X = 0$ [25]. However, as in the case of the Einstein equation, or any other fully covariant system of equations, the physics can be understood fully only if one chooses an observer. In the 1 + 3 approach this is done basically choosing a velocity field, but in the 1 + 1 + 2 framework this is not sufficient. One has to give also a particular form of "*radial*" coordinate. This in turn will define a specific form for the "*hat*" derivative. As we will see in the later sections there is a natural choice for this coordinate given by the geometry of our problem and we will use it to find exact spherically symmetric solutions for some specific $f(R)$ gravity models.

VI. COVARIANT RESULTS FOR THE SPHERICALLY SYMMETRIC SYSTEM

From the structure of (59)–(62), we can already deduce some important results for spherically symmetric static solutions in a general $f(R)$ gravity in an absolute coordinate independent manner. These results are important because they can be used as guidelines to understand the behavior of any proposed $f(R)$ model in this setting and to design new ones.

A. Necessary condition for existence of solutions with vanishing Ricci scalar

It is evident from Eqs. (59)–(62) above, the function f must be of class C^3 at $R = 0$, which implies

$$|f'(0)| < +\infty, \qquad |f''(0)| < +\infty, \qquad |f'''(0)| < +\infty. \tag{63}$$

Also, we impose the conditions

$$f(0) = 0, \qquad R = 0. \tag{64}$$

Note that the condition of vanishing of the Ricci scalar throughout the manifold automatically implies $X = 0$.

Now there are two possibilities:

(a) $f'(0) \neq 0$: In this case we see the system reduces to the following:

$$\hat{\phi} + \phi(\tfrac{1}{2}\phi - \mathcal{A}) = 0, \tag{65}$$

$$\hat{\mathcal{A}} + \mathcal{A}(\mathcal{A} + \phi) = 0. \tag{66}$$

It can be easily checked that the conditions $R = 0$, $f(0) = 0$, and $f'(0) \neq 0$ makes the Einstein tensor G_{ab} vanish and therefore Schwarzschild solution is the only spherically symmetric static solution. This then allows us to state a generalization of *Birkhoff's theorem* in higher order gravity.

For all functions $f(R)$ which are of class C^3 at $R = 0$ and $f(0) = 0$ while $f'(0) \neq 0$, the Schwarzschild solution is the only static spherically symmetric vacuum solution with vanishing Ricci scalar.

It is also interesting to note that the above result is consistent with the conditions $f' > 0$ and $f'' > 0$, which guarantee the attractive nature of the gravitational interaction and the absence of tachyons [12]. This shows that there may be a connection between this solution and the very nature of the gravitational interaction.

The presence of this solution, can have interesting consequences on the validity of these models on the Solar System level. In particular, if one concludes that the Sun behaves very close to a Schwarzschild solution, the experimental data of the solar system would help constraining these models.

(b) $f'(0) = 0$, $f(0) = 0$: In this case (59)–(62) are identically satisfied for all values of ϕ and $\mathcal{A}$ that guarantees $R = 0$ and hence $X = 0$ [26]. Hence, for all models with $f'(0) = 0$, any solution with vanishing Ricci scalar in general relativity would be a solution to the above system. This is interesting as it shows that fourth-order gravity in this context can present the same solutions of GR plus additional solutions. For example, the Reissner-Nordström solution which represents the spacetime outside a spherically symmetric charged body, is a solution to the system (59)–(62) even if no electric charge is present. Similarly a static spherically symmetric solution for a perfect fluid interior with equation of state $p = (1/3)\rho$ (for example, the Hajj-Boutros solution or the special case of the Whittaker solution [28]) can be a solution of this system in the absence of any standard fluid.

The presence of solutions of type (*b*) shows that when the conditions given in paragraph (*a*) are not satisfied the Schwarzschild solution is not a unique static spherically symmetric solution. Such results hint toward disproving the general Birkhoff theorem in its classical form for fourth-order gravity.

B. Necessary condition for existence of solutions with constant scalar curvature

Solutions with constant Ricci scalar are characterized by the fact that $R = R_0 = \text{const}$ and, as consequence, $X, \hat{X} = 0$. Imposing these conditions on (59)–(62) and supposing it to be at least of class C^3 in $R = R_0$ one obtains

$$f_0'[\hat{\phi} + \phi(\tfrac{1}{2}\phi - \mathcal{A}) = \tfrac{1}{3}R_0 f_0' - \tfrac{2}{3}f_0\,, \tag{67}$$

$$f_0'[\hat{\mathcal{A}} + \mathcal{A}(\mathcal{A} + \phi)] = \tfrac{1}{6}f_0 - \tfrac{1}{3}R_0 f'\,, \tag{68}$$

$$0 = -R_0 f_0' + 2f_0\,, \tag{69}$$

where $f'(R_0) = f_0'$, etc. A first solution exists if

$$f_0' \neq 0, \qquad f_0 \neq 0, \qquad 2f_0 - R_0 f_0' = 0. \tag{70}$$

Instead in the case $f_0' \neq 0$, $f_0 = 0$ one obtains again the Schwarzschild solution ($R_0 = 0$). Finally, another solution can be achieved if

$$f_0' = 0, \qquad f_0 = 0, \qquad R = R_0\,, \qquad X, \qquad \hat{X} = 0 \tag{71}$$

is satisfied. As in the previous subsection, in this case also, any constant Ricci scalar solution in GR would identically be a solution to the system.

The relation (70) was already found by Barrow and Ottewill [29] in the cosmological context and later rediscovered in [30]. It relates the value of the constant Ricci scalar with the universal constants in the action. For example if we have the Lagrangian as $R - 2\Lambda$, which is the Lagrangian for GR with the cosmological constant, we must have, as is well known, the relation $R_0 = 4\Lambda$.

C. The curious case of R^2 gravity

As we have already explained, the condition for existence of solutions with covariantly constant scalar curvature connects the constant curvature with the universal constants of the Lagrangian. However, this is not the case for $f(R) = KR^2$. In fact, for this type of Lagrangian the third condition of (70) is identically satisfied. This means that we can have a constant curvature solution for any value of the curvature. Thus for R^2 gravity, the "*cosmological*" constant term in a Schwarzschild-de Sitter/anti-de Sittter (dS/AdS) spacetime becomes a local constant of integration just like the mass. Hence, in this theory we can have two distant stars behaving like two different Schwarzschild-dS/AdS objects with different values of the constant. Unfortunately this case is rather pathological since it corresponds to the case in which the trace of the field equations in vacuum, $3\Box f' + f'R - 2f = 0$ is satisfied *identically* for constant Ricci scalar, whereas usually it may be satisfied for special values of R. Also, this model is ruled out by Solar System experiments (see [31]).

VII. CHOOSING A COORDINATE SYSTEM AND RELATION BETWEEN THE COVARIANT VARIABLES AND THE METRIC

The most natural way to choose the proper radial coordinate in spherically symmetric static spacetimes, is to make the Gaussian curvature "K" of the spherical sheets to be proportional to the inverse square of the radius. In that case, this coordinate "r" becomes the *area radius* of the sheets. This gives a geometrical definition to the "*hat*" derivative. As we have already seen, $\hat{K} = -\phi K$, therefore the most natural way to define the hat derivative of any scalar M would be

$$\hat{M} = \frac{1}{2}r\phi\frac{dM}{dr}. \tag{72}$$

With this choice, the system of Eqs. (59)–(62) becomes

$$f'\left(\frac{1}{2}r\phi\frac{d\phi}{dr} + \frac{1}{2}\phi^2 - A\phi\right) = \frac{1}{3}Rf' - \frac{2}{3}f + (\phi + 2\mathcal{A})f''X, \tag{73}$$

$$f'\left(\frac{1}{2}r\phi\frac{d\mathcal{A}}{dr} + \mathcal{A}^2 + \mathcal{A}\phi\right) = \frac{1}{6}f - \frac{1}{3}Rf' - f''X, \tag{74}$$

$$\frac{1}{2}r\phi\frac{dR}{dr} = X, \tag{75}$$

$$f''\frac{1}{2}r\phi\frac{dX}{dr} = -\frac{1}{3}Rf' + \frac{2}{3}f - f'''X^2 - (X\phi + X\mathcal{A})f''. \tag{76}$$

In the r coordinate above the most general spherically symmetric static metric is

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{77}$$

Now, from the properties of the four-velocity u^a and the radial vector e^a, i.e. $u^a u_a = -1$ and $e^a e_a = 1$, we find that

$$u^t = \sqrt{A(r)}, \qquad e^r = \sqrt{B(r)}, \tag{78}$$

also, from the definitions of different covariant scalars, we get

$$\mathcal{A} = -u^b u^a \nabla_b e_a = \frac{1}{2A}\frac{dA}{dr}\sqrt{B}, \tag{79}$$

$$\phi = N^b{}_a \nabla_b e^a = \frac{2}{r}\sqrt{B}. \tag{80}$$

R can be found in the usual way as a contraction of the Riemann tensor, and X is derived from it as in (75). Thus, we see any solution to the Eqs. (73)–(76), would uniquely determine the metric for the spacetime.

VIII. AN EXAMPLE: SOME EXACT SOLUTIONS FOR R^n GRAVITY

In this section we present, as an example, a few exact solutions for R^n gravity, in absence of standard matter. Specializing the choice of $f(R) = R^n$, Eqs. (73)–(76) becomes

$$\frac{1}{2}nr\phi\frac{d\phi}{dr}R^{n-1} = \left(A - \frac{1}{2}\phi\right)\phi R^{n-1} + \frac{n-2}{3n}R^n + (n-1)R^{n-2}X(\phi + 2\mathcal{A}), \tag{81}$$

$$\frac{1}{2}nr\phi\frac{d\mathcal{A}}{dr}R^{n-1} = -(\mathcal{A} + \phi)\mathcal{A}R^{n-1} + \frac{1-2n}{6n}R^n - (n-1)R^{n-2}X\mathcal{A}, \tag{82}$$

$$\frac{1}{2}r\phi\frac{dR}{dr} = X, \tag{83}$$

$$\frac{1}{2}r\phi n(n-1)\frac{dX}{dr}R^{n-2} = \frac{2-n}{3}R^n - X(\phi + \mathcal{A}) - n(n-1)(n-2)R^{n-3}X^2. \tag{84}$$

A. Schwarzschild solution

Substituting $R = 0$, $dR/dr = 0$ in the above set of equations, we see that the equations are satisfied trivially provided that $n = 1, 2, \geq 3$. However, since $R = 0$ is by itself a differential constraint involving ϕ and $\mathcal{A}$, hence any ϕ and $\mathcal{A}$ that ensures a zero Ricci scalar would solve the system. As we know the following solution

$$\phi = \frac{2}{r}\sqrt{1 - \frac{2m}{r}}, \qquad \mathcal{A} = \frac{m}{r^2}\left[1 - \frac{2m}{r}\right]^{-(1/2)}, \tag{85}$$

with Eqs. (79) and (80) gives the usual Schwarzschild metric in (t, r, θ, ϕ) coordinates that has a zero Ricci scalar; hence, the above solution is the solution of the system.

B. A solution with constant nonzero Ricci scalar

As described before if we substitute $X = 0$, $R = R_0 \neq 0$ in the above system of equations then a solution is possible if and only if $n = 2$. In that case the solutions of the other two functions are

$$\phi = \frac{2}{r}\sqrt{1 - \frac{2m}{r} + \frac{R_0}{3}r^2}, \tag{86}$$

$$\mathcal{A} = \frac{m + R_0 r^2}{r^2}\left[1 - \frac{2m}{r} + \frac{R_0}{3}r^2\right]^{-(1/2)}. \tag{87}$$

This is the usual Schwarzschild-dS/AdS solution depending on the sign of R_0.

C. A solution with nonconstant Ricci scalar vanishing at infinity

To find more nontrivial solutions of the above system of equations, let us use a Schwarzschild like ansatz,

$$\phi = \sqrt{C_1 r^\alpha + C_2 r^\beta}, \qquad R = C_3/r^\gamma (\gamma > 0), \tag{88}$$

such that the Ricci scalar vanishes at infinity. We use these ansatz in the system of equations and then algebraically solve for the powers and coefficients such that the system is identically satisfied. With this choice we get the following solution:

$$\mathcal{A} = \frac{-C(5-4n)r^{((4n^2-11n+9)/(n-2))} + (4n^2-6n+2)r^{-1}}{2(2-n)} \times \left(\frac{(1+2n-2n^2)(7-10n+4n^2)(1+Cr^{(-(7-10n+4n^2)/(2-n))}}{(2-n)^2}\right)^{-(1/2)}, \tag{89}$$

$$\phi = \frac{2}{r}\left(\frac{(1+2n-2n^2)(7-10n+4n^2)}{(2-n)^2(1+Cr^{(-(7-10n+4n^2)/(2-n))})}\right)^{-(1/2)}, \tag{90}$$

$$R = \frac{6n(n-1)}{(2n(n-1)-1)r^2}, \tag{91}$$

$$X = -\frac{12n(n-1)}{(2n(n-1)-1)r^3} \times \left(\frac{(1+2n-2n^2)(7-10n+4n^2)}{(2-n)^2(1+Cr^{(-(7-10n+4n^2)/(2-n))}}\right)^{-(1/2)}. \quad (92)$$

Now solving for the metric coefficients, we get

$$A(r) = r^{(2n-2)((2n-1)/(2-n))} + \frac{C}{r^{((5-4n)/(2-n))}};$$
$$\frac{1}{B(r)} = \frac{(2-n)^2}{(7-10n+4n^2)(1+2n-2n^2))} \times \left(1 + \frac{C}{r^{((7-10n+4n^2)/(2-n))}}\right). \quad (93)$$

This solution was originally found by Clifton [32]. The solution reduces to Schwarzschild for $n = 1$ and valid for $n < (1+\sqrt{3})/2$ beyond which the metric has unphysical signature. However, for $n \in (1, (1+\sqrt{3})/2)$ the Ricci scalar is negative and hence the action is only real valued if n is an even rational number. That is in its lowest form the numerator of the fraction is even. This problem can be also be resolved by assuming the absolute value of the Ricci scalar in the action. However, in that case the Schwarzschild limit at $n = 1$ is not possible as $|R|$ is not differentiable at $R = 0$ and hence does not belong to the class C^3 functions. It is also interesting to note that in spite of the Ricci scalar vanishing at infinity this solution is *not* asymptotically flat.

IX. CONCLUSION

In this paper we have analyzed static spherically symmetric metrics within the $f(R)$ gravity framework. Using the $1+1+2$ formalism we were able to derive equations describing these metrics for a general form of the function f and a pointlike source. These equations have been used to obtain a set of general conditions for the existence of certain types of static spherically symmetric solutions. It is important to note that our system of equations are much simpler than what one obtains when writing the Einstein field equations in terms of metric components and therefore it is much easier to find new solutions and general covariant results. In particular, the results when $f(0) = 0$ and $f'(0) = 0$ are much more transparent using this approach.

Our results show that the presence of solutions with constant Ricci scalar is influenced by the properties of the derivatives of the function f up to the third order. This implies that two $f(R)$ models are indistinguishable in this framework if differences only arise after the third derivative of f at the given value of the Ricci scalar. Also one can probe in general that a form of the Birkhoff theorem exists for $f(R)$ gravity only if $f(0) = 0$ and $f'(0) \neq 0$, while in general there is more than a static and spherically symmetric solution for the field equations.

It is also interesting to note that the conditions $f' > 0$, $f'' > 0$, which guarantee the attractive nature of the gravitational interaction and the absence of tachyons [12] are consistent with the form of Birkhoff's theorem stated here. This suggests a possible link between these conditions and the Birkhoff theorem, something which definitely deserves further study.

In order to extract observable results the $1+1+2$ equations need to be further specialized choosing a specific form of the radial coordinate. This is equivalent to the choice of an observer in the $1+1+2$ formalism, which, unlike the $1+3$ case, requires not only the specification of a velocity field but also a specific spatial direction. The requirement that the Gauss curvature has an inverse square dependence offers a natural choice for this coordinate.

Once this is done given any $f(R)$ theory of gravity (and sufficient ingenuity) one can derive static and spherically symmetric solution(s) for this theory. We have used a R^n gravity as an example to derive some exact solutions. As expected from our general considerations, since for this class of models $f'(0)$, $f(0) = 0$, the system (59)–(62) does not admit a unique solution and Birkhoff's theorem is violated.

It is worth to stress, however, that such considerations are limited to the case of pointlike sources. It is known that the situation can be really different in the case of extended ones [33]. Such issues will be treated elsewhere.

As a final comment we would like to point out that if one admits the fact that in this framework the Birkhoff theorem is violated, any Newtonian limit of a background solution will give, in principle, a different results [34]. This means that there is no way to calculate the physical Newtonian potential without knowing the *exact* background which characterizes the entire Universe. This is not surprising because in these theories the relation between local physics and the rest of the Universe is much tighter that in general relativity due to their relation with Mach's principle.

ACKNOWLEDGMENTS

S. C. was funded by Generalitat de Catalunya through the Beatriu de Pinos under Contract No. 2007BP-B1 00136. We thank the National Research Foundation (South Africa) for financial support and T. Clifton for useful comments. The University of Cape Town provided support by a grant for R. G. and the National Astrophysics and Space Science Program supported A. M. N.

APPENDIX: USEFUL RELATIONS

In this appendix we give some useful relation needed for the calculations performed in the text.

The full covariant derivatives of e^a and u^a are

$$\nabla_a e_b = -\mathcal{A} u_a u_b - u_a \alpha_b + (\Sigma + \tfrac{1}{3}\Theta) e_a u_b + (\Sigma_a - \varepsilon_{ac}\Omega^c) u_b + e_a a_b + \tfrac{1}{2}\phi N_{ab} + \xi \varepsilon_{ab} + \zeta_{ab}, \tag{A1}$$

$$\nabla_a u_b = -u_a(\mathcal{A} e_b + \mathcal{A}_b) + e_a e_b(\tfrac{1}{3}\Theta + \Sigma) + e_a(\Sigma_b + \varepsilon_{bc}\Omega^c) + (\Sigma_a - \varepsilon_{ac}\Omega^c) e_b + N_{ab}(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma) + \Omega \varepsilon_{ab} + \Sigma_{ab}. \tag{A2}$$

The covariant time derivative of e^a is given by

$$\dot{e}_a = \mathcal{A} u_a + \alpha_a, \quad \text{where } \mathcal{A} = e^a \dot{u}_a, \tag{A3}$$

and α_a is the component lying in the sheet.

The new variables a_a, ϕ, ξ, ζ_{ab} and α_a are fundamental objects of the spacetime, and their dynamics give us information about the spacetime geometry. The spatial covariant derivative of a scalar Ψ is defined as

$$D_a \Psi = \hat{\Psi} e_a + \delta_a \Psi, \tag{A4}$$

while for any vector Ψ^a orthogonal to both u^a and e^a (i.e. Ψ^a lies in the sheet), the various parts of its spatial derivative may be decomposed as follows (Note that a bar on a particular index indicates that the vector or tensor lies in the sheet.):

$$D_a \Psi_b = -e_a e_b \Psi_c a^c - e_b[\tfrac{1}{2}\phi \Psi_a + (\xi \varepsilon_{ac} + \zeta_{ac})\Psi^c] + e_a \hat{\Psi}_{\bar{b}} + \delta_a \Psi_b. \tag{A5}$$

Similarly, for a tensor Ψ_{ab} (where $\Psi_{ab} = \Psi_{\{ab\}}$):

$$D_a \Psi_{bc} = -2 e_a e_{(b}\Psi_{c)d} a^d + e_a \hat{\Psi}_{bc} + \delta_a \Psi_{bc} - 2 e_{(b}[\tfrac{1}{2}\phi \Psi_{c)a} + \Psi_{c)}{}^d(\xi \varepsilon_{ad} + \zeta_{ad})]. \tag{A6}$$

For the Levi-Civita 2-tensor, we have

$$\varepsilon_{ab} e^b = 0 = \varepsilon_{(ab)}, \tag{A7}$$

$$\varepsilon_{abc} = e_a \varepsilon_{bc} - e_b \varepsilon_{ac} + e_c \varepsilon_{ab}, \tag{A8}$$

$$\varepsilon_{ab}\varepsilon^{cd} = N_a{}^c N_b{}^d - N_a{}^d N_b{}^c, \tag{A9}$$

$$\varepsilon_a{}^c \varepsilon_{bc} = N_{ab}, \qquad \varepsilon^{ab}\varepsilon_{ab} = 2, \tag{A10}$$

and

$$\dot{\varepsilon}_{ab} = -2u_{[a}\varepsilon_{b]c}\mathcal{A}^c + 2e_{[a}\epsilon_{b]c}\alpha^c, \qquad \hat{\varepsilon}_{ab} = 2e_{[a}\varepsilon_{b]c}a^c, \qquad \delta_c \varepsilon_{ab} = 0. \tag{A11}$$

For the projection tensor, we have

$$\dot{N}_{ab} = 2u_{(a}\dot{u}_{b)} - 2e_{(a}\dot{e}_{b)} = 2u_{(a}\mathcal{A}_{b)} - 2e_{(a}\alpha_{b)}, \qquad \hat{N}_{ab} = -2e_{(a}a_{b)}, \qquad \delta_c N_{ab} = 0. \tag{A12}$$

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[26] It has been noted by several authors that the situation $f(0) = f'(0) = 0$ is somewhat pathological, since the scalar degree of freedom of this theory, $f'(R)$ corresponds to a Brans-Dicke scalar field in the equivalent Brans-Dicke representation, with Brans-Dicke parameter $\omega = 0$, it also corresponds (apart from a constant) to the inverse effective gravitational coupling of the theory. Therefore, $f' = 0$ corresponds to infinite gravitational coupling $G_{\text{effective}} = G/f'$ and to a singularity of the field equations. However, one can formally set $f' \equiv 0$ and look for solutions of the field equations with this constant value of f'. A similar situation has been pointed out to occur in scalar-tensor gravity [27].

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