

Notes on entropy force in general spherically symmetric spacetimesRong-Gen Cai,^{1,*} Li-Ming Cao,^{2,†} and Nobuyoshi Ohta^{2,‡}¹*Key Laboratory of Frontiers in Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, P.O. Box 2735, Beijing 100190, China*²*Department of Physics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan*

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In a recent paper [arXiv:1001.0785], Verlinde has shown that the Newton gravity appears as an entropy force. In this paper we show how gravity appears as entropy force in Einstein's equation of gravitational field in a general spherically symmetric spacetime. We mainly focus on the trapping horizon of the spacetime. We find that when matter fields are absent, the change of entropy associated with the trapping horizon indeed can be identified with an entropy force. When matter fields are present, we see that heat flux of matter fields also leads to the change of entropy. Applying arguments made by Verlinde and Smolin, respectively, to the trapping horizon, we find that the entropy force is given by the surface gravity of the horizon. The cases in the untrapped region of the spacetime are also discussed.

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I. INTRODUCTION

Quantum mechanics together with general relativity predicts that a black hole behaves like a black body, emitting thermal radiations, with a temperature proportional to its surface gravity at the black hole horizon and with an entropy proportional to its horizon area [1,2]. The Hawking temperature and horizon entropy together with the black hole mass obey the first law of black hole thermodynamics [3]. Since these seminal works in the 1970s, the relation between thermodynamics and spacetime horizons has been widely discussed, and further developments can be found in a nice review [4].

The study on the relation between thermodynamics and gravity theory can be classified into two categories: One is to discuss thermodynamics associated with spacetime horizons in the Einstein general relativity or in generalized gravity theories. The study of stationary black hole thermodynamics belongs to this category. Recently the discussions on thermodynamic properties associated with event horizon of stationary black holes have been generalized to various horizons of dynamical spacetimes [5]. For example, it has been shown that there also exists Hawking radiation associated with an apparent horizon of a Friedmann-Robertson-Walker (FRW) universe [6]. The other is more interesting: to derive equations of motion of the gravitational field from thermodynamics. In 1995, Jacobson derived the Einstein equation by employing the fundamental Clausius relation $\delta Q = TdS$ together with the equivalence principle [7]. Here the key idea is to demand that this relation should hold for all the local Rindler causal horizons through each spacetime point, with δQ and T interpreted as the energy flux and Unruh

temperature seen by an accelerated observer just inside the horizon. The entropy S is assumed to be proportional to the area of the Rindler horizon. In this way, the Einstein equation is nothing but an equation of state of spacetime. In addition, assuming the apparent horizon of a FRW universe has temperature T and entropy S satisfying $T = 1/(2\pi\tilde{r}_A)$ and $S = A/(4G)$, where \tilde{r}_A is the radius of the apparent horizon and A is the area of the apparent horizon, Cai and Kim [8] are able to derive Friedmann equations of the FRW universe with any spatial curvature by applying the Clausius relation to the apparent horizon of the FRW universe. This approach also holds for Gauss-Bonnet gravity and the more general Lovelock gravity. For more discussions on the relation between the first law of thermodynamics and Friedmann equations in diverse gravity theories, see [9,10] and references therein. In the black hole spacetimes, the relation between the first law of thermodynamics and gravitational field equations has also been studied [11]. For a recent review on this topic and some relevant issues, see [12].

In a recent paper by Verlinde [13], with the holographic assumption, gravity is explained as an entropic force caused by changes in the information associated with the positions of material bodies. Among various interesting observations made by Verlinde, here we mention two of them. One is that with the assumption of the entropic force together with the Unruh temperature [14], Verlinde is able to derive the second law of Newton. The other is that the assumption of the entropic force together with the holographic principle and the equipartition law of energy leads to the Newton law of gravitation. Similar observations are also made by Padmanabhan [15]. He observed that the equipartition law of energy for the horizon degrees of freedom combined with the thermodynamic relation $S = E/2T$ also leads to the Newton law of gravity. Here S and T are thermodynamic entropy and temperature associated with the horizon and E is the active gravitational mass

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producing the gravitational acceleration in the spacetime [16]. Some very recent discussions on entropic properties of gravity can be found in [17–30].

On the other hand, it is well known that the Einstein general relativity describes gravity quite well, at least classically. Therefore the Einstein equation should imply some implications of gravity as an entropy force. Note that various discussions made by Verlinde are focused on the Newtonian gravity, namely, in the nonrelativistic case. Therefore it is quite interesting and important to see how gravity appears as an entropic force in the relativistic gravity theory. In this paper we will focus on the Einstein theory of gravity, namely, general relativity.

Note that the entropy force for a system (with many degrees of freedom) is a macroscopic force, and it is induced by the statistical tendency to increase the entropy of the system. So a natural starting point to consider in the Einstein general relativity is causal horizon of spacetimes because there exist well-understood temperature and entropy associated with the causal horizon. In this paper, we will mainly focus on the trapping horizon of a general spherically symmetric dynamical spacetime and explore how the Einstein equation shows as an entropy-force-like equation. We will also discuss the case away from the trapping horizon.

This paper is organized as follows: In Sec. II, starting from the Einstein equation, we show thermodynamics associated with the trapping horizon in a general spherically symmetric dynamical spacetime. In Sec. III, we define a gravitational potential by employing the Kodama vector and generalize Verlinde’s argument to dynamical spacetimes, which relates the gravitational potential to the surface gravity of the trapping horizon. In Sec. IV, we discuss how the gravity on the trapping horizon appears as an entropy force. In Sec. V, we use a Smarr-like formula and the holographic assumption of horizon entropy to derive the Newton gravity. In Sec. VI, following Verlinde [13] and Smolin [20], we give some discussions on the entropy force from the point of view of quantum fluctuation. Section VII is devoted to the conclusion and discussion.

II. GENERAL SPHERICALLY SYMMETRIC SPACETIME

Let us consider a general spherically symmetric spacetime $(\mathcal{M}^n, g_{\mu\nu})$ with the metric

$$g = h_{ab}dx^a dx^b + r^2(x)d\Omega_{n-2}^2, \quad (2.1)$$

where $d\Omega_{n-2}^2$ is the line element of an $(n-2)$ -sphere, and x^a , $a = 1, 2$ are coordinates of the two-dimensional spacetime which is normal to the sphere. Assume the connection of the two-dimensional space is D_a (which is associated with the two-dimensional metric h_{ab}). In this spacetime the Einstein equation can be decomposed as

$$\begin{aligned} G_{ab} &= -\frac{n-2}{r}D_a D_b r - \left[\frac{1}{2}(n-2)(n-3) \right. \\ &\quad \left. \times \left(\frac{1 - D_c r D^c r}{r^2} \right) - \frac{n-2}{r}D_c D^c r \right] h_{ab} \\ &= 8\pi G_n T_{ab}, \end{aligned} \quad (2.2)$$

$$\begin{aligned} G_j^i &= \left[-\frac{1}{2}R^{(2)} - \frac{(n-3)(n-4)}{2} \left(\frac{1 - D_c r D^c r}{r^2} \right) \right. \\ &\quad \left. + \frac{n-3}{r}D_c D^c r \right] \delta_j^i \\ &= 8\pi G_n T_j^i, \end{aligned} \quad (2.3)$$

where G_n is the n -dimensional Newton constant, and $T_{\mu\nu} = (T_{ab}, T_{ij})$ is the energy-momentum tensor. The term $R^{(2)}$ in Eq. (2.3) is a scalar curvature of the two-dimensional spacetime described by h_{ab} . It is obvious that one has $T_{ab} = T_{ab}(x)$ and $T_j^i = \sigma(x)\delta_j^i$ in this case. Substituting the relation

$$\begin{aligned} R^{(2)} &= R + \frac{2(n-2)}{r}D_c D^c r - (n-2)(n-3) \\ &\quad \times \frac{1 - D_c r D^c r}{r^2}, \end{aligned} \quad (2.4)$$

one can find that Eq. (2.3) is trivially satisfied if Eq. (2.2) holds. So Eq. (2.2) is the master equation we will analyze.

The Misner-Sharp energy inside the sphere with radius r is given by [31]

$$E = \frac{1}{16\pi G_n}(n-2)\Omega_{n-2}r^{n-3}(1 - D_a r D^a r). \quad (2.5)$$

This is active energy inside the sphere. The properties of this energy are discussed in some detail in Refs. [31,32]. With the energy-momentum tensor T_{ab} , one can define two physical quantities:

$$w = -\frac{1}{2}T_a^a, \quad (2.6)$$

which is called work density, and

$$\psi_a = T_a^b D_b r + w D_a r, \quad (2.7)$$

which is called energy supply. It follows from (2.2) and (2.3) that

$$\begin{aligned} w &= \frac{1}{16\pi G_n} \left[-\frac{n-2}{r}D_c D^c r + (n-2)(n-3) \right. \\ &\quad \left. \times \left(\frac{1 - D_c r D^c r}{r^2} \right) \right], \end{aligned} \quad (2.8)$$

and

$$\psi_a = \frac{1}{16\pi G_n} \frac{n-2}{r} [(D_b D^b r) D_a r - D_a (D^b r D_b r)]. \quad (2.9)$$

Combing Eqs. (2.5), (2.8), and (2.9), one has [33,34]

$$D_a E = A \psi_a + w D_a V, \quad (2.10)$$

where $A = \Omega_{n-2} r^{n-2}$ and $V = \Omega_{n-2} r^{n-1} / (n-1)$ are area and volume of the sphere with radius r , respectively. We can also express this equation in the form $dE = A \psi + w dV$ by defining one-form $\psi = \psi_a dx^a$ and differential operator $d = dx^a D_a$.

To study causal structure of the spacetime (2.1), it is convenient to introduce two null vector fields ℓ_a and k_a , and write the two-dimensional metric as $h_{ab} = -\ell_a k_b - k_a \ell_b$, where $\ell_a k^a = -1$. By calculating the extrinsic curvature of the $(n-2)$ -sphere, one gets the value of the extrinsic curvature along the ℓ_a and k_a directions, and then gets the expansions of the corresponding null congruences. These two expansions are denoted by $\theta_{(\ell)}$ and $\theta_{(k)}$, respectively.

An $(n-2)$ -dimensional sphere is called marginal if $\theta_{(\ell)} \theta_{(k)} = 0$. Similarly, an untrapped sphere is given by $\theta_{(\ell)} \theta_{(k)} < 0$, and a trapped sphere is given by $\theta_{(\ell)} \theta_{(k)} > 0$. It is found that $\theta_{(\ell)} \theta_{(k)} \sim -D_a r D^a r$. Therefore the marginal sphere satisfies $D_a r D^a r = 0$. The hypersurface foliated by the marginal spheres is called a trapping horizon. This means that $D_a r D^a r$ always vanishes on this hypersurface. Let ξ be a vector field which is tangent to the trapping horizon. We then have

$$\mathcal{L}_\xi (D_b r D^b r) = \xi^a D_a (D_b r D^b r) = 0. \quad (2.11)$$

Considering Eq. (2.9), on the trapping horizon, we find

$$A \psi_a \xi^a = \frac{\kappa_H}{2\pi} \mathcal{L}_\xi S, \quad (2.12)$$

where \mathcal{L}_ξ is a Lie derivative along ξ , and

$$\kappa_H = \frac{1}{2} D_a D^a r, \quad S = \frac{A_H}{4G_n}. \quad (2.13)$$

The κ_H is called surface gravity [33] and A_H is the area of the trapping horizon. By defining $T_H = \kappa_H / 2\pi$, along the vector ξ , we have

$$\mathcal{L}_\xi E = T_H \mathcal{L}_\xi S + w \mathcal{L}_\xi V. \quad (2.14)$$

This is the first law of the dynamical black holes [33]. S and T are the Bekenstein-Hawking entropy and Hawking temperature associated with the trapping horizon [35].

The surface gravity (2.13) can also be understood from the so-called Kodama vector field [36]:

$$K^a = -\epsilon^{ab} D_b r, \quad K_a K^a = -D_a r D^a r. \quad (2.15)$$

So the Kodama vector is null on the trapping horizon and timelike in the untrapped region. In addition, on the trapping horizon one has $K_a = D_a r$. By this vector, one can define a surface gravity on the trapping horizon as

$$K^b D_{[b} K_{a]} = \kappa_H K_a. \quad (2.16)$$

A straightforward calculation shows that this gives the same result of κ_H as (2.13). In the following discussions,

we will also use the notation $\kappa = (1/2) D_a D^a r$. κ reduces to the surface gravity (κ_H) on the trapping horizon, while the physical meaning of κ will be shown shortly.

III. SURFACE GRAVITY AND GRAVITATIONAL POTENTIAL

In this section, we discuss the relation between surface gravity and gravitational potential. We note that Eq. (2.9) can be rewritten as

$$\kappa D_a r - \frac{1}{2} D_a (D_b r D^b r) = \frac{8\pi G_n}{n-2} r \psi_a. \quad (3.1)$$

This equation holds not only on the trapping horizon but also in the untrapped region. Actually, it is just a part of the Einstein equation and is valid at each point of spacetime.

Let us first consider the vacuum case in which the energy-momentum tensor vanishes. Then Eq. (3.1) gives

$$\kappa D_a r = \frac{1}{2} D_a (D_b r D^b r) = \frac{1}{2} D_a (-K_b K^b). \quad (3.2)$$

Defining $e^{2\phi} \equiv -K^a K_a = D^a r D_a r$, we have

$$\kappa D_a r = e^{2\phi} D_a \phi, \quad \text{or} \quad \kappa = D^a r D_a \phi. \quad (3.3)$$

In the static case, the Kodama vector reduces to a timelike Killing vector, and ϕ is the generalized Newton potential in general relativity. Here, by using the Kodama vector, we have generalized to the dynamical spacetime from the static one discussed by Verlinde [13], where a timelike Killing vector is employed to relate the Newton potential to gravitational acceleration.

Let n_a be an arbitrary vector field. We then have

$$\begin{aligned} \kappa \mathcal{L}_n r &= e^{2\phi} \mathcal{L}_n \phi, \\ \left(\frac{\kappa}{2\pi}\right) \mathcal{L}_n S &= A n^a \left[\frac{n-2}{8\pi G_n} \left(\frac{e^{2\phi}}{r}\right) D_a \phi \right], \end{aligned} \quad (3.4)$$

where S is given by $A/(4G_n)$.

Next, to discuss the general case with matter fields, let us assume that the metric h_{ab} and the energy-momentum tensor T_{ab} can be written as

$$h_{ab} = -u_a u_b + v_a v_b, \quad (3.5)$$

and

$$T_{ab} = \alpha u_a u_b + \beta (u_a v_b + v_a u_b) + \gamma v_a v_b, \quad (3.6)$$

where $u_a u^a = -1$, $v_a v^a = 1$, and $u_a v^a = 0$. The quantities α , β , and γ are functions of the two-dimensional coordinates x^a . In that case we have

$$w = \frac{1}{2}(\alpha - \gamma), \quad (3.7)$$

$$\psi_a u^a = -\frac{1}{2}(\alpha + \gamma) \mathcal{L}_u r - \beta \mathcal{L}_v r, \quad (3.8)$$

$$\psi_a v^a = \frac{1}{2}(\alpha + \gamma) \mathcal{L}_v r + \beta \mathcal{L}_u r. \quad (3.9)$$

Further we can obtain

$$\kappa \mathcal{L}_u r = e^{2\phi} \mathcal{L}_u \phi - \frac{4\pi G_n}{n-2} r [(\alpha + \gamma) \mathcal{L}_u r + 2\beta \mathcal{L}_v r], \quad (3.10)$$

$$\kappa \mathcal{L}_v r = e^{2\phi} \mathcal{L}_v \phi + \frac{4\pi G_n}{n-2} r [(\alpha + \gamma) \mathcal{L}_v r + 2\beta \mathcal{L}_u r], \quad (3.11)$$

and then

$$\begin{aligned} \mathcal{L}_u E &= -\gamma \mathcal{L}_u V - \beta \mathcal{L}_v V, \\ \mathcal{L}_v E &= \alpha \mathcal{L}_v V + \beta \mathcal{L}_u V. \end{aligned} \quad (3.12)$$

Thus, when the energy-momentum tensor does not vanish, the relation between the surface gravity κ and gravitational potential in Eq. (3.4) has to be modified. One has to consider the contribution of the matter fields.

In a general case, ψ_a 's do not vanish. In some special cases, for example, Reissner-Nordström spacetime, however, one has vanishing ψ_a with a nonvanishing w (see the Appendix). Note that for a FRW universe with $\beta = 0$ and $\alpha + \gamma = 0$, one has also $\psi_a = 0$. In those special cases, (3.4) still holds, although matter fields are not absent.

IV. GRAVITY AS ENTROPY FORCE

On the trapping horizon, Eq. (3.4) implies some relation between the change of entropy and the gravitational potential. For an arbitrary vector field n , from Eq. (3.1), we find

$$\mathcal{L}_n S = (\varphi_g + \varphi_m) A, \quad (4.1)$$

where

$$\varphi_g = s_g^a n_a, \quad \varphi_m = s_m^a n_a. \quad (4.2)$$

Here φ_m is the value of entropy flux s_m^a induced by the matter field along the n direction [37,38], and s_m^a is defined as

$$\left(\frac{\kappa}{2\pi}\right) s_m^a = \psi^a. \quad (4.3)$$

Similarly φ_g can be understood as the entropy flux s_g^a given by the change of gravitational potential along the n direction. s_g^a is defined by

$$\left(\frac{\kappa}{2\pi}\right) s_g^a = \left(\frac{n-2}{8\pi G_n}\right) \left(\frac{e^{2\phi}}{r}\right) D^a \phi. \quad (4.4)$$

We may understand that the term $A\varphi_g$ corresponds to the work done by gravity. The reason is that the gravitational potential will change along the n direction for an arbitrary n . This suggests that on the trapping horizon we have

$$T_H \mathcal{L}_n S = n^a F_a + \delta_n Q, \quad (4.5)$$

where $\delta_n Q = A\psi_a n^a$. To understand the meaning of F_a in this equation, let us define

$$U_a = e^{2\phi} D_a \phi. \quad (4.6)$$

Obviously, this U_a has a dimension of gravitational acceleration. Note that on the trapping horizon, we have

$$E = \frac{1}{16\pi G_n} (n-2) \Omega_{n-2} r^{n-3}, \quad (4.7)$$

which leads to

$$n^a F_a = n^a (2EU_a). \quad (4.8)$$

This suggests that F_a is a force—the force acting on the active energy inside the marginal sphere.

Equation (4.5) is valid on the trapping horizon only. The term $\delta_n Q = A\psi_a n^a$ is nothing, but heat flux caused by the matter fields.

Here some remarks are in order.

- (1) When the vector field n is tangent to a surface with a fixed gravity potential (equipotential surface, the trapping horizon is a kind of equipotential surface), the force along the n direction does not exist. In this case, under the Lie derivative \mathcal{L}_n , the marginal sphere changes to another marginal sphere (of course inside the trapping horizon). The modified Clausius relation (4.5) becomes a normal one, i.e., $T_H \mathcal{L}_n S = \delta_n Q$.
- (2) If n has a component which is normal to the trapping horizon, the marginal sphere tends to change to an untrapped sphere. There is a change of gravitational potential along the n direction. In this case, the work term of gravity appears. In other words the force F_a is present in this case. It is clear from (4.5) that the force appears when the entropy associated with the trapping horizon changes. In this way the force can be understood as an entropy force in the spirit of arguments by Verlinde.
- (3) At the moment, it is not clear whether it is valid that the force acts as an entropy force on the untrapped sphere because in that case it is not clear whether the surface gravity κ and $A/(4G_n)$ have the interpretation as temperature and entropy for an untrapped sphere. On this point we will have more discussions below.
- (4) Combing Eqs. (2.10) and (4.5), we have

$$\mathcal{L}_n E = T_H \mathcal{L}_n S + w \mathcal{L}_n V - n^a F_a. \quad (4.9)$$

This equation is a consequence of the Einstein equation on the trapping horizon. Along an arbitrary vector field n , we should consider not only the work done by the matter fields, i.e., $w \mathcal{L}_n V$, but also the work made by gravity, which is just the term $n^a F_a$.

(5) Since the surface gravity can be expressed as

$$\begin{aligned} \frac{\kappa}{2\pi} &= \frac{1}{4\pi} D_a D^a r \\ &= \frac{4G_n}{n-2} \left[\left(\frac{n-3}{\Omega_{n-2}} \right) \left(\frac{E}{r^{n-2}} \right) - rw \right], \end{aligned} \quad (4.10)$$

we may define a new surface gravity $\bar{\kappa}$ as

$$\bar{\kappa} = \kappa + \frac{8\pi G_n}{n-2} rw = \frac{8\pi G_n}{\Omega_{n-2}} \left(\frac{n-3}{n-2} \right) \left(\frac{E}{r^{n-2}} \right). \quad (4.11)$$

On the tapping horizon, this new surface gravity is just the so-called ‘‘effective surface gravity’’ proposed by Ashtekar *et. al.* [5]. From the definition (4.11), this effective surface gravity reduces to the Newton surface gravity if we take a nonrelativistic limit of E (that is, replacing the energy E by mass M times c^2). Thus, the term $w \mathcal{L}_n V$ in Eq. (4.9) can be absorbed into $T_H \mathcal{L}_n S$ to give $\bar{T}_H \mathcal{L}_n S$ with definition

$$\bar{T}_H = \frac{\bar{\kappa}_H}{2\pi} = \frac{(n-3)}{4\pi r_H}.$$

The first law (4.9) then becomes

$$\mathcal{L}_n E = \bar{T}_H \mathcal{L}_n S - n^a F_a. \quad (4.12)$$

Along the trapping horizon, the force disappears and this equation changes to

$$\mathcal{L}_n E = \bar{T}_H \mathcal{L}_n S. \quad (4.13)$$

Everything becomes simple with this effective surface gravity $\bar{\kappa}_H$. Although this effective surface gravity does not reduce in the static limit to the standard surface gravity of static black holes, for example, Reissner-Nordström black holes (see the Appendix), it is enlightening when studying dynamical spacetimes. With this effective surface gravity, one immediately has $\mathcal{L}_n E = \delta_n Q$, and the work term $w \mathcal{L}_n V$ disappears. Unfortunately, for dynamical black holes, the definitions of the surface gravity are far from clear so far [39]. Namely it is still not very clear which definition of surface gravity is indeed related to Hawking temperature associated with a trapping horizon.

(6) From Eqs. (4.5) and (4.8), when ψ_a vanish, we have

$$T_H \mathcal{L}_n S = n^a F_a = n^a (2EU_a). \quad (4.14)$$

This clearly indicates that the gravity comes from the entropy force: gravity appears as a change of entropy. However, when the energy support ψ_a do not vanish, one has to consider the contribution of the heat flux $\delta_n Q$, which also causes the change of entropy. In addition, on the trapping horizon, if the Kodama vector is a Killing vector, the gravity indeed appears as an entropy force.

In a general case when matter fields are present, on the trapping horizon, we have

$$\begin{aligned} T_H \mathcal{L}_n S &= n^a F_a + \delta_n Q = n^a (2EU_a) + \delta_n Q \\ &= 2\kappa_H E \mathcal{L}_n r. \end{aligned} \quad (4.15)$$

It should be stressed here that we have

$$T_H \mathcal{L}_n S = 2\kappa_H E \mathcal{L}_n r, \quad (4.16)$$

even when ψ_a do not vanish. In this case, however, the term $2\kappa_H E \mathcal{L}_n r$ includes the contributions from the heat flux given by the matter fields and the work done by gravity. Therefore only when ψ_a vanish, $2\kappa_H E \mathcal{L}_n r$ stands for the work done by gravity.

(7) To further understand the gravitational acceleration U_a , let us consider the case without matter. This means $\psi_a = w = 0$. In this case, E is a constant [this can be seen from Eq. (2.10) or (3.12)]. The gravitational acceleration U_a can be expressed as

$$\begin{aligned} U_a &= e^{2\phi} D_a \phi = \frac{1}{2} D_a (D_b r D^b r) \\ &= -\frac{8\pi G_n}{(n-2)\Omega_{n-2}} D_a \left(\frac{E}{r^{n-3}} \right). \end{aligned} \quad (4.17)$$

Considering E is a constant, we have

$$U_a = \left(\frac{n-3}{n-2} \right) \frac{8\pi G_n}{\Omega_{n-2}} \left(\frac{E}{r^{n-2}} \right) D_a r. \quad (4.18)$$

This is the gravitational acceleration on the trapping horizon provided by the energy E . In the nonrelativistic limit, E reduces to the Newton mass M (the light velocity is set to be unity). So U_a indeed gives us the correct gravitational acceleration. Note that this calculation is also valid in the untrapped region. On the trapping horizon, we have

$$\begin{aligned} n^a F_a &= n^a (2EU_a) \\ &= 2 \left(\frac{n-3}{n-2} \right) \left(\frac{8\pi G_n}{\Omega_{n-2}} \right) \left(\frac{E^2}{r^{n-2}} \right) \mathcal{L}_n r. \end{aligned} \quad (4.19)$$

Remarkably, this force has the form of the Newton gravity if we take the nonrelativistic limit where E is replaced by mass M . But there is an additional factor 2 in the second and last terms, compared to the standard form of the Newton gravity. The factor 2 might come from the self-gravitating effect here since the Newton force appears as a probe approximation.

From the above discussions, we can conclude that gravity indeed can be viewed as an entropy force on the trapping horizon; it is particularly clear when the energy supply ψ_a is absent on the trapping horizon. This conclusion is based on the definition of the quasilocal energy E in Eq. (2.5) and the temperature T_H in Eq. (2.13). Furthermore, if one uses the effective surface gravity $\bar{\kappa}_H$

and the corresponding temperature \bar{T}_H when matter fields are present, gravity can be viewed as an entropy force if the variation of the energy $\mathcal{L}_n E$ vanishes on the trapping horizon.

V. SMARR-LIKE EQUATIONS AND HOLOGRAPHIC ASSUMPTION OF ENTROPY

It is interesting to note that the relation among the thermodynamical quantities discussed in the previous sections can be put in a simple form. On the trapping horizon, we find from the expression of the surface gravity (4.10) that

$$\frac{\kappa_H}{2\pi} = \frac{4G_n}{n-2} \left[\left(\frac{n-3}{\Omega_{n-2}} \right) \left(\frac{E}{r^{n-2}} \right) - rw \right]. \quad (5.1)$$

A straightforward calculation gives

$$\left(\frac{\kappa_H}{2\pi} \right) \left(\frac{A}{4G_n} \right) = \left(\frac{n-3}{n-2} \right) E - \left(\frac{n-1}{n-2} \right) wV. \quad (5.2)$$

Identifying $T_H = \kappa_H/2\pi$ and $S = A_H/4G_n$, we get

$$(n-2)T_H S = (n-3)E - (n-1)wV. \quad (5.3)$$

Further if we use the effective surface gravity $\bar{\kappa}$ instead of κ , Eq. (5.3) changes to

$$(n-2)\bar{T}_H S = (n-3)E. \quad (5.4)$$

Equations (5.3) and (5.4) are very similar to the Smarr formula in general relativity, and we call them Smarr-like equations. Note here that they take the quasilocal form. These relations among energy, temperature, and entropy give us a lot of implications. For instance, since the entropy is determined by the area of the marginal sphere, it means that the physical degrees of freedom are determined by the marginal surface. One can imagine that there are some bits living on the marginal sphere which give the same amount of the entropy. If we further assume there are no interactions among these bits, from statistic physics, at least at high temperature, we can use the equipartition of energy to link the energy E and T_H . This idea is used to investigate gravity as an entropy force by Padmanabhan [15] and Verlinde [13].

Now let us consider the case without matter with $w = 0$. Assume there are N bits associated with the marginal surface with [13]

$$N = \frac{1}{2} \left(\frac{n-2}{n-3} \right) \left(\frac{A}{G_n} \right). \quad (5.5)$$

The relation of the entropy and N is given by

$$S = \frac{1}{2} \left(\frac{n-3}{n-2} \right) N k_B, \quad (5.6)$$

where the Boltzmann constant k_B is recovered. Because N bits have the energy $(1/2)Nk_B T_H$, this gives

$$\frac{1}{2} N k_B T_H = \left(\frac{n-2}{n-3} \right) T_H S = E. \quad (5.7)$$

Thus when the matter fields are absent, the assumption of the equipartition of energy is consistent with the Smarr-like equation.

In the presence of matter, the law of the equipartition of energy is broken by the term including wV . However, since the Smarr-like equation (5.3) is always satisfied, with the holographic assumption that the entropy is given by (5.6), we still have

$$\frac{1}{2} N k_B T_H = \left(\frac{n-2}{n-3} \right) T_H S = E - \left(\frac{n-1}{n-3} \right) wV. \quad (5.8)$$

It follows from Eq. (5.4) that the equipartition of energy can be always used if we use the effective surface gravity $\bar{\kappa}_H$ and the corresponding temperature \bar{T}_H .

It is interesting to study the entropy force by using this description of the physical degrees of freedom. Can we get the relation (4.16) just from the above holographic scenario? We can imagine that there are N bits living on the marginal surface. Every bit has energy $\frac{1}{2}k_B T_H$. So along the direction normal to the trapping horizon, it will feel force $\frac{1}{2}\kappa_H k_B T_H$, and the total force is given by

$$N \left(\frac{1}{2} \kappa_H k_B T_H \right) = \kappa_H E \neq 2\kappa_H E. \quad (5.9)$$

However, it is different from the result (4.16) by a factor 2. In fact, it is expected because the simple counting does not include the self-gravitating effect and the result (5.9) appears in the probe approximation.

VI. ENTROPY FORCE FROM QUANTUM FLUCTUATION

When $\psi_a = 0$, it can be clearly seen that the gravitational force appears as an entropy force (4.14) in the Einstein general relativity. However, it should be noted that this is just a consequence of the Einstein equation together with thermodynamic properties of the horizon. The change of the entropy in (4.14) actually comes from the change of the area of the marginal sphere. So the variation of the entropy is geometric, and has no direct relation to quantum behavior of the black hole horizon. However, it is clear that there must exist underlying microscopic degrees of freedom associated with black hole horizon entropy.

We now apply a similar discussion made by Verlinde [13] (the so-called thought experiment of Bekenstein) to the trapping horizon of the dynamical spacetime. Consider a test particle (with rest mass m) staying just outside the trapping horizon with some distance, Δx , from the trapping horizon. As a test particle, we assume that it will not change the background geometry. However, if the value of the distance Δx is about the Compton wavelength of the particle, this particle should be viewed as a part of the dynamical black hole. Then the entropy of the black hole

has a fluctuation ΔS even though the geometry of the trapping horizon is not supposed to change. Here let us concentrate only on the four-dimensional case, with generalization to other dimensions being straightforward. The change of the entropy is given by

$$\Delta S = 2\pi k_B \frac{mc}{\hbar} \Delta x. \quad (6.1)$$

The temperature of the system is given by

$$k_B T_H = 2 \left(\frac{E}{N} \right). \quad (6.2)$$

This is correct only for the case without matter field. Note that the number N is given by

$$N = \frac{A_H c^3}{G_4 \hbar}. \quad (6.3)$$

Substituting the expression of N and T_H into Eq. (6.1), we have

$$T_H \Delta S = G_4 \frac{mE/c^2}{r^2} \Delta x. \quad (6.4)$$

Thus, one can identify the right-hand side of the above equation as the work done by some entropy force. It is clear that this force is similar to the Newton force

$$f = G_4 \frac{mE/c^2}{r^2}. \quad (6.5)$$

We emphasize that our discussion is restricted to the region near the trapping horizon because T_H is the Hawking temperature of the dynamical black hole.

Note that in the case without matter, the relation $2(\kappa/2\pi)S = E$ is also valid in the untrapped region. If we assume that there is a ‘‘temperature’’ T associated with κ and the entropy is $S = A/4G$, by using a similar reasoning, we find the corresponding gravitational acceleration is given by κ/c^2 which is similar to (4.18):

$$G_4 \frac{E/c^2}{r^2}. \quad (6.6)$$

This is just the Newton gravitational acceleration provided by the active energy E inside the untrapped sphere with radius r . We will give more discussions on this temperature T later.

Let us now turn to the case with matter. In this case, the equipartition law of energy is violated, but the Smarr-like formula together with the holographic assumption of the entropy can be used. For the case of $n = 4$, Eq. (5.8) becomes

$$2T_H S = \frac{1}{2} N k_B T_H = E - 3wV. \quad (6.7)$$

By using a similar discussion, we arrive at

$$\begin{aligned} f &= G_4 \frac{m(E - 3wV)/c^2}{r^2} = G_4 m \left(\frac{E}{r^2} - 4\pi r w \right) / c^2 \\ &= m \kappa_H / c^2, \end{aligned} \quad (6.8)$$

where we have used $V = 4\pi r^3/3$. So the force is nothing but the surface gravity κ_H times the rest mass m . Note that we have used the unit of $c = 1$ in the discussion of previous sections. It should be noted that we have not used any nonrelativistic limit till now.

Our conclusion (in units of $c = 1$) is that near the trapping horizon, for the test particle with rest mass m , the entropy force given by Verlinde is just the Newtonian force $\kappa_H m/c^2$ with gravitational acceleration κ_H/c^2 .

We can understand the entropy force from another point of view which is similar to the discussion by Smolin [20]: Suppose that there is a particle with mass m ‘‘pulled away’’ from the horizon. This pulling away should be understood as quantum fluctuations. Assume that the distance of the particle from horizon is given by Δx . This Δx should be within the Compton wavelength of the particle, so the particle still belongs to the horizon. So the passive energy of this system will not change, and the change of the active energy E is then just given by

$$\Delta E = F \Delta x = T_H \Delta S. \quad (6.9)$$

Assume the change of the entropy is still given by Eq. (6.1). Then similar discussions can lead to

$$F = G_4 \frac{(E - 3wV)/c^2}{r^2} m = m \kappa_H / c^2. \quad (6.10)$$

Assume that the mass m carries q ($q \ll N$) bits of information, and we then have

$$m c^2 = \frac{1}{2} q k_B T_H, \quad (6.11)$$

and we obtain the force per bit as

$$f = \frac{F}{q} = \left(\frac{1}{2} k_B T_H \right) \kappa_H / c^4. \quad (6.12)$$

Namely one bit information on the horizon will feel a force $(\frac{1}{2} k_B T_H) \kappa_H / c^4$. As we have mentioned at the end of Sec. V, there is a factor 2 difference between this kind of probe approximation and our result (4.16). In this holographic description, the procedure to get results (4.16) should be understood as extract all bits on the marginal sphere to a very nearby untrapped sphere. Obviously, the probe (or test particle) approximation is not valid in this case.

VII. CONCLUSION AND DISCUSSION

In this paper we mainly investigated the issue that gravity appears as an entropy force from Einstein’s equation in a general spherically symmetric spacetime. On the trapping horizon of the spacetime, we found that the gravity acting on the marginal sphere indeed can be identified as an entropy force when the energy supply ψ_a vanishes on

the trapping horizon. We noticed that when matter fields are present, the heat flux also leads to the change of entropy. With the holographic assumption of the entropy, following Verlinde and Smolin, we showed that the entropy force induced by quantum fluctuations is measured by the surface gravity on the trapping horizon.

We give further remarks here.

In most parts of this paper, we have focused on the trapping horizon, because on the trapping horizon, we have well-defined temperature and entropy associated with the trapping horizon. Off the trapping horizon, although the surface gravity (4.10) on the untrapped sphere has a dimension of gravitational acceleration, its physical meaning is still not very clear. In the static case, the temperature at a fixed r surface outside the horizon is given by the Tolman redshift relation $e^{-\phi}T_H$ with a redshift factor $e^{-\phi}$, provided the system is in a thermal equilibrium. This is the temperature measured by a static observer at the constant r surface. Obviously, it is not equal to $T = \kappa/2\pi$, while in the case without matter, this κ indeed reduces in the nonrelativistic limit to the Newton gravity acceleration at the surface with radius r . Therefore the temperature $T = \kappa/2\pi$ can be understood as a local Unruh temperature given by a Rindler observer with acceleration κ . Indeed, as it is well known that some curved spacetimes like a Schwarzschild spacetime can be embedded in a higher-dimensional flat spacetime, a static observer at the constant r surface is mapped into a Rindler observer in the higher-dimensional spacetime. The surface gravity κ is just the Unruh temperature measured by the Rindler observer in the higher-dimensional spacetime. Once accepting T as a real temperature in the above sense, we have from Einstein's equation that

$$(n-2)TS = (n-3)E - (n-1)wV, \quad (7.1)$$

which reduces to Eq. (5.3) on the trapping horizon. Here now E is just the Misner-Sharp energy given by (2.5), and $S = A/4G_n$ for the sphere with an arbitrary radius r . Employing (7.1) with the holographic assumption of S , one can find the force $m\kappa/c^2$ acting on a test particle with mass m . From the expression (4.10), we find that this is just the Newton gravity produced by E when the matter fields are absent.

Ashtekar's effective surface gravity $\bar{\kappa}$ (4.10) is simple, which has a similar form to the Newton gravity acceleration if we take the nonrelativistic limit of E . With this definition of surface gravity, the corresponding temperature \bar{T}_H , entropy and energy satisfy the relation (5.4). In this case since we have (5.4), the equipartition law of energy holds and on the trapping horizon, we can obtain

$$m\bar{\kappa}_H/c^2 = G_4 \frac{mE/c^2}{r^2}. \quad (7.2)$$

This is just the Newton gravity produced by the active energy E . With the same argument as the above, defining

the temperature $\bar{T} = \bar{\kappa}/2\pi$ on the untrapped surface, and still employing the relation (5.4), we can obtain the Newton gravity (7.2) on an arbitrary sphere with radius r .

In addition, in this paper, we have focused on the future outer trapping horizon of spacetime, and there the surface gravity $\kappa_H > 0$. In fact, our discussion is also valid on the inner trapping horizon, where surface gravity κ_H is negative. The apparent horizon of the FRW universe is just this case [6,8]. In this case, the temperature is defined as $T_H = |\kappa_H|/2\pi$, and Eq. (4.15) then becomes

$$T_H \mathcal{L}_n S = n^a (2|\kappa_H|EU_a) - \delta_n Q. \quad (7.3)$$

The entropy force also appears naturally. This is particularly clear in the case of the de Sitter spacetime, in which case $\psi_a = 0$ and $\delta_n Q = 0$, and gravity appears as an entropy force.

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APPENDIX: REISSNER-NORDSTRÖM BLACK HOLES

In this Appendix, we consider the Reissner-Nordström spacetime as an example to show some results discussed in this paper. In this case, the trapping horizon coincides with the event horizon of the black hole. The two-dimensional part of the solution is

$$h_{ab}dx^a dx^b = -f dt^2 + \frac{1}{f} dr^2, \quad (A1)$$

where f is given by

$$f = 1 - \frac{2M}{r^{n-3}} + \frac{e^2}{r^{2n-6}}. \quad (A2)$$

The Kodama vector is just the timelike Killing vector ∂_t and the trapping horizon is given by $f = 0$. The horizon radius is given by $r_H^{n-3} = M + \sqrt{M^2 - e^2}$. Further it is easy to show

$$\begin{aligned} T_H &= \frac{\kappa_H}{2\pi} = \frac{n-3}{4\pi r_H} \left(1 - \frac{e^2}{r_H^{2n-6}} \right), \\ w &= \frac{1}{16\pi G_n} \frac{(n-2)(n-3)e^2}{r_H^{2n-4}}. \end{aligned} \quad (A3)$$

The energy supply ψ_a vanishes on the horizon. The Smarr-like equation is given by

$$(n-2)T_H S = (n-3)E - (n-1)wV \\ = \frac{(n-2)(n-3)\Omega_{n-2}\sqrt{M^2 - e^2}}{8\pi G_n}, \quad (\text{A4})$$

where E is just the Misner-Sharp energy inside the horizon, which is given by ($c = 1$)

$$E = \frac{1}{16\pi G_n}(n-2)\Omega_{n-2}r_H^{n-3} \\ = \frac{(n-2)\Omega_{n-2}}{16\pi G_n}M\left(1 + \sqrt{1 - \frac{e^2}{M^2}}\right). \quad (\text{A5})$$

Obviously, this energy includes the contribution of the Maxwell field. From this relation and holographic assumption of entropy, we can obtain the entropy force given by $m\kappa_H$, where m is the rest mass of the test particle.

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