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Estimating energy-momentum and angular momentum near null infinity

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The energy-momentum and angular momentum contained in a spacelike two-surface of spherical topology are estimated by joining the two-surface to null infinity via an approximate no-incoming-radiation condition. The result is a set of gauge-invariant formulas for energy-momentum and angular momentum which should be applicable to much numerical work; it also gives estimates of the finite-size effects.

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I. INTRODUCTION

The exciting successes in recent years of numerical treatments of general-relativistic systems have given point to long-standing theoretical challenges involved in explicating Einstein's theory. Numerical relativists are now able to calculate with good accuracy solutions to the field equations. But how is one to extract from all these data the key physically significant quantities?

Energy-momentum and angular momentum would be among the most important, and the strongest hope is that good definitions of these will exist at the *quasilocal* level, that is, on generic acausal two-surfaces (perhaps restricted to have spherical topology). If such definitions do exist, and if, moreover, they can be formulated in such a way that it is possible to meaningfully compare the quasilocal quantities associated with different surfaces, they would be a powerful tool for understanding the exchange of energy-momentum and angular momentum between strongly generally relativistic astrophysical systems. At present, however, the problems involved in developing a quasilocal kinematics seem so profound that one is driven to look for more modest approaches which will still provide useful results in broad categories of cases of current interest.

The goal of this paper is to get working approximations for the total energy-momentum and angular momentum of isolated systems which are suitable for contemporary numerical use. Most numerical codes only evolve the system throughout a finite volume of space-time, so one has available data on large but finite two-surface $\mathcal{S}$. I shall show that there is a reasonable ''poor man's'' no-incoming-radiation condition which can be used to extrapolate the data on $\mathcal{S}$ to future null infinity $\mathcal{I}^+$, where the Bondi-Sachs definition of energy-momentum and twistorial treatment of angular momentum apply. It turns out—nontrivially—that this approach allows for a comparison of the energy-momentum and angular momentum as $\mathcal{S}$ is moved forward in time; one thus has measures of the energy-momentum and angular momentum emitted in gravitational radiation.

While a certain amount of work is required to derive these measures, the result is a compact set of gauge-independent formulas which should be usable by numerical relativists.

A. No-incoming-radiation condition

The state of a general-relativistic system is specified not only by its material degrees of freedom but by its gravitational data. In principle, for most modeling one would like to fix those data to correspond to no incoming radiation.

To implement the no-incoming-radiation condition exactly would be very difficult. (One would have to solve a hard inverse problem, finding what constraints on Cauchy data led to the required behavior of the solution in the distant past. Numerical workers typically specify the data as well as they can before there are strong interactions, and then discard any early transients as potentially due to spurious incoming radiation.) However, this is a practical difficulty and does not affect the validity of the condition as the correct restriction on the data.

We may make use of this observation at the two-surface $\mathcal{S}$ (assumed to be large and approximately spherical), as follows. Consider the null hypersurface $\mathcal{N}_{\text{phys}}$ orthogonal outward from $\mathcal{S}$. Radiation incoming to the future of $\mathcal{S}$ would leave its profile on this surface, the transverse component being measured by the Weyl component Ψ_0 in the standard Newman-Penrose formalism. (One has $\Psi_0 = \Psi_{ABCD} o^A o^B o^C o^D$, where o^A is a tangent spinor to $\mathcal{N}_{\text{phys}}$, that is, one has $l^{AA'} = o^A o^{A'}$ for l^a the null tangent to $\mathcal{N}_{\text{phys}}$.) We may thus take as an approximate, poor man's, no-incoming-radiation condition that

$$\Psi_0 = 0 \text{ on the null hypersurface outgoing from } \mathcal{S} \quad (1)$$

(and there is no matter crossing this hypersurface). More precisely, we consider embedding $\mathcal{S}$ in a space-time with the same first and second fundamental forms, but we discard $\mathcal{N}_{\text{phys}}$ and replace it with a null hypersurface $\mathcal{N}$ (orthogonally outward from $\mathcal{S}$) with Ψ_0 identically zero on $\mathcal{N}$. [Note that condition (1) is imposed only on the single hypersurface $\mathcal{N}$ determined by $\mathcal{S}$. We shall take up below the question of what to do when $\mathcal{S}$ is evolved.]

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The spin-coefficient equations that effect transport outward from $\mathcal{S}$ along $\mathcal{N}$ can be integrated, so one can work out the asymptotic behavior of the fields as one approaches null infinity on $\mathcal{S}$. One can then use the Bondi-Sachs energy-momentum and the twistorial angular momentum.

A number of remarks should be made about this.

First, according to the Sachs peeling property, it is the component Ψ_0 that has the most rapid falloff along outgoing null geodesics ($\Psi_0 \sim r^{-5}$ where r is an affine parameter), and so one has good reason to think that if $\mathcal{S}$ can be regarded as in the asymptotic regime, then $\Psi_0 = 0$ should be a good approximation.

Second, another way of viewing the nature of the approximation is that, because $\mathcal{N}_{\text{phys}}$ is at a finite location, the incoming waves need not be exactly transverse. That is, the extent to which $\Psi_0 = 0$ fails to implement the exact no-incoming-radiation condition is the extent to which incoming waves would not be transverse at $\mathcal{N}_{\text{phys}}$. (This means, in particular, that the approximation might be a poor one if $\mathcal{S}$ were substantially wrinkled. However, in numerical work one typically has surfaces $\mathcal{S}$ which are very nearly spherical. If one did have to deal with substantially wrinkly surfaces, one could generalize the present method by considering an outgoing null congruence other than the one orthogonal to $\mathcal{S}$, adapted to the ambient geometry.)

Finally, one might consider using the asymptotics of the field along $\mathcal{N}$ for waveform extraction; cf. e.g. [1]. [Since these asymptotics only give the field at an instant of retarded time determined by $\mathcal{S}$, to really extract a waveform one would need to apply the procedures here for an evolving family of surfaces $\mathcal{S}(\eta)$ with associated null hypersurfaces $\mathcal{N}(\eta)$, for η in some interval J.] Such a procedure would not be exact, of course; its validity would be limited by the applicability of (1). While this is certainly natural, it raises issues beyond those treated in this paper. This is because the points which are presently problematic in the case of waveform extraction and in the case of computation of energy-momentum and angular momentum are different. For energy-momentum and angular momentum, resolving gauge ambiguities is essential, and condition (1) allows us to do this (as a mathematical procedure, irrespective of the degree to which the condition accurately models the physical space-time); for waveforms, the gauge choices for the observer are trivial. Thus the definitions here allow one both to compute the energy-momentum and angular momentum, and to estimate the finite-size effects involved, from data on surfaces $\mathcal{S}(\eta)$, but the problem of waveform extraction necessarily involves additionally the question of how accurately condition (1) models the geometry of the physical space-time. One should also bear in mind that, for present work, computations of (say) the energy-momentum or angular momentum emitted in gravitational waves which were accurate to a few per cent would typically be quite adequate; whereas extraction of physical information from waveforms may require rather more accurate modeling. Because of this sensitivity, if the ideas used for asymptotics here are to be helpful to the waveform extraction problem, they will probably be so when combined with other physical insights. One should probably use data, not just on the surfaces $\mathcal{S}(\eta)$, but from the portion of the physical space-time interior to them, to extrapolate the waveforms.

B. The role of idealization

The procedure used here turns on embedding the two-surface $\mathcal{S}$ of interest in a mathematically constructed space-time and then evaluating that space-time's kinematics. One might think at first that this is less desirable than actually working out the asymptotics along $\mathcal{N}_{\text{phys}}$ to infer the Bondi-Sachs energy-momentum and twistorial angular momentum in the physical space-time. For most purposes, this is not the case, however.

The point is that the Bondi-Sachs treatment applies to idealized isolated systems; for practical purposes one must choose which portion of a real (or numerical) system is to be regarded as the isolated component, and there may be several such choices. Consider, for instance, a system which at several widely separated intervals emits bursts of gravitational radiation. Each of these bursts contains, not just the outgoing transverse wave front, but smaller trailing, nontransverse, pieces; also each of these bursts will generate a certain amount of backscatter via nonlinearities. If one wanted to truly be in the mathematically exact Bondi-Sachs asymptotic regime for this system, one would have to go outward so far along $\mathcal{N}_{\text{phys}}$ that one had passed any slight trailing fields and backscatters due to very early emissions—in principle, one would have to know the history of the system in the arbitrarily distant past to do this. One rarely wants to do this; one would rather think of the system as to good approximation isolated in the interval around one burst—and if necessary then worry about the fact that the isolation is not perfect.

Thus in most situations the task is not to construct the null infinity and kinematics of the entire space-time, but to determine how to measure the kinematics of a large but finite accessible region. On the other hand, the main reason for considering null infinity (together with the Bondi-Sachs energy-momentum and the twistorial angular momentum)—that it provided an invariant treatment of quantities of interest—remains valid. So our approach is based on constructing a strictly well-defined null infinity from the data on the finite surface $\mathcal{S}$, in order to have an invariant energy-momentum and angular momentum. These are the energy-momentum and angular momentum which would be ascribed to $\mathcal{S}$, were it in a space-time satisfying (1).

The question of whether one really is in a regime which satisfies the Bondi-Sachs asymptotics to a given approximation—which is the question of whether the energy-momentum and angular momentum constructed here are

stable when $\mathcal{S}$ is moved outward along $\mathcal{N}_{\text{phys}}$—is important but lies beyond the scope of this paper, for one cannot investigate this stability based on data at a single surface $\mathcal{S}$ [nor from a finite family of $\mathcal{S}(\eta)$'s].

C. Contemporary numerical work

Numerical relativists who work with codes aimed at treating generic space-times have recognized the importance of invariant and theoretically justified measures of energy-momentum and angular momentum. Indeed, this concern plays a part in the choice of some groups to use characteristic (or mixed Cauchy-characteristic) codes. Such codes, if cast in a form admitting a clean extension to null infinity, ought to allow the extraction of the Bondi-Sachs energy-momentum and twistorial angular momentum.

Characteristic (or mixed) code computations of the Bondi energy (often referred to in the literature as the Bondi mass, for historical reasons) have been done in a number of cases, although typically they are hampered by the fact that the codes themselves are not usually cast in Bondi coordinates, so that nontrivial gauge transformations are required. These also make the computation of the Bondi momentum and the angular momentum difficult. (See Ref. [2] for a review.) The results here may help streamline such computations, for the formalism to be developed automatically produces the required gauges.

A great body of numerical work, however, is based on "3 + 1" formalisms rather than characteristic or mixed ones. For these formalisms, the problem of gauge invariance for energy-momentum and angular momentum has been more severe.

Most contemporary attempts to extract information about the total energy-momentum in the 3 + 1 formalisms can be usefully thought of as based on the Bondi-Sachs energy-momentum loss formula.[1] (They have often been justified by other means; however, the Bondi-Sachs formula would be the broadest and most theoretically secure starting point.) This formula identifies the rate of change of the Bondi-Sachs energy-momentum, with respect to Bondi retarded time, as an integral of the squared modulus of the "news function" with respect to the measure induced on an asymptotically large sphere by the Bondi coordinates. Thus to use this formula to recover the radiated energy-momentum one must know (a) that one is in the asymptotic regime; (b) the Bondi coordinate system, both to identify the measure on the sphere correctly and to do the integral over retarded time; and (c) the news function. (The news function can be given as the integral of a curvature component—the component depending on the Bondi coordinates—with respect to Bondi retarded time.)

Numerical work in the 3 + 1 formalism has not yet implemented any systematic transition to Bondi coordinates. Thus what is actually done is to use the numerical angle and time coordinates to compute the curvature component and integrals required for the energy-momentum-loss formula and its integral (e.g. [3,4]). Consistency checks are then done by studying the stability of the result as the extraction radius is increased. However, the lack of gauge control makes it impossible to know what these numbers really signify. The stability of the actual, gauge-invariant, energy-momentum could be either better or worse than the cited numbers, depending on whether the gauge freedoms exacerbate or mask extraction problems.

The present approach overcomes the concerns about gauge by giving formulas for the Bondi-Sachs energy-momentum and its evolution in terms of gauge-invariant quantities on the extraction surface $\mathcal{S}$. As far as the question of the stability of the results with increasing extraction radius goes, this is an issue which one can only investigate directly, by considering larger and larger surfaces. However, the present work does give one the confidence that in such an investigation other potential error sources have been controlled.

The situation for angular momentum has been more difficult than for energy-momentum. In the first place, there has been for some time no really theoretically satisfactory formula; and in the second, the angular momentum depends on curvature terms deeper in the asymptotic expansion, which are still more sensitive to the correct choice of the Bondi frame. The approach given here overcomes these difficulties. We use the recent twistorial definition of angular momentum, which appears to be theoretically satisfactory. We identify the Bondi frame exactly on $\mathcal{N}$, and the finite-size contributions may be read off directly from the formulas here.

D. Recent theoretical work

The problem of estimating kinematic quantities in terms of numerical data has been taken up by two sets of authors recently.

Gallo, Lehner, and Moreschi [5] (see also [6]) raised many of the concerns motivating the present work. They emphasized the importance of extracting invariant information, and also of considering finite-size effects. They gave an approach to estimating the Bondi momentum which (while presented somewhat differently) can be viewed as assuming that the two-surface $\mathcal{S}$ is only infinitesimally separated from $\mathcal{I}^+$, and computing the Bondi-Sachs energy-momentum at the corresponding cut. (While it may seem odd to speak of a two-surface infinitesimally separated from infinity, it is a well-defined concept from

[1]When the objects of interest are black holes, there are also some specialized techniques for estimating their masses. However, these rely strongly on stationarity. The problem of identifying the corresponding energy-momenta as elements of a suitable space in the asymptotic regime is unaddressed, too, so one does not have a good way of comparing the energy-momenta of several interacting holes, or even one radiating one at different times.

the point of view of the conformally completed manifold, and amounts to assuming that one only needs leading terms in the appropriate asymptotic expansions.) Their results therefore correspond to a limiting case of some of those here.

Deadman and Stewart [7] recently discussed the estimation of the Bondi-Sachs energy from numerical data. Their approach is rather different; it is based on constructing a transformation from the coordinates of the numerical evolution to Bondi-like coordinates. As these formulas are deduced by considering finitely many terms in the asymptotic expansions and extrapolating, this work could also be viewed as based on the notion that $\mathcal{S}$ was infinitesimally separated from $\mathcal{I}^+$.

E. Some technical points

A significant feature of the present approach is that, with it, all the quantities of interest can be expressed in terms of standard Geroch-Held-Penrose (GHP) (the boost-weight-covariant version of the Newman-Penrose calculus) quantities at $\mathcal{S}$, and the energy-momentum and angular momentum are given as natural integrals of these at $\mathcal{S}$. The formulas derived here include within themselves all necessary changes to refer to Bondi frames; no separate computation of Bondi coordinates is necessary.

The question of how accurately the Geroch-Held-Penrose quantities can be computed at $\mathcal{S}$ of course depends on the particular code. Presumably the most difficult one to measure accurately is Ψ_1, which is central in computing the angular momentum (and also, because of finite-size effects, contributes to the energy-momentum). It should be emphasized that the question here is only that of the computation of Ψ_1 at $\mathcal{S}$, not of its inferred asymptotic value (an issue raised by Deadman and Stewart [7]).

A second point is that we need not take up the delicate questions of just what degree of smoothness or peeling is encoded in the numerical solution. This is because we have separated the question of computing the energy-momentum and angular momentum from the question of finding their limiting values at $\mathcal{I}^+$: our results are given entirely in terms of data at $\mathcal{S}$.

The approach here also allows one to quantify finite-size effects, and so provides a useful consistency check on the degree to which $\mathcal{S}$ is ''effectively at infinity'' (using only data at $\mathcal{S}$). The integrals for the energy-momentum and angular momentum at null infinity are given in terms of the asymptotic values of the curvature quantities Ψ_1, Ψ_2, and Ψ_3; here, those asymptotic values appear as the values at $\mathcal{S}$ (suitably scaled) plus correction terms. Those correction terms, then, are a measure of how removed $\mathcal{S}$ is from null infinity. We also, as importantly, are able to quantify how strongly the inferred structure of null infinity is subject to finite-size effects as evolution proceeds.

In numerical work, the surfaces $\mathcal{S}$ are typically large coordinate spheres, and their first and second fundamental forms appear as slight perturbations of the values they would have for large spheres in Minkowski space. In particular, the convergence $\rho_{\mathcal{S}}$ of the outgoing congruence is a slight perturbation of $-R^{-1}$ [by $O(R^{-2})$ or less], and the shear $\sigma_{\mathcal{S}}$ is expected to be $O(R^{-2})$ if there is no incoming radiation; here R is the radius of the sphere. Thus to good approximation

$$|\sigma_{\mathcal{S}}| \ll |\rho_{\mathcal{S}}|. \tag{2}$$

(If there were incoming radiation, one would expect $\sigma_{\mathcal{S}}$ to go like a dimensionless number—the news function of that radiation—over R. Thus a small amount of incoming radiation would not upset this inequality.) This is helpful, for the asymptotic forms of Ψ_2, Ψ_3 on $\mathcal{N}$ are simplified in this case, and we make this approximation in computing them. If more accuracy is needed for particular work, probably the most efficient approach would be to compute Ψ_2, Ψ_3 perturbatively in $\sigma_{\mathcal{S}}/\rho_{\mathcal{S}}$ to the required order. (In such computations, note that while $|\sigma_{\mathcal{S}}| \ll |\rho_{\mathcal{S}}|$, the angular derivatives of $\rho_{\mathcal{S}}$ and $\sigma_{\mathcal{S}}$ may very well be of the same size. Thus one must be careful not to discard at one stage terms whose angular derivatives may be essential later.) Curiously, while Ψ_2, Ψ_3 are very complicated, all other elements of the calculation are manageable; in particular, even the exact forms of the asymptotic spinors and twistors are simple.

F. Evolution

I have so far emphasized that, by casting the problem of measuring energy-momentum and angular momentum at a finite two-surface in a certain form, one can defer the difficult questions surrounding quasilocal kinematics. However, these questions must be faced to some degree when we consider the evolution of the system, and the question of how to compare the energy-momenta and angular momenta at two different two-surfaces. This is because the auxiliary space-times constructed via the poor man's no-incoming-radiation condition from the two surfaces are not the same, and so it is not obvious how to identify the spaces on which their energy-momenta and angular momenta take values. Indeed, we must anticipate on physical grounds that unless the extraction surfaces are large enough there will be no way of identifying their auxiliary null infinities which preserves all of the usual structures.

This is an instance of a more general problem for quasilocal kinematics: how is one to compare the kinematic quantities associated with different two-surfaces? Quasilocal kinematic proposals are not well enough developed at present to take up this problem, but the degree to which quasilocal kinematics will be useful depends very largely on the degree to which it can be solved.

It turns out that, for us, there is a natural approach to this problem which fits well with structures previously developed for treating angular momentum at null infinity. We

shall see that there is a natural way to compare the null infinities from two surfaces $\mathcal{S}, \mathcal{S}'$ infinitesimally separated in time; this procedure can then be integrated. Potential finite-size effects show up in that the identifications of the null infinities are not via Bondi-Metzner-Sachs transformations, but via more complicated motions, unless the extraction surfaces are large enough. Thus one can say that while a single null infinity is being used, to the extent that finite-size effects are important the null infinity has a weaker structure than is conventional.

More precisely, we suppose we have a one-parameter family of two-surfaces $\mathcal{S}(\eta)$ (for η in some interval J) foliating a timelike three-surface T, with η increasing toward the future. Each two-surface $\mathcal{S}(\eta)$ is embedded (with the same first and second fundamental forms) in an auxiliary space-time $M(\eta)$ defined by taking $\Psi_0 = 0$ along a null surface $\mathcal{N}(\eta)$ orthogonally outward from $\mathcal{S}(\eta)$. Then the constructions already described give a null infinity $\mathcal{I}^+(\eta)$ for each $M(\eta)$, with the null geodesics orthogonally outward from $\mathcal{S}(\eta)$ defining a preferred cut $\mathcal{C}(\eta) \subset \mathcal{I}^+(\eta)$. Thus really we have a bundle of space-times $\{M(\eta) \mid \eta \in J\}$.[2] (We do *not* have a single space-time for which the condition $\Psi_0 = 0$ holds on a local foliation of null surfaces; that would generally be far too restrictive a condition to impose.)

With these structures, it turns out that there is a natural way to identify the null infinities $\mathcal{I}^+(\eta)$ for the different η's. The key step is to make the identifications at an infinitesimal level; one can then integrate. The main issue then comes down to understanding how one should define the cut of $\mathcal{I}^+(\eta)$ corresponding to the null geodesics orthogonally outward from the two-surface $\mathcal{S}(\eta + d\eta)$, that is, *for $\eta + d\eta$ infinitesimally differing from η*. Once this is done, one can fix both the identification of the generators (because one gets a point-to-point mapping of the cuts) and the supertranslation freedom.

The actual identification we need arises from natural isomorphisms. Let us begin with a vector field w^a in M_{phys} connecting $\mathcal{S}(\eta)$ to $\mathcal{S}(\eta + d\eta)$. (That is, the field w^a is tangent to T and $w^a \nabla_a \eta = 1$.) At each point of $\mathcal{S}(\eta)$, we may consider the Jacobi field along the outgoing null geodesic in M_{phys} whose initial value is w^a at that point [and whose initial velocity is chosen to make the field represent a null geodesic outward orthogonal to $\mathcal{S}(\eta + d\eta)$]. We thus get a family of Jacobi fields, over $\mathcal{S}(\eta)$, which represent null geodesics orthogonally outward from $\mathcal{S}(\eta + d\eta)$. All of this so far is in M_{phys}.

Now, any Jacobi field is specified by its initial data. By the construction of $M(\eta)$, there is a natural isomorphism of the tangent bundles $T(M_{\text{phys}})|_{\mathcal{S}(\eta)} \cong T(M(\eta))|_{\mathcal{S}(\eta)}$. [See footnote 2. The symbol T which occurs as part of the notation $T(X)$, the tangent bundle of X, should not be confused with the isolated T representing the timelike three-surface foliated by the $\mathcal{S}(\eta)$'s.] Thus we may naturally identify the Jacobi fields we found above with fields *in the auxiliary space-time $M(\eta)$*. The limiting values of these fields at $\mathcal{I}^+(\eta)$ we take to define the displacement of the cut corresponding to $\mathcal{S}(\eta + d\eta)$ from $\mathcal{C}(\eta)$. (This will be well defined independent of questions about the asymptotic behavior of the original Jacobi fields in the physical space-time.) It is this definition we required for the identification of $\mathcal{I}^+(\eta)$ with $\mathcal{I}^+(\eta + d\eta)$, and its subsequent integration to give identifications of the null infinities $\mathcal{I}^+(\eta')$ for different values of η'.

As noted above, these identifications will not be perfect Bondi-Metzner-Sachs motions, because of finite-size effects. The formulas we derive for the evolution of energy-momentum and angular momentum apply even in this case. However, for purposes of extracting the *total* energy-momentum and angular momentum, substantial finite-size effects (that is, non-Bondi-Metzner-Sachs identifications) should be regarded as signaling that the extraction surfaces have not been taken to be distant enough. The finite-size results are more of interest at present in that they may provide clues about how to develop quasilocal kinematics generally.

Having discovered that there is a well-defined null infinity (if with somewhat weaker than usual properties) for the family of extraction surfaces $\{\mathcal{S}(\eta) \mid \eta \in J\}$, it is natural to ask if one cannot construct a single asymptotic regime for the physical space-time M_{phys} to which this null infinity is attached? In some sense, this is provided by the bundle $\{\mathcal{N}(\eta) \mid \eta \in J\}$, which can be attached to M_{phys} along T. However, this bundle is not usually a space-time [it does not admit a metric structure compatible with the geometry of the $\mathcal{N}(\eta)$'s, since the condition $\Psi_0 = 0$ is generally too strong to impose on a foliating family of hypersurfaces]. While such constructions might be of some interest in the general problem of defining asymptotic regimes, in this paper there is no reason to make use of them; considering the individual auxiliary space-times $M(\eta)$ and the identifications of their null infinities will be what is relevant.

G. Outline and conventions

The next section of this paper outlines the integration of the Newman-Penrose equations under the poor man's no-incoming-radiation condition. In Sec. III, the asymptotic reference frames of Bondi and Sachs are introduced. Section IV gives the computation of asymptotically con-

[2] One technicality may be worth mentioning: As usual, when one writes of "preserving the second fundamental form" under an embedding [here $\mathcal{S}(\eta) \to M(\eta)$], one really means that the embedding is accompanied by an isomorphism of the ambient tangent spaces [here $\phi(\eta)$: $T(M(\eta))|_{\mathcal{S}(\eta)} \to T(M_{\text{phys}})|_{\mathcal{S}(\eta)}$, an isomorphism of spatially and temporally oriented Lorentzian vector bundles]. Thus strictly speaking we consider, for each $\eta \in J$, not just the space-time $M(\eta)$ but a pair $(M(\eta), \phi(\eta))$, and we should work with the bundle $\{(M(\eta), \phi(\eta)) \mid \eta \in J\}$ of such pairs. However, we shall not need such an explicit formalism.

stant spinors and the kinematic twistor. Section V gives the Bondi-Sachs energy-momentum, and Sec. VI the twistorial angular momentum. Section VII derives the formulas for comparing the energy-momentum and angular momentum (as well as the deformation of the numerical coordinates relative to the Bondi coordinates) as the two-surface is moved forward in time. Section VIII is a users' guide to the results; it summarizes what procedures and equations would be needed in numerical work.

A reader wishing simply to use the results here can use Sec. VIII as an index to the paper. (Since the twistorial angular momentum is new, the reader wishing to use this will probably want to read the introduction to Sec. VI and also Sec. VI A.)

All material necessary for understanding this paper and not otherwise cited can be found in Ref. [8], whose notation and conventions are used. Reference [9] gives the twistorial treatment of angular momentum at $\mathcal{I}^+$; Ref. [10] is an account of it for nonspecialists.

The standard literature uses the symbol λ for three different things: a spin coefficient, a rescaling factor for the spinor dyad, and the angular potential for the shear. We shall use λ_{B} for the angular potential for the shear and ƛ for the rescaling $o^A \mapsto \text{ƛ} o^A$. (The spin coefficient will be $-\sigma'$; it will play little explicit role in this paper.)

The metric signature is $+ - - -$. The symbol $\sim$ will be used to denote asymptotic equality as one moves outward along null geodesics. The symbol $\approx$ stands for equality modulo $o(|\sigma_{\mathcal{S}}|/|\rho_{\mathcal{S}}|)$. All logarithms are natural.

II. ASYMPTOTICS FROM LOCAL DATA

The aim here is to compute those asymptotic quantities we will need—enough to find the energy-momentum and angular momentum at $\mathcal{I}^+$ along $\mathcal{N}$—in terms of local data on $\mathcal{S}$. This can be reduced to a series of integration problems. Some of the details of the (lengthy, but straightforward) integrations are omitted.

What we require are the asymptotic forms of the curvature quantities Ψ_1, Ψ_2, and Ψ_3, as well as the operators $\eth$, $\eth'$, and the shear σ. (We also must verify that certain other spin coefficients have an appropriate asymptotic decay, even if we do not use their values; however, these decays will be obvious from the general forms of the integrals determining them, except in one case, which we shall treat explicitly.) And we must identify the correct Bondi frame, that is, we must be sure that when we take the asymptotic limits, by looking at two-surfaces receding to infinity in null directions orthogonally outward from $\mathcal{S}$, their null inward normals are compatible with those used in the standard analysis.

In this section, the subscript $\mathcal{S}$ is used to indicate the value of a quantity at $\mathcal{S}$ rather than at an arbitrary point of $\mathcal{N}$. However, having found the expressions for all quantities of interest in terms of data at $\mathcal{S}$, in later sections almost all computations will be expressed in terms of these data, and the subscript will be omitted.

A. The integration scheme

Let $\mathcal{S}$ be a spacelike two-surface of spherical topology in a vacuum region of space-time, and let o^A, ι^A and $l^{AA'} = o^A o^{A'}$, $m^{AA'} = o^A \iota^{A'}$, $\bar{m}^{AA'} = \iota^A o^{A'}$, and $n^{AA'} = \iota^A \iota^{A'}$ be a spinor dyad and vector tetrad associated with it, so l^a is the outgoing null congruence and n^a is the ingoing null congruence. Assume next that this surface is embedded in a space-time with the same first and second fundamental forms, and with the same values of Ψ_n for $1 \le n \le 4$ at $\mathcal{S}$. (Actually, the value of Ψ_4 will not enter.) We further assume that o^A, ι^A are propagated parallel along the outgoing null congruence from $\mathcal{S}$, and that Ψ_0 vanishes along this congruence. This leaves a freedom $o^A \mapsto \text{ƛ} o^A$, $\iota^A \mapsto \text{ƛ}^{-1} \iota^A$, where ƛ is a function of the generator only.

We have then that $\rho = \bar{\rho}$ on the null congruence, and also that $\rho'_{\mathcal{S}} = \bar{\rho}'_{\mathcal{S}}$, where the subscript indicates restriction to $\mathcal{S}$. The conditions for o^A, ι^A to be propagated parallel along l^a mean the spin coefficients ϵ, κ, γ', and τ' all vanish. The choices so far apply to the spin frame on $\mathcal{N}$, but it will also be convenient to restrict its behavior to first order off $\mathcal{N}$. We require that $\nabla_{[a} l_{b]} = 0$, which implies $\tau = \bar{\alpha} + \beta$.

The restrictions $\epsilon = 0$, $\gamma' = 0$, and $\tau = \bar{\alpha} + \beta$ break the strict boost invariance of the Geroch-Held-Penrose calculus. However, a modified invariance still holds. If we consider rescaling $o^A \mapsto \text{ƛ} o^A$, $\iota^A \mapsto \text{ƛ}^{-1} \iota^A$, where ƛ is a nonzero complex-valued function on $\mathcal{N}$ which depends on the generator only, then the conditions $\epsilon = 0$, $\gamma' = 0$ are preserved. If we consider that, accompanying any such rescaling the spinor field o^A is changed to first order off $\mathcal{N}$ by $D' o^A \mapsto \bar{\text{ƛ}}^{-1}(D' o^A - \bar{\text{ƛ}}^{-2}(\eth \text{ƛ} \bar{\text{ƛ}}) \iota^A)$, then we find $\tau \mapsto \text{ƛ} \bar{\text{ƛ}}^{-1}(\tau + (\text{ƛ} \bar{\text{ƛ}})^{-1} \eth \text{ƛ} \bar{\text{ƛ}})$, $\bar{\alpha} + \beta \mapsto \text{ƛ} \bar{\text{ƛ}}^{-1} \times (\bar{\alpha} + \beta) + \bar{\text{ƛ}}^{-2} \eth \text{ƛ} \bar{\text{ƛ}}$, and $\tau = \bar{\alpha} + \beta$ is preserved. Since this modified transformation law for the spinor dyad differs from the simple rescaling of the GHP scheme only by altering the derivative $D' o^A$ by a multiple of ι^A, the only spin coefficient inhomogeneously affected on $\mathcal{N}$ is τ; also the operators $\eth$, $\eth'$ on $\mathcal{N}$ retain their usual GHP transformation rules. (The behavior of τ adopted here is natural within the context of the characteristic initial-value problem; see [11,12].)

Let s be an affine parameter along the outgoing null geodesics normalized so that $l^a \nabla_a s = 1$ and vanishing at $\mathcal{S}$. We may think of s as having Geroch-Held-Penrose type $\{p, q\} = \{-1, -1\}$ (given our restrictions $D o^A = 0, D \iota^A = 0$). (It is conventional to use r for an affine parameter if the boost freedom in the spin frame is fixed to give the standard Bondi-Sachs asymptotics. However, as we shall have not fixed the spin frame in this way, we use s to avoid potential confusions.)

All of the computations are built on the integration of the optical equations, which can be written as

$$D\begin{bmatrix} \rho & \sigma \\ \bar{\sigma} & \rho \end{bmatrix} = \begin{bmatrix} \rho & \sigma \\ \bar{\sigma} & \rho \end{bmatrix}^2, \tag{3}$$

since $\Psi_0 = 0$ and $\Phi_{00} = 0$. The solution to this is

$$\begin{bmatrix} \rho & \sigma \\ \bar{\sigma} & \rho \end{bmatrix} = \left(-s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \rho & \sigma \\ \bar{\sigma} & \rho \end{bmatrix}_{\mathcal{S}}^{-1}\right)^{-1}. \tag{4}$$

Notice that these matrices, for different values of s, all commute, since they lie in the commutative algebra generated by the identity and the single matrix

$$\begin{bmatrix} 0 & \sigma \\ \bar{\sigma} & 0 \end{bmatrix}_{\mathcal{S}}.$$

It is this which allows the integration of the system explicitly.

It is easy to see that ρ and σ will be nonsingular for $s \geq 0$ if and only if $\rho_{\mathcal{S}} < -|\sigma_{\mathcal{S}}| \leq 0$, which is the condition that no conjugate points develop. In practice one expects $\rho_{\mathcal{S}}$ to fall off as $1/r$ and $\sigma_{\mathcal{S}}$ to fall off as $1/r^2$ where peeling holds (where r is the affine parameter in the physical space-time, not the mathematical stand-in with $\Psi_0 = 0$), so for large enough surfaces $\mathcal{S}$ which are close enough to spheres one should have $|\sigma_{\mathcal{S}}| \ll |\rho_{\mathcal{S}}|$. This means that for such surfaces it should be a good approximation to neglect the effects of σ relative to ρ on propagation outward. In this approximation, we have

$$\begin{bmatrix} \rho & \sigma \\ \bar{\sigma} & \rho \end{bmatrix} \approx \begin{bmatrix} \rho_{\mathcal{S}}(1 - s\rho_{\mathcal{S}})^{-1} & \sigma_{\mathcal{S}}(1 - s\rho_{\mathcal{S}})^{-2} \\ \bar{\sigma}_{\mathcal{S}}(1 - s\rho_{\mathcal{S}})^{-2} & \rho_{\mathcal{S}}(1 - s\rho_{\mathcal{S}})^{-1} \end{bmatrix}. \tag{5}$$

Here and throughout, we shall use $\approx$ to indicate the condition $|\sigma_{\mathcal{S}}| \ll |\rho_{\mathcal{S}}|$.

The remaining equations are integrated successively, as follows. (The results are given in Tables I and II.) From the equation

$$D\Psi_1 = 4\rho\Psi_1, \tag{6}$$

one finds Ψ_1. With that in hand, one takes up that for the spin coefficients α and β, which we write as

$$D\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \rho & \bar{\sigma} \\ \sigma & \rho \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} 0 \\ \Psi_1 \end{bmatrix}. \tag{7}$$

This, and others to follow, can be integrated using the result

$$\exp\int_0^s \begin{bmatrix} \rho & \bar{\sigma} \\ \sigma & \rho \end{bmatrix}(s')ds' = \begin{bmatrix} \rho & \bar{\sigma} \\ \sigma & \rho \end{bmatrix}\begin{bmatrix} \rho & \bar{\sigma} \\ \sigma & \rho \end{bmatrix}_{\mathcal{S}}^{-1}. \tag{8}$$

(Recall that the matrices in the exponential all commute, so there is no need to take a path-ordered exponential.)

One next takes up the transport of the operators δ, δ' up the generators. Lie transport along l^a, the vector tangent to these, establishes a canonical diffeomorphism of the outgoing null surface $\mathcal{N}$ with $\mathcal{S} \times \{s \mid s \geq 0\}$. Using this diffeomorphism, we may extend $\delta_{\mathcal{S}} = m^a_{\mathcal{S}}\nabla_a$ from its definition on $\mathcal{S}$ to $\mathcal{N}$; this is equivalent to extending it by requiring it to be Lie transported along l^a. If we put $m^a_{\mathcal{S}} = Am^a + B\bar{m}^a + Cl^a$, then using the standard spin-coefficient commutators we find the conditions being Lie transported are

$$D\begin{bmatrix} A \\ B \\ C \end{bmatrix} = -\begin{bmatrix} \rho & \bar{\sigma} & 0 \\ \sigma & \rho & 0 \\ -(\beta + \bar{\alpha}) & -(\bar{\beta} + \alpha) & 0 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix}, \tag{9}$$

which can be integrated using (8) and the initial conditions $A = 1$, $B = 0$, and $C = 0$.

With the results of these integrations, one can find the remaining quantities of interest. For Ψ_2, we integrate

$$D\Psi_2 = (\delta' - 2\alpha)\Psi_1 + 3\rho\Psi_2. \tag{10}$$

With this, we can integrate the transport equation for the optical scalars of the ingoing congruence:

$$D\begin{bmatrix} \rho' \\ \sigma' \end{bmatrix} = \begin{bmatrix} \rho & \sigma \\ \bar{\sigma} & \rho \end{bmatrix}\begin{bmatrix} \rho' \\ \sigma' \end{bmatrix} + \begin{bmatrix} -\Psi_2 \\ 0 \end{bmatrix}. \tag{11}$$

In fact, of these, we only need σ' as a datum for the next equation, and we need to note that ρ' and σ' will have, by virtue of (8), the asymptotic behavior $O(s^{-1})$. Then one can integrate

$$D\Psi_3 = \delta'\Psi_2 + 2\rho\Psi_3 + 2\sigma'\Psi_1. \tag{12}$$

Recall that we are interested in the asymptotic forms of those spin coefficients and operators necessary to define the asymptotic spinors and twistors, and also the curvature quantities Ψ_1, Ψ_2, and Ψ_3. Table I gives these results. The asymptotic values of the quantities α, β, A, B, and C enter into the determination of the asymptotic spinors and twistors, and there is little point in discussing them before those results are at hand. The asymptotic values of Ψ_1, Ψ_2, and Ψ_3 are of some interest, however. For Ψ_2 and Ψ_3, we see a leading term which is the value on $\mathcal{S}$ multiplied by the power of $(1 - s\rho)$ dictated by peeling, plus correction terms. One can check that, if the usual peeling assumptions hold in the physical space-time, then the scaling of these correction terms with the position of $\mathcal{S}$ is subdominant to that of the leading term. Thus the terms represent finite-size effects due to the distance of $\mathcal{S}$ from $\mathcal{I}^+$, and, despite their complexity, should be small for large enough $\mathcal{S}$. The complicated nonlinear forms of these corrections are also of some interest.

While we expect the results of Table I to suffice for most numerical work, it is of some conceptual interest to extend these by dropping the assumption $|\sigma_{\mathcal{S}}| \ll |\rho_{\mathcal{S}}|$. This is done for those quantities needed to determine the asymptotic spinors and twistors in Table II. Expressions for the curvature quantities Ψ_2 and Ψ_3 in this case are prohibitively lengthy; if any are needed, the best approach is to do the corresponding integrals as a power series in $\sigma_{\mathcal{S}}/\rho_{\mathcal{S}}$, keeping as many terms as required.

B. Transformation to a Newman-Unti frame

The asymptotics of the tetrad and curvature components computed above can all be examined as $s \to +\infty$, and

TABLE I. Asymptotic forms of the relevant spin coefficients, coefficients of operators, and curvature, in the parallel-transported frame, in the case $|\sigma_{\mathcal{S}}| \ll |\rho_{\mathcal{S}}|$. The left-hand column gives the asymptotic form of each quantity X for large affine parameter s in terms of a leading coefficient X^0; the coefficient is given in the right-hand column. In the right-hand column, all quantities are to be evaluated at $\mathcal{S}$ (and so would, in the notation of this section, ordinarily carry the subscript $\mathcal{S}$, but this is omitted here).

Quantity X	Leading coefficient X^0, assuming $\lvert\sigma_{\mathcal{S}}\rvert \ll \lvert\rho_{\mathcal{S}}\rvert$
$\alpha \sim \alpha^0 s^{-1}$	$(-\rho)^{-1}\alpha$
$\beta \sim \beta^0 s^{-1}$	$(-\rho)^{-1}\{\beta - (2\rho)^{-1}\Psi_1\}$
$A \sim A^0 s$	$-\rho$
$B \sim B^0 s$	0
$C \sim C^0 s$	$\bar{\alpha} + \beta$
$\rho \sim \rho^0 s^{-1}$	-1
$\sigma \sim \sigma^0 s^{-2}$	$(-\rho)^{-2}\sigma$
$\Psi_1 \sim \Psi_1^0 s^{-4}$	$(-\rho)^{-4}\Psi_1$
$\Psi_2 \sim \Psi_2^0 s^{-3}$	$(-\rho)^{-3}\{\Psi_2 - \rho^{-1}\eth'\Psi_1 + 2\rho^{-2}\Psi_1\eth'\rho + 2(3\rho)^{-2}\lvert\Psi_1\rvert^2\}$
$\Psi_3 \sim \Psi_3^0 s^{-2}$	$(-\rho)^{-2}\{\Psi_3 + \rho^{-1}\sigma'\Psi_1 + (1/6)\eth'[-4\rho^{-3}\Psi_1\eth'\rho + 3\rho^{-2}\eth'\Psi_1 - \rho^{-3}\lvert\Psi_1\rvert^2 - \rho^{-1}\Psi_2] + (30\rho^4)^{-1}(\eth'\rho)\times$ $[15\Psi_1\eth'\rho - 10\rho\eth'\Psi_1 + 4\lvert\Psi_1\rvert^2 + 15\rho^2\Psi_2] + (30\rho^4)^{-1}[15\lvert\Psi_1\rvert^2\eth'\rho + 4\Psi_1(\overline{\Psi_1})^2 - 10\rho\Psi_1\eth'\Psi_1 + 15\rho^2\overline{\Psi_1}\Psi_2]\}$

compared with the requirements for a Bondi-Sachs frame, as given, for example, in Ref. [8]. There are three sorts of adaptations which are necessary to make the tetrad accord with the standard formulas.

One of these is to shift the zero of s to eliminate the s^{-2} term in the asymptotic expansion of ρ. However, this only alters subdominant terms in the expansions, and we will only need the dominant terms, so we omit this. The second change is to replace the spinor ι^A by one which becomes tangent to future null infinity as $s \to \infty$; this can be accomplished by a null rotation. The final change is to break the local boost invariance of the dyad so as to achieve the standard asymptotic scaling $\rho' \sim (2s)^{-1}$. This requires solving an elliptic equation on $\mathcal{S}$ (equivalent to conformally uniformizing the sphere).

Because of the additional computational resources required to solve the elliptic equation on the sphere, we will distinguish here between the two stages of the passage to the Bondi-Sachs frame. When the frame has been adjusted by a null rotation so one spinor is tangent to null infinity, we call it a *Newman-Unti* frame and indicate it by the postscript NU; after the rescaling, the Bondi-Sachs frame is indicated by B. Formulas valid in Newman-Unti frames will thus automatically be valid in Bondi-Sachs frames. We shall give the transformation to a Newman-Unti frame in this section; the next section will cover Bondi-Sachs frames.

The spinor ι^A representing the ingoing null congruence has been fixed by the geometry of the two-surface $\mathcal{S}$, and propagated outward along null geodesics; there is no reason to expect it to be asymptotically tangent to null infinity. We must therefore anticipate making a null rotation $\iota^A \to \iota^A_{\mathrm{NU}} = \iota^A + Qo^A$ in order to achieve this. To preserve the parallel-propagation condition $D\iota^A_{\mathrm{NU}} = 0$, we shall need $DQ = 0$.

The equation determining Q comes from the fact that (with the correct choice of ι^A_{NU}) the quantity τ must vanish at least as $O(s^{-2})$ as $s \to \infty$ (cf. [8,13]). (Note that this means Q will not transform homogeneously under rescalings of the dyad.) Since we have $\tau = o^A D' o_A$, we find $\tau_{\mathrm{NU}} = o^A(D + Q\delta + \bar{Q}\delta')o_A = \tau + Q\sigma + \bar{Q}\rho$. Setting the lead asymptotic term of this to zero and using the asymptotic forms of ρ, σ, and τ, we see that

$$Q = (\rho^2 - |\sigma|^2)^{-1}\left\{\begin{bmatrix}\rho & \sigma\end{bmatrix}\begin{bmatrix}\bar{\alpha} + \beta \\ -\alpha - \bar{\beta}\end{bmatrix} - \sigma\left(\frac{\rho^2 - |\sigma|^2}{4|\sigma|^3}\log\frac{\rho + |\sigma|}{\rho - |\sigma|} - \frac{\rho}{2|\sigma|^2}\right)\overline{\Psi_1}\right\}\Bigg|_{\mathcal{S}}. \quad (13)$$

TABLE II. Exact leading spin coefficients and coefficients of operators in the parallel-transported frame. In the right-hand column, all quantities are to be evaluated at the surface $\mathcal{S}$ (and so would, in the notation of this section, ordinarily carry the subscript $\mathcal{S}$, but this is omitted here).

Coefficient	Value in terms of data at $\mathcal{S}$
$\begin{bmatrix}\alpha^0 \\ \beta^0\end{bmatrix}$	$-(\rho^2 - \lvert\sigma\rvert^2)^{-1}\begin{bmatrix}\rho & -\bar{\sigma} \\ -\sigma & \rho\end{bmatrix}\{\begin{bmatrix}\alpha \\ \beta\end{bmatrix} + (\frac{\rho^2 - \lvert\sigma\rvert^2}{4\lvert\sigma\rvert^3}\log\frac{\rho+\lvert\sigma\rvert}{\rho-\lvert\sigma\rvert} - \frac{\rho}{2\lvert\sigma\rvert^2})\begin{bmatrix}0 \\ \Psi_1\end{bmatrix} - (\frac{\rho}{4\lvert\sigma\rvert^3}\log\frac{\rho+\lvert\sigma\rvert}{\rho-\lvert\sigma\rvert} - \frac{1}{2\lvert\sigma\rvert^2})\begin{bmatrix}\bar{\sigma} \\ \rho\end{bmatrix}\Psi_1\}$
$\begin{bmatrix}A^0 \\ B^0\end{bmatrix}$	$-\begin{bmatrix}\rho \\ \sigma\end{bmatrix}$
C^0	$\{\bar{\alpha} + \beta - (\rho\Psi_1 + \sigma\overline{\Psi_1})(\frac{\rho}{4\lvert\sigma\rvert^3}\log\frac{\rho+\lvert\sigma\rvert}{\rho-\lvert\sigma\rvert} - \frac{1}{2\lvert\sigma\rvert^2}) + (\frac{\rho^2-\lvert\sigma\rvert^2}{4\lvert\sigma\rvert^3}\log\frac{\rho+\lvert\sigma\rvert}{\rho-\lvert\sigma\rvert} - \frac{\rho}{2\lvert\sigma\rvert^2})\Psi_1\}$
ρ^0	-1
σ^0	$(\rho^2 - \lvert\sigma\rvert^2)^{-1}\sigma$

Now let us work out the operator δ for this frame, which we denote $\delta_{\rm NU} = \delta + \bar{Q}D$. We first invert the definition of $\delta_{\mathcal{S}}$ to get

$$\begin{aligned}\begin{bmatrix} \delta \\ \delta' \end{bmatrix} &= (|A|^2 - |B|^2)^{-1}\begin{bmatrix} A & -B \\ -\bar{B} & \bar{A} \end{bmatrix}\begin{bmatrix} \delta_{\mathcal{S}} - CD \\ \delta'_{\mathcal{S}} - \bar{C}D \end{bmatrix} \\ &\sim -s^{-1}(\rho_{\mathcal{S}}^2 - |\sigma_{\mathcal{S}}|^2)^{-1}\begin{bmatrix} \rho & -\sigma \\ -\bar{\sigma} & \rho \end{bmatrix}_{\mathcal{S}} \\ &\quad\times\begin{bmatrix} \delta_{\mathcal{S}} - CD \\ \delta'_{\mathcal{S}} - \bar{C}D \end{bmatrix} \end{aligned} \tag{14}$$

(where $\sim$ denotes asymptotic equality as $s \to +\infty$), so

$$\begin{aligned}\begin{bmatrix} \delta_{\rm NU} \\ \delta'_{\rm NU} \end{bmatrix} &\sim -s^{-1}(\rho_{\mathcal{S}}^2 - |\sigma_{\mathcal{S}}|^2)^{-1}\begin{bmatrix} \rho & -\sigma \\ -\bar{\sigma} & \rho \end{bmatrix}_{\mathcal{S}} \\ &\quad\times\begin{bmatrix} \delta_{\mathcal{S}} - CD \\ \delta'_{\mathcal{S}} - \bar{C}D \end{bmatrix} + \begin{bmatrix} \bar{Q} \\ Q \end{bmatrix}D. \end{aligned} \tag{15}$$

Inserting the asymptotic values from Table II, we find that the coefficients of D cancel, and we are left simply with

$$\begin{bmatrix} \delta_{\rm NU} \\ \delta'_{\rm NU} \end{bmatrix} \sim -s^{-1}(\rho_{\mathcal{S}}^2 - |\sigma_{\mathcal{S}}|^2)^{-1}\begin{bmatrix} \rho & -\sigma \\ -\bar{\sigma} & \rho \end{bmatrix}_{\mathcal{S}}\begin{bmatrix} \delta_{\mathcal{S}} \\ \delta'_{\mathcal{S}} \end{bmatrix}. \tag{16}$$

This cancellation is not a surprise; it arises from the fact that the $s =$ const surfaces can be taken as the approaching cuts of $\mathcal{I}^+$.

We now find the spin coefficients $\alpha_{\rm NU}$, $\beta_{\rm NU}$ in the Newman-Unti frame. (We shall not use these explicitly, but we give them both for completeness and for purposes of comparison with other work.) The transformation rules for these are

$$\alpha_{\rm NU} = \alpha + Q\rho, \tag{17}$$

$$\beta_{\rm NU} = \beta + Q\sigma. \tag{18}$$

Using the results from Table II, we find

$$\alpha_{\rm NU} \sim -s^{-1}(\rho_{\mathcal{S}}^2 - |\sigma_{\mathcal{S}}|^2)^{-1}\{-\rho\bar{\beta} + \overline{\sigma\alpha} + (1/2)\overline{\Psi_1}\}|_{\mathcal{S}}, \tag{19}$$

$$\beta_{\rm NU} \sim -s^{-1}(\rho_{\mathcal{S}}^2 - |\sigma_{\mathcal{S}}|^2)^{-1}\{\rho\beta - \sigma\alpha - (1/2)\Psi_1\}|_{\mathcal{S}}; \tag{20}$$

the results in the approximation $|\sigma_{\mathcal{S}}| \ll |\rho_{\mathcal{S}}|$ are listed in Table III.

The fact that Q remains bounded as $s \to \infty$ (in fact, Q is constant along the generators of $\mathcal{N}$), together with the peeling property, means that transformation to the Newman-Unti frame does not affect the asymptotic forms of Ψ_1, Ψ_2, and Ψ_3. Also $\rho_{\rm NU} = \rho$ and $\sigma_{\rm NU} = \sigma$, since these only involve derivatives of o^A.

The transformations for the optical coefficients for the ingoing congruence are more complex. We have $\rho'_{\rm NU} = \rho' - \eth Q - Q^2\sigma$, and $\sigma'_{\rm NU} = \sigma' - \eth' Q - Q^2\rho$. We shall not need these, but for completeness their asymptotic forms are given in Table III.

III. BONDI-SACHS FRAMES

In this section, we complete the transformation to a Bondi-Sachs frame, and also establish some of the calculus of these frames which will be required for the analysis of energy-momentum and angular momentum. The Bondi-Sachs frames are essentially equivalent to the notion of an "asymptotic laboratory frame."

From this point on, we will omit the subscript $\mathcal{S}$ for spin-coefficient quantities (including operators δ, δ', $\eth$, and $\eth'$) at the two-surface, unless explicitly indicated otherwise. Quantities considered at other points on $\mathcal{N}$ will either be in the Newman-Unti or the Bondi-Sachs frames and will be indicated by subscripts NU or B.

A. Transformation to a Bondi-Sachs frame

We recall that the Newman-Unti frame established in the previous sections is very nearly a Bondi-Sachs frame; what remains is to adjust the spin frame, or equivalently conformally transform, in order to make the $s =$ const cross sections unit spheres.

Let us begin with the metric structure. This is characterized by the intrinsic δ operators of the $s =$ const surfaces. Equation (16) expressed these in terms of the structure at $\mathcal{S}$; let us put $\delta_{\rm NU} = M^a\nabla_a$, so

TABLE III. Leading forms of the spin coefficients and operators in the Newman-Unti frame, under the assumption $|\sigma_{\mathcal{S}}| \ll |\rho_{\mathcal{S}}|$. Each of these quantities falls off as s^{-1} with the corresponding coefficient. Thus $\alpha_{\rm NU} \sim s^{-1}\alpha^0$, etc. In the right-hand column, all quantities are to be evaluated at $\mathcal{S}$.

Quantity	Value
δ_0	$-\rho^{-1}\delta_{\mathcal{S}}$
δ'_0	$-\rho^{-1}\delta'_{\mathcal{S}}$
α^0	$-\rho^{-1}\{-\bar{\beta} + (2\rho)^{-1}\overline{\Psi_1}\}$
β^0	$-\rho^{-1}\{\beta - (2\rho)^{-1}\Psi_1\}$
$(\rho')^0$	$\rho^{-1}\{-\rho' - 2(3\rho^2)^{-1}\lvert\Psi_1\rvert^2 + (2\rho^2)^{-1}(\eth'\Psi_1 + \eth\overline{\Psi_1}) - (2\rho)^{-1}(\Psi_2 + \overline{\Psi_2}) - 5(6\rho^3)^{-1}(\Psi_1\eth'\rho + \overline{\Psi_1}\eth\rho) - (2\rho)^{-1}\times$ $(\eth\bar{\tau} + \eth'\tau) + (2\rho^2)^{-1}(\bar{\tau}\eth\rho + \tau\eth'\rho) + (2\rho^2)^{-1}(\Psi_1\bar{\tau} + \overline{\Psi_1}\tau) + (2\rho^2)^{-1}(\tau\eth\bar{\sigma} + \bar{\tau}\eth'\sigma) - (6\rho^3)^{-1}(\Psi_1\eth\bar{\sigma} + \overline{\Psi_1}\eth'\sigma)\}$
$(\sigma')^0$	$-\rho^{-1}\{\sigma' - \eth((2\rho^2)^{-1}\overline{\Psi_1} - \rho^{-1}\tau) - \rho((2\rho^2)^{-1}\overline{\Psi_1} - \rho^{-1}\tau)^2\}$

$$M^a \sim -s^{-1}(\rho^2 - |\sigma|^2)^{-1}(\rho m^a - \sigma \bar{m}^a). \tag{21}$$

Then $-2M^{(a}\bar{M}^{b)}$ gives the intrinsic inverse metric on the $s = \text{const}$ surfaces. The intrinsic metric is $-2M_{(a}\bar{M}_{b)}$, where M_a, $\bar{M}_a$ are *not* defined via lowering with g_{ab}, but via the duality relations $M_a\bar{M}^a = -1$, $M_aM^a = 0$; explicitly

$$M_a \sim s(-\rho m_a - \sigma \bar{m}_a), \tag{22}$$

and

$$-2M_{(a}\bar{M}_{b)} \sim -2s^2(\rho m_{(a} + \sigma \bar{m}_{(a})(\rho \bar{m}_{b)} + \bar{\sigma} m_{b)}). \tag{23}$$

Under a change of scale $o^A \mapsto \lambda o^A$, $\iota^A \mapsto \lambda^{-1}\iota^A$, the factor $(\rho m_{(a} + \sigma \bar{m}_{(a})(\rho \bar{m}_{b)} + \bar{\sigma} m_{b)})$ will be multiplied by $|\lambda|^4$; we may therefore regard the change of scale as equivalent to a conformal transformation by $\Omega^2 = |\lambda|^4$. More precisely, for each choice of scale we have a family of metrics on the $s = \text{const}$ surfaces which are scalar multiples of each other. When we change the scale we also change the choices of the $s = \text{const}$ surfaces (since $s \mapsto |\lambda|^{-2}s$), and it is the metrics on these (pulled back to $\mathcal{S}$) which are conformally rescaled by $|\lambda|^4$ (relative to the pullback of the metric on the original surface at the same numerical value of s).

The conformal structure is characterized by the complex structure. We may introduce a complex stereographic (antiholomorphic) coordinate[3] ζ on the $s = \text{const}$ surfaces by requiring $M^a\nabla_a\zeta = 0$, that is,

$$(\rho\delta - \sigma\delta')\zeta = 0 \tag{24}$$

and requiring that ζ be regular over $\mathcal{S}$ except for a simple pole (equivalently, that ζ^{-1} vanishes at a single point and in the limit of approach to this point its argument has winding number -1, the minus sign on account of its antiholomorphic character).

The coordinate ζ is unique up to a fractional linear transformation. In order to keep its interpretation as direct as possible, when $\mathcal{S}$ is approximately a round sphere the coordinate ζ should be taken to be close to a stereographic coordinate $e^{i\phi}\cot(\theta/2)$ on $\mathcal{S}$. One way to fix the freedom would be to require the pole to lie on the $+z$ coordinate axis, the zero to lie on the $-z$ axis, and the point $\zeta = 1$ to lie on the $+x$ axis. With these choices we effectively fix an asymptotic laboratory frame. (The time axis is fixed by the requirement that $|\zeta| = 1$ be a great circle.)

For numerical work, it may be more convenient to recast Eq. (24) in terms of regular quantities. If we let ζ_0 be any smooth function on $\mathcal{S}$ with a simple pole and simple zero of the required type and write $\zeta = \hat{\zeta}\zeta_0$, then $\hat{\zeta}$ is smooth over $\mathcal{S}$ and satisfies the everywhere-regular equation $M^a\nabla_a\hat{\zeta} = -\hat{\zeta}M^a\nabla_a\log\zeta_0$. There will be a one-complex-dimensional space of everywhere-regular nowhere-vanishing solutions to the equation for $\hat{\zeta}$. These solutions will give $\zeta = \hat{\zeta}\zeta_0$ the same pole and zero as ζ_0; thus, if these have been chosen as in the previous paragraph, one has simply to adjust the multiplicative constant in $\hat{\zeta}$ to achieve the final normalization $\zeta = 1$ on the $+x$ axis. We shall assume from now on that a solution ζ to (24) has been found.

Now let us turn to the metric structure. As is conventional, put $M_a = -\bar{P}^{-1}d\zeta$, so that

$$-1 = \bar{M}^aM_a \sim \bar{P}^{-1}s^{-1}(\rho^2 - |\sigma|^2)^{-1}(\rho\delta' - \bar{\sigma}\delta)\zeta \sim \bar{P}^{-1}s^{-1}\rho^{-1}\delta'\zeta$$

and

$$P \sim -s^{-1}\rho^{-1}\delta\bar{\zeta}. \tag{25}$$

Then the metric is $\sim -2|P|^{-2}d\zeta d\bar{\zeta}$.

Now let us consider a change of scale to achieve a Bondi-Sachs frame. Let this be $o^A \mapsto o^A_{\mathrm{B}} = \lambda o^A$. We may keep ζ unchanged (it is a conformal invariant); we have then $P \mapsto P_{\mathrm{B}} = -s_{\mathrm{B}}^{-1}\rho_{\mathrm{B}}^{-1}\delta_{\mathrm{B}}\bar{\zeta} \sim -\bar{\lambda}^{-2}s_{\mathrm{B}}^{-1}\rho^{-1}\delta\bar{\zeta}$. We rescale the metric to a sphere of radius s_{B} with $P_{\mathrm{B}} = 2^{-1/2}s_{\mathrm{B}}^{-1}(1 + |\zeta|^2)$. This will align the time axis of the Bondi-Sachs system with that of the laboratory frame, and give the standard spin frame adapted to the coordinate ζ. We then find

$$\lambda^2 = -2^{1/2}\rho^{-1}(1 + |\zeta|^2)^{-1}\delta'\zeta. \tag{26}$$

With λ known, we may read off the value of any spin- and boost-weighted quantity in the Bondi-Sachs frame from its value in a Newman-Unti frame. (Note that the original o^A, ι^A may be *any* spin frame for which m^a is tangent to $\mathcal{S}$; Eq. (26) provides the correct transformation to the frame adapted to ζ.)

In what follows, we shall need the shear in a Bondi-Sachs frame. We have, from Table II, that $\sigma_{\mathrm{NU}} \sim s^{-2}(\rho^2 - |\sigma|^2)^{-1}\sigma$. Inserting the appropriate rescalings, we find

$$\sigma_{\mathrm{B}} \sim s_{\mathrm{B}}^{-2}\lambda\bar{\lambda}^{-3}(\rho^2 - |\sigma|^2)^{-1}\sigma \tag{27}$$

with λ given by (26). In particular, the *Bondi shear* is the coefficient of s_{B}^{-2}, that is,

$$\sigma^0_{\mathrm{B}} = \lambda\bar{\lambda}^{-3}(\rho^2 - |\sigma|^2)^{-1}\sigma. \tag{28}$$

Finally, we remark that an alternative (and somewhat more traditional) route to fixing the conformal factor is to require that the Gaussian curvature of the $s = \text{const}$ surfaces becomes asymptotically constant. Since the Gaussian curvature is $\sim -2\rho_{\mathrm{NU}}\rho'_{\mathrm{NU}}$ and $\rho_{\mathrm{NU}} \sim -s^{-1}$, this leads to the requirement that $\rho'_{\mathrm{NU}} \sim (2s)^{-1}$. One can use Table III and the transformation rules discussed at the beginning of Sec. II A to write this as a second-order partial differential

[3]The orientation on the sphere induced from its embedding in space-time (or in a spacelike hypersurface with its induced orientation) is opposite to the one used in ordinary complex analysis, and so what is antiholomorphic from the space-time point of view is holomorphic for ordinary complex analysis.

equation for $|\text{ð}\lambda|$, which is equivalent to the usual equation for finding a conformal transformation uniformizing the Gaussian curvature. The formulas for this are rather more complicated than those given in the present section, however.

B. The angular potential for the shear

The Bondi shear, being a spin-weight two quantity, admits an angular potential λ_B such that

$$s_B^2 \text{ð}_B^2 \lambda_B = \sigma_B^0 \tag{29}$$

(or equivalently $\text{ð}_B^2 \lambda_B = \sigma_B$). The use of this potential facilitates the computation of the Bondi-Sachs energy, and the potential also plays a central role in the analysis of angular momentum. The *electric* and *magnetic* parts of the shear are $\sigma_{el} = \text{ð}_B^2 \,\mathrm{Re}\lambda_B$ and $\sigma_{mag} = i\text{ð}_B^2 \,\mathrm{Im}\lambda_B$.

Equation (29) is easily solved when $P_B = 2^{-1/2} s_B^{-1}(1 + |\zeta|^2)$. In this case, it can be written as

$$(1/2)\partial_{\bar\zeta}(1 + \zeta\bar\zeta)^2 \partial_{\bar\zeta}\lambda_B = \sigma_B^0, \tag{30}$$

and a Green's function for the operator can easily be derived from the relation $\partial_{\bar\zeta}\bar\zeta^{-1} = \pi\delta^{(2)}(\zeta, \bar\zeta)$ (where the right-hand side is the usual δ function in the ζ plane). We find

$$\lambda_B(\zeta, \bar\zeta) = \int G(\zeta, \bar\zeta; \acute\zeta, \acute{\bar\zeta})\sigma_B^0(\acute\zeta, \acute{\bar\zeta}) d\acute\zeta \wedge d\acute{\bar\zeta}/(2i), \tag{31}$$

where

$$G(\zeta, \bar\zeta; \acute\zeta, \acute{\bar\zeta}) = -\pi^{-1}(\zeta - \acute\zeta)^{-1}\left(\frac{\bar\zeta}{1 + |\zeta|^2} - \frac{\acute{\bar\zeta}}{1 + |\acute\zeta|^2}\right), \tag{32}$$

and

$$(2i)^{-1} d\zeta \wedge d\bar\zeta = (1 - |\sigma/\rho|^2)|\delta\bar\zeta|^2 d\mathcal{S}. \tag{33}$$

An alternative means of solving Eq. (29) would be to resolve σ_B^0 into spin-weighted spherical harmonics ${}_2Y_{j,m}$ and then use the relation

$$\text{ð}_B^2 \,{}_0Y_{j,m} = (1/2)\sqrt{(j-1)j(j+1)(j+2)}\,{}_2Y_{j,m}$$

to infer the corresponding resolution of λ_B. Thus one would have

$$\sigma^0_{j,m} = 2^{1/2}\int \overline{{}_2Y_{j,m}}(1 + |\zeta|^2)^{-1}\sqrt{\frac{(\delta'\zeta)^3}{(\delta\bar\zeta)}}\rho^{-1}\sigma d\mathcal{S}, \tag{34}$$

$$\lambda_{j,m} = 2((j-1)j(j+1)(j+2))^{-1/2}\sigma^0_{j,m} \quad (j \geq 2), \tag{35}$$

$$\lambda_B = \sum \lambda_{j,m}\,{}_0Y_{j,m}. \tag{36}$$

The terms $\lambda_{j,m}$ with $j = 0, 1$ are freely specifiable. We take these terms to vanish, which will simplify the coordinatization of the twistor space, below.

The phases of the spin-weighted spherical harmonics depend on the spin frame. There are two common choices: that adapted to the complex coordinate ζ, and that adapted to θ, ϕ. Because the analysis has been given here in terms of ζ, it is that spin frame and those spherical harmonics which are used in Eq. (34). To use the harmonics with respect to θ, ϕ one must, besides replacing ${}_{-2}Y_{j,m}(\zeta, \bar\zeta)$ with ${}_{-2}Y_{j,m}(\theta, \phi)$, also include a factor of $(\zeta/\bar\zeta)^2$ in the integrand of (34). (See the Appendix; see Ref. [14] for a detailed discussion of the harmonics.)

IV. ASYMPTOTIC TWISTORS AND SPINORS

The Bondi-Sachs energy-momentum is a covector in a certain vector space, the space of asymptotically constant covectors. This space is most easily constructed from the space of asymptotically constant spinors. Similarly, the twistorial angular momentum is defined on the space of asymptotic twistors. In fact, the asymptotic spinors are naturally defined in terms of a canonical fibration of twistor space, so we shall start by constructing the twistors and then specialize to the spinors. We conclude this section by giving the kinematic twistor, in terms of which the energy-momentum and angular momentum will be defined.

A. The twistor space

The twistor space $\mathbb{T}(\mathcal{C}(\mathcal{S}))$ of the cut $\mathcal{C}(\mathcal{S})$ of null infinity associated with $\mathcal{S}$ is the set of solutions of the two-surfaces twistor equation at $\mathcal{C}(\mathcal{S})$. These equations are

$$\text{ð}'_B \tilde\omega^0_B = 0, \tag{37}$$

$$s\text{ð}_B \omega^1_B = \sigma^0_B \tilde\omega^0_B, \tag{38}$$

where $\tilde\omega^0_B = s^{-1}\omega^0$ is rescaled to attain a finite limit at $\mathcal{C}(\mathcal{S})$, and $s\text{ð}_B$ tends to an operator depending on angle only; cf. Eq. (25). There is a four-complex-dimensional space of solutions to these which we shall give shortly. For completeness, we note the forms of these equations in terms of the spin coefficients at $\mathcal{S}$ are

$$-(\rho\text{ð}' - \bar\sigma\text{ð} + (1/2)\overline{\Psi_1})\tilde\omega^0 = 0, \tag{39}$$

$$-(\rho\text{ð} - \sigma\text{ð}' + (1/2)\Psi_1)\omega^1 = \sigma\tilde\omega^0. \tag{40}$$

(The minus signs are included because ρ is negative.)

B. Solving the twistor equation

Solutions to the twistor equation are easily found. Equation (37) for $\tilde\omega^0$ has as its space of solutions the spherical harmonics of spin weight $-1/2$; thus

$$\tilde\omega^0_B = 2^{1/2}(1 + |\zeta|^2)^{-1/2}(Z^3 + Z^2\bar\zeta), \tag{41}$$

where Z^2 and Z^3 are constants. To solve the remaining

equation, we adopt a device of Tod and set

$$\omega^1_{\mathrm{B}} = \tilde{\omega}^0_{\mathrm{B}}\eth_{\mathrm{B}}\lambda_{\mathrm{B}} - \lambda_{\mathrm{B}}\eth_{\mathrm{B}}\tilde{\omega}^0_{\mathrm{B}} + \xi_{\mathrm{B}}, \tag{42}$$

where λ_{B} is an angular potential for the shear (Sec. III B), and ξ_{B} is a spin-weight $+1/2$ quantity to be determined. We note that

$$\eth_{\mathrm{B}}\tilde{\omega}^0_{\mathrm{B}} = (1 + |\zeta|^2)^{-1/2}(Z^2 - Z^3\zeta). \tag{43}$$

Then the remaining equation (38) is equivalent to $\eth'_{\mathrm{B}}\xi_{\mathrm{B}} = 0$, and the solutions to this are

$$\xi_{\mathrm{B}} = -i(1 + |\zeta|^2)^{-1/2}(-Z^0 + Z^1\zeta). \tag{44}$$

Thus (Z^0, Z^1, Z^2, Z^3) coordinatize the twistor space; the factors have been chosen to make them in accord with those induced from the standard Cartesian basis of Minkowski space if the cut of null infinity is gotten from the light cone of the origin (cf. [14], Sec. 4.15 and [8], Sec. 6.1).[4]

C. Structures on twistor space

There are three important structures on twistor space: a fibration, an infinity twistor, and a certain reality structure. The fibration and the infinity twistor allow the definition of asymptotic spinors; the reality structure defines the null geodesics which play the roles of origins for the definition of angular momentum. Finally, the reality structure and the infinity twistor combine to define a certain twistor operation, the ''hook,'' which enters in the definition of energy-momentum.

1. The fibration, the infinity twistor, and spinors

The most primitive and important structure on twistor space is the fibration, which is defined by simply keeping the $\tilde{\omega}^0$ field of the twistor. This is just $(Z^0, Z^1, Z^2, Z^3) \mapsto (Z^2, Z^3)$ in our coordinates, since $\tilde{\omega}^0$ is specified by Z^2 and Z^3. We see then that the space of fibers is a two-complex-dimensional space; it is naturally identifiable with the space of (dual, primed) *asymptotically constant spinors* $\mathbb{S}_{A'}$. Asymptotic spinors of other valences and asymptotic vectors and tensors are defined as usual by tensor operations from $\mathbb{S}_{A'}$.

Closely related to the fibration is the *infinity twistor* $I(Z, \acute{Z}) = I_{\alpha\beta}Z^\alpha\acute{Z}^\beta$. In our coordinates, it is simply given by

$$I(Z, \acute{Z}) = Z^2\acute{Z}^3 - Z^3\acute{Z}^2. \tag{45}$$

This evidently defines a skew form on $\mathbb{S}_{A'}$, which represents the asymptotically constant spinor $\epsilon^{A'B'}$. Its negative inverse is $\epsilon_{A'B'}$; the spinor $\epsilon_{AB}\epsilon_{A'B'}$ represents the asymptotic metric $g_{ab} = g_{AA'BB'}$. One often puts $Z^2 = \pi_{0'}$, $Z^3 = \pi_{1'}$ and then one has

$$I(Z, \acute{Z}) = \epsilon^{A'B'}\pi_{A'}\acute{\pi}_{B'} \tag{46}$$

with $\epsilon^{0'1'} = 1$ as usual.

One can also define spin space $\mathbb{S}^A$ directly in twistor terms, as the kernel of the fibration; thus $\mathbb{S}^A$ is identified with the set of spinors whose coordinates are $(Z^0, Z^1, 0, 0)$. However, it will be more natural for us to work with $\mathbb{S}_{A'}$, especially when we take up evolution.

2. The reality structure

The twistor space for Minkowski space is equipped with a sesquilinear form of signature $++--$ whose zero set is the set of *real twistors*. In general relativity, there is also a reality structure, but it is more nonlinear when spin is present. The analytic manifestation of this is that the candidate expression for the norm

$$i(\bar{\omega}^{1'}_{\mathrm{B}}\eth_{\mathrm{B}}\tilde{\omega}^0_{\mathrm{B}} - \tilde{\omega}^0_{\mathrm{B}}\eth_{\mathrm{B}}\bar{\omega}^{1'}_{\mathrm{B}} + \bar{\tilde{\omega}}^{0'}_{\mathrm{B}}\eth'_{\mathrm{B}}\omega^1_{\mathrm{B}} - \omega^1_{\mathrm{B}}\eth'_{\mathrm{B}}\bar{\tilde{\omega}}^{0'}_{\mathrm{B}}) \tag{47}$$

is not in general constant over $\mathcal{S}$. It turns out that we get a good theory of angular momentum, however, by simply evaluating this expression at the point on the sphere for which $\tilde{\omega}^0_{\mathrm{B}}$, or equivalently $\tilde{\omega}^0$, vanishes. (There always will be a unique such point, unless the field $\tilde{\omega}^0$ vanishes identically, in which case the result is taken to be zero by a continuity argument.) We will denote the point at which $\tilde{\omega}^0$ vanishes by $\gamma(\tilde{\omega}^0)$, the restriction of (47) to $\gamma(\tilde{\omega}^0)$ by $\Phi(Z)$. A twistor is *real* if and only if $\Phi(Z) = 0$. The real twistors correspond to real null geodesics meeting null infinity; they take the place of space-time points as ''origins'' for the evaluation of angular momentum.

Using the formulas above we find that

$$\Phi(Z) = [-2\,\mathrm{Im}\lambda|\eth_{\mathrm{B}}\tilde{\omega}^0_{\mathrm{B}}|^2 + i(\bar{\xi}\eth_{\mathrm{B}}\tilde{\omega}^0_{\mathrm{B}} + \xi\eth'_{\mathrm{B}}\bar{\tilde{\omega}}^{0'}_{\mathrm{B}})]|_{\gamma(\tilde{\omega}^0)}. \tag{48}$$

The stereographic coordinate of $\gamma(\tilde{\omega}^0)$ is $\zeta = -\overline{Z^3/Z^2}$ and a brief calculation gives

$$\eth_{\mathrm{B}}\tilde{\omega}^0|_{\gamma(\tilde{\omega}^0)} = \frac{Z^2}{|Z^2|}\sqrt{|Z^2|^2 + |Z^3|^2}. \tag{49}$$

Using these, we find

$$\Phi(Z) = -2\,\mathrm{Im}\lambda|_{\gamma(\tilde{\omega}^0)}(|Z^2|^2 + |Z^3|^2) + Z^0\overline{Z^2} + Z^1\overline{Z^3} + Z^2\overline{Z^0} + Z^3\overline{Z^1}. \tag{50}$$

Here the second line is the usual twistor norm in special relativity; the contribution on the first line is an essentially general-relativistic effect. We recall that $\mathrm{Im}\lambda$ is the angular potential for the magnetic part of the shear; it is this magnetic shear which distorts the twistor norm. (The magnetic shear is found to be the $j \geq 2$ components of the specific—that is, per unit mass—spin.)

[4]This differs slightly from the basis common for work near null infinity, with our o^A being $2^{-1/4}$ of that one and our ι^A being $2^{1/4}$ of that one.

3. The hook operation

While, as discussed above, the special-relativistic twistor norm does not extend to general relativity, enough of the structure does survive that a certain antilinear operation does carry over to general relativity. This is the *hook* of a twistor Z^α, denoted by $I^{\beta\alpha}\bar{Z}_\beta$. (Owing to the nonexistence of a norm, the quantity $\bar{Z}_\beta$ does not have any separate meaning for us, but we shall keep the special-relativistic notation.) If we set $\acute{Z}^\alpha = I^{\beta\alpha}\bar{Z}_\beta$, then the definition of the hook is

$$\acute{\tilde{\omega}}^{\,0} = 0, \tag{51}$$

$$\acute{\tilde{\omega}}^{\,1} = -i\bar{\tilde{\omega}}^{0'}. \tag{52}$$

In our coordinates, the hook operation is $(Z^0, Z^1, Z^2, Z^3) \mapsto (-\overline{Z^3}, \overline{Z^2}, 0, 0)$. (The hook of any twistor is an element of $\mathbb{S}^A$.)

D. The kinematic twistor

In twistor theory, the energy-momentum and angular momentum are encoded in a *kinematic twistor* $A(Z) = A_{\alpha\beta}Z^\alpha Z^\beta$, defined by

$$A(Z) = -i(4\pi G)^{-1}\oint\{\Psi_1^{\mathrm{NU}}(\omega_{\mathrm{NU}}^0)^2 + 2\Psi_2^{\mathrm{NU}}\omega_{\mathrm{NU}}^0\omega_{\mathrm{NU}}^1 + \Psi_3^{\mathrm{NU}}(\omega_{\mathrm{NU}}^1)^2\}dS_{\mathrm{NU}}, \tag{53}$$

where the limit as the surface tends to null infinity is understood. The kinematic twistor satisfies the Hermiticity property $A_{\alpha\beta}\acute{Z}^\alpha I^{\gamma\beta}\bar{Z}_\gamma = \overline{A_{\alpha\beta}Z^\alpha I^{\gamma\beta}\acute{\bar{Z}}_\gamma}$, which in our coordinates is

$$A_{00} = A_{01} = A_{11} = 0, \tag{54}$$

$$A_{02} = -\overline{A_{13}}, \qquad A_{12} = \overline{A_{12}}, \qquad A_{03} = \overline{A_{03}}. \tag{55}$$

To work out the kinematic twistor explicitly, let us first insert the asymptotic forms of the quantities; we find

$$A(Z) = -i(4\pi G)\oint(\rho^2 - |\sigma|^2)\{\Psi_1^0(\tilde{\omega}^0)^2 + 2\Psi_2^0\tilde{\omega}^0\omega^1 + \Psi_3^0(\omega^1)^2\}d\mathcal{S}, \tag{56}$$

in terms of data on $\mathcal{S}$.

We next express the twistors in terms of their forms in the Bondi-Sachs frame, taking into account the rescaling relative to the Newman-Unti one (recall that $\tilde{\omega}^0$, ω^1 have Newman-Penrose types $\{0, 1\}$, $\{1, 0\}$, respectively):

$$A(Z) = -i(4\pi G)\oint\{\Psi_1^0\bar{\lambda}^{-2}(\tilde{\omega}_{\mathrm{B}}^0)^2 + 2\Psi_2^0(\lambda\bar{\lambda})^{-1}\tilde{\omega}_{\mathrm{B}}^0\omega_{\mathrm{B}}^1 + \Psi_3^0\lambda^{-2}(\omega_{\mathrm{B}}^1)^2\}(\rho^2 - |\sigma|^2)d\mathcal{S}. \tag{57}$$

At this point, we may substitute the explicit forms of the solutions of the twistor equation given in this section to compute the components $A_{\alpha\beta}$ of the kinematic twistor. Because of round-off errors (and also the approximation $|\sigma| \ll |\rho|$, if used), the numerical computation of $A_{\alpha\beta}$ directly from (57) could fail to satisfy the Hermiticity conditions (54) and (55); we therefore enforce these conditions at the levels of the integrands. The results of this are given in Table IV.

V. ENERGY-MOMENTUM

The Bondi-Sachs energy-momentum P_a is most naturally viewed as a function $P^{AA'}\bar{\pi}_A\pi_{A'}$ on the space $\mathbb{S}_{A'}$ of asymptotically constant spinors. In twistor terms, this is

$$P^{AA'}\bar{\pi}_A\pi_{A'} = A_{\alpha\beta}Z^\alpha I^{\gamma\beta}\bar{Z}_\gamma, \tag{58}$$

where the right-hand side represents the contraction of the kinematic twistor once with Z^α and once with its hook $I^{\gamma\beta}\bar{Z}_\gamma$; the result (58) is real and depends only on the projection $\pi_{A'}$ of the twistor Z; that is, if the twistor components are $(Z^0, Z^1, Z^2 = \pi_{0'}, Z^3 = \pi_{1'})$, the choice of Z^0 and Z^1 is immaterial.

Using the formula for the hook map in coordinates [just below Eq. (52)], we find explicitly

TABLE IV. Integrands for the computation of the potentially nonzero components $A_{\alpha\beta} = A_{\beta\alpha}$ of the kinematic twistor. Each term on the right is to be multiplied by the common factor $-i(4\pi G)^{-1}(1 + |\zeta|^2)^{-1}(\rho^2 - |\sigma|^2)$ and integrated with respect to $d\mathcal{S}$ over the sphere. In this table, the Hermiticity conditions $A_{12} = \overline{A_{12}}$, $A_{03} = \overline{A_{03}}$, and $A_{02} = -\overline{A_{13}}$ have been enforced at the level of the integrands.

Component	Quantity to be multiplied by $-i(4\pi G)^{-1}(1 + \lvert\zeta\rvert^2)^{-1}(\rho^2 - \lvert\sigma\rvert^2)$ and integrated with respect to $d\mathcal{S}$
$A_{02} = -\overline{A_{13}}$	$2^{1/2}i\bar{\zeta}\lvert\lambda\rvert^{-2}\mathrm{Re}\Psi_2^0 + i\bar{\zeta}\mathrm{Re}(\Psi_3^0\lambda^{-2}2^{1/2}\eth_{\mathrm{B}}\lambda_{\mathrm{B}}) + 2^{-1}i(\bar{\zeta}^2 - 1)\mathrm{Re}(\Psi_3^0\lambda^{-2}\lambda_{\mathrm{B}}) - 2^{-1}(\bar{\zeta}^2 + 1)\mathrm{Im}(\Psi_3^0\lambda^{-2}\lambda_{\mathrm{B}})$
A_{03}	$2^{1/2}i\lvert\lambda\rvert^{-2}\mathrm{Re}\Psi_2^0 + i\mathrm{Re}(\Psi_3^0\lambda(2^{1/2}\eth_{\mathrm{B}}\lambda_{\mathrm{B}} + \zeta\lambda_{\mathrm{B}}))$
A_{12}	$-2^{1/2}i\lvert\zeta\rvert^2\lvert\lambda\rvert^{-2}\mathrm{Re}\Psi_2^0 - i\mathrm{Re}(\Psi_3^0\zeta(2^{1/2}\bar{\zeta}\eth_{\mathrm{B}}\lambda_{\mathrm{B}} - \lambda_{\mathrm{B}}))$
A_{22}	$2\Psi_1^0\bar{\lambda}^{-2}\bar{\zeta}^2 + 2^{3/2}\Psi_2^0\lvert\lambda\rvert^{-2}\bar{\zeta}(2^{1/2}\bar{\zeta}\eth_{\mathrm{B}}\lambda_{\mathrm{B}} - \lambda_{\mathrm{B}}) + \Psi_3^0\lambda^{-2}(2^{1/2}\bar{\zeta}\eth_{\mathrm{B}}\lambda_B - \lambda_{\mathrm{B}})^2$
A_{23}	$2\Psi_1^0\bar{\lambda}^{-2}\bar{\zeta} + 2^{1/2}\Psi_2^0\lvert\lambda\rvert^{-2}(2^{3/2}\bar{\zeta}\eth_{\mathrm{B}}\lambda_{\mathrm{B}} + (\lvert\zeta\rvert^2 - 1)\lambda_{\mathrm{B}}) + \Psi_3^0\lambda^{-2}(2^{1/2}\eth_{\mathrm{B}}\lambda_{\mathrm{B}} + \zeta\lambda_{\mathrm{B}})(2^{1/2}\bar{\zeta}\eth_{\mathrm{B}}\lambda_{\mathrm{B}} - \lambda_{\mathrm{B}})$
A_{33}	$2\Psi_1^0\bar{\lambda}^{-2} + 2^{3/2}\Psi_2^0\lvert\lambda\rvert^{-2}(2^{1/2}\eth_{\mathrm{B}}\lambda_{\mathrm{B}} + \zeta\lambda_{\mathrm{B}}) + \Psi_3^0\lambda^{-2}(2^{1/2}\eth_{\mathrm{B}}\lambda_{\mathrm{B}} + \zeta\lambda_{\mathrm{B}})^2$

$$\begin{bmatrix} P^{00'} & P^{01'} \\ P^{10'} & P^{11'} \end{bmatrix} = 2^{-1/2}\begin{bmatrix} P^t + P^z & P^x + iP^y \\ P^x - iP^y & P^t - P^z \end{bmatrix} = \begin{bmatrix} A_{12} & A_{13} \\ -A_{02} & -A_{03} \end{bmatrix}. \tag{59}$$

VI. ANGULAR MOMENTUM

In special relativity, the angular momentum is a position-dependent skew tensor field $M_{ab}(x)$, or equivalently a spinor $\mu_{A'B'}(x)$, where $M_{ab}(x) = \mu_{A'B'}(x)\epsilon_{AB} + \bar{\mu}_{AB}(x)\epsilon_{A'B'}$. In general relativity, as is well known, there is no good set of "asymptotic origins" for the measurement of angular momentum, and thus the special-relativistic treatment does not apply.

A good treatment of angular momentum in general relativity is possible, if we adjust our perspective a bit. We first note that if $\boldsymbol{\gamma}$ is a null geodesic in Minkowski space and $\pi_{A'}$ is a tangent spinor to it, then the component $\mu^{A'B'}(x)\pi_{A'}\pi_{B'}$ of the angular momentum is (by the change-of-origin formula) independent of x, as long as x lies on $\boldsymbol{\gamma}$. Thus we may think of the angular momentum either as a spinor-valued function on space-time, or as $\mu(\boldsymbol{\gamma}, \pi_{A'}) = \mu^{A'B'}(x)\pi_{A'}\pi_{B'}$, a function on the space of null geodesics together with their tangent spinors. While angular momentum does not extend to general relativity as a spinor-valued function, it does extend as a function of the null geodesic and the tangent spinor. Indeed, the expression is very simple: we have

$$\mu(\boldsymbol{\gamma}, \pi_{A'}) = (2i)^{-1}A_{\alpha\beta}Z^\alpha Z^\beta, \tag{60}$$

where $Z \leftrightarrow (\boldsymbol{\gamma}, \pi_{A'})$ is the real twistor defined by the null geodesic $\boldsymbol{\gamma}$ and the tangent spinor $\pi_{A'}$.

While the general-relativistic angular momentum is thus an extension of the special-relativistic concept, some features which are prominent on the special-relativistic case do not extend to general relativity, and others which are usually viewed as secondary become central in the general-relativistic setting.

The root of this is that the general-relativistic angular momentum is defined on the space $\{(\boldsymbol{\gamma}, \pi_{A'})\}$ of null geodesics together with their tangent spinors. In the asymptotic regime, this space naturally fibers over the space $\mathbb{S}_{A'}$ of spinors, because there is a well-defined asymptotic spin space. This contrasts with the usual view of $\mu^{A'B'}(x)\pi_{A'}\pi_{B'}$ being defined on the spin bundle $\{(x, \pi_{A'})\}$ of Minkowski space, where the *base* space is Minkowski space and the *fibers* are copies of $\mathbb{S}_{A'}$. In practical terms, this means that while the component of the angular momentum $\mu(\boldsymbol{\gamma}, \pi_{A'})$ in a direction corresponding to $\boldsymbol{\gamma}$ and $\pi_{A'}$ will be well defined, there will be no natural way of simultaneously varying $\boldsymbol{\gamma}$ and $\pi_{A'}$ so the angular momentum is specified by a pure $j = 1$ representation $\mu^{A'B'}$ of the Lorentz group; the angular momentum will inevitably (if a magnetic part of the shear is present) have $j \geq 2$ parts as well.

The reason for the appearance of these $j \geq 2$ components is that general relativity unifies the "ordinary" ($j = 1$) angular momentum with gravitational radiation. The $j \geq 2$ parts of the angular momentum correspond exactly to the shear (times the Bondi mass). Because there is no split of the angular momentum into $j = 1$ and $j \geq 2$ parts with the appropriate geometric invariance (invariance under the Bondi-Metzner-Sachs group), one must, to get an invariant theory, consider all of the $j \geq 1$ parts of the angular momentum.

While the angular momentum does depend on the pairs $(\boldsymbol{\gamma}, \pi_{A'})$, the dependence is not arbitrary: there is a general-relativistic analog of the change-of-origin formula, which says that the angular momenta at different points in a fiber differ by appropriate multiples of the components of the energy-momentum. Thus the full information in the angular momentum can be recovered by choosing any cross section $\boldsymbol{\gamma}(\pi_{A'})$ of the fibration and computing $\mu(\boldsymbol{\gamma}(\pi_{A'}), \pi_{A'})$ as $\pi_{A'}$ varies. Because this is a homogeneous function (of degree 2) in $\pi_{A'}$, the essential information in the angular momentum is that in one spin weight minus one function on the sphere.

It is natural for us to choose the cross section to be given by the congruence of null geodesics meeting the cut of null infinity orthogonally (that is, the congruence specified by l^a). This congruence then serves as a sort of origin for the computation. (However, the present prescription differs essentially from previous attempts to use cuts as origins.) Then the electric and magnetic parts of $\mu(\boldsymbol{\gamma}(\pi_{A'}), \pi_{A'})$ represent the energy moments and spatial angular momentum, respectively, of the system with respect to the asymptotic laboratory frame.

We will also want a general-relativistic extension of the "polarized" form $\mu^{A'B'}\pi_{A'}\acute{\pi}_{B'}$. This corresponds to a two-point function

$$\mu((\boldsymbol{\gamma}, \pi_{A'}), (\acute{\boldsymbol{\gamma}}, \acute{\pi}_{A'})) = (2i)^{-1}A_{\alpha\beta}Z^\alpha \acute{Z}^\beta \tag{61}$$

on twistor space, where $Z \leftrightarrow (\boldsymbol{\gamma}, \pi_{A'})$, $\acute{Z} \leftrightarrow (\acute{\boldsymbol{\gamma}}, \acute{\pi}_{A'})$. In special relativity we would have $(2i)^{-1}A(Z, \acute{Z}) = \mu^{A'B'}(x_{\text{av}})\pi_{A'}\acute{\pi}_{B'}$, where x_{av} is any point on the world line defined by "averaging" the geodesics $\boldsymbol{\gamma}$, $\acute{\boldsymbol{\gamma}}$ with respect to the energy-momentum [$x_{\text{av}} = (x + \acute{x})/2$ where $x \in \boldsymbol{\gamma}$, $\acute{x} \in \acute{\boldsymbol{\gamma}}$ satisfy $(x^a - \acute{x}^a)P_a = 0$] [9]. While in general relativity there is no similar invariant notion of averaging null geodesics, that is a limitation only on interpreting the origin of the angular momentum in direct space-time terms and not on its well definition as a conserved quantity.

If we fix a cross section of twistor space, then the angular momentum $\mu((\boldsymbol{\gamma}(\pi_{A'}), \pi_{A'}), (\boldsymbol{\gamma}(\acute{\pi}_{A'}), \acute{\pi}_{A'}))$ can be thought of as a two-point function on the sphere. However, the essential information in it corresponds to functions of one point on the sphere, not two. This is because the condition (54) implies that the higher-j terms enter only in tensor products with $j = 1/2$, $s = -1/2$ representations; that is,

the angular momentum is in fact the symmetrized tensor product of a single spin weight minus one-half function with an ordinary spinor.

The intrinsic spin may be computed by passing to a boosted frame in which the time axis is aligned with the Bondi-Sachs energy-momentum; then the magnetic part of the angular momentum is the spin. (The electric part of the angular momentum in this frame has a natural interpretation, too. The twistorial construction makes the cut appear as if it were a supertranslated cut in a stationary spacetime; the electric part is the Bondi mass times this supertranslation.)

A. Reporting the angular momentum

Besides the need to accommodate $j \geq 1$ representations, there is another issue to address in choosing how to report the angular momentum, which is a trade-off between invariance and intuitive familiarity. This issue already occurs in special relativity, where the angular momentum is invariantly an element of the $j = 1$, $s = -1$ representation, but when a reference frame is chosen we usually think of it as two spatial vectors (really the elements of a complex $j = 1$, $s = 0$ representation). We shall opt for the familiar presentation, so that we can speak of the spatial angular momentum and energy-moment (both with contributions for $j \geq 1$) parts of the full angular momentum.

Consider for a moment special relativity. Let t^a be a unit future-pointing timelike vector, and let z^a be a unit spacelike vector orthogonal to it. (For this paragraph only it will be convenient to regard z^a as a variable direction on the sphere.) Then $2^{-1/2}(t^a + z^a)$ is a future-directed null vector, say $\bar{\pi}^A \pi^{A'}$ (normalized by $t_{AA'} \bar{\pi}^A \pi^{A'} = 2^{-1/2}$), and $t_A{}^{(A'} \pi^{B')} \bar{\pi}^A = t^{A(A'} z^{B')B} \epsilon_{AB}$. It follows that

$$\begin{aligned} 2^{1/2} \mu_{A'B'} t_A{}^{A'} \bar{\pi}^A \pi^{B'} &= M_{ab} t^{A(A'} z^{B')B} \\ &= M_{ab}(1/2)(t^a z^b - (i/2)\epsilon^{abcd} t_c z_d) \\ &= (1/2)(M_{tz} - iM_{xy}), \end{aligned} \tag{62}$$

where (x, y, z) form a right-handed spatial triad. Thus, having fixed t^a, as $\pi_{A'}$ varies, one gets the energy moment M_{tz} and spatial angular momentum M_{xy} in the direction $z^{AA'} = 2^{1/2} \bar{\pi}^A \pi^{A'} - t^{AA'}$ it determines.

We shall do the same thing in general relativity. We take for t^a the time direction determined by the asymptotic laboratory frame, we allow $Z^2 = \pi_{0'}$, $Z^3 = \pi_{1'}$ to vary (normalized to $|Z^2|^2 + |Z^3|^2 = 1$), and we take $\acute{\pi}_{A'} = 2^{1/2} t_{AA'} \bar{\pi}^A$, that is $\acute{\pi}_{0'} = \bar{\pi}_1$, $\acute{\pi}_{1'} = -\bar{\pi}_0$. With these restrictions $\mu(\pi_{A'}) = \mu((\boldsymbol{\gamma}(\pi_{A'}), \pi_{A'}), (\boldsymbol{\gamma}(\acute{\pi}_{A'}), \acute{\pi}_{A'}))$ becomes a spin-weight zero function on the sphere, with $\mu + \bar{\mu}$ giving the energy moment, and $i\mu - i\bar{\mu}$ giving the spatial angular momentum, in the direction determined by $\pi_{A'}$ (and t^a). These may be reported as real functions on the sphere, or resolved into spherical harmonics.

(Of course, there is some freedom in choosing how to extend the terminology appropriate to a purely $j = 1$ quantity to a $j \geq 1$ one. For instance, what one chooses to call the $j \geq 2$ energy moments and spatial angular momenta could be taken to be some function of j times the ones used here. Such differences are unimportant here, since what we are interested in is simply extracting the invariant information.)

B. Derivation of the formula

The twistorial formula for the angular momentum is simply

$$\mu = (2i)^{-1} A_{\alpha\beta} Z^\alpha \acute{Z}^\beta, \tag{63}$$

where $A_{\alpha\beta}$ is the kinematic twistor introduced earlier and Z^α, $\acute{Z}^\alpha$ are twistors whose null geodesics meet the cut of null infinity orthogonally (and satisfy $|Z^2|^2 + |Z^3|^2 = 1$, $\acute{Z}^2 = \overline{Z^3}$, and $\acute{Z}^3 = -\overline{Z^2}$). The condition that Z^α (say) meets the cut at a point is that the fields $(\tilde{\omega}^0_{\mathrm{B}}, \omega^1_{\mathrm{B}})$ vanish there; that the meeting be orthogonal means that the tangent spinor to the geodesic must lie in the $o_{A'}$ direction.

Let the point in question on the cut have stereographic coordinate ζ. Then from the formulas (41)–(44) for the twistor fields, we deduce that the conditions for $\boldsymbol{\gamma}$ to meet the cut are

$$\zeta = -\overline{Z^3/Z^2}, \tag{64}$$

$$\lambda_{\mathrm{B}}(|Z^2|^2 + |Z^3|^2) = i(Z^0 \overline{Z^2} + Z^1 \overline{Z^3}). \tag{65}$$

The component of the tangent spinor in the $\iota^{A'}_{\mathrm{B}}$ direction, which we require to vanish, was computed in Eq. (29) of Ref. [9]. It is

$$\begin{aligned} &(|Z^2|^2 + |Z^3|^2)^{-1/2} |Z^2|^{-1} (i(\eth'_{\mathrm{B}} \lambda_{\mathrm{B}})(|Z^2|^2 + |Z^3|^2) Z^2 \\ &\quad + 2^{-1/2}(Z^0 Z^3 - Z^1 Z^2)\overline{Z^2}), \end{aligned} \tag{66}$$

where $\eth'_{\mathrm{B}} \lambda_{\mathrm{B}}$ is evaluated at (64).

After a little algebra, we find the equations for Z^0, Z^1 in terms of $Z^2 = \pi_{0'}$, $Z^3 = \pi_{1'}$:

$$Z^0 = -i(2^{1/2} Z^2 \overline{Z^3} \eth'_{\mathrm{B}} \lambda_{\mathrm{B}} + |Z^2|^2 \lambda_{\mathrm{B}})/\overline{Z^2}, \tag{67}$$

$$Z^1 = i(2^{1/2} Z^2 \eth'_{\mathrm{B}} \lambda_{\mathrm{B}} - Z^3 \lambda_{\mathrm{B}}), \tag{68}$$

where λ_{B} and $\eth'_{\mathrm{B}} \lambda_{\mathrm{B}} = 2^{-1/2}(1 + |\zeta|^2) \partial \lambda_{\mathrm{B}} / \partial \zeta$ are evaluated at ζ. Equations (67) and (68) determine the cross section of twistor space as a bundle over spin space.

The same analysis applies, of course, with Z^α replaced by $\acute{Z}^\alpha$. We note that $\acute{\zeta} = -1/\bar{\zeta}$ is the point antipodal to ζ on the sphere, and the quantities λ_{B}, $\eth_{\mathrm{B}} \lambda_{\mathrm{B}}$ appearing in the formulas for $\acute{Z}^0$, $\acute{Z}^1$ must be evaluated at this antipodal point.

Substituting these formulas into Eq. (63) and collecting like terms, we find (with the normalization $|Z^2|^2 + |Z^3|^2 = 1$) that

$$\mu = 2^{-1}(1+|\zeta|^2)^{-1}\{-[A_{13}+(A_{12}+A_{03})\zeta+A_{02}\zeta^2]2^{1/2}\eth'_{\mathrm{B}}\lambda_{\mathrm{B}}(\zeta,\bar{\zeta})+[A_{03}+A_{02}\zeta-A_{13}\bar{\zeta}-A_{12}|\zeta|^2]\lambda_{\mathrm{B}}(\zeta,\bar{\zeta})$$
$$+[A_{02}-(A_{03}+A_{12})\bar{\zeta}+A_{13}\bar{\zeta}^2](\zeta/\bar{\zeta})2^{1/2}\eth'_{\mathrm{B}}\lambda_{\mathrm{B}}(-1/\bar{\zeta},-1/\zeta)+[A_{12}+A_{02}\zeta-A_{13}\bar{\zeta}-A_{03}|\zeta|^2]\lambda_{\mathrm{B}}(-1/\bar{\zeta},-1/\zeta)$$
$$+iA_{22}\zeta-iA_{23}(|\zeta|^2-1)-iA_{33}\bar{\zeta}\}. \quad (69)$$

[Here we have followed the standard convention of writing a nonholomorphic function f of the complex coordinate ζ as $f(\zeta,\bar{\zeta})$.] While the formula is lengthy, this is mostly due to the appropriate factors of ζ, $\bar{\zeta}$ for weighting the components $A_{\alpha\beta}$ of the kinematic twistor; the only complicated expressions are those involving λ_{B}, which encode the shear.

Thus Eq. (69) allows one to report the angular momentum as a function of the direction (with $2\mathrm{Re}\mu$ giving the energy moment in the direction, and $-2\mathrm{Im}\mu$ the spatial angular momentum about the axis, specified by ζ).

If the resolution of λ_{B} in spherical harmonics is known, one can use it to find the resolution of μ into spherical harmonics, by identifying the explicit functions of ζ in (69) with particular spin-weighted harmonics and applying tensor product formulas ("Clebsch-Gordan decompositions"). The computation is lengthy but straightforward using the formulas derived in the Appendix; we find

$$\mu = \sum \hat{\mu}_{j,m\,0}Y_{j,m} + (2i)^{-1}\sqrt{2\pi/3}(A_{22\,0}Y_{1,1} + 2^{1/2}A_{23\,0}Y_{1,0} + A_{33\,0}Y_{1,-1}), \quad (70)$$

where the last three terms are the λ_{B}-independent ones, and the coefficient $\hat{\mu}_{j,m}$ is a sum of terms, each of which is $\lambda_{j'm'}$ for $j' = j-1, j, j+1$, $m' = m-1, m$, and $m+1$ times a factor; these are given in Table V (for j even) and Table VI (for j odd).

TABLE V. Terms contributing to the λ_{B}-dependent part of the angular momentum proportional to ${}_0Y_{jm}$ for *even* j. Each term is the product of the component $\lambda_{j',m'}$ in the left column by the quantity in the right column. (We understand the term is zero unless $j' = 0, 1, 2, \ldots, m' = -j', \ldots, j'$.)

Component $\lambda_{j',m'}$	Factor it multiplies
$\lambda_{j,m-1}$	$2^{-1}A_{02}((j-m+1)(j+m))^{1/2}$
$\lambda_{j,m}$	$-2^{-1}(A_{03}+A_{12})m$
$\lambda_{j,m+1}$	$-2^{-1}A_{13}((j+m+1)(j-m))^{1/2}$

VII. EVOLUTION

The usefulness of a definition of energy-momentum or angular momentum in radiation problems depends considerably on whether it admits a well-defined notion of evolution. At null infinity, it is well known that the Bondi-Sachs energy-momenta at two cuts can be compared. Many proposed definitions of angular momentum at null infinity took values in cut-dependent spaces, making tracking their evolution problematic; the twistor-based definition solves this problem. Here, however, because we are dealing with energy-momentum and angular momentum at large but *finite* spheres $\mathcal{S}$, we must take up the problem anew, both for energy-momentum and angular momentum.

Suppose we have a one-parameter family of surfaces $\mathcal{S}(\eta)$ (for η in some interval J) foliating a timelike surface T, with η increasing toward the future (precisely, we require $v^a\nabla_a\eta > 0$ for every future-causal vector v^a tangent to T). We may compute the energy-momentum and angular momentum on each of these; the difficulty is that these quantities are naturally defined on the null infinities $\mathcal{I}^+(\eta)$ of the different auxiliary space-times $M(\eta)$ [each defined by taking $\Psi_0 = 0$ along the null hypersurface $\mathcal{N}(\eta)$ outward from $\mathcal{S}(\eta)$]. In order to compare the energy-momenta and angular momenta for different η, then, we must find a natural way of identifying the null infinities $\mathcal{I}^+(\eta)$ for different η.

We could express this as a problem of finding transition functions. The constructions above determine a preferred Bondi coordinate system $(\zeta_\eta, \overline{\zeta_\eta}, u_\eta)$ on $\mathcal{I}^+(\eta)$. [We have shown how to fix, for each $\mathcal{S}(\eta)$, a complex stereographic coordinate $\zeta = \zeta_\eta$ on $\mathcal{I}^+(\eta)$, and we have chosen an associated Bondi-Sachs frame by fixing the factor P_{B}. This determines the Bondi retarded time $u = u_\eta$ up to a

TABLE VI. Terms contributing to the λ_{B}-dependent part of the angular momentum proportional to ${}_0Y_{jm}$ for *odd* j. Each term is the product of the component $\lambda_{j',m'}$ in the left column by the quantity in the right column. (We understand the term is zero unless $j' = 0, 1, 2, \ldots, m' = -j', \ldots, j'$.)

Component $\lambda_{j',m'}$	Factor it multiplies
$\lambda_{j-1,m-1}$	$2^{-1}A_{02}(j-2)((j+m)(j-m+1)/(4j^2-1))^{1/2}$
$\lambda_{j+1,m-1}$	$2^{-1}A_{02}(j+1)((j-m+2)(j-m+1)/(4(j+1)^2-1))^{1/2}$
$\lambda_{j,m}$	$2^{-1}(A_{03}-A_{12})$
$\lambda_{j-1,m+1}$	$2^{-1}A_{13}(j-2)((j-m)(j-m-1)/(4j^2-1))^{1/2}$
$\lambda_{j+1,m+1}$	$2^{-1}A_{13}(j+1)((j+m+2)(j+m+1)/(4(j+1)^2-1))^{1/2}$

supertranslation; we fix this by requiring the preferred cut $\mathcal{C}(\eta)$ of $\mathcal{I}^+(\eta)$—the limit of $\mathcal{N}(\eta)$—have $u_\eta = 0$.] Then we wish to find formulas for $(\zeta_{\eta_2}, \overline{\zeta_{\eta_2}}, u_{\eta_2})$ in terms of $(\zeta_{\eta_1}, \overline{\zeta_{\eta_1}}, u_{\eta_1})$.

Structures preserved by the identifications.—I will discuss the way these identifications are made shortly. More important, though, is the question, what structures are preserved by the identifications? If the surfaces $\mathcal{S}(\eta)$ were actually cuts of the null infinity of the physical space-time, we should expect the usual structures to be preserved and the identifications to be Bondi-Metzner-Sachs transformations. However, here we must expect a weaker structure due to finite-size effects. The extent to which these effects are significant should be interpreted as the extent to which the extraction surfaces $\mathcal{S}(\eta)$ are insufficiently distant to capture the full radiative structure (or, if the effects persist for arbitrarily large surfaces, the extent to which the Bondi-Sachs asymptotics fail for the physical space-time). Even with this interpretation, though, in order to precisely quantify the finite-size effects we must work out the structure in general.

The usual intrinsic structure of null infinity may be regarded as determined by three elements: its set of generators, each with an affine structure; a conformal structure on the set of generators (it is a remarkable feature of the construction that the set does have a well-defined conformal structure); and the ''strong conformal geometry'' (which links the scales of vectors up the generators with the scales of the area forms transverse to them). In our case, two of these three elements survive: there are natural invariant definitions of the generators and of the strong conformal geometry, but the conformal structure on the space of generators is *not* preserved under the identifications.

A priori, while for each value of η each generator of $\mathcal{I}^+(\eta)$ has an affine structure, it is not evident that there is a preferred way of identifying these for different values of η. However, because the strong conformal geometry and the space of generators are well defined, the identifications will extend in a natural way to vectors tangent to the generators. This means that supertranslations are well defined. We shall see that there is a natural way of measuring the supertranslation relating $u_{\eta_1} = 0$ to $u_{\eta_2} = 0$, and this will allow us to appropriately account for the change of section of the twistor space when we compare angular momentum.

The failure of an invariant conformal structure to exist on the space of generators (that is, to be preserved under evolution) turns out to mean that in comparing the energy-momenta at different cuts higher-j representations appear.

To see this, let us first recall that, since the space of generators has naturally the smooth structure of an oriented sphere, a conformal structure on it is equivalent to a complex structure. In the Bondi-Sachs case, the transition functions $(\zeta_{\eta_1}, \overline{\zeta_{\eta_1}}) \mapsto (\zeta_{\eta_2}, \overline{\zeta_{\eta_2}})$ preserve this complex structure, and are thus fractional linear transformations. On the other hand, these fractional linear transformations are isomorphic to the (proper, isochronous) Lorentz transformations. Thus in the Bondi-Sachs case, the admissible coordinate changes induce Lorentz transformations on the space of generators. One builds up the spaces of asymptotically constant spinors, vectors, etc., as functions on the space of generators, and it is the fact that the conformal transformations induce Lorentz motions which is responsible for these fields breaking neatly into Lorentz-invariant representations.

Now let us turn to the present, non-Bondi-Sachs case. If we compute the component $P(\eta_1; \zeta_{\eta_1}, \overline{\zeta_{\eta_1}})$ of the energy-momentum at η_1 along the null vector determined by the Bondi stereographic coordinates $(\zeta_{\eta_1}, \overline{\zeta_{\eta_1}})$ in the chart at η_1, we find, as usual, that, as a function of $(\zeta_{\eta_1}, \overline{\zeta_{\eta_1}})$, the energy-momentum consists of $j = 0$ and $j = 1$ components, forming a covector. However, if we want to compare the energy-momenta at η_1 and η_2, we must express them both in a common chart, say the chart $(\zeta_{\eta_2}, \overline{\zeta_{\eta_2}})$ on the space of generators for $\mathcal{I}^+(\eta_2)$. Then of course $P(\eta_2; \zeta_{\eta_2}, \overline{\zeta_{\eta_2}})$ will have only $j = 0$ and $j = 1$ components, but, because the change of variables $(\zeta_{\eta_1}, \overline{\zeta_{\eta_1}}) \mapsto (\zeta_{\eta_2}, \overline{\zeta_{\eta_2}})$ will not be a fractional linear transformation, the expression for $P(\eta_1)$ in terms of $(\zeta_{\eta_2}, \overline{\zeta_{\eta_2}})$ will generally contain not just $j = 0$ and $j = 1$ components, but those for all integral j. So comparison of energy-momenta at different surfaces $\mathcal{S}(\eta_1)$, $\mathcal{S}(\eta_2)$, will require higher-j representations. (This sort of behavior occurs even strictly at null infinity for angular momentum, but is a finite-size effect for energy-momentum.)

Again, this potential failure of a linear identification of asymptotic covectors (and elements of the spin-tensor algebra generally) as η changes is a finite-size effect; it will become negligible if one takes the family $\mathcal{S}(\eta)$ of extraction surfaces distant enough (that is, close enough to the physical space-time's null infinity), assuming that the system is indeed isolated. Thus the occurrence of these nonlinearities in a numerical computation would be a signal that the extraction surface had not been taken large enough that a model null infinity, with the usual Bondi-Sachs structure, stable under evolution existed.

Some further, more technical, discussion of structure is given in Sec. VII E.

How the identifications are made.—A few words now about how the identifications of $\mathcal{I}^+(\eta)$ for different η are made. They all grow out of two considerations, which we have already used extensively. The first of these is that at any point $p(\eta) \in \mathcal{S}(\eta)$, there is a null geodesic orthogonally outward in $M(\eta)$ whose end point lies on $\mathcal{I}^+(\eta)$; holding η fixed but varying $p(\eta)$ we get a preferred cut $\mathcal{C}(\eta)$ of $\mathcal{I}^+(\eta)$. The second is that at any $p(\eta) \in \mathcal{S}(\eta)$, there is a canonical isomorphism $T_p(M_{\text{phys}}) \cong T_p(M(\eta))$ between the tangent space of the physical space-time and that of the auxiliary space $M(\eta)$ (cf. footnote 2).

We define the identifications at the infinitesimal level (that is, for infinitesimally separated η), and then integrate. At the infinitesimal level, the problem comes down to understanding how the structures on $\mathcal{I}^+(\eta + d\eta)$ are represented by quantities on $\mathcal{I}^+(\eta)$.

Let us consider a one-parameter family of points $p(\eta) \in \mathcal{S}(\eta)$ (with η the parameter) in M_{phys}. At each point we have the null normal l^a orthogonally outward from $\mathcal{S}(\eta)$, and its associated null geodesic in M_{phys}. As η varies, the vector u^a_{phys} connecting this family of geodesics is a Jacobi field, which in turn is determined by its initial data $(u^a_{\text{phys}}, \dot{u}^b_{\text{phys}})$ at $\mathcal{S}(\eta)$. On the other hand, we may use the isomorphism of tangent spaces to regard these as initial data for a Jacobi field u^a in $M(\eta)$. We will take the limiting value of *this* field at $\mathcal{I}^+(\eta)$ as the *definition* of the rate of change of the end point of the null geodesic orthogonally outwards from $p(\eta)$ as η varies. The important thing to note here is that this vector, while it lies in the tangent space to $\mathcal{I}^+(\eta)$, represents the end point of a geodesic with base point $p(\eta + d\eta)$ for an infinitesimally differing value of η. This is the root of all of the identifications.

We may apply this in several ways. If, for example, we require the curve $p(\eta)$ to be such that the corresponding vector at $\mathcal{I}^+(\eta)$ points along a generator, we may say the generator of null infinity does not change with $p(\eta)$; this gives the identification of the generators of null infinity for different values of η. Or if we imagine a congruence of curves, say $p(\zeta, \bar{\zeta}, \eta)$ with stereographic coordinate ζ, we get a vector at each point of the cut of $\mathcal{I}^+(\eta)$ labeled by ζ, and this vector field over the cut gives a measure of the supertranslation induced by changing η infinitesimally, that is, in passing from $\mathcal{S}(\eta)$ to $\mathcal{S}(\eta + d\eta)$.

Outline of this section.— Section VII A gives the computation of the Jacobi fields, and Sec. VII B the main formulas for comparing the null infinities. Then Secs. VII C and VII D give the formulas for treating the evolution of the energy-momentum and angular momentum. The final section discusses some technical aspects of the structure of the null infinities; these are not needed for the computations in this paper but are given for completeness of the conceptual framework.

Coordinates.—In what follows, we will be comparing structure on the timelike hypersurface T with that on $\mathcal{I}^+(\eta)$, and also structure on $\mathcal{I}^+(\eta_1)$ and $\mathcal{I}^+(\eta_2)$. As a ready reference, here is a summary of the coordinate systems to be used.

Recall that, for each η, we have already defined coordinates $(\zeta, \bar{\zeta}) = (\zeta_\eta, \overline{\zeta_\eta})$ on $\mathcal{S}(\eta)$. We may regard $(\zeta, \bar{\zeta})$ then as defined over the whole of T.

We shall eventually use coordinates $(\eta, \zeta, \bar{\zeta})$ on T. However, in the next section it will be convenient to briefly use coordinates $(\eta, \xi, \bar{\xi})$, where ξ need not be simply related to ζ.

As already indicated, we will have Bondi coordinates $(\zeta_\eta, \overline{\zeta_\eta}, u_\eta)$ on $\mathcal{I}^+(\eta)$.

A. The Jacobi fields

We have a family of spacelike surfaces $\mathcal{S}(\eta)$ forming a timelike hypersurface T, with $v^a \nabla_a \eta > 0$ for any future-pointing vector v^a tangent to T. In practice, it is convenient to represent the evolution from one surface to the next by a connecting vector field w^a, that is, a field tangent to T with $w^a \nabla_a \eta = 1$. The freedom in choosing w^a is the freedom to add a vector field which, at each η, is tangent to $\mathcal{S}(\eta)$, that is, is a linear combination of m^a and $\bar{m}^a$. We shall see below that there is a natural way to fix this freedom, but for now we leave it unspecified.

It will be helpful to briefly use coordinates adapted to the foliation of T by η and the integral curves of w^a. Near any point of interest on T, fix η_0 and let $(\xi, \bar{\xi})$ be a complex coordinatization of $\mathcal{S}(\eta_0)$. (The use of a complex coordinate is only for brevity of treatment; the coordinate ξ need not be a holomorphic function of ζ, or have any other special relation to it.) We may let the integral curve of w^a with coordinates $(\xi, \bar{\xi})$ at η_0 be $p(\eta, \xi, \bar{\xi})$. If we Lie drag ξ along w^a (so $w^a \nabla_a \xi = 0$), then $(\eta, \xi, \bar{\xi})$ provides a coordinatization of a portion of T. In these coordinates, we have $w^a = \partial/\partial\eta$, and $\partial/\partial\xi$ is a linear combination of m^a and $\bar{m}^a$.

1. Definition of the fields, their initial data

In this section, we work in the physical space-time. To avoid cumbersome notation, however, the Jacobi field is denoted simply u^a, rather than u^a_{phys}.

Along each integral curve $p(\eta, \xi, \bar{\xi})$ of w^a (holding ξ fixed), let $\gamma(\eta, s, \xi, \bar{\xi})$ be the affinely parametrized null geodesic outward from $\mathcal{S}(\eta)$ in M_{phys}, so that $l^a = \partial_s \gamma(\eta, s, \xi, \bar{\xi})$ and $\gamma(\eta, 0, \xi, \bar{\xi}) = p(\eta, \xi, \bar{\xi})$. (The scale of the affine parametrization will not matter.) Then differentiating along η we get a Jacobi field $u^a = \partial_\eta \gamma(\eta, s, \xi, \bar{\xi})$ connecting these geodesics.

It is this Jacobi field u^a we wish to work out; more precisely, we wish to work out u^a modulo terms proportional to l^a. The field is determined by its initial data. One of those data is simply $u^a|_{s=0} = w^a$; the other is $l^b \nabla_b u^a|_{s=0}$. One constraint on this second datum is that $l_a l^b \nabla_b u^a = 0$. [This follows by differentiating $l_a l^a = 0$: we have $0 = u^b \nabla_b (l^a l_a) = 2 l_a u^b \nabla_b l^a = 2 l_a l^b \nabla_b l^a$.] The other constraint (affecting one complex degree of freedom) comes from requiring that the geodesics meet $\mathcal{S}(\eta)$ orthogonally.

The condition that the geodesics $\gamma(\eta, s, \xi, \bar{\xi})$ meet $\mathcal{S}(\eta)$ orthogonally is that $(\partial_\xi p)^a l_a = 0$. If we differentiate this along w^a, and we apply the conditions $w^b \nabla_b (\partial_\xi p)^a = (\partial_\xi p)^b \nabla_b w^a$ [which holds since $w^a = \partial/\partial\eta$ in the $(\eta, \xi, \bar{\xi})$ coordinates on T] and $w^b \nabla_b l^a = u^b \nabla_b l^a = l^b \nabla_b u^a$ (which holds because u^a is a connecting vector for the geodesics with tangents l^a), we find

$$((\partial_\xi p)^b \nabla_b w^a) l_a + (\partial_\xi p)_a l^b \nabla_b u^a = 0. \tag{71}$$

However, since $(\partial_\xi p)^b$ is a complex basis vector spanning the tangent space to $\mathcal{S}(\eta)$, so the equation is equivalent to one with this vector replaced by any other complex basis vector for this tangent space; in particular, it is equivalent to

$$(m^b\nabla_b w^a)l_a + m_a l^b\nabla_b u^a = 0, \tag{72}$$

which determines the complex datum $m_a l^b\nabla_b u^a|_{s=0}$.

We shall not need the coordinate ξ in what follows.

2. Solving the Jacobi equation

We have derived the initial data for the Jacobi fields on the physical space-time M_{phys}. We now make use of the isomorphism of tangent spaces at $\mathcal{S}(\eta)$ to regard these same data as determining Jacobi fields in the auxiliary space-time $M(\eta)$, and solve the Jacobi equation there. [Since this isomorphism is $u^a_{\text{phys}} \mapsto u^a$ and we have already dropped the "phys" subscript, this amounts to simply using the formulas derived above for their data at $\mathcal{S}(\eta)$.] Ultimately we are interested in the vectors on $\mathcal{I}^+(\eta)$ determined by the asymptotic forms of the Jacobi fields.

In this section, we work in a frame parallel propagated along the null geodesics, and express the Jacobi fields in terms of their initial data. In the next one, we will transform the asymptotic form of the Jacobi field to the Bondi frame.

It will be convenient to put

$$u^a = u^{00'}l^a + u^{01'}m^a + u^{10'}\bar{m}^a + u^{11'}n^a. \tag{73}$$

Then the constraints we have worked out above will be the initial conditions for the geodesic deviation equation:

$$u^a|_{s=0} = w^a \quad \text{and} \quad l^a\nabla_a u^{10'}|_{s=0} = l^b m^a \nabla_a w_b. \tag{74}$$

Recall that w^a is the vector field connecting $\mathcal{S}(\eta)$ to $\mathcal{S}(\eta + d\eta)$, so the quantities on the right-hand side in Eq. (74) are known. As noted above, that the Jacobi field represents a null geodesic entails additionally

$$l^a\nabla_a u^{11'} = 0. \tag{75}$$

The geodesic deviation equation itself ($l^a\nabla_a l^b\nabla_b u^c = l^p l^q R_{prq}{}^c u^r$) becomes in terms of the components

$$\ddot{u}^{00'} = \Psi_1 u^{01'} + \Psi_2 u^{11'} + \text{conjugate}, \tag{76}$$

$$\ddot{u}^{10'} = -\Psi_1 u^{11'}, \tag{77}$$

$$\ddot{u}^{11'} = 0, \tag{78}$$

where the dots are differentiation with respect to s. [Here of course the components of u^a and also the curvatures Ψ_1, Ψ_2 are evaluated at points along the geodesic, not at $\mathcal{S} = \mathcal{S}(\eta)$; we temporarily violate the convention that quantities unsubscripted by NU or B are evaluated at $\mathcal{S}$.] Integrating these with the initial conditions [using the explicit form of Ψ_1 provided by integrating (6) using (4)], we find

$$u^{11'}(s) = w^{11'}, \tag{79}$$

and

$$\begin{aligned} u^{10'}(s) = w^{10'} + s l^b m^a \nabla_a w_b + w^{11'}\Psi_1 (4|\sigma|^2)^{-1}\{&-\log(1 \\ &- s(\rho + |\sigma|))(1 - s(\rho - |\sigma|)) - 2\rho s \\ &- |\sigma|^{-1}(\rho - |\sigma|)(1 - s(\rho + |\sigma|)\log(1 - s(\rho \\ &+ |\sigma|)) + |\sigma|^{-1}(\rho + |\sigma|)(1 - s(\rho - |\sigma|)\log(1 \\ &- s(\rho - |\sigma|))\}, \end{aligned} \tag{80}$$

where, on the right-hand side, the spin coefficients and the curvature Ψ_1 (as well as the field w^a) are evaluated at $\mathcal{S}(\eta)$ (we restore the convention about subscripting). We note the asymptotic behavior

$$u^{10'} \sim s(l^b m^a \nabla_a w_b + w^{11'} X), \tag{81}$$

where

$$\begin{aligned} X = \Psi_1 \Big(&-2^{-1}|\sigma|^{-2}\rho + 4^{-1}|\sigma|^{-3}(\rho^2 - |\sigma|^2) \\ &\times \log\frac{\rho + |\sigma|}{\rho - |\sigma|}\Big). \end{aligned} \tag{82}$$

[In the limit $|\sigma| \ll |\rho|$, we have $X \approx -(3\rho)^{-1}\Psi_1$.]

We will not need $u^{00'}$.

3. Transformation to the Bondi frame

In the previous section, we found the Jacobi field (modulo terms tangent to the geodesic) in a parallel-propagated frame. We here transform the asymptotic form of this field to the Bondi frame previously constructed for $M(\eta)$.

We have

$$u^a \text{ modulo } l^a = u^{01'}m^a + u^{10'}\bar{m}^a + u^{11'}n^a. \tag{83}$$

Now m^a differs from m^a_{NU} by a term proportional to l^a, and so we may replace m^a by m^a_{NU} in this expression. As $s \to \infty$, we may also, to leading order, replace n^a by n^a_{NU}. To see this, first note that we have seen that $u^{01'}$, $u^{10'} = O(s)$, and $u^{11'} = O(1)$ as $s \to \infty$. On the other hand, we have $n^a = n^a_{\text{NU}} - Qm^a - \bar{Q}\bar{m}^a + Q\bar{Q}l^a$. Since Q is $O(1)$, making this substitution in Eq. (83) would only change the coefficients of m^a, $\bar{m}^a$ (or m^a_{NU}, $\bar{m}^a_{\text{NU}}$) by subdominant terms. Thus

$$
\begin{aligned}
u^a \text{ modulo } l^a &\sim u^{01'} m^a_{\mathrm{NU}} + u^{10'} \bar{m}^a_{\mathrm{NU}} + u^{11'} n^a_{\mathrm{NU}} \\
&\sim u^{01'} P \frac{\partial}{\partial \overline{\zeta_\eta}} + u^{10'} \bar{P} \frac{\partial}{\partial \zeta_\eta} + u^{11'} n^a_{\mathrm{NU}} \\
&\sim -(l^b \bar{m}^a \nabla_a w_b + w^{11'} \bar{X}) \rho^{-1} (\delta \bar{\zeta}) \frac{\partial}{\partial \overline{\zeta_\eta}} \\
&\quad - (l^b m^a \nabla_a w_b + w^{11'} X) \rho^{-1} (\delta' \zeta) \frac{\partial}{\partial \zeta_\eta} \\
&\quad + w^{11'} |\lambda|^2 \frac{\partial}{\partial u_\eta}. \qquad (84)
\end{aligned}
$$

Here $\partial/\partial u_\eta = n^a_{\mathrm{B}}$, where u_η is a Bondi retarded time coordinate for $\mathcal{I}^+(\eta)$ adapted to the frame defined by $(\zeta_\eta, \overline{\zeta_\eta})$ (and P_{B}).

Equation (84) represents the displacement, in $\mathcal{I}^+(\eta)$, of the cut formed from the null vectors orthogonally outward, as the two-surface moves from $\mathcal{S}(\eta)$ to $\mathcal{S}(\eta + d\eta)$ along the vector field w^a. It thus codes the relation between the null infinities $\mathcal{I}^+(\eta)$ and $\mathcal{I}^+(\eta + d\eta)$; our next task is to develop this into formulas for transition functions.

B. Comparison of null infinities

We now have the tools to compare the null infinities $\mathcal{I}^+(\eta)$ associated with different values of η. We recall that (for each η) the invariant structures of $\mathcal{I}^+(\eta)$ are its space of generators, the conformal structure on that space, and the strong conformal geometry; we shall see that the first and last of these can be identified under changes of η, but not the conformal structure on the space of generators.

1. Identification of the space of generators

We have a family of two-surfaces $\mathcal{S}(\eta)$, and for each of these the null geodesics outward determine a cut of the corresponding null infinity $\mathcal{I}^+(\eta)$. We saw in the last section, however, that we could represent the cut of an $\mathcal{S}(\eta + d\eta)$ infinitesimally perturbed from $\mathcal{S}(\eta)$ by a vector field in $\mathcal{I}^+(\eta)$. Precisely, if w^a was a vector field at $\mathcal{S}(\eta)$ with $w^a \nabla_a \eta = 1$, then Eq. (84) gave the corresponding apparent displacement of the cut.

We may use this to identify the generators of $\mathcal{I}^+(\eta)$ for different values of η. The connecting field w^a will preserve the generator if the corresponding field at $\mathcal{I}^+(\eta)$ points purely up the generator, which, from Eq. (84), is evidently if

$$
l^b m^a \nabla_a w_b + w^{11'} X = 0. \qquad (85)
$$

Expanding $l^b m^a \nabla_a w_b$ in spin coefficients we find

$$
\eth w^{11'} - \sigma w^{01'} - \rho w^{10'} = w^{11'} X. \qquad (86)
$$

Here $w^{11'} = w^a l_a$ depends only on the displacement of $\mathcal{S}(\eta + d\eta)$ relative to $\mathcal{S}(\eta)$; it is insensitive to the horizontal components of w^a, which are $w^{01'}$ and its conjugate. We may thus use (86) as an equation to determine the horizontal components of w^a from the condition that (81) vanishes. After a little algebra, we find

$$
w^{10'} = (\rho^2 - |\sigma|^2)^{-1} \begin{bmatrix} \rho & -\sigma \end{bmatrix} \begin{bmatrix} \eth w^{11'} + X w^{11'} \\ \eth' w^{11'} + \bar{X} w^{11'} \end{bmatrix}. \qquad (87)
$$

We may therefore determine a vector field w^a on T by requiring $w^a \nabla_a \eta = 1$ and its components tangential to $\mathcal{S}(\eta)$ to be given by Eq. (87). Each integral curve of this vector field corresponds to a generator of null infinity, in the sense that under the identification of the $\mathcal{I}^+(\eta)$ for different η's described here, the null geodesics orthogonally outward from $\mathcal{S}(\eta)$ along this curve are all considered to strike the same generator.

An equivalent way of expressing this is in terms of transition functions for the angular coordinates. Let us write ζ_{η_0} for the stereographic coordinate on $\mathcal{I}^+(\eta_0)$ determined by restricting the stereographic coordinate ζ to $\mathcal{S}(\eta_0)$ [and identifying the cut of $\mathcal{I}^+(\eta_0)$ with $\mathcal{S}(\eta_0)$ by using the ideal end points of the null geodesics orthogonally outward]. We may then extend ζ_{η_0} to $\mathcal{I}^+(\eta)$ for all η by requiring $w^a \nabla_a \zeta_{\eta_0} = 0$.

Now let $\mathfrak{z}(\zeta_0, \overline{\zeta_0}, \eta_0, \eta)$ be the value of $\zeta = \zeta_\eta$ at $\mathcal{S}(\eta)$ corresponding to the same generator as does the value ζ_0 at $\mathcal{S}(\eta_0)$. Then $\mathfrak{z}(\zeta_0, \overline{\zeta_0}, \eta_0, \eta) = \zeta_\eta \circ \zeta_{\eta_0}^{-1}$ can be regarded as the transition function from ζ_{η_0} to ζ_η, with η, η_0 parametrizing the particular choices of the coordinate function of interest.

The derivative of $\mathfrak{z}$ with respect to η will be the component w^ζ of w^a, in the coordinates $(\eta, \zeta, \bar{\zeta})$, if w^a is chosen to preserve the generators of null infinity. It will be conceptually useful to put $w^a = w^a_{\mathrm{v}} + w^a_{\mathrm{h}}$, where $w^a_{\mathrm{v}} = w^{00'} l^a + w^{11'} n^a$ are the "vertical" components and $w^a_{\mathrm{h}} = w^{01'} m^a + w^{10'} \bar{m}^a$ are the "horizontal" components (with respect to the foliation of T by η and the induced metric). Then $w^\zeta = w^a \nabla_a \zeta = w^\zeta_{\mathrm{v}} + w^\zeta_{\mathrm{h}}$, where $w^\zeta_{\mathrm{v,h}} = w^a_{\mathrm{v,h}} \nabla_a \zeta$. We have

$$
\begin{aligned}
w^a_{\mathrm{h}} &= w^{10'} \bar{m}^a + \text{conjugate} \\
&= -w^{10'} s (\rho M^a + \bar{\sigma} \bar{M}^a) + \text{conjugate} \\
&= -w^{10'} s \left(\rho P \frac{\partial}{\partial \zeta} + \bar{\sigma} \bar{P} \frac{\partial}{\partial \bar{\zeta}} \right) + \text{conjugate} \\
&= -s (w^{10'} \rho + w^{01'} \sigma) P \frac{\partial}{\partial \zeta} + \text{conjugate} \\
&= -(w^{10'} \rho + w^{01'} \sigma) \rho^{-1} \delta \bar{\zeta} \frac{\partial}{\partial \zeta} + \text{conjugate}, \qquad (88)
\end{aligned}
$$

and so $w^\zeta_{\mathrm{h}} = -(w^{10'} \rho + w^{01'} \sigma) \rho^{-1} \delta \bar{\zeta}$. Using Eq. (87), this becomes $w^\zeta_{\mathrm{h}} = -(\eta w^{11'} + X w^{11'}) \rho^{-1} \delta \bar{\zeta}$, and so the transition function $\mathfrak{z}$ is determined by

$$
\frac{d\mathfrak{z}}{d\eta} = -(\eth w^{11'} + X w^{11'}) \rho^{-1} \delta \bar{\zeta} + w^\zeta_{\mathrm{v}}, \qquad (89a)
$$

$$
\mathfrak{z}(\zeta_0, \bar{\zeta}_0, \eta_0, \eta_0) = \zeta_0. \qquad (89b)
$$

(This has been written as a family of ordinary differential equations parametrized by ζ_0, $\overline{\zeta_0}$, and η to emphasize that it is a simple evolution equation; one could write $\partial/\partial\eta$ in place of $d/d\eta$ just as correctly.)

In Eq. (89a), the two terms on the right represent contributions which are, respectively, intrinsic and extrinsic to $\mathcal{S}(\eta)$ in their dependence on the coordinate ζ. The first term can be thought of as representing the rate of deformation of the coordinate ζ due to changes in the geometry of the surface as η is increased. The second may contribute to this but also takes into account the freedom in specifying ζ on successive surfaces. [Recall that ζ is determined only up to a fractional linear transformation by the geometry; it was suggested the remaining freedom in ζ be fixed by comparison with the numerical coordinate system in order to have as direct an interpretation as possible. See the paragraph following the one containing Eq. (24).]

2. The conformal structure of the generators

As emphasized above, because of finite-size effects, the conformal structure of the space of generators is *not* preserved under evolution. The question of how severe this issue is governs the degree to which there is a sense of asymptotically constant spinors, vectors, and tensors which is stable under evolution. This section discusses the obstruction.

The space of generators has naturally the structure of a smooth oriented two-sphere, and a conformal structure on this is equivalent to a choice of complex structure, and this in turn is equivalent to giving a complex basis vector (or covector) up to proportionality. In our case, we could take the basis vector to be M^a; then $\mathcal{L}_w M^a$, modulo terms proportional to M^a, would give a measure of the rate of change in the complex structure. Equivalently, the shear $M_a \mathcal{L}_w M^a$ of M^a along w^a would measure the rate of change. This quantity can be readily computed but it is not directly useful here.

The more direct way of accounting for the change in complex structure is the part of $d\eth/d\eta$ [Eq. (89a)] antiholomorphic in ζ; if $d\eth/d\eta$ were holomorphic, the complex structure would be unchanged at first order in η and $d\eth/d\eta$ would induce an infinitesimal Lorentz motion.

In practice, it is likely that $d\eth/d\eta = w^\zeta$ will be close to holomorphic. We saw above that there are two contributions to it, one $w^\zeta_{\mathrm{h}} = -(\eth w^{11'} + X w^{11'})\rho^{-1}\delta\bar{\zeta}$ depending on ζ intrinsic to $\mathcal{S}(\eta)$ and one w^ζ_{v} depending on the extension of ζ off $\mathcal{S}(\eta)$. For the intrinsic one, we have $X \approx -(3\rho)^{-1}\Psi_1$, and so, if $\mathcal{S}(\eta)$ is in fact within the peeling regime we will have $X \sim O(R^{-3})$. If the sphere is nearly round and $w^{11'}$ is nearly constant, then w^ζ_{h} will be small.

The quantity w^ζ_{v} represents the rate of change of the stereographic coordinate as one moves normal to $\mathcal{S}(\eta)$ along w^a. This means that w^ζ_{v} depends on just how ζ is extended off $\mathcal{S}(\eta)$, which in turn depends on how the underlying numerical coordinates extend to the future of $\mathcal{S}(\eta)$. As typically the extraction surfaces are very distant, nearly round, and these features are reflected to good approximation in the numerical coordinates (and preserved under evolution), we expect that w^ζ_{v} will give something which is close to an infinitesimal fractional linear transformation.

3. The strong conformal geometry

The strong conformal geometry of null infinity links the scales of vectors along the generators with the scales of those transverse to the generators [8]. It can be characterized by the quantity

$$\sqrt{(2i)^{-1}|P_{\mathrm{B}}|^{-2} d\zeta \wedge d\bar{\zeta}}\,\frac{\partial}{\partial u}, \tag{90}$$

where the forms are defined on the space of tangent vectors to $\mathcal{I}^+(\eta)$ modulo the vectors tangent to the generators, and the square root is taken in the sense of a line-bundle-valued quantity. (The root taken is irrelevant; there are also other equivalent ways of characterizing the structure.) This quantity is independent of the Bondi frame.

We have so far constructed, for each $\mathcal{I}^+(\eta)$, a stereographic coordinate ζ_η on the space of its generators, and we have transition functions $\zeta_\eta \circ \zeta_{\eta_0}^{-1}$ relating these for different values of η. On any $\mathcal{I}^+(\eta)$ we may define a Bondi coordinate u_η with respect to the Bondi frame defined by ζ_η (and P_{B}), fixing the zero of u_η to lie on the preferred cut. Thus the quantity (90) is well defined, and requiring it to be preserved under changes of η leads to a transformation law for the vectors $\partial/\partial u_\eta$.

We have

$$\begin{aligned}(2i)^{-1}|P_{\mathrm{B}}|^{-2} d\zeta_\eta \wedge d\overline{\zeta_\eta} &= (2i)^{-1}(1+|\zeta_\eta|^2)^{-1} d\zeta_\eta \wedge d\overline{\zeta_\eta} \\ &= \frac{(1+|\zeta_{\eta_0}|^2)^2}{(1+|\zeta_\eta|^2)^2}\left|\frac{\partial\zeta_\eta}{\partial\zeta_{\eta_0}}\right|^2 (2i)^{-1} \\ &\quad \times (1+|\zeta_{\eta_0}|^2)^{-1} d\zeta_{\eta_0} \wedge d\overline{\zeta_{\eta_0}},\end{aligned} \tag{91}$$

and so we have

$$\frac{\partial}{\partial u_\eta} = \frac{1+|\zeta_\eta|^2}{1+|\zeta_{\eta_0}|^2}\left|\frac{\partial\zeta_{\eta_0}}{\partial\zeta}\right|\frac{\partial}{\partial u_{\eta_0}}. \tag{92}$$

4. Identification of the generators

We now turn to the supertranslations identifying the generators of $\mathcal{I}^+(\eta)$ for different values of η.

We saw above [Eq. (84)] that $|\text{ð}|^2 w^b l_b n^a_\eta$ represents the apparent displacement of the cut corresponding to $\mathcal{S}(\eta + d\eta)$ with respect to the preferred cut in $\mathcal{I}^+(\eta)$. To integrate these infinitesimal displacements, however, we must express them all as elements of the same vector space. We may do this by using the transformation rule (92) relating $n^a_\eta = \partial/\partial u_\eta$ to $n^a_{\eta_0} = \partial/\partial u_{\eta_0}$ derived above; the infinitesimal displacement, expressed as a vector at $\mathcal{I}^+(\eta_0)$ for

some fixed η_0, is $(1+|\zeta_{\eta_0}|^2)^{-1}(1+|\zeta_\eta|^2)|\partial\zeta_{\eta_0}/\partial\zeta_\eta|\times|\text{ƛ}|^2 w^b l_b n^a_{\eta_0}$. Thus the supertranslation taking the cut labeled by η_1 to that labeled by η_2 will be, in $\mathcal{I}^+(\eta_0)$,

$$\Delta u_{\eta_0}(\eta_2,\eta_1)=\int_{\eta_1}^{\eta_2}\frac{1+|\zeta_\eta|^2}{1+|\zeta_{\eta_0}|^2}\left|\frac{\partial\zeta_{\eta_0}}{\partial\zeta_\eta}\right||\text{ƛ}|^2 w^a l_a d\eta. \tag{93}$$

In this integral, the coordinates $(\zeta_{\eta_0},\overline{\zeta_{\eta_0}})$ are held fixed and $\zeta_\eta=\eth(\zeta_{\eta_0},\overline{\zeta_{\eta_0}})$.

The full transformation law for the Bondi retarded times will be, from Eq. (92) again and this,

$$u_{\eta_2}=\frac{1+|\zeta_{\eta_1}|^2}{1+|\zeta_{\eta_2}|^2}\left|\frac{\partial\zeta_{\eta_2}}{\partial\zeta_{\eta_1}}\right|u_{\eta_1}+\Delta u_{\eta_2}(\eta_2,\eta_1). \tag{94}$$

One can verify directly that these transition functions are compatible, that is, computing u_{η_3} either directly from u_{η_1}, or from u_{η_2} in terms of u_{η_1}, gives the same answer.

What we shall actually need is to refer the angular momenta at different values of η to a single value η_0. We therefore define $\mathfrak{u}=u_\eta\circ u_{\eta_0}^{-1}|_{\eta_0=0}=\Delta u_\eta(\eta,\eta_0)$. Explicitly,

$$\mathfrak{u}(\eta_1)=\int_{\eta_0}^{\eta_1}\frac{1+|\zeta_\eta|^2}{1+|\zeta_{\eta_1}|^2}\left|\frac{\partial\zeta_{\eta_1}}{\partial\zeta_\eta}\right||\text{ƛ}|^2 w^a l_a d\eta. \tag{95}$$

In this integral, we have $\zeta_\eta=\eth(\zeta_0,\overline{\zeta_0},\eta_0,\eta)$ in order to keep the generator of null infinity fixed. This applies not only to the explicit factors of ζ_η, ζ_{η_1}, but also to the dependences $|\text{ƛ}|^2 w^a l_a$ and the derivative terms. [So $\partial\zeta_{\eta_1}/\partial\zeta_\eta=(\partial\zeta_\eta/\partial\zeta_{\eta_0})^{-1}(\partial\zeta_{\eta_1}/\partial\zeta_{\eta_0})=(\partial\eth(\zeta_0,\overline{\zeta_0},\eta_0,\eta)/\partial\zeta_0)^{-1}(\partial\eth(\zeta_0,\overline{\zeta_0},\eta_1,\eta)/\partial\zeta_0)$.]

C. Evolution of energy-momentum

To compare the energy-momentum at $\mathcal{S}(\eta)$ with that at $\mathcal{S}(\eta_0)$, then, we express $P^{AA'}_{\mathcal{S}(\eta)}\bar\pi_A\pi_{A'}$ by giving $\pi_{A'}$ as a function of ζ; we then insert for this function $\eth$. We have

$$\begin{aligned}P^{AA'}\bar\pi_A\pi_{A'}&=P^{00'}\bar\pi_0\pi_{0'}+P^{01'}\bar\pi_0\pi_{1'}+P^{10'}\bar\pi_1\pi_{0'}\\&\quad+P^{11'}\bar\pi_1\pi_{1'}\\&=(1/2)(P^{00'}+P^{11'})+(1/2)(P^{00'}-P^{11'})\\&\quad\times\frac{1-|\eth|^2}{1+|\eth|^2}-P^{01'}\frac{\bar\eth}{1+|\eth|^2}-P^{10'}\frac{\eth}{1+|\eth|^2}.\end{aligned} \tag{96}$$

In this formula, the components of P^a are evaluated at $\mathcal{S}(\eta)$ as in Sec. V. The formula here gives the component of the energy-momentum at $\mathcal{S}(\eta)$ in the direction specified by ζ at $\mathcal{S}(\eta_0)$.

As noted above, if the extraction surfaces $\mathcal{S}(\eta)$ are far enough away that they provide good models of cuts of null infinity, then $\eth$ will be a fractional linear transformation representing a Lorentz transformation and $P^{AA'}\bar\pi_A\pi_{A'}$ will be interpretable as the component of a covector along the null direction specified by ζ. Another way of saying this is that energy-momentum (96) will have only $j=0$ and $j=1$ components. Because of finite-size effects, however, we cannot expect this to hold exactly, and there is a question of principle of how to extract the (co)vectorial part of the energy-momentum when these effects cannot be neglected.

The natural thing to do is find the boost relative to which the energy-momentum (96) has zero dipole moment ($j=1$ component), project the energy-momentum in this frame (that is, keep only the $j=0$ component in this frame), and then boost back to the asymptotic laboratory frame at $\mathcal{S}(\eta_0)$. (There will be a unique frame in which the dipole moment is zero [15].) While there is no simple closed-form expression for this, there is an iterative procedure which one would expect to converge rapidly.

The projection of the energy-momentum relative to the frame defined by a unit future-pointing vector $t^{AA'}$ will be

$$\begin{aligned}P^{AA'}(t)=2^{5/2}\pi^{-1}\int\begin{bmatrix}|\zeta|^2 & \zeta\\ \bar\zeta & 1\end{bmatrix}(1-|\sigma|^2/\rho^2)|\delta\zeta|^2\\\times\frac{P^{BB'}\bar\pi_B\pi_{B'}}{(t^{00'}-\bar\zeta t^{01'}-\zeta t^{10'}+|\zeta|^2 t^{11'})^3}d\mathcal{S},\end{aligned} \tag{97}$$

where $P^{BB'}\bar\pi_B\pi_{B'}$ is given by (96). Thus if, starting from any $t_0^{AA'}$, we define

$$t_{n+1}^{AA'}=(P^{00'}(t_n)P^{11'}(t_n)-|P^{01'}(t_n)|^2)^{-1}P^{AA'}(t_n), \tag{98}$$

then the sequence $t_n^{AA'}$ will converge to the time direction determined by the Bondi-Sachs energy-momentum, with $P^{AA'}(t_n)$ converging to that energy-momentum. If $\eth(\zeta,\bar\zeta)$ is close to ζ (as would often be expected), then it would be natural to choose $t_0^a=P^a/\sqrt{P_bP^b}$. It may not even be necessary to move to further t_n^a in the sequence to attain the accuracy required in many situations.

D. Evolution of angular momentum

The strategy for comparing the angular momenta at different cuts is similar to that for comparing the energy-momenta. The essential difference is that we must refer all angular momenta to the same origin (that is, the same cross section of $\mathcal{I}^+$). We have already found that $\mathfrak{u}$ is the supertranslation relating the measurement of the angular momentum at $\mathcal{S}(\eta)$ to that at $\mathcal{S}(\eta_0)$. Since a supertranslation acts on the twistors by simply being added to λ_B in the parametrization (67) and (68), in order to refer the angular momentum back to the original cut $\mathcal{C}(\eta_0)$, we need to replace λ_B by $\lambda_B-\mathfrak{u}$ in Eq. (69), as well as replacing ζ by $\eth$.

Since, even at a single cut, the angular momentum in general relativity is given by an object with components for arbitrary $j\geq 1$, the angular momentum cannot be reduced to a vectorial object and so the sort of projection procedure which was used for the energy-momentum is not needed.

On the other hand, if it is desired to compute the components of μ for different values of j, this can certainly be done. However, there is no simple algebraic (that is, involving only finitely many operations for each term) transformation taking a resolution of μ in spherical harmonics at $\mathcal{S}(\eta)$ [such as might be found via Eq. (70) and Tables V and VI] and transporting it to $\mathcal{S}(\eta_0)$, even if $\eth(\zeta, \bar{\zeta})$ is given by a fractional linear transformation, because the asymptotic laboratory frames at $\mathcal{S}(\eta)$ and $\mathcal{S}(\eta_0)$ might be relatively boosted, and boosts mix infinitely many j values. (Of course, if $\eth$ can be approximated as differing from the identity only to a finite order, then an algebraic transformation can be derived.)

E. Two technical points

The formulas derived above for the evolution of the energy-momentum and angular momentum were the present paper's goals. As emphasized above, these results include possible finite-size corrections, which, if significant, should be interpreted as signs that the extraction surfaces are not distant enough to give a stable model of null infinity.

I mention here two further issues related to these finite-size effects, points which do not figure in the results above but would be relevant if one were to try to draw broader lessons for the development of quasilocal kinematics from these results.

The first is that in general the comparisons of energy-momentum and angular momentum at $\mathcal{S}(\eta_0)$ and $\mathcal{S}(\eta_1)$ depend not just on these surfaces themselves but on the intermediate ones $\mathcal{S}(\eta)$, $\eta_0 \leq \eta \leq \eta_1$. In other words, because of finite-size effects, one would not expect an *integrable* comparison. This issue, of course, would disappear if the surfaces were actually at null infinity.

The second issue is that the structures discussed here do not actually determine how to evolve the phase of a twistor or spinor. This does not lead to any difficulties in the formalism given here, but it would be a point to keep in mind in developing a more general theory.

VIII. USERS' GUIDE

The preceding sections covered the derivations of formulas for the energy-momentum, angular momentum, and comparisons of them at different times. The aim of the present section is to give a users' guide to the results.

The starting point is a spacelike two-surface $\mathcal{S}$ of spherical topology [or, for evolution, a one-parameter family $\mathcal{S}(\eta)$ of such surfaces, with $\nabla_a \eta$ timelike]. We assume that a null tetrad adapted to $\mathcal{S}$ has been chosen, and the Newman-Penrose quantities at $\mathcal{S}$ are available in terms of this tetrad.

The first step is to find a complex stereographic coordinate ζ on $\mathcal{S}$; see the paragraph containing Eq. (24) and the two paragraphs thereafter. With this known, the factor λ giving the rescaling to a Bondi-Sachs frame is determined by Eq. (26). This defines an *asymptotic laboratory frame*.

The second step is to compute the angular potential λ_B for the Bondi shear. This may be done either via a Green's function [using Eqs. (28), (31), and (32)], or by resolution in spin-weighted spherical harmonics [Eqs. (34)–(36); see also the last paragraph of Sec. III for phase conventions].

The third step is to compute the components $A_{\alpha\beta}$ of the kinematic twistor. These are given by integrals over $\mathcal{S}$. Table IV lists the integrands, which require, besides the quantities already discussed, the asymptotic forms Ψ_1^0, Ψ_2^0, and Ψ_3^0 of the Weyl curvature components in the poor man's no-incoming-radiation approximation; these asymptotic forms are given in Table I in terms of quantities on $\mathcal{S}$. (As noted in the text, Table I lists these under the assumption $|\sigma| \ll |\rho|$ on $\mathcal{S}$, which should be very good for most purposes. Section II shows how to compute them more accurately, if required.)

With the components of $A_{\alpha\beta}$ known, the Bondi-Sachs energy-momentum may be read off directly in the asymptotic laboratory frame: see Eq. (59).

The angular momentum is reported as a function $\mu(\zeta, \bar{\zeta})$ of the asymptotic direction with respect to the asymptotic reference frame, with $\mu + \bar{\mu}$ giving the energy-moment in the direction and $i\mu - i\bar{\mu}$ the spatial angular momentum about that axis. One could choose to either present this function directly or to give its resolution in spherical harmonics. For a direct presentation, the function is given by Eq. (69), and this can be resolved into spherical harmonics by standard means. If the components of λ_B in spherical harmonics have already been computed, then the resolution of μ is given by Eq. (70), which makes use of Tables V and VI.

If the spin and center of mass are required, one must transform to a boosted asymptotic frame in which the time axis lies along the Bondi-Sachs energy-momentum [9].

To study the evolution of the energy-momentum and the angular momentum, one must, besides computing them on the different surfaces $\mathcal{S}(\eta)$, also give an invariant method for relating the quantities on one surface to those on another. If w^a is a vector from $\mathcal{S}(\eta)$ to $\mathcal{S}(\eta + d\eta)$, one solves Eqs. (89) and (95), to find the functions $\eth(\zeta, \bar{\zeta}, \eta)$ and $\mathfrak{u}(\zeta, \bar{\zeta}, \eta)$ expressing the appearance of the cut $\mathcal{C}(\mathcal{S}(\eta))$ relative to $\mathcal{C}(\mathcal{S}(\eta_0))$. Then Eq. (96) [with $P^{AA'}$ defined by (59) evaluated at $\mathcal{S}(\eta)$] gives the energy-momentum at $\mathcal{S}(\eta)$ in the directions as specified by the angular variables at $\mathcal{S}(\eta_0)$. This function may, because of finite-size effects, in general not be simply a vector but will have components in all $j \geq 0$ representations. To project the vectorial part invariantly, one solves (97) and (98) iteratively. The angular momentum μ at $\mathcal{S}(\eta)$ but referred to the Bondi coordinates constructed at $\mathcal{S}(\eta_0)$ is given by replacing λ_B by $\lambda_B - \mathfrak{u}$ and ζ by $\eth$ in Eq. (69). There is no simple formula for the evolution of the components of the

angular momentum in spherical harmonics in the most general case; one must compute the components from the evolved μ (cf. Sec. VII D).

APPENDIX: SOME PROPERTIES OF SPIN-WEIGHTED SPHERICAL HARMONICS

This paper relies on some technical properties of spin-weighted spherical harmonics, which are derived here. The conventions are those of Ref. [14].

1. Definitions

The spin-weighted spherical harmonics are determined as follows. Fix a spin frame $\hat{o}^A$, $\hat{\iota}^A$ (normalized with $\hat{o}_A\hat{\iota}^A = 1$ and with $2^{1/2}t^{AA'} = \hat{o}^A\hat{o}^{A'} + \hat{\iota}^A\hat{\iota}^{A'}$). Put

$$Z(j,m)_{B\cdots CD\cdots E} = \underbrace{\hat{o}_{(B}\cdots\hat{o}_C}_{j-m}\underbrace{\hat{\iota}_D\cdots\hat{\iota}_{E)}}_{j+m}. \tag{A1}$$

Now let o^A, ι^A be a second normalized frame, which is considered to vary and to determine a point on the sphere (corresponding to the null vector $o^Ao^{A'}$, say). Then one puts

$${}_sZ_{j,m} = Z(j,m)_{B\cdots E}\underbrace{o^B\cdots o^C}_{j+s}\underbrace{\iota^D\cdots\iota^E}_{j-s}, \tag{A2}$$

$${}_sY_{j,m} = (-1)^{j+m}{}_sZ_{j,m}\sqrt{\frac{(2j+1)!(2j)!}{4\pi(j+s)!(j-s)!(j+m)!(j-m)!}}. \tag{A3}$$

One has $\overline{{}_sY_{j,m}} = (-1)^{m+s}{}_{-s}Y_{j,-m}$.

The spin-weighted spherical harmonics as defined above are functions on certain line bundles over the sphere. However, it is common to represent them by ordinary functions, by giving their values on preferred sections. There are two main conventions for this. In the first, the spin frame is adapted to the complex stereographic coordinate ζ and given by

$$o^A(\zeta,\bar{\zeta}) = i(1+|\zeta|^2)^{-1/2}(-\zeta\hat{o}^A - \hat{\iota}^A), \tag{A4}$$

$$\iota^A(\zeta,\bar{\zeta}) = i(1+|\zeta|^2)^{-1/2}(-\hat{o}^A + \bar{\zeta}\hat{\iota}^A); \tag{A5}$$

in the second, the adaptation is to the polar coordinates θ, ϕ, and

$$o^A(\theta,\phi) = e^{i\phi/2}\cos(\theta/2)\hat{o}^A + e^{-i\phi/2}\sin(\theta/2)\hat{\iota}^A, \tag{A6}$$

$$\iota^A(\theta,\phi) = -e^{i\phi/2}\sin(\theta/2)\hat{o}^A + e^{-i\phi/2}\cos(\theta/2)\hat{\iota}^A. \tag{A7}$$

These frames differ by a phase only; one has $\hat{o}^A(\theta,\phi) = -e^{-i\phi/2}\hat{o}^A(\zeta,\bar{\zeta}) = -(\bar{\zeta}/\zeta)^{1/2}o^A(\zeta,\bar{\zeta})$. Note that this means that, viewed as ordinary functions, we have

$${}_sY_{j,m}(\theta,\phi) = (-(\bar{\zeta}/\zeta)^{1/2})^{2s}{}_sY_{j,m}(\zeta,\bar{\zeta}). \tag{A8}$$

In this paper, the spin-weighted harmonics are applied at two stages. First, they are used in the solution of the twistor equation; in this case, the variable frame is determined by o^{B}_A (and the fixed frame by the asymptotic reference frame). The second occurrence of the harmonics is in parametrizing the $\pi_{A'}$ spinors appearing in the definitions of energy-momentum and angular momentum. In those cases, it is $\bar{\pi}_A$ which takes on the role of the variable spinor o_A in the spherical harmonics. We note that in this case

$$\pi_{0'} = \overline{\bar{\pi}_A\hat{o}^A} = {}_{-1/2}Z_{1/2,-1/2}, \tag{A9}$$

$$\pi_{1'} = \overline{\bar{\pi}_A\hat{\iota}^A} = -{}_{-1/2}Y_{1/2,1/2}, \tag{A10}$$

and similarly

$$\pi_{0'}\pi_{0'} = {}_{-1}Z_{1,1} = \sqrt{4\pi/3}\,{}_{-1}Y_{1,1}, \tag{A11}$$

$$\pi_{0'}\pi_{1'} = -{}_{-1}Z_{1,0} = -\sqrt{4\pi/6}\,{}_{-1}Y_{1,0}, \tag{A12}$$

$$\pi_{1'}\pi_{1'} = {}_{-1}Z_{1,-1} = \sqrt{4\pi/3}\,{}_{-1}Y_{1,-1}, \tag{A13}$$

$$\pi_{0'}\bar{\pi}_1 = {}_0Z_{1,1} = \sqrt{4\pi/6}\,{}_0Y_{1,1}, \tag{A14}$$

$$\pi_{0'}\bar{\pi}_0 - \pi_{1'}\bar{\pi}_1 = 2{}_0Z_{1,0} = \sqrt{4\pi/3}\,{}_0Y_{1,0}, \tag{A15}$$

$$\pi_{1'}\bar{\pi}_0 = -{}_0Z_{1,-1} = -\sqrt{4\pi/6}\,{}_0Y_{1,-1}. \tag{A16}$$

2. Behavior under inversion

Each of the spin-weighted spherical harmonics is defined as a function of the spinor o_A. We consider here how the harmonics change when the spinor is acted on by a spatial inversion.

There is a choice of sign in lifting the inversion from vectors to spinors; we use $\alpha_A \mapsto 2^{1/2}t_{AA'}\bar{\alpha}^{A'}$. (Then the spinor $\hat{\pi}_{A'}$ appearing in the treatment of angular momentum is the image of $\pi_{A'}$ under inversion.)

Each of the harmonics is given as a (normalization factor times a) function $Z_{A\cdots CD\cdots F}o^A\cdots o^C\iota^D\cdots\iota^F$, where there are $j+s$ omicrons and $j-s$ iotas. The antipodal map gives $o_A \mapsto -\iota_A$, $\iota_A \mapsto o_A$. This will evidently effect a change ${}_sY_{j,m} \mapsto (-1)^{j+s}{}_{-s}Y_{j,m}$. [Reversing the sign in the definition of the antipodal map on spinors would change the action on spin-weighted spherical harmonics to ${}_sY_{j,m} \mapsto (-1)^{j-s}{}_{-s}Y_{j,m}$, that is, would contribute an extra minus sign for half-integral spin weights. In this paper, since all final quantities have integral spin weights, the sign convention, as long as it is kept fixed, is unimportant.]

However, when one represents the harmonics by ordinary functions, their behavior under inversions appears more complicated, because the sections used to effect the

trivialization of the bundles are not invariant under inversions. Indeed, evaluating the sections at the antipodal point (whose stereographic coordinate is $-\bar{\zeta}^{-1}$), we find

$$o^A(-\bar{\zeta}^{-1}, -\zeta^{-1}) = -(\zeta/\bar{\zeta})^{1/2}\iota^A(\zeta, \bar{\zeta}), \tag{A17}$$

$$\iota^A(-\bar{\zeta}^{-1}, -\zeta^{-1}) = (\bar{\zeta}/\zeta)^{1/2}o^A(\zeta, \bar{\zeta}). \tag{A18}$$

Thus

$${}_sY_{j,m}(-\bar{\zeta}^{-1}, -\zeta^{-1}) = (-1)^{j+s}(\zeta/\bar{\zeta})^s {}_{-s}Y_{j,m}(\zeta, \bar{\zeta}), \tag{A19}$$

where the extra factor is the conversion from the gauge at one point to its antipodal point.

For the frame adapted to the polar system, one has

$$o^A(\pi - \theta, \phi + \pi) = -i\iota^A(\theta, \phi), \tag{A20}$$

$$\iota^A(\pi - \theta, \phi + \pi) = -io^A(\theta, \phi), \tag{A21}$$

and so

$${}_sY_{j,m}(\pi - \theta, \phi + \pi) = (-i)^{2j} {}_{-s}Y_{j,m}(\theta, \phi). \tag{A22}$$

3. Tensor products

We derive here the resolutions of certain products of spin-weighted spherical harmonics (Clebsch-Gordan decompositions) which are used in the text. Because in all cases one of the factors has small values of j and s (indeed, $j = 0, 1$, $s = -1, 0, 1$) it is easiest to proceed iteratively.

Our starting point is the identity

$$\alpha_A\beta_{B\cdots DE} = \alpha_{(A}\beta_{B\cdots DE)} - \frac{2j}{2j+1}\epsilon_{A(B}\beta_{C\cdots E)F}\alpha^F, \tag{A23}$$

where α_A is any spinor and $\beta_{B\cdots DE}$ is any totally symmetric spinor of valence $2j$. Taking $\alpha_A = \hat{o}_A$ or $\alpha = \hat{\iota}_A$ and $\beta_{B\cdots E} = Z(j, m)_{C\cdots E}$, we find

$$\begin{aligned} Z(1/2, -1/2)_A Z(j, m)_{B\cdots E} &= Z(j + 1/2, m - 1/2)_{AB\cdots E} \\ &\quad + \frac{j+m}{2j+1}\epsilon_{A(B}Z(j - 1/2, m - 1/2)_{C\cdots E)} \end{aligned} \tag{A24}$$

and

$$\begin{aligned} Z(1/2,1/2)_A Z(j, m)_{B\cdots E} &= Z(j + 1/2, m + 1/2)_{AB\cdots E} \\ &\quad - \frac{j-m}{2j+1}\epsilon_{A(B}Z(j - 1/2, m + 1/2)_{C\cdots E)}. \end{aligned} \tag{A25}$$

Contracting now with either o^A or ι^A, and with $o^B \cdots o^C$ ($j + s$ times) and $\iota^D \cdots \iota^E$ ($j - s$ times), we find

$${}_{1/2}Z_{1/2,\pm 1/2}\,{}_sZ_{j,m} = {}_{s+1/2}Z_{j+1/2,m\pm 1/2} \mp \frac{(j \mp m)(j - s)}{(2j)(2j+1)}\,{}_{s+1/2}Z_{j-1/2,m\pm 1/2}, \tag{A26}$$

and

$${}_{-1/2}Z_{1/2,\pm 1/2}\,{}_sZ_{j,m} = {}_{s-1/2}Z_{j+1/2,m\pm 1/2} \pm \frac{(j \mp m)(j + s)}{(2j)(2j+1)}\,{}_{s-1/2}Z_{j-1/2,m\pm 1/2}. \tag{A27}$$

We may by repeated application of these build up all the tensor decompositions. The cases we need are as follows.

We have ${}_{-1}Z_{1,0} = {}_{-1/2}Z_{1/2,1/2}\,{}_{-1/2}Z_{1/2,-1/2}$. Using this, we find, after some algebra

$${}_{-1}Z_{1,0}\,{}_1Z_{j,m} = {}_0Z_{j+1,m} - \frac{m}{2j}{}_0Z_{j,m} - \frac{(j^2 - m^2)(j+1)}{4j(4j^2-1)}{}_0Z_{j-1,m}. \tag{A28}$$

Similarly

$${}_{-1}Z_{1,\pm 1} = ({}_{-1/2}Z_{1/2,\pm 1/2})^2,$$

from which

$${}_{-1}Z_{1,\pm 1}\,{}_1Z_{j,m} = {}_0Z_{j+1,m\pm 1} \pm \frac{j \mp m}{2j}{}_0Z_{j,m\pm 1} + \frac{(j+1)(j \mp m)(j \mp m - 1)}{4j(4j^2-1)}{}_0Z_{j-1,m\pm 1}. \tag{A29}$$

Using

$${}_0Z_{1,\pm 1} = {}_{1/2}Z_{1/2,\pm 1/2}\,{}_{-1/2}Z_{1/2,\pm 1/2},$$

we find

$${}_0Z_{1,\pm 1}\,{}_0Z_{j,m} = {}_0Z_{j+1,m\pm 1} - \frac{(j \mp m)(j \mp m - 1)}{4(4j^2-1)}{}_0Z_{j-1,m\pm 1}, \tag{A30}$$

and, using

$${}_0Z_{1,0} = (1/2)_{1/2}Z_{1/2,1/2}\,{}_{-1/2}Z_{1/2,-1/2} + (1/2)_{-1/2}Z_{1/2,1/2}\,{}_{-1/2}Z_{1/2,-1/2},$$

$${}_0Z_{1,0}\,{}_0Z_{j,m} = {}_0Z_{j+1,m} + \frac{j^2 - m^2}{4j(4j^2-1)}{}_0Z_{j-1,m}. \tag{A31}$$

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