# Indications of de Sitter spacetime from classical sequential growth dynamics of causal sets

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A large class of the dynamical laws for causal sets described by a classical process of sequential growth yields a cyclic universe, whose cycles of expansion and contraction are punctuated by single ''origin elements'' of the causal set. We present evidence that the effective dynamics of the immediate future of one of these origin elements, within the context of the sequential growth dynamics, yields an initial period of de Sitter-like exponential expansion, and argue that the resulting picture has many attractive features as a model of the early universe, with the potential to solve some of the standard model puzzles without any fine-tuning.

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## I. INTRODUCTION

Cosmology is the natural playground for theories of quantum gravity for many reasons. The most obvious is the fact that all quantum gravity theories are formulated at such high energies (or small distances) that it is practically impossible for an earth based laboratory or accelerator to test them. However, the early universe can provide such a laboratory. In addition, cosmology is one of the most natural applications of general relativity, which is exactly what a theory of quantum gravity hopes to ''unify'' with quantum theory. On the other hand, the standard model of cosmology [1,2] also suffers from some problems/puzzles [3] that warrant extensions to it. Most of these have their origins in our lack of understanding of the physics of very high energies and thus a successful theory of quantum gravity should be able to address these problems. Quantum gravity has recently begun to shed some light on the puzzles of cosmology, such as resolution of the cosmic singularity [4–6], providing some hints as to alternatives to inflation [7], and providing potential explanations of density perturbations [8]. Additionally the prediction of a nonzero cosmological constant arises naturally from the discreteness expected from quantum gravity [9]. Thus quantum gravity is beginning to show some promise in resolving some of the paradoxes of the standard model. However, there is far from any consensus in this regard, so cosmologists generally look for alternative explanations.

The most common path taken is an extension via particle physics that introduces a scalar field with very special properties in the early universe. If the potential of the field satisfies certain conditions it can be arranged that the universe undergoes rapid expansion in a ''de Sitter-like'' phase. This rapid expansion, called the inflationary era or inflation, gets rid of many of the standard model puzzles/problems such as the so-called ''horizon problem,'' ''the flatness puzzle,'' ''the monopole problem,'' etc. For a more complete discussion see Refs. [10,11]. Despite the fact that this scenario has ''problems'' of its own, as has been pointed out by many authors [7,12], the resulting features are very attractive and hard to ignore. On the other hand, the lack of any competing model leaves a scientific void that needs to be filled if the case for or against inflation has to be decided.

Fortunately, causal set theory has reached a stage in its evolution where some predictions about cosmology have come out [9,13]. Based on a mixture of ''classical dynamics'' (referred to as the *classical sequential growth* [14] or *CSG models*) and some expectations about ''quantum dynamics,'' causal set theory predicts fluctuations in the cosmological term around a mean value. If this mean value is taken to be zero, the fluctuations are of the right magnitude to explain the present observations. Computer simulations of the behavior of the universe with such a fluctuating cosmological term have shown that the energy density in $\Lambda$ follows the total energy density of the universe and is roughly of the same order. This is the first time that such testable predictions have come out of a fundamental theory of quantum gravity.

Causal set theory is also different from most other quantum gravity theories in the sense that it assumes fundamental discreteness. There have been many arguments for discreteness at the most basic level, but all of them have either been merely philosophical, or regarded only as methods of regulating the infinities and singularities in particle physics and general relativity. In the absence of real predictions it has always been difficult to see if these considerations have more than a philosophical value. Prediction about fluctuations in $\Lambda$, however, draws life directly from fundamental discreteness, and it is but natural to wonder if causal set theory has more to say about cosmology. In this paper we will present evidence that many of the CSG models produce a de Sitter-like early universe, and thus may prove helpful towards solving another puzzle—why is the universe so large when it is not so old?

A general question which naturally arises in causal set theory is whether causal sets which are well approximated by continua arise dynamically. It has been shown that the sequential growth models possess continuum limits, as $N \to \infty$ and $p \to 0$, however the resulting continua look nothing like spacetime manifolds of dimension $>1$ [15]. However, it may still be the case that something resembling a spacetime arises at finite $p$. We consider this latter question here.

This paper is organized as follows. In Sec. II we briefly describe that portion of causal set theory which is relevant to the current work. In Sec. III we describe the behavior of the ''originary percolation'' dynamics, which arises as an effective dynamics of the ''early universe'' of CSG models. Then in Sec. IV we compute the spacetime volume of ''Alexandrov neighborhoods'' (''causal diamonds'') in de Sitter space of arbitrary (integer) dimension. In Sec. V we describe the particular simulation we perform, with results in Sec. VI, and wrap up with some concluding remarks in Sec. VII.

## II. CAUSAL SETS

A causal set, or ''causet'' for short, is a locally finite partially ordered set, whose elements can be thought of as irreducible[1] ''atoms of spacetime.'' A partially ordered set $C$ consists of a ''ground set,'' which one generally labels with integers from 0 to $N-1$ ($N$ can be infinite), along with a binary relation $\prec$ which is irreflexive ($x \not\prec x$) and transitive ($x \prec y \prec z \Rightarrow x \prec z$). Local finiteness is the condition that every *order interval* (or simply *interval*) $[x, y] = \{y | x \prec y \prec z\}\ \forall\, x, z \in C$ has finite cardinality.

### A. Kinematics

The connection to macroscopic spacetime arises via the notion of a ''sprinkling,'' in which one selects events of a spacetime at random by a Poisson process, identifies them with causal set elements, and then deduces a partial ordering among the elements from the causal structure of the spacetime. One regards a continuum spacetime as being a good approximation to an underlying causal set if that causal set is likely to have arisen from a sprinkling into that spacetime. For an extensive review of the causal set program, see [16–19].

The connection between familiar concepts from continuum geometry and their discrete counterparts on the causal set is the domain of causal set kinematics. We will make extensive use of two results in this regard.

The first is stated as a definition at this stage of the theory's development, and is implicit in the description in the previous paragraph of the correspondence of the causal set with the continuum. In order for a causal set to be likely to arise from a sprinkling, it must be the case that the number of elements sprinkled into any region of spacetime with volume $V$ is Poisson distributed, with a mean of $V$. This Poisson fluctuation in the correspondence between spacetime volume and number of elements plays a crucial role in the prediction of a fluctuating cosmological constant [9].

The second result relates to the correspondence between the length of chains and proper time. A *chain* is a subset of a causal set for which each pair of elements is related. The *length* $L$ of a chain is the number of elements in the chain minus 1. In Minkowski space of any dimension, it has been proven that the length of the longest chain between any pair of elements is proportional to the proper time between the events at which they are sprinkled, in the limit of infinite sprinkling density [20,21]. In [22] the proportionality is claimed to hold for any spacetime. Following [21], we define $m$ to be the constant of proportionality, so that

$$\tau = mL. \tag{2.1}$$

### B. Dynamics

There are a number of approaches to constructing a dynamical law for causal sets. Perhaps the most developed to date is the classical sequential growth model [14,23,24] mentioned in the Introduction. It describes the causal set as growing via a sort of ''cosmological accretion'' process, in which elements of the causet arise one at a time, each selecting some subset of the causal set to be its past. The process of growth in the model is stochastic; each newborn element selects a ''precursor set'' at random, with probabilities which satisfy a discrete analog of general covariance and a causality condition akin to that used to derive the Bell inequalities. This randomness is regarded as fundamental, and yet purely classical in nature, because it does not allow for any quantum interference among alternative outcomes. Given the classical nature of the probability distribution, the dynamics is incomplete, but can be seen as a stepping stone toward formulating a fully quantum process, which could then be regarded as a generalization of classical probability theory. Although the dynamically generated causal sets do not lead to orders which are readily approximated by smooth spacetime manifolds, they do have a number of striking cosmological features, which we explore further in this paper.

The sequential growth dynamics is described to take place in ''stages,'' though it is important to emphasize that the discrete general covariance condition enforces that this ordering in which the causet elements arise is

[1]It may be necessary to add matter degrees of freedom to the causet elements, and at some stage it may be important to ''coarse grain'' the causet so that a single element may stand in for many, but for the moment we can think of the elements as not containing any internal information.

"pure gauge"—it has no effect on the probability of forming a particular (order equivalence class of) causal set. At stage $n$ the causal set has $n$ elements "so far," and the task is to select a precursor set for the new element which arises in this stage. The probabilities of the CSG model derive from a sequence of non-negative "coupling constants" $(t_n)$, $n \geq 0$. With these weights, the probability for selecting a precursor set $S$ is proportional to $t_{|S|}$.[2] Thus the probability to choose a particular set $S$ is

$$Pr(S) = \frac{t_{|S|}}{\sum_{i=0}^{n} \binom{n}{i} t_i}.$$

Once a precursor is chosen, to be to the past of the new element, all the relations implied by transitivity are included as well. Thus it is the "past closure" of $S$ which forms the past of the newly generated element.

The particular sequence $t_n = t^n$, for a single non-negative real number $t$, gives rise to a dynamics called *transitive percolation* [14]. This sequence plays an important role, as we will see in a moment. The rule for deciding which elements to select for the past of a new element is particularly simple for transitive percolation. The newborn element simply considers each already existing element in turn, and selects it to be to its past with a fixed probability $p = t/(1 + t)$. It then adds to its past every element which precedes any of the originally selected elements, to maintain transitivity.

### C. Cosmic renormalization

Consider an element of a causal set, called in the combinatorics literature a "post," which is related to every other element of the causal set. This would resemble an initial or final singularity of a universe, in that the entirety of the universe is causally related to it. Now for any finite $p$, it has been proven that a (an infinite) causet generated by transitive percolation almost surely contains an infinite number of posts [25]. It is the large scale behavior of the universe subsequent to one of these posts which is the subject of this paper. We will present evidence that the period immediately following a post is one of rapid expansion of spacetime volume with respect to proper time. Thus at the largest scales the causal sets generated by transitive percolation resemble a bouncing universe, which periodically undergoes collapse down to a final singularity and an ensuing reexpansion.

It has further been shown that a large class of CSG models, which includes the sequence $t_n = (\alpha/\ln(n))^n$ for $n > 0$, $\alpha > \pi^2/3$ also lead to causets which almost surely contain an infinite number of posts [26]. Now the presence of posts in the dynamical model suggests an interesting possibility, as described in [5], that the dynamics following a post can be regarded as growing an entirely new universe, except with coupling constants which are "renormalized" with respect to those of the previous era. Much is now known about the flow of the coupling constants $(t_n)$ under this "cosmic renormalization" [27,28], in particular, that the transitive percolation dynamics family $t_n = t^n$ forms a unique attractive fixed point. The sequence $t_n = t^n/n!$ has also been studied in some detail [5,29]. There it is shown that the region immediately subsequent to a post behaves like transitive percolation, with a parameter $t$ which gets driven toward zero under the cosmic renormalization [$t \to \sqrt{t/N}$ for $N$ elements to the past of the current era's post (or "origin element")].

## III. ORIGINARY PERCOLATION AND RANDOM TREES

Note that the effective dynamics following a post comes with a caveat: each element is required to be related to the post, by definition. Therefore we find an *originary dynamics*, for which the possibility of being born unrelated to any other element is excluded, and all remaining probabilities are normalized correspondingly. Thus the probabilities of an originary dynamics are equal to those of an ordinary CSG model, conditioned on the event that the newborn element connects to at least one other element. (The originary dynamics is in fact one of the general class of solutions to the covariance and causality conditions on sequential growth, described in [30].)

*Originary percolation* is the originary version of transitive percolation. As mentioned in Sec. II B, at each stage of the growth process, the newborn element considers each existing element $x$ in turn, and selects $x$ to be in its past with probability $p$. In order to maintain transitivity of the order, if it chooses $x$ for part of its past, it includes all ancestors of $x$ as well. In the event that no element $x$ is selected in this process, it simply "tries again," so as to maintain the condition of originarity. At stage $n$ (meaning that there are $n$ elements currently in the causet), the probability to select a particular subset $S$ of existing elements is

$$Pr(S) = \frac{p^{|S|} q^{n-|S|}}{1 - q^n},$$

where $q = 1 - p$, and the factor $1/(1 - q^n)$ accounts for the originary condition, which excludes the possibility of not connecting to anything (which occurs with probability $q^n$). Once a set $S$ is chosen, the past closure of $S$ becomes the past of the newborn element $x$.

### A. Random tree era

For small values of $p$, the early universe of originary percolation, by which we mean the structure of that portion

[2]The definition of precursor set used here differs from that in [14,23]. There only precursor sets which contained their own past were included. This definition allows for a much simpler expression of the transition probabilities (e.g. as given in [15]), which better reveals the physical meaning of the $t_n$.

of the causal set which is born shortly after the origin element, forms a random tree, with high probability. To see why this occurs, consider the probability that the selected precursor set contains $m$ elements is given by

$$Pr(|S| = m) = \binom{n}{m} \frac{p^{|S|} q^{n-|S|}}{1 - q^n}.$$

For small $p$ this becomes vanishingly small for any $m > 0$. However, the case $m = 0$ is excluded by the originarity condition, while it remains true that, as long as $p \ll 1/n$, the transitions with $m = 1$ will be much more likely than any of the others. In these transitions one element is chosen at random from those already present, with a uniform distribution. This behavior yields a simple model of a random tree. It persists until $n \sim 1/p$, at which point we get a percolation phase transition, which heralds the end of the random tree era, and the beginning of a phase of de Sitter-like expansion, which we describe below. Note that the structure of this earliest random tree era of the universe is independent of $p$, save in determining how long it lasts.

We can begin our study of the early universe of originary percolation by studying this simple random tree process. To get some feel for the initial rate of expansion of the universe, we ask what is the expected number of elements which arise in "level $t$." which we define to be those elements whose longest chain to the root element is of length $t$. With certainty, the first element appears in level 0, and the second in level 1. At stage $n$, the probability of joining level $t$ is proportional to the number of elements in level $t - 1$. The same exact process has been studied in the combinatorics literature, under the name "random recursive trees." There similar questions have been studied, such as the probability distribution of the level of an element chosen uniformly at random from the tree [31].

Despite the simple recursion obeyed by the "joining probability" above, this problem is not easy to solve, e.g. because it involves an infinite sequence of distributions. Rather than analyze this problem in detail here, we simply observe that, after forming a random tree with $N$ elements, the mean cardinality of level $t$ looks very much like a multiple of a Poisson distribution in $t$. Thus the mean number $N_t$ of elements in level $t$ is very well fit by the function

$$N_t = \frac{A\lambda^t e^{-\lambda}}{t!}, \tag{3.1}$$

where the normalization factor $A \gtrsim N$ and $\lambda \sim \ln N$. An example is shown in Fig. 1. Despite the excellent fit of Fig. 1, the relation cannot be exact because, for example, $N_t$ must be exactly zero for $t > N$, which does not occur in (3.1).

The random tree era will continue until $n \sim 1/p$. After this stage a newborn element becomes as likely to choose more than one parent as not. Thus we expect that the $N$ of the random tree era is $\sim 1/p$. As far as the level population

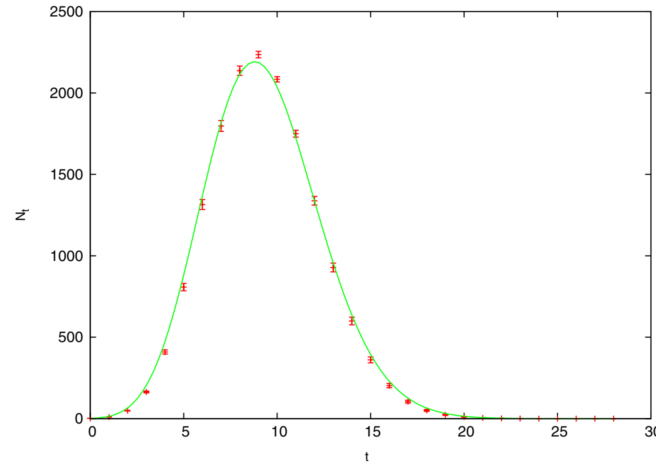


FIG. 1 (color online). Mean population of levels for an $N = 16\,384$ element random tree. The mean and its error are measured from 100 samples of the random tree process. The fitting function proportional to a Poisson distribution is shown. Here $A = 16\,687 \pm 85$ and $\lambda = 9.306 \pm 0.022$.

discussion goes, this is not the end of the story, as it is possible for such a "nontree element" to select all its parents from elements of the tree at an early level $t$, and thus itself to join an early level, say one much earlier than the maximum of (3.1), which is $\sim\lambda \sim \ln(1/p)$. Thus (3.1) provides only a lower bound on the cardinality of early levels.

It is interesting to note that Gerhard 't Hooft predicted almost this exact scenario in 1978, cf. Fig. 10 of [19].

### B. Originary percolation

To get a better handle on the initial rate of expansion, we perform simulations of the full originary percolation dynamics. This task is greatly simplified through the use of the CAUSAL SETS toolkit within CACTUS framework [32]. All we need to do is write a "thorn" (module) which counts the number of elements in each level, and counts the number of elements and longest chains in each order interval, as explained in Sec. V. The ability to generate causal sets via the originary percolation dynamics is already provided within the toolkit.

As an illustration, we show in Fig. 2 a small example causal set generated by originary percolation with $N = 16$, $p = 0.2$. The past of any element is the set of elements which can be reached from it by traversing the lines ("links") downward. The origin element/post is at the bottom. The red squares are elements which are part of the tree era.

Results for originary percolation at $p = 0.001$, $N = 11\,585$, are depicted in Fig. 3. In addition to the cardinality of each level mentioned above, we compute the cardinality of a "foliation" of the causal set by inextendible antichains. This is a more appropriate analogue to the (edgeless) spatial hypersurfaces of general relativity. An

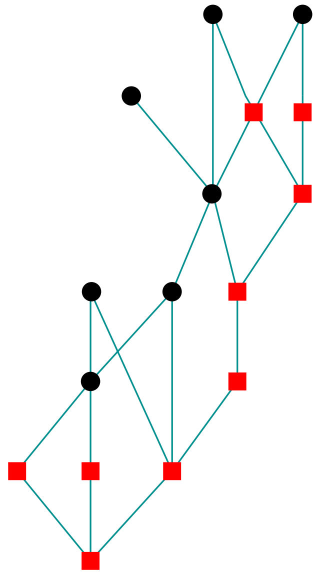

FIG. 2 (color online). A sample causal set generated by originary percolation with $N = 16$, $p = 0.2$. Elements of the initial tree are shown by red squares.

*antichain* is a subset of the causal set which contains no relations. An *inextendible antichain* is one which is maximal in the sense that no elements can be added to it while remaining an antichain, i.e. every other element of the causal set is to the future or past of one of its elements. The inextendible antichains we employ here are defined as follows. The level $t$ as defined above forms an antichain, but in general it will not be inextendible. We can extend it by adjoining the maximal elements (ones which have no elements to their future) of that portion of the causal set which is unrelated to any element of level $t$. It is easy to show that this will always yield an inextendible antichain. Note that all of the subcausal set which is unrelated to the level $t$ antichain lives in levels $<t$, for otherwise there would be a past directed chain to some element of level $t$. This fact motivates the choice of using the maximal elements to form the inextendible antichain.[3]

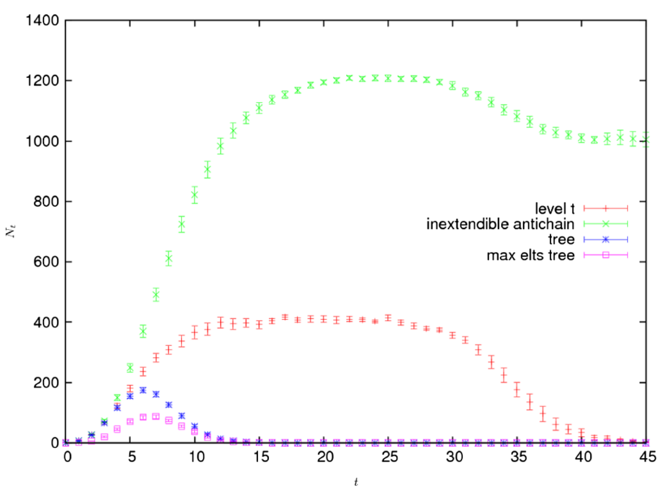


FIG. 3 (color online). Mean "spatial volume" of ten originary percolated causal sets with $N = 11\,585$, $p = 0.001$. Shown is the cardinality of level $t$ in red, the cardinality of an inextendible antichain containing level $t$ in green, the number of elements of the initial tree at level $t$ in blue, and the number of maximal elements of the initial tree, at level $t$, in magenta.

A final question that we consider before turning our attention to de Sitter spacetime regards how the initial random tree sits within the larger percolated causal set. To this end we define an element to be within the "tree era" if the order interval between it and the origin element is a chain. In Fig. 3 we plot, in addition to the cardinality of the antichains discussed above, the number of elements in each layer that are part of the tree era. We see that if $N \gg 1/p$ then the initial tree sits in the very early part of the percolated causal set. The exponential expansion extends well beyond the tree era, and thus the initial exponential growth of (3.1) is indeed only a precursor to an ensuing exponential growth involving a much larger portion of the causal set.

Before closing this section, it is important to note that the future of every element of a percolated causal set is itself an instance of originary percolation. This is simply because percolation is completely homogeneous—the future of an element is the same (in probability) as that of any other element. However, by discussing the future of an element $x$, one is conditioning on each element being to the future of $x$, which is exactly the condition of originary percolation. Thus originary percolation describes a homogeneous universe, for which the future of every element is exponentially expanding. This sounds a lot like de Sitter space.

## IV. VOLUME OF AN ALEXANDROV NEIGHBORHOOD IN DE SITTER SPACETIME

For a spacetime respecting "the cosmological principle," an exponential expansion means *the de Sitter spacetime*. If the universe is described by something like a causal set, the early universe region that we consider is very young. It does not look like a spacetime yet in the sense that it does not render itself easily to many of the familiar concepts of the continuum. This is particularly clear if one considers, for example, the random tree era. It is not possible to define the notion of "spacelike distance" in a random tree as no two elements have a common future [33]. Similarly, it is difficult to see what curvature would mean in this case. On the other hand, the notions of the length of the longest chain between two causal set elements ($L$), which is proportional to the proper time between the two events ($\tau$), and the number of causal set elements $N_\diamond$

[3] It turns out that this inextendible antichain is equivalent to the one which arises by taking the maximal elements of those whose level is $\leq t$.

which are causally between two given elements,[4] which is proportional to the volume of the Alexandrov neighborhood[5] $V_\diamond$ formed by the two elements, is still defined. We try to see if $L$ and $N_\diamond$ follow the same relationship as $\tau$ and $V_\diamond$ in $D+1$-dimensional de Sitter spacetime.

We use

$$ds^2 = -dt^2 + e^{2t/\ell}(dr^2 + r^2 d\Omega_D^2)$$

as the $D+1$-dimensional de Sitter metric [34], where $\ell$ is the radius of curvature and all other symbols have their usual meaning. As we have spherical symmetry in this case we can represent the Alexandrov neighborhood of two events in $t$ and $r$ space as sketched in Fig. 4. The spacetime volume of this region can be written as

$$V_\diamond = \int_{-t_1}^{0} dt e^{Dt/\ell} \int_0^{r_o} dr r^{D-1} \int d\Omega_D + \int_0^{t_2} dt e^{Dt/\ell} \int_0^{r_i} dr r^{D-1} \int d\Omega_D. \quad (4.1)$$

As the light cones in de Sitter space follow $\dot{r} = \pm e^{-t/\ell}$ and we choose outgoing $r_o$ and ingoing $r_i$ radial coordinates such that $r_o(0) = r_i(0) = R$, we can write $r_o = R + \ell(1 - e^{-t/\ell})$ and $r_i = R + \ell(e^{-t/\ell} - 1)$. Using these we can write (4.1) as

$$V_\diamond = \mathcal{C}_D \ell^D \bigg[ \int_{-t_1}^{0} dt \bigg( \frac{R+\ell}{\ell} e^{t/\ell} - 1 \bigg)^D + \int_0^{t_2} dt \bigg( \frac{R-\ell}{\ell} e^{t/\ell} + 1 \bigg)^D \bigg], \quad (4.2)$$

where $\mathcal{C}_D$ is the volume of a $D$-dimensional unit ball. Using $t_1 = \ell \ln\frac{\ell+R}{\ell}$, $t_2 = -\ell \ln\frac{\ell-R}{\ell}$, and $t_1 + t_2 = \tau$, we can simplify (4.2) to

$$V_\diamond = \mathcal{C}_D \ell^{D+1} \bigg[ \ln\cosh^2\bigg(\frac{\tau}{2\ell}\bigg) + \sum_{i=1}^{D} \frac{(-1)^{i+1}}{i} \binom{D}{i} \times \bigg( \bigg(1 + \tanh\bigg(\frac{\tau}{2\ell}\bigg)\bigg)^i + \bigg(1 - \tanh\bigg(\frac{\tau}{2\ell}\bigg)\bigg)^i - 2 \bigg) \bigg] \quad (4.3)$$

for $D$ odd and

$$V_\diamond = \mathcal{C}_D \ell^{D+1} \bigg[ \frac{\tau}{\ell} + \sum_{i=1}^{D} \frac{(-1)^i}{i} \binom{D}{i} \bigg( \bigg(1 + \tanh\bigg(\frac{\tau}{2\ell}\bigg)\bigg)^i - \bigg(1 - \tanh\bigg(\frac{\tau}{2\ell}\bigg)\bigg)^i \bigg) \bigg] \quad (4.4)$$

for $D$ even. One obvious case of interest is $D = 3$. Using the above-mentioned expressions and the fact that $\mathcal{C}_3 = 4\pi/3$, it turns out that

$$V_\diamond = \frac{4\pi}{3} \ell^4 \bigg( \ln\cosh^2\bigg(\frac{\tau}{2\ell}\bigg) - \tanh^2\bigg(\frac{\tau}{2\ell}\bigg) \bigg)$$

for a four-dimensional de Sitter spacetime. It should be noted that $V_\diamond \sim \tau^{D+1}$ for $\frac{\tau}{\ell} \ll 1$ as every spacetime looks locally like Minkowski space of the same dimension and $\sim\tau$ for $\frac{\tau}{\ell} \gg 1$. For the four-dimensional de Sitter space $V_\diamond = \frac{\pi}{24}\tau^4 + O(\tau^5)$ for $\tau \ll \ell$ and $\approx 4\pi/3(\tau - \ln 4e)$ for $\tau \gg \ell$).

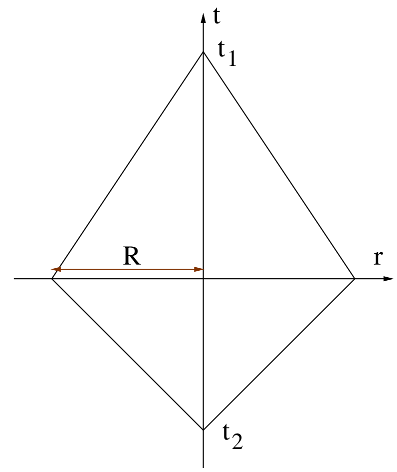


FIG. 4 (color online). An Alexandrov neighborhood (order interval) in de Sitter space.

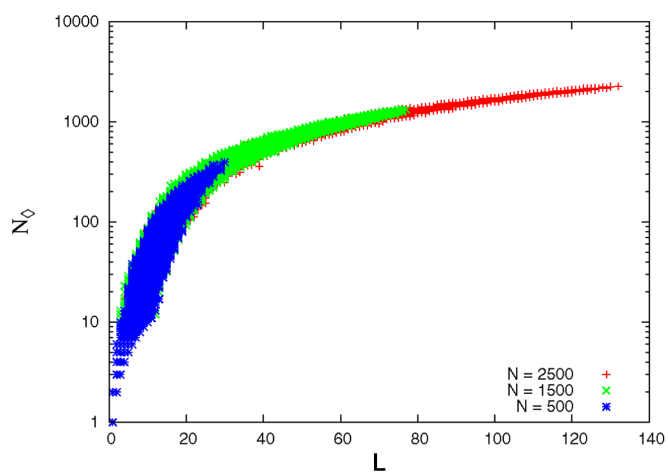


FIG. 5 (color online). The set of all pairs $(L, N_\diamond)$ for each related pair of elements, in three causal sets. Each causet was generated with the same value of $p = 0.025$ but three different values of $N = 500$, 1500, and 2500. (To make the figure size manageable, we plot only every 4th point for $N = 500$, every 10th for $N = 1500$, and every 200th point for $N = 2500$.) Note that the smaller data sets are a subset of the larger ones, and that at some point the maximum $N_\diamond(L)$ no longer increases with $N$.

[4]For an order interval $[x, y]$, we define $N_\diamond = |\{z | x \prec z \prec y\}| + 1$, where $|\cdot|$ indicates set cardinality. The $+1$ allows $N_\diamond = L$ for an interval which is a chain.

[5]The Alexandrov neighborhood of two events is the overlap of the past of the futuremost event with the future of the other.

## V. SIMULATION DETAILS

We want to compare the relationship between $V_\diamond$ and $\tau$ given by Eqs. (4.3) and (4.4) with that produced by originary percolation between $N_\diamond$ and $L$. In a given simulation we generate a causal set via originary percolation, with a given number of elements $N$ and the percolation parameter $p \in [0, 1]$. We then calculate the lengths of the longest chains $L$ between all pairs of elements and the corresponding number of elements $N_\diamond$ that are connected to both of these elements and lie causally between them. For this exercise the typical values of $N$ lie between 1000 and 50 000 and of $p$ between 0.0001 and 0.03. The primary computational constraint is run time, as finding the length of the longest chain in an interval involves an $O(N^2)$ algorithm, and there are $O(N^2)$ intervals to check.

For a given causal set, we collect a large number of pairs of numbers $(L, N_\diamond)$, one for every related pair of elements in the causet. The set of such pairs for three causal sets is plotted in Fig. 5. We wish to compare these data points with the functional forms (4.3) and (4.4), for some value of the dimension $D$. If these causal sets are exactly represented by de Sitter space, i.e. if they arose from a Poisson sprinkling of a region of de Sitter of spatial dimension $D$, then one would expect the data points to be scattered about the curve (4.3) or (4.4), with Poisson fluctuations. There are indications that, in spacetime dimensions larger than 3, the fluctuations in the length of the longest chain in a sprinkled interval of Minkowski space grows only logarithmically with $L$ [35], so one might guess that we would see data points distributed roughly uniformly above the curve (4.3) or (4.4). However, for reasons we do not fully understand, it turns out that the data points all seem to fall *below* the curve (4.3) or (4.4), such that the *maximum* value of $N_\diamond$ for a given $L$, for an appropriate range of values of $L$, gives an excellent match to one of the functions (4.3) or (4.4).

It is important to notice that almost all of the physics in this scenario is dictated by the choice of $p$, as long as $N \gg 1/p$. This can be easily seen from Fig. 5, by observing that the data points for smaller $N$ are effectively a subset of those for a larger value of $N$. Notice, in particular, that the maximum values of $N_\diamond$ for the $N = 1500$ causet are the same as those for the $N = 2500$ causet. Thus, as long as $N$ is large enough to capture the relevant region of exponential expansion, increasing $N$ further will have no effect on the results of interest.[6] In particular, this means that the dimension $D$ which gives the best fit, for example, will only depend on $p$. If $N$ is too small, on the other hand, then the causal set is not large enough to "sample the region of interest," and we will get poor results. This is manifested in Fig. 5 by the fact that the maximum $N_\diamond$ for $N = 500$ are substantially smaller than those for the larger causets.

The reader may be concerned that we use the maximum $N_\diamond$ for a given value of $L$, rather than the mean. This is an indication that the percolated causal set is not exactly manifoldlike. However this is not too surprising, as we already know that the CSG models do not have nontrivial spacetimes as their continuum limits [15]. Another indication that these are not quite manifoldlike is that at the smallest scales they are trees, as explained in Sec. III A, and thus one dimensional (because the shortest intervals will always be chains). This failure of the mean to give good results may be expected, in that it gets contributions from all sorts of intervals, including ones that might be "close to a boundary," such that they have small $N_\diamond$ for large $L$. In a sense we are considering only intervals as measured by observers which are stationary in the cosmic rest frame, so that they can get the most elements for a given proper time separation.

Since each causal set only provides a single maximum $N_\diamond$ for each $L$, we repeat the computation for a number of causal sets, and from these compute a mean maximum $N_\diamond$ with its error. We then fit each such data set with the expressions given in Eqs. (4.3) and (4.4), with $\tau$ replaced by $L/m$. $\ell$ and $m$ are used as fitting parameters. For $D = 3$ the fitting expression looks like

$$V_\diamond = \frac{4\pi}{3}\ell^4\left(\ln\cosh^2\left(\frac{\tau}{2\ell}\right) - \tanh^2\left(\frac{\tau}{2\ell}\right)\right). \tag{5.1}$$

As mentioned above, at the smallest scales $L$ the causal set behaves like a tree and is therefore, in the sense of order intervals, $0 + 1$ dimensional. At the largest scales the intervals "see the infrared cutoff" $N$, and therefore are not expected to give good results. We thus only fit our data within a range of $L$ values, as shown in Table I. Furthermore, since the error bars are much smaller for the small intervals than for the large ones, fitting directly to the forms above would strongly favor the small scales, and tend to ignore the data for larger scales. We handle this by fitting (the log of the maximum $N_\diamond$) to the log of the functions above such as (5.1), which has the effect of fitting to the relative error in the maximum $N_\diamond$.

At no point have we ever mentioned any number for the dimension, in expressing the dynamics. Thus we have no idea what dimension of de Sitter space to expect from our results. We therefore fit our data to every (spatial) dimension, usually from 1 to 9, and take the one which fits best.

## VI. RESULTS

Figure 6 shows a typical behavior of the plot of the maximum number of elements in an order interval and the corresponding longest chains, for $N = 15\,000$ and $p = 0.001$. Interestingly enough, the best fit was achieved by the function for $3 + 1$ dimensional de Sitter space, which is

[6] It is true that originary percolation for fixed $p$ and infinite $N$ contains every finite partial order as a suborder. Thus somewhere in that infinite causet is an interval with height $L$ and cardinality arbitrarily large. However, we do not send $N \to \infty$ for fixed $p$, we are only interested in the "early universe" of originary percolation, with $N$ no larger than say $1/p^3$. In such a regime the maximum $N_\diamond$ is effectively independent of $N$.

TABLE I. Fitting parameters for some values of $p$.

| $p$ | $N$ | $D+1$ | $\ell$ | $m$ | $\chi$ | Fitting range in $L$ | Number of runs |
|---|---|---|---|---|---|---|---|
| 0.0001 | 50 000 | 4 | $8.7 \pm 1.8$ | $2.105 \pm 0.028$ | 2.32 | 9–25 | 6 |
| 0.0002 | 30 000 | 4 | $6.81 \pm 0.72$ | $1.926 \pm 0.023$ | 3.07 | 8–27 | 3 |
| 0.0005 | 20 000 | 4 | $7.81 \pm 0.57$ | $1.787 \pm 0.022$ | 5.59 | 7–32 | 10 |
| 0.0008 | 15 000 | 4 | $6.86 \pm 0.22$ | $1.749 \pm 0.019$ | 4.69 | 6–35 | 4 |
| 0.001 | 15 000 | 4 | $6.20 \pm 0.12$ | $1.710 \pm 0.013$ | 4.97 | 7–35 | 23 |
| 0.003 | 15 000 | 4 | $3.73 \pm 0.013$ | $1.483 \pm 0.009$ | 5.12 | 6–100 | 20 |
| 0.005 | 15 000 | 4 | $3.097 \pm 0.009$ | $1.388 \pm 0.009$ | 4.69 | 5–150 | 20 |
| 0.01 | 2000 | 3 | $4.086 \pm 0.028$ | $1.136 \pm 0.006$ | 2.75 | 5–39 | 50 |
| 0.03 | 1000 | 3 | $2.331 \pm 0.011$ | $1.046 \pm 0.006$ | 0.663 | 5–53 | 5 |

shown in black. The best fits for the two neighboring dimensions are also shown, to give some indication of the robustness of the dimension "measurement."

The results for all our runs, for a variety of values of $p$, are summarized in Table I. All fits are performed with the GNUPLOT fit function. For each value of $p$ we have considered, Table I provides the value of $N$ we have used, the best fit spacetime dimension, the best fit values for $\ell$ and $m$ with their errors, $\chi = \sqrt{\sum_L\{[N_{\diamond,L} - V_\diamond(L)]^2/(N_L-2)\sigma^2_{N_\diamond,L}\}}$ (where $N_L$ is the number of data points fit), the range of $L$ values we fit, and also the number of causal sets generated. All reported errors are as given by GNUPLOT.

As discussed in Sec. II A, proper times are expected to be related to length of the longest chain by (2.1). If we assume that the largest intervals of our causal sets do indeed behave like intervals of de Sitter space, then the fits of Table I serve as an alternate measurement of $m$, in de Sitter spacetime of three and four dimensions. It is interesting to see that the values come out comparable to those for Minkowski space, which fall between 1.77 and 2.62 [21]. The $\ell$ measurements indicate that we can grow a universe which is roughly $2m\ell = 36$ elements "across."

Figure 6 contains the results from our largest data set (largest number of causal sets generated with those parameters). The plot for our smallest value of $p$ is shown in Fig. 7. There the range of $L$ values available for the fit is smaller, because one needs a very large causal set to get large chains with such a small $p$. The curve for 3 + 1 de Sitter continues to make an excellent fit, and better than curves for different dimensions of de Sitter space. Figure 8 portrays another example of the fits, this time in log scale. Our final example, Fig. 9, comes from a larger value for $p$. Here the universe is quite small, with a radius of curvature of just $\sim$4 in fundamental units. The best fit dimension is only 2 + 1 for this tiny universe.

Before concluding, we return attention to the issue of fitting the mean vs the maximum $N_\diamond$. In Fig. 10 we directly compare the two on the same data set, generated from four $N = 15\,000$ $p = 0.0008$ causal sets. It is clear that the

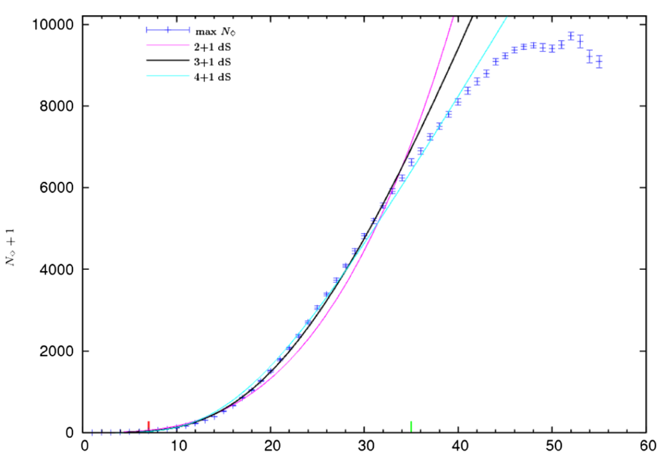


FIG. 6 (color online). A plot of the maximum values of $N_\diamond$ as a function of $L$ for $p = 0.001$ and $N = 15\,000$, along with best fit curves for de Sitter space of three different dimensions. The vertical line on the left marks the end of the tree era, while the one on the right separates the points that "see" the finiteness of the causal set. The overall best fit is achieved from the curve for 3 + 1 dimensions, and is shown in black.

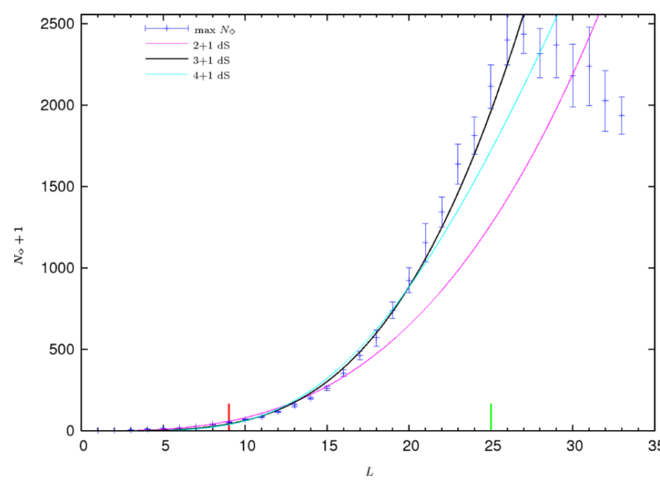


FIG. 7 (color online). A plot of the maximum values of $N_\diamond$ as a function of $L$ for $p = 0.0001$ and $N = 50\,000$, along with best fit curves for de Sitter space of three different dimensions. The overall best fit is achieved from the curve for 3 + 1 dimensions.

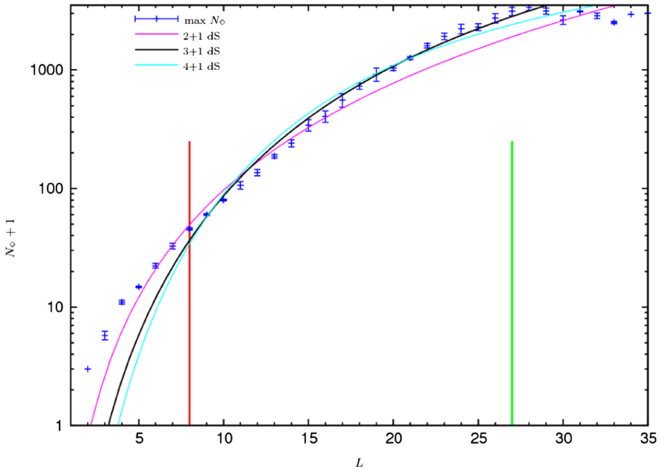


FIG. 8 (color online). A plot of the maximum values of $N_\diamond$ as a function of $L$ for $p = 0.0002$ and $N = 30\,000$, in log scale, along with best fit curves for de Sitter space of three different dimensions. Again the overall best fit is achieved from the curve for $3 + 1$ dimensions.

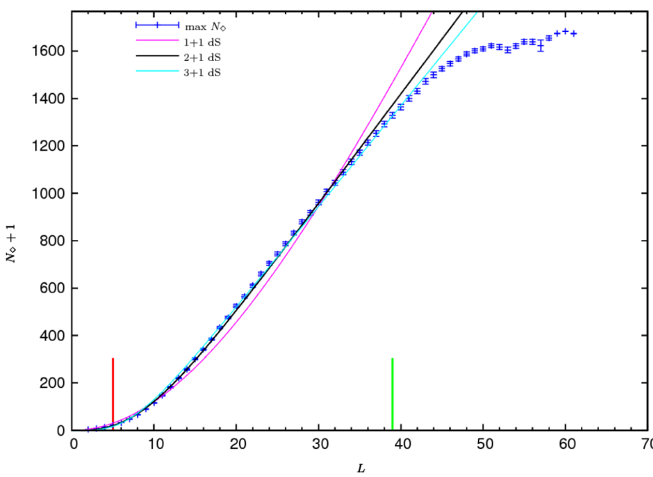


FIG. 9 (color online). A plot of the maximum values of $N_\diamond$ as a function of $L$ for $p = 0.01$ and $N = 2000$, along with best fit curves for de Sitter space of three different dimensions. Here the overall best fit is achieved from the curve for $2 + 1$ dimensions.

maximums fit the $3 + 1$ de Sitter form quite well, while the mean values do not. We omit the errors in the means, because they are extremely small for the small intervals, and strongly bias the fits away from the data at larger $L$. In any event the fit is poor for the means, for any dimension (though again $3 + 1$ fits the best there as well).

## VII. SUMMARY AND CONCLUSIONS

After motivating the study of originary percolation as an appropriate dynamical model for the early universe of causet set theory, at least within the context of classical sequential growth models, we explored a number of indications that it yields an exponentially expanding universe. In particular, for $p \ll 1$, we saw that, after a post, the

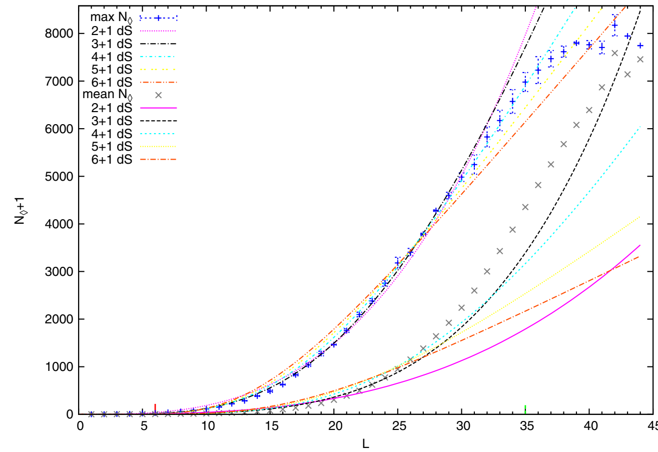


FIG. 10 (color online). Comparing behavior of the mean $N_\diamond$ versus the maximum. The results come from causal sets generated with $p = 0.0008$, $N = 15\,000$.

universe begins with a random tree era, followed by a period of de Sitter-like exponential expansion. More specifically, for $p \ll 1$ and $N \gg p^{-1}$, the largest intervals in the post-tree era resemble de Sitter spacetime insofar as spacetime volume as a function of proper time is concerned. Furthermore, the expression that best fits the data has $D = 3$ for a significant range of $p$ (at least one and a half decades). If $D$ continues to vary monotonically with $p$, then our results are compatible with $D = 3$ all the way down to physically realistic values, say $10^{-84}$ as needed to explain the initial large size of the universe [5]. Does this indicate how the observed number of spatial dimensions will emerge dynamically from quantum gravity? One must wait for the full quantum theory to be sure, but the dynamical appearance of $3 + 1$ dimensions, without being put into the theory in any way, is intriguing. It is important to gain a deeper analytical understanding as to why the sequential growth models are exhibiting such features in common with continuum spacetime.

There are many arguments that motivate the assumption of a discrete structure for our universe at the most fundamental level, and causal sets are a very simple and clearly defined theory that does just that. Some of these arguments which are mostly philosophical in nature are powerful and have been around for a very long time but the lack of any observational effects of discreteness has left the idea as a beautiful orphan that few want to adopt. It has only been recently that through arguments that derive life from causal set theory have we been able to predict some observational effects of discreteness as well. Fluctuations in the cosmological "constant" are one such prediction. Now we have shown that the universe generated by (many of) the *CSG* models not only exhibits some very desirable cosmological properties but may help solve some of the toughest problems of the standard cosmology, such as:

(i) The standard model of cosmology does not tell us from where the universe comes. In fact, if the theory of general relativity is supposed to be valid all the way to time $t = 0$, the universe ends up in a singularity, where not only the physical laws do not apply but it is impossible to get any information from $t < 0$. Thus it is impossible to know what happens "before" the singularity. On the other hand, if causal set cosmology is taken seriously, one still has a "beginning" or a big bang in the model but the singularity is not a problem anymore. The post is like any other element in the theory and thus discreteness can "resolve" the singularities. In fact, the same post is the big crunch singularity of the previous cycle of the universe. One can in principle calculate the probability of post occurrence in any of the *CSG* models, and it is easy to see what happens to the universe before and after the post is formed.

(ii) Every time the universe collapses (to a post) and then bounces back, the effective behavior of the expansion can be described as if the whole causal set started with that post with renormalized coupling constants. Since the percolation dynamics is an attractive fixed point under this renormalization flow in the space of CSG models that have posts, one may start the universe generically in any of these models, and it eventually will end up arbitrarily close to percolation. This makes percolation the natural candidate for the study and also guarantees the results are free of any kind of fine-tuning in the space of models.

(iii) The universe in the percolation model has two clearly separable eras early on. The first of these resembles a random tree, where the spatial volume of the universe increases exponentially with the "cosmological time." As the universe accumulates $1/p$ elements after the post, where $p$ is the parameter of the percolation, it enters a de Sitter-like phase.

(iv) One of the most unsettling problems of the standard cosmology is the fact that the universe appears very homogeneous on large scales—something that can be seen directly in the cosmic microwave background temperature isotropy. The percolation universe as it emerges from its early phase is very homogeneous in the sense that any neighborhood looks like any other. Every element has the same sort of past and future and the same number of nearest neighbors. Thus the model has a very strong potential for solving the homogeneity problem as it naturally favors a homogeneity in the initial conditions. This is particularly true if the matter is generated by the structure in the causal set itself. On the other hand, if we put external degrees of freedom on the causal set, it may happen that, even if we start with different initial conditions for these degrees of freedom, the de Sitter-like expansion gets rid of this inhomogeneity. Of course there are random fluctuations that cause deviations away from homogeneity. These fluctuations might prove helpful in solving another extremely important puzzle in the early universe, namely, the origin of density perturbations that seed the late time structure formation.

(v) Another puzzle is the large size of the universe compared to, say, the Planck length, when the universe is still very young, say, $O(100)$ Planck times old. This is related to both the horizon problem and the flatness puzzle. Models with percolation dynamics naturally generate a large size of the universe. If we start a percolation model with parameter $p$, the spatial volume becomes of the order of $p^{-1}$ within $\ln p^{-1}$ time steps. Depending on how small $p$ is, the universe can be made arbitrarily large. Since cosmic renormalization provides a mechanism which can drive the effective value of $p$ to arbitrarily small values if one waits long enough, there is no fine-tuning involved.

It may be the case that the quantum mechanism which drives the cosmological constant to zero [9] is the same mechanism which causes a smooth continuum to emerge from the discrete partial order. In this case, one may not be so surprised that the CSG models do not lead to smooth continuum like manifolds. However, it is possible that they capture some new physics at cosmological scales, given their discrete nature. Here we have demonstrated that CSG models are easily capable of describing a rapidly expanding universe which is much like our own, at least at the largest scales. Could the locally Minkowskian light cone structure of continuum spacetime be an effect which arises only at an intermediate scale, much larger than the discreteness scale, and thus is not a good description of our universe until after an initial period of de Sitter-like expansion?

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