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Statistical isotropy violation of the CMB brightness fluctuations

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Certain anomalies at large angular scales in the cosmic microwave background (CMB) measured by WMAP have been suggested as possible evidence of breakdown of statistical isotropy (SI). SI violation of cosmological perturbations is a generic feature of ultralarge scale structure of the cosmos and breakdown of global symmetries. Most CMB photons free-stream to the present from the surface of last scattering. It is thus reasonable to expect statistical isotropy violation in the CMB photon distribution observed now to have originated from SI violation in the baryon-photon fluid at last scattering, in addition to anisotropy of the primordial power spectrum studied earlier in the literature. We consider the generalized anisotropic brightness distribution fluctuations, $\Delta(\vec{k}, \hat{n}, \tau)$ (at conformal time τ) in contrast to the SI case where it is simply a function of $|\vec{k}|$ and $\hat{k}\cdot\hat{n}$. The brightness fluctuations expanded in bipolar spherical harmonic (BipoSH) series can then be written as $\Delta^{LM}_{\ell_1\ell_2}(k, \tau)$, where $L > 0$ terms encode deviations from statistical isotropy. Violation of SI encoded in the present off-diagonal elements of the harmonic space correlation $\langle a_{\ell m} a^*_{\ell' m'}\rangle$, equivalently, the BipoSH coefficients $A^{LM}_{\ell\ell'}$, are then related to the generalized BipoSH brightness fluctuation terms at present. We study the evolution of $\Delta^{LM}_{\ell_1\ell_2}(k, \tau)$ from nonzero terms $\Delta^{LM}_{\ell_3\ell_4}(k, \tau_s)$ at last scattering, in the free-streaming regime. We show that the terms with given BipoSH multipole LM evolve independently. Moreover, similar to the SI case, power at small spherical harmonic (SH) multipoles of $\Delta^{LM}_{\ell_3\ell_4}(k, \tau_s)$ at the last scattering is transferred to $\Delta^{LM}_{\ell_1\ell_2}(k, \tau)$ at larger SH multipoles. The structural similarity is more apparent in the asymptotic expression for large values of the final SH multipoles. This formalism allows an elegant identification of any SI violation observed today to a possible origin in SI violating physics present in the baryon-photon fluid. This is illustrated for the known result of SI violating angular correlations due to the presence of a homogeneous magnetic field in the baryon-photon fluid.

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I. INTRODUCTION

The cosmic microwave background (CMB) anisotropy is a very powerful observational probe of cosmology. In standard cosmology, the CMB anisotropy signal is expected to be statistically isotropic, i.e., statistical expectation values of the temperature fluctuations $\Delta T(\hat{n}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n})$ are preserved under rotations of the sky. The condition for statistical isotropy (SI), in spherical harmonic space translates to a diagonal $\langle a_{\ell m} a^*_{\ell' m'}\rangle = C_\ell \delta_{\ell\ell'}\delta_{mm'}$, where C_ℓ is the widely used angular power spectrum of the CMB anisotropy.

After the release of first year data of the Wilkinson Microwave Anisotropy Probe (WMAP), statistical isotropy of the CMB anisotropy attracted considerable attention. The study of full sky maps from the WMAP 5 yr data [1–3] and the very recent WMAP 7 yr data [4] has led to some intriguing anomalies which seem to suggest that the assumption of statistical isotropy is broken on the largest angular scales [5–9]. Broken isotropy would have profound implications for the standard cosmological model as statistical isotropy underlies all cosmological inferences.

It was pointed out that the suppression of power in the quadrupole and octopole are aligned in the form of the ‘‘axis of evil’’ [10–14]. Further ‘‘multipole-vector’’ directions associated with these multipoles (and some other low multipoles as well) appear to be anomalously correlated [6,15,16]. There are indications of asymmetry in the power spectrum at low multipoles in opposite hemispheres, the ‘‘north-south asymmetry’’ [7,8,17–19]. Possibly related, are the results of tests of Gaussianity that show asymmetry in the amplitude of the measured genus amplitude (at about 2 to 3σ significance) between the north and south galactic hemispheres [20–22]. Analysis of the distribution of extrema in WMAP sky maps has indicated non-Gaussianity, and to some extent violation of SI [23].

An observed map of CMB anisotropy $\Delta T(\hat{n})^{\rm obs}$ contains the true CMB temperature $\Delta T(\hat{n})$ fluctuations, convolved with the beam and instrumental noise and foreground contaminations. Breakdown of statistical isotropy can occur in any of these parts and can be categorized as

(i) Theoretically motivated effects which are intrinsic to the true CMB sky, $\Delta T(\hat{n})$ include nontrivial cosmic topology [24,25], Bianchi models [26–29], and primordial magnetic fields [30,31]. A recent article [32] claims that the solution to the cosmological vacuum energy can be explained as a result of the interaction of the infrared sector of the effective theory of

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gravity with standard model fields. This theory predicts the violation of cosmological isotropy.

(ii) Although a possible source of SI breakdown, residual foreground contamination would need to be of the order of the intrinsic CMB temperature anisotropy to account for an appreciable effect [33–35].

(iii) It would be erroneous to assume that the true CMB temperature fluctuations are completely extracted from the observed map. Observational artifacts such as noncircular beam, inhomogeneous noise correlation, and residual striping patterns could be potential sources of SI breakdown.

Violation of statistical isotropy of CMB anisotropy and its measurement has been discussed in the literature earlier [36–41] by defining an estimator where SI breakdown in an observed CMB anisotropy sky map is indicated by the nonzero value of this estimator. Studies have also been done by implementing a directional dependent inflationary power spectrum $P(\vec{k})$ which gives rise to off-diagonal terms in the covariance matrix [42,43]; spontaneous breakdown of SI in the CMB by a nonlinear response to long-wavelength field fluctuations that appear as a gradient locally to the observer [44] or locally through a modulation field [45]; incorporating an initial period of kinetic energy domination in single field inflation [46]. Deviation from SI in the CMB has also been studied as a direct consequence of breakdown of homogeneity and isotropy of the inflationary background from quasiclassical perturbations permanently generated at early stages of inflation [47,48].

In this paper we present a new formulation that relates the breakdown of SI in the CMB photon fluctuations at last scattering, and evolving them to find the effect of the modes at present epoch, hence, the CMB $\langle a_{\ell m} a_{\ell' m'} \rangle$ today. We also find the bipolar spherical harmonic coefficients (BipoSH) [39,40] which are linear combinations of off-diagonal elements of the covariance matrix. BipoSH expansion completely represents the information of the covariance matrix thus being the most general way of studying two point correlation functions of CMB anisotropy. These BipoSH coefficients are mathematically complete measures of SI violation on a sphere.

II. REVIEW OF STATISTICALLY ISOTROPIC CMB BRIGHTNESS FLUCTUATIONS

A. Boltzmann equations, inhomogeneities, and anisotropies

In the smooth background universe, thermalized photons being distributed homogeneously and isotropically, the temperature T is independent of $\vec{x}$ and direction of propagation $\hat{p}$, respectively. To describe perturbations about this smooth universe, we allow inhomogeneities in the photon distribution and anisotropies.

Before recombination, $z_{\rm rec} \approx 1100$, the photons were tightly coupled to the electrons and protons; all together they can be described as a single fluid, the baryon-photon fluid. After recombination, photons free-stream from the surface of last scattering to the present epoch.

Given the cosmological perturbations to the photons at recombination, one can predict the anisotropy spectrum today. The main motivation next is to relate the moments today to the moments at recombination using the photon distribution function.

B. Fluctuations of CMB photon distribution

In the Boltzmann equation for photons $df/dt = C[f]$, we expand the photon distribution function $f(\vec{x}, p, \hat{n}, \tau)$ about its zero-order Bose-Einstein value $T(\tau)$ [49], where $\vec{p} = p\hat{n}$. The distribution function of the photons changes with the perturbed temperature as

$$f(\vec{x}, p, \hat{n}, \tau) = \left[\exp\left\{\frac{p}{T(\tau)[1 + \Delta(\vec{x}, \hat{n}, \tau)]}\right\} - 1\right]^{-1}. \quad (1)$$

The perturbation to the distribution function is characterized by $\Delta \equiv \delta T/T$ termed as CMB brightness fluctuations henceforth. Since the perturbation Δ is small, we can expand $f(\vec{x}, p, \hat{n}, \tau)$ keeping only terms up to first order to get

$$\Delta(\vec{x}, \hat{n}, \tau) \equiv \left(\frac{\partial f^0}{\partial \ln p}\right)^{-1} \delta f, \quad (2)$$

where f^0 is the zero-order photon distribution function. $\Delta(\vec{x}, \hat{n}, \tau)$ depends on $\vec{x}$, $\hat{n}$, and τ and not on the magnitude of momentum p; this is a valid assumption since the temperature of the plasma is very small compared to the rest energy of the electrons which undergo scattering, elastic Thomson scattering has a negligible effect on the magnitude of the photon momentum.

Perturbations to the CMB remain small at all cosmological epochs; evolution of the largest scales being in the linear regime. In solving the linear evolution equations, it is simplest to work with Fourier transforms since every Fourier mode evolves independently:

$$\Delta(\vec{x}, \hat{n}, \tau) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \tilde{\Delta}(\vec{k}, \hat{n}, \tau)\phi(\vec{k}), \quad (3)$$

where $\phi(\vec{k})$ is the primordial density fluctuations.

With statistical isotropy assumption,

$$\begin{aligned} \tilde{\Delta}(\vec{k}, \hat{n}, \tau) &= \tilde{\Delta}(k, \hat{k}\cdot\hat{n}, \tau) \\ &= \sum_{\ell} (-i)^{\ell} (2\ell + 1) \tilde{\Delta}_{\ell}(k, \tau) P_{\ell}(\hat{k}\cdot\hat{n}) \\ &= 4\pi \sum_{\ell m} (-i)^{\ell} \tilde{\Delta}_{\ell}(k, \tau) Y_{\ell m}(\hat{k}) Y_{\ell m}(\hat{n}). \end{aligned} \quad (4)$$

$\tilde{\Delta}_{\ell}(k, \tau)$ are the moments of the CMB brightness fluctuation. The monopole $\tilde{\Delta}_0$ is related to the density perturbations while the dipole $\tilde{\Delta}_1 \propto \hat{n}\cdot\vec{v}$ gives the velocity term for baryons.

C. Correlations

The observed anisotropy in multipole space can be written in terms of the CMB brightness fluctuation as

$$\Delta(\vec{x}=0,\hat{n},\tau)=\sum_{\ell m}a_{\ell m}Y_{\ell m}(\hat{n}). \tag{5}$$

Using the orthonormality property of the spherical harmonics $Y_{\ell m}$, the spherical harmonic (SH) coefficients $a_{\ell m}$ become

$$a_{\ell m}=4\pi(-i)^{\ell}\int\frac{d^3k}{(2\pi)^3}\phi(\vec{k})\tilde{\Delta}_{\ell}(k,\tau)Y^*_{\ell m}(\hat{k}). \tag{6}$$

The angular correlation can be expressed as

$$\langle a_{\ell m}a^*_{\ell' m'}\rangle=4\pi\int\frac{dk}{k}\frac{k^3P_0(k)}{2\pi^2}|\Delta_{\ell}(k,\tau)|^2\delta_{\ell\ell'}\delta_{mm'}$$
$$C_{\ell}=4\pi\int\frac{dk}{k}\mathcal{P}_0(k)|\Delta_{\ell}(k,\tau)|^2, \tag{7}$$

where correlation of the primordial density fluctuations $\langle\phi(\vec{k})\phi^*(\vec{k}')\rangle=P_0(k)\delta(\vec{k}-\vec{k}')$ and $\mathcal{P}_0(k)=k^3P_0(k)/2\pi^2$ is the primordial power spectrum per logarithmic interval generated by the inflationary model and the second term is the radiative transport kernel in the post-recombination universe given by cosmological parameters.

D. Evolution of CMB brightness fluctuation in the free-streaming regime

The evolution of $\tilde{\Delta}(\vec{k},\hat{n},\tau)$ in the free-streaming regime can be written as

$$\tilde{\Delta}(\vec{k},\hat{n},\tau)=e^{i\vec{k}\cdot\hat{n}(\tau-\tau_s)}\tilde{\Delta}(\vec{k},\hat{n},\tau_s), \tag{8}$$

where τ is well inside the free-streaming regime, i.e. $\tau_s<\tau<\tau_0$, τ_s and τ_0 being the conformal time at last scattering and today, respectively [50].

Using the expansion

$$e^{i\vec{k}\cdot\hat{n}\Delta\tau}=\sum_l(-i)^l(2l+1)j_l(k\Delta\tau)P_l(\hat{k}\cdot\hat{n}), \tag{9}$$

and defining $\Delta\tau=\tau-\tau_s$, the evolution equation for $\tilde{\Delta}_{\ell}(k,\tau)$ can be written as

$$\tilde{\Delta}_{\ell}(k,\tau)=\sum_{l\ell'}(-i)^{\ell+\ell'-l}(2\ell'+1)j_l(k\Delta\tau)\times[\mathcal{C}^{l0}_{\ell 0\ell' 0}]^2\tilde{\Delta}_{\ell'}(k,\tau_s). \tag{10}$$

$j_l(k\Delta\tau)$ and $P_l(\hat{k}\cdot\hat{n})$ are the ℓth order spherical Bessel function and Legendre polynomial, respectively. Equation (10) is the well-known "free-streaming " equation in CMB literature [50]. $\mathcal{C}^{LM}_{\ell_1m_1\ell_2m_2}$ are the Clebsch-Gordan coefficient which satisfies the triangle inequalities [51] putting a constraint $|\ell_1-\ell_2|\le L\le\ell_1+\ell_2$ and $m_1+m_2=M$.

III. STATISTICAL ISOTROPY BREAKDOWN IN THE CMB BRIGHTNESS FLUCTUATION

In this paper we take into account the SI violation of the CMB anisotropy which is seeded due to the inherent SI breakdown in the CMB photon distribution. We consider the general form of the CMB brightness fluctuation, allowing for anisotropy in $\hat{k}$, i.e., $\tilde{\Delta}(\vec{k},\hat{n},\tau)\not\equiv\tilde{\Delta}(k,\hat{k}\cdot\hat{n},\tau)$.

A. Generalized CMB brightness fluctuations

The most general CMB brightness fluctuation is not simply a function of $|\vec{k}|$ and $\hat{k}\cdot\hat{n}$. In this case the physical situation of anisotropic fluctuations demands the brightness fluctuations to be expanded in bipolar spherical harmonic series (not just a Legendre series as in the statistical isotropic case). The brightness fluctuation in multipole space is $\Delta^{LM}_{\ell_1\ell_2}(\vec{k},\tau)$, where the $L>0$ term incorporates deviation from statistical isotropy:

$$\begin{aligned}\tilde{\Delta}(\vec{k},\hat{n},\tau)&=4\pi\sum_{\ell_1\ell_2LM}(i)^{(\ell_1+\ell_2)/2}\sqrt{\ell_1+\ell_2+1}\tilde{\Delta}^{LM}_{\ell_1\ell_2}(k,\tau)\\&\quad\times\{Y_{\ell_1}(\hat{k})\otimes Y_{\ell_2}(\hat{n})\}_{LM}\\&=4\pi\sum_{\ell_1\ell_2LM}\beta_{\ell_1\ell_2}\tilde{\Delta}^{LM}_{\ell_1\ell_2}(k,\tau)\{Y_{\ell_1}(\hat{k})\otimes Y_{\ell_2}(\hat{n})\}_{LM},\end{aligned} \tag{11}$$

where $\beta_{\ell_1\ell_2}=(i)^{(\ell_1+\ell_2)/2}\sqrt{\ell_1+\ell_2+1}$ has been defined for convenient notational simplicity.

The tensor product in the BipoSH function is defined as

$$\{Y_{\ell_1}(\hat{k})\otimes Y_{\ell_2}(\hat{n})\}_{LM}=\sum_{m_1m_2}\mathcal{C}^{LM}_{\ell_1m_1\ell_2m_2}Y_{\ell_1m_1}(\hat{k})Y_{\ell_2m_2}(\hat{n}).$$

The prefactors in Eq. (11) are the normalization terms associated with the CMB brightness fluctuation. The deviations from statistical isotropy are associated with nonzero values of L, $L>0$. To check for the statistical isotropy limit, i.e. $L=0$ and $M=0$, we use Eq. (8.5.1) from [51],

$$\mathcal{C}^{00}_{\ell_1m_1\ell_2m_2}=(-1)^{\ell_1-m_1}\frac{\delta_{\ell_1\ell_2}\delta_{m_1-m_2}}{\sqrt{2\ell_1+1}}, \tag{12}$$

to recover Eq. (4) in Sec. II B.

B. Angular correlations

Starting with Eq. (5) and the Fourier transform relation from Eq. (3), the SH coefficients $a_{\ell m}$ for the general case are

$$a_{\ell m}=4\pi\int\frac{d^3k}{(2\pi)^3}\phi(\vec{k})\sum_{\ell_1m_1LM}\beta_{\ell_1\ell}\tilde{\Delta}^{LM}_{\ell_1\ell}(k,\tau)\times\mathcal{C}^{LM}_{\ell_1m_1\ell m}Y_{\ell_1m_1}(\hat{k}). \tag{13}$$

The angular correlations turn out to be

$$\langle a_{\ell m}a^*_{\ell' m'}\rangle = (4\pi)^2\iint\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\langle\phi(\vec{k})\phi^*(\vec{k}')\rangle\sum_{\ell_1 m_1 LM}\sum_{\ell_2 m_2 L'M'}\beta_{\ell_1\ell}\beta^*_{\ell_2\ell'}\mathcal{C}^{LM}_{\ell_1 m_1\ell m}\mathcal{C}^{L'M'}_{\ell_2 m_2\ell' m'}Y_{\ell_1 m_1}(\hat{k})Y^*_{\ell_2 m_2}(\hat{k}')\tilde{\Delta}^{LM}_{\ell_1\ell}(k,\tau)\times[\tilde{\Delta}^{L'M'}_{\ell_2\ell'}(k',\tau)]^*. \quad (14)$$

The most general power spectrum (under statistical homogeneity) depends on the direction $\hat{k}$,

$$\langle\phi(\vec{k})\phi^*(\vec{k}')\rangle = P(\vec{k})\delta(\vec{k}-\vec{k}'). \quad (15)$$

Further, it is useful to parametrize the directional dependence of $\hat{k}$ in $P(\vec{k})$ as [43]

$$P(\vec{k}) = P_0(k)\left[1+\sum_{\mathfrak{l}>0}\sum_{\mathfrak{m}=-\mathfrak{l}}^{\mathfrak{l}} g_{\mathfrak{l}\mathfrak{m}}(k)Y_{\mathfrak{l}\mathfrak{m}}(\hat{k})\right], \quad (16)$$

where the first term with $\mathfrak{l}=0$ represents the statistical homogeneous and isotropic primordial power spectrum.

For a directional dependent power spectrum, the angular correlations of temperature anisotropy can be written as

$$\begin{aligned}\langle a_{\ell m}a^*_{\ell' m'}\rangle = 4\pi\int\frac{k^2dk}{2\pi^2}P_0(k)\sum_{\ell_1 m_1 LML'M'}&[D_1\tilde{\Delta}^{LM}_{\ell_1\ell}(k,\tau)\\ &\times[\tilde{\Delta}^{L'M'}_{\ell_1\ell'}(k,\tau)]^* + \sum_{\ell_2 m_2\mathfrak{l}\mathfrak{m}}D_2 g_{\mathfrak{l}\mathfrak{m}}(k)\tilde{\Delta}^{LM}_{\ell_1\ell}(k,\tau)\\ &\times[\tilde{\Delta}^{L'M'}_{\ell_2\ell'}(k,\tau)]^*\mathcal{Y}^{\ell_2 m_2}_{\ell_1 m_1,\mathfrak{l}\mathfrak{m}}],\end{aligned} \quad (17)$$

where

$$\begin{aligned}D_1 &= \beta_{\ell_1\ell}\beta^*_{\ell_1\ell'}\mathcal{C}^{LM}_{\ell_1 m_1\ell m}\mathcal{C}^{L'M'}_{\ell_1 m_1\ell' m'},\\ D_2 &= \beta_{\ell_1\ell}\beta^*_{\ell_2\ell'}\mathcal{C}^{LM}_{\ell_1 m_1\ell m}\mathcal{C}^{L'M'}_{\ell_2 m_2\ell' m'},\\ \mathcal{Y}^{\ell_2 m_2}_{\ell_1 m_1,\mathfrak{l}\mathfrak{m}} &= \int d\Omega_{\hat{k}}Y_{\ell_1 m_1}(k)Y^*_{\ell_2 m_2}(k)Y_{\mathfrak{l}\mathfrak{m}}(k)\\ &= \frac{\Pi_{\ell_1\mathfrak{l}}}{\sqrt{4\pi}\Pi_{\ell_2}}\mathcal{C}^{\ell_2 0}_{\ell_1 0\mathfrak{l}0}\mathcal{C}^{\ell_2 m_2}_{\ell_1 m_1\mathfrak{l}\mathfrak{m}},\end{aligned}$$

where we have used the expression for an integral of three spherical harmonics as in Eq. (5.9.4) from [51].

Here $\Pi_{\ell_1\ell_2\ldots\ell_n} = [(2\ell_1+1)(2\ell_2+1)\ldots(2\ell_n+1)]^{1/2}$ has been defined for convenient notational simplicity.

The case $L=0$ reduces Eq. (17) to that in the analysis [43]

$$\langle a_{\ell m}a^*_{\ell' m'}\rangle = \frac{(-1)^\ell}{\Pi_\ell}A^{00}_{\ell\ell'}\delta_{\ell\ell'}\delta_{mm'} + (-1)^{\ell'+m'}\sum_{\mathfrak{l},\mathfrak{m}}A^{\mathfrak{l}\mathfrak{m}}_{\ell\ell'}\mathcal{C}^{\mathfrak{l}-\mathfrak{m}}_{\ell m\ell' m'}, \quad (18)$$

where statistical anisotropy is quantified by the BipoSH coefficients [37,39–41], defined as a tensor product of the spherical harmonic coefficients $a_{\ell m}$ and $a_{\ell' m'}$,

$$\begin{aligned}A^{JN}_{\ell\ell'} &= \sum_{mm'}\langle a_{\ell m}a^*_{\ell' m'}\rangle(-1)^{m'}\mathcal{C}^{JN}_{\ell m\ell'-m'} = \{a_\ell\otimes a_{\ell'}\}_{JN}\\ A^{00}_{\ell\ell} &= (-1)^\ell\Pi_\ell C_\ell.\end{aligned} \quad (19)$$

Here, C_ℓ is the usual CMB power spectrum for the SI case. Directional dependent $P(\hat{k})$ [43] introduces the second term in Eq. (18):

$$A^{\mathfrak{l}\mathfrak{m}}_{\ell\ell'} = \sqrt{4\pi}(-i)^{\ell+\ell'}\frac{\Pi_\ell\Pi_{\ell'}}{\Pi_{\mathfrak{l}}}\int\frac{k^2dk}{2\pi^2}P_0(k)\times g_{\mathfrak{l}\mathfrak{m}}\mathcal{C}^{\mathfrak{l}0}_{\ell 0\ell' 0}\{\tilde{\Delta}^0_{\ell\ell}\otimes\tilde{\Delta}^0_{\ell'\ell'}\}_{00}, \quad (20)$$

where a corresponding tensor product in bipolar harmonic space for the indices L and L' of CMB brightness fluctuations is defined as

$$\{\tilde{\Delta}^L_{\ell_1\ell}\otimes\tilde{\Delta}^{L'}_{\ell_2\ell'}\}_{JN} = \sum_{MM'}(-1)^{M'}\mathcal{C}^{JN}_{LML'-M'}\tilde{\Delta}^{LM}_{\ell_1\ell}(k,\tau)\times[\tilde{\Delta}^{L'M'}_{\ell_2\ell'}(k,\tau)]^*. \quad (21)$$

In general for SI violations ($L>0$), the BipoSH coefficients can be expressed using the angular correlations in Eq. (17) as shown in Appendix A as

$$\begin{aligned}A^{JN}_{\ell\ell'} = (A^{JN}_{\ell\ell'})_{\mathfrak{l}=0} + \sqrt{4\pi}\int\frac{k^2dk}{2\pi^2}P_0(k)&\times\sum_{\mathfrak{l}\ell_1\ell_2LL'}\Pi_{\mathfrak{l}\ell_1LL'}(-1)^{\mathfrak{l}+L'+\ell'+\ell_1-\ell_2}\beta_{\ell_1\ell}\beta^*_{\ell_2\ell'}\mathcal{C}^{\ell_2 0}_{\ell_1 0\mathfrak{l}0}\\ &\times\sum_{\mathfrak{m}}g_{\mathfrak{l}\mathfrak{m}}\sum_{\ell_3 m_3}\begin{Bmatrix}J & \ell_3 & \mathfrak{l}\\ \ell & L & \ell_1\\ \ell' & L' & \ell_2\end{Bmatrix}\\ &\times\mathcal{C}^{JN}_{\ell_3 m_3\mathfrak{l}\mathfrak{m}}\{\tilde{\Delta}^L_{\ell_1\ell}\otimes\tilde{\Delta}^{L'}_{\ell_2\ell'}\}_{\ell_3 m_3}.\end{aligned} \quad (22)$$

The first term on the right-hand side of Eq. (22) is the contribution due to statistically isotropic primordial power spectrum, i.e. $\mathfrak{l}=0$ terms in Eq. (16). The second term gives the contribution to the bipolar coefficients due to $\mathfrak{l}>0$ in Eq. (16). The term in the first braces is the Wigner-9j symbol [51] which is related to the coefficients of transformations between different coupling schemes of four angular momenta and satisfies the triangular conditions for the triads $(J\ell_3\mathfrak{l})$, $(\ell L\ell_1)$, $(\ell' L'\ell_2)$, $(J\ell\ell')$, $(\ell_3 LL')$, and $(\mathfrak{l}\ell_1\ell_2)$.

The case for statistically isotropic brightness fluctuations, i.e. $L=0$, reduces the BipoSH coefficients in Eq. (22) to Eq. (20).

To evaluate the first term $(A^{JN}_{\ell\ell'})_{\mathfrak{l}=0}$ in Eq. (22), we consider statistically isotropic primordial perturbations

$\langle\phi(\vec{k})\phi^*(\vec{k}')\rangle = P_0(k)\delta(\vec{k}-\vec{k}')$ and express the angular correlations as

$$\langle a_{\ell m}a^*_{\ell' m'}\rangle = 4\pi\int\frac{k^2dk}{2\pi^2}P_0(k)\sum_{\ell_1 m_1 LML'M'}\beta_{\ell_1\ell}\beta^*_{\ell_1\ell'}\tilde{\Delta}^{LM}_{\ell_1\ell}(k,\tau)\times[\tilde{\Delta}^{L'M'}_{\ell_1\ell'}(k,\tau)]^*\mathcal{C}^{LM}_{\ell_1 m_1\ell m}\mathcal{C}^{L'M'}_{\ell_1 m_1\ell' m'}. \quad (23)$$

To check for the statistical isotropic case, we put $L=0$ and $M=0$ and recover Eq. (7) in Sec. II C.

As shown in detail in Appendix A, the BipoSH coefficients in Eq. (19) can be expressed using the angular correlations in Eq. (23) for statistically isotropic primordial perturbations as

$$(A^{JN}_{\ell\ell'})_{\mathrm{I}=0} = 4\pi\int\frac{k^2dk}{2\pi^2}P_0(k)\sum_{\ell_1 LL'}(-1)^{\ell_1+L+L'+J}\times\Pi_{LL'}\beta_{\ell_1\ell}\beta^*_{\ell_1\ell'}\begin{Bmatrix}L & J & L'\\ \ell' & \ell_1 & \ell\end{Bmatrix}\times\{\tilde{\Delta}^{L}_{\ell_1\ell}\otimes\tilde{\Delta}^{L'}_{\ell_1\ell'}\}_{JN}. \quad (24)$$

The first term in braces is the Wigner-6j symbol which is related to the coefficients of transformations between different coupling schemes of three angular momenta. These vanish unless the triangular conditions [51] are fulfilled for the triads (LJL'), $(L'\ell'\ell_1)$, $(J\ell'\ell)$, and $(\ell L\ell_1)$.

We consider low bipolar deviations from SI, i.e. L, L' and $J \ll \ell, \ell', \ell_1$ in Eq. (24) and use the asymptotic relation for Wigner-6j functions given by Eq. (9.9.1) from [51]. We find that the asymptotic limit to these BipoSH coefficients are

$$(A^{JN}_{\ell\ell'})_{\mathrm{I}=0} \approx 4\pi\int\frac{k^2dk}{2\pi^2}P_0(k)\sum_{\ell_1 LL'}(-1)^{\ell+\ell'+\ell_1+L+L'}\times\frac{\Pi_{LL'}}{\sqrt{2\ell_1}\Pi_J}\beta_{\ell_1\ell}\beta^*_{\ell_1\ell'}\mathcal{C}^{J(\ell-\ell')}_{L(\ell-\ell_1)L'(\ell_1-\ell')}\times\{\tilde{\Delta}^{L}_{\ell_1\ell}\otimes\tilde{\Delta}^{L'}_{\ell_1\ell'}\}_{JN}. \quad (25)$$

For diagonal brightness fluctuations, i.e. $\ell_1=\ell$ and $\ell_1=\ell'$, the BipoSH coefficients themselves turn out to be diagonal in multipole space:

$$(A^{JN}_{\ell\ell})_{\mathrm{I}=0} \approx 4\pi\int\frac{k^2dk}{2\pi^2}P_0(k)\sum_{LL'}(-1)^{L+L'+1}\times\frac{\Pi_{\ell\ell LL'}}{\Pi_J}\mathcal{C}^{J0}_{L0L'0}\{\tilde{\Delta}^{L}_{\ell\ell}\otimes\tilde{\Delta}^{L'}_{\ell\ell}\}_{JN}. \quad (26)$$

IV. EVOLUTION IN THE FREE-STREAMING REGIME

A. Generalized evolution equation

We find the moments of the CMB brightness fluctuation to be

$$\tilde{\Delta}^{LM}_{\ell_1\ell_2}(k,\tau) = \frac{1}{4\pi}\frac{1}{\beta_{\ell_1\ell_2}}\iint d\Omega_{\hat{n}}d\Omega_{\hat{k}}\tilde{\Delta}(\vec{k},\hat{n},\tau)\times\{Y_{\ell_1}(\hat{k})\otimes Y_{\ell_2}(\hat{n})\}_{LM}, \quad (27)$$

starting with Eq. (11) and the orthonormality condition of BipoSH,

$$\iint d\Omega_{\hat{n}}d\Omega_{\hat{k}}\{Y_{\ell_1}(\hat{k})\otimes Y_{\ell_2}(\hat{n})\}_{LM}\{Y_{\ell_3}(\hat{k})\otimes Y_{\ell_4}(\hat{n})\}^*_{L'M'} = \delta_{\ell_1\ell_3}\delta_{\ell_2\ell_4}\delta_{LL'}\delta_{MM'}. \quad (28)$$

In the free-streaming regime, using the plane wave approximation as in Eqs. (8) and (9), the most general evolution equation turns out to be

$$\tilde{\Delta}^{LM}_{\ell_1\ell_2}(k,\tau) = \frac{1}{\beta_{\ell_1\ell_2}}\sum_{\ell\ell_3\ell_4 L'M'}(-i)^{\ell}\Pi_{\ell\ell}j_\ell(k\Delta\tau)\beta_{\ell_3\ell_4}\times\sum_{m_1m_2m_3m_4}\tilde{\Delta}^{LM}_{\ell_3\ell_4}(k,\tau_s)\mathcal{C}^{LM}_{\ell_1m_1\ell_2m_2}\mathcal{C}^{L'M'}_{\ell_3m_3\ell_4m_4}\times\iint d\Omega_{\hat{n}}d\hat{k}P_\ell(\hat{k}\cdot\hat{n})Y^*_{\ell_1m_1}(\hat{k})Y^*_{\ell_2m_2}(\hat{n})\times Y_{\ell_3m_3}(\hat{k})Y_{\ell_4m_4}(\hat{n}). \quad (29)$$

As shown in detail in Appendix B, this can be further simplified to

$$\tilde{\Delta}^{LM}_{\ell_1\ell_2}(k,\tau) = \sum_{\ell\ell_3\ell_4}(-i)^{\ell}\Pi_{\ell\ell\ell_1\ell_4}\frac{\beta_{\ell_3\ell_4}}{\beta_{\ell_1\ell_2}}(-1)^{\ell_3+\ell_4+L}\times j_\ell(k\Delta\tau)\tilde{\Delta}^{LM}_{\ell_3\ell_4}(k,\tau_s)\mathcal{C}^{\ell_3 0}_{\ell 0\ell_1 0}\mathcal{C}^{\ell_2 0}_{\ell 0\ell_4 0}\times\begin{Bmatrix}\ell_1 & L & \ell_2\\ \ell_4 & \ell & \ell_3\end{Bmatrix}. \quad (30)$$

The generalized evolution equation thus can be expressed so as to structurally resemble the evolution equation for the SI case in Eq. (10):

$$\tilde{\Delta}^{LM}_{\ell_1\ell_2}(k,\tau) = \sum_{\ell\ell_3\ell_4}C(L,\ell,\ell_1,\ell_2)j_\ell(k\Delta\tau)\tilde{\Delta}^{LM}_{\ell_3\ell_4}(k,\tau_s), \quad (31)$$

where

$$C(L,\ell,\ell_1,\ell_2) = (-i)^{\ell}\Pi_{\ell\ell\ell_1\ell_4}\frac{\beta_{\ell_3\ell_4}}{\beta_{\ell_1\ell_2}}(-1)^{\ell_3+\ell_4+L}\times\mathcal{C}^{\ell_3 0}_{\ell 0\ell_1 0}\mathcal{C}^{\ell_2 0}_{\ell 0\ell_4 0}\begin{Bmatrix}\ell_1 & L & \ell_2\\ \ell_4 & \ell & \ell_3\end{Bmatrix}. \quad (32)$$

Setting $L=0$, $M=0$, we recover the statistical isotropic case as in Eq. (10), Sec. II D.

The transfer of power of the statistical anisotropic terms to higher SH multipoles ℓ_1 and ℓ_2 due to free-streaming is illustrated in Figs. 1 and 2. Starting from a unit normalized $\tilde{\Delta}^{LM}_{\ell_3,\ell_4}(k,\tau_s)$, we plot the evolution of the coefficients in Eq. (32), $C(L,\ell,\ell_1,\ell_2)$ with ℓ_1 for specific values of the SH multipole moments ℓ_3 and ℓ_4. We find that the values of

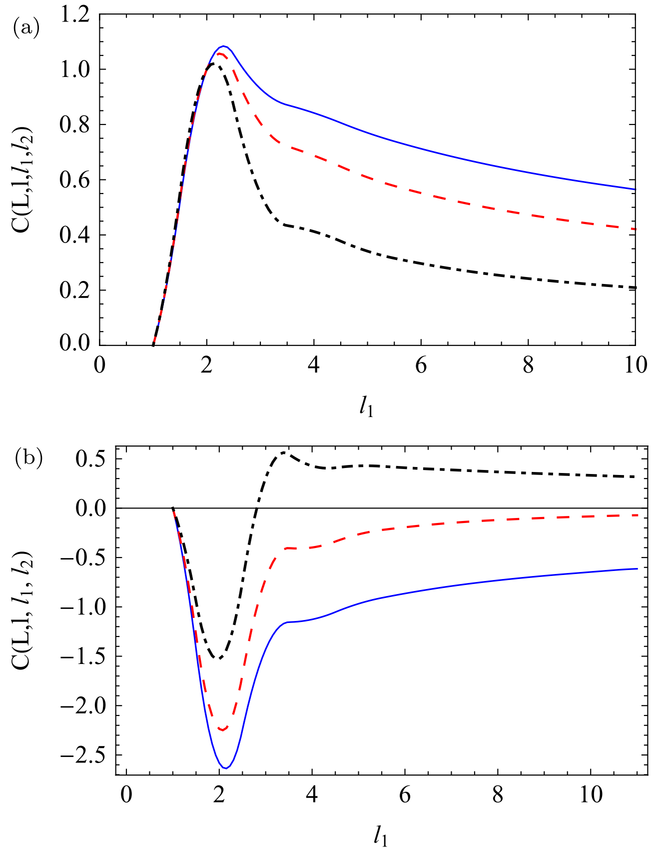

FIG. 1 (color online). Evolution of the coefficient of the spherical Bessel functions, $C(L, \ell, \ell_1, \ell_2)$ with multipole moment ℓ_1 as in Eq. (31), for nonzero diagonal terms of a unit normalized $\tilde{\Delta}^{LM}_{\ell_3\ell_4}(k, \tau_s)$. (a) Case 1: Plot of $C(L, \ell, \ell_1, \ell_2)$ for $\tilde{\Delta}^{LM}_{11}(k, \tau_s)$; with initial SH multipoles $\ell_3 = \ell_4 = 1$, the final SH multipoles being $\ell = \ell_1 - 1 (= \ell_2 - 1)$ and the index of deviation from statistical isotropy $L = 0$ (blue, solid), $L = 1$ (red, dashed), and $L = 2$ (black, dot dashed) respectively. (b) Case 2: Plot of $C(L, \ell, \ell_1, \ell_2)$ for $\tilde{\Delta}^{LM}_{22}(k, \tau_s)$; with initial multipoles $\ell_3 = \ell_4 = 2$, the final multipoles being $\ell = \ell_1 (= \ell_2)$ and the index of deviation from statistical isotropy $L = 0$ (blue, solid), $L = 1$ (red, dashed), and $L = 2$ (black, dot dashed), respectively.

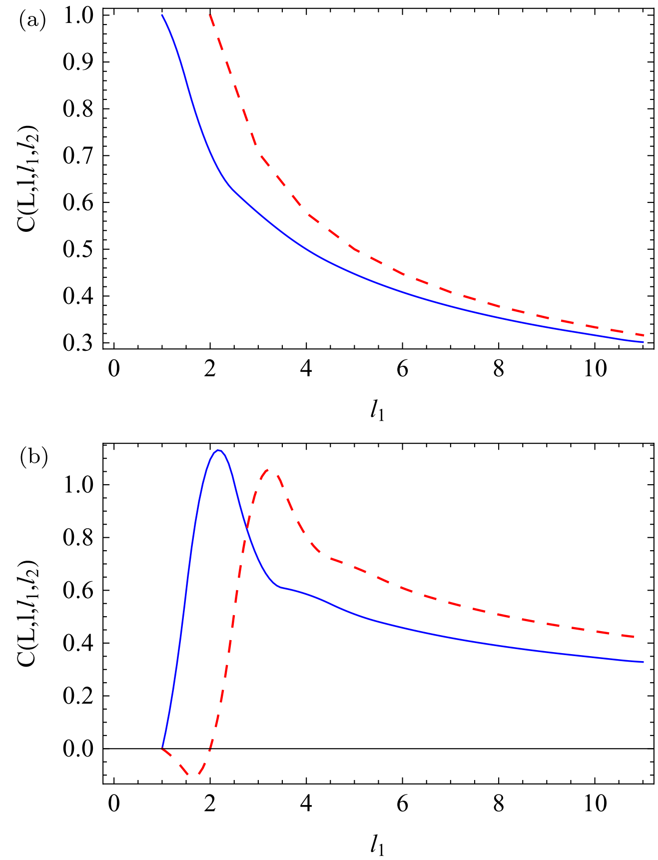

FIG. 2 (color online). Evolution of the coefficient of the spherical Bessel functions, $C(L, \ell, \ell_1, \ell_2)$ with multipole moment ℓ_1 as in Eq. (31), for nonzero off-diagonal terms of a unit normalized $\tilde{\Delta}^{LM}_{\ell_3\ell_4}(k, \tau_s)$. (a) Case 1: Plot of $C(L, \ell, \ell_1, \ell_2)$ for $\tilde{\Delta}^{LM}_{10}(k, \tau_s)$; with initial multipoles $\ell_3 = 1$, $\ell_4 = 0$, the final multipoles being $\ell = \ell_2 = \ell_1 + 1$ (blue, solid) and $\ell = \ell_2 = \ell_1 - 1$ (red, dashed) respectively. The terms are nonzero only when the index of deviation from statistical isotropy $L = |\ell_3 - \ell_4| = 1$. (b) Case 2: Plot of $C(L, \ell, \ell_1, \ell_2)$ for $\tilde{\Delta}^{LM}_{20}(k, \tau_s)$; with initial multipoles $\ell_3 = 2$, $\ell_4 = 0$, the final multipoles being $\ell = \ell_2 = \ell_1$ (blue, solid) and $\ell = \ell_2 = \ell_1 - 2$ (red, dashed) respectively. The terms are nonzero only when the index of deviation from statistical isotropy $L = |\ell_3 - \ell_4| = 2$.

ℓ and ℓ_2 are constrained by the values of ℓ_1 due to the triangular inequalities of the Clebsch-Gordan coefficients.

Figure 1 shows the evolution of these coefficients for diagonal, unit normalized CMB brightness fluctuations, namely $\tilde{\Delta}^{LM}_{11}(k, \tau_s)$ and $\tilde{\Delta}^{LM}_{22}(k, \tau_s)$ with $L = 0, 1, 2$, for possible values of ℓ and ℓ_2.

Figure 2 shows the evolution of the coefficients for off-diagonal, unit normalized $\tilde{\Delta}^{LM}_{10}(k, \tau_s)$ and $\tilde{\Delta}^{LM}_{20}(k, \tau_s)$, for possible values of L, ℓ, and ℓ_2. The coefficients for off-diagonal $\tilde{\Delta}^{LM}_{\ell_3,\ell_4}(k, \tau_s)$ vanish for $L = 0$, i.e. the statistical isotropic case.

B. SI violation at large multipoles

The ability to measure violation of statistical isotropy at low SH multipoles is largely compromised by cosmic variance. At larger SH multipoles, the effect due to violation of SI would become prominent. The previous section shows that power is transferred from small to large SH multipoles during free streaming. This opens the door to more readily measurable SI violations arising from SI violation induced due to physical processes (e.g., presence of magnetic fields, or other breakdown of rotational symmetries) in the baryon-photon plasma.

Because of the tight coupling in the baryon-photon plasma prior to τ_s, primordial SI violation can be expected to be limited to small SH multipoles. It is illuminating then to obtain an expression for the free streaming of BipoSH brightness fluctuations at small SH multipole moments at the last scattering to large SH multipoles at the present epoch for which we essentially evaluate the asymptotic limit of the CMB brightness fluctuations today.

In Eq. (31) we take ℓ_1 and ℓ_2 to be large compared to ℓ_3 and ℓ_4. From symmetry of Wigner-6j symbols,

$$\begin{Bmatrix} \ell_1 & \ell & \ell_3 \\ \ell_4 & L & \ell_2 \end{Bmatrix} = \begin{Bmatrix} \ell_4 & L & \ell_3 \\ \ell_1 & \ell & \ell_2 \end{Bmatrix}. \tag{33}$$

We evaluate the asymptotic limit with arbitrary values of ℓ, ℓ_1, ℓ_2, ℓ_3, and ℓ_4 using Eq. (9.9.1) from [51] as,

$$\begin{Bmatrix} \ell_4 & L & \ell_3 \\ R+d & R+e & R+f \end{Bmatrix} \approx \frac{(-1)^{\ell_4+L+d+e}}{\sqrt{2\ell}\Pi_{\ell_3}} \mathcal{C}^{\ell_3(d-e)}_{\ell_4(f-e)L(d-f)}, \tag{34}$$

where R is large and

$$d = \ell_1 - R \qquad e = \ell - R \qquad f = \ell_2 - R.$$

Thus for $\ell_1, \ell_2 \gg \ell_3, \ell_4$, Eq. (34) becomes

$$\begin{Bmatrix} \ell_4 & L & \ell_3 \\ \ell_1 & \ell & \ell_2 \end{Bmatrix} \approx \frac{(-1)^{\ell_4+L+\ell_1+\ell-2R}}{\sqrt{2\ell}\Pi_{\ell_3}} \mathcal{C}^{\ell_3(\ell_1-\ell)}_{\ell_4(\ell_2-\ell)L(\ell_1-\ell_2)}. \tag{35}$$

Putting the expression for the Wigner-6j symbols from Eq. (35), the asymptotic limit of the generalized evolution equation (31) takes the form

$$\tilde{\Delta}^{LM}_{\ell_1\ell_2}(k,\tau) = \sum_{\ell\ell_3\ell_4} (-i)^{\ell} \frac{\Pi_{\ell\ell\ell_1\ell_4}}{\sqrt{2\ell}\Pi_{\ell_3}} \frac{\beta_{\ell_3\ell_4}}{\beta_{\ell_1\ell_2}} (-1)^{\ell+\ell_1+\ell_3} j_\ell(k\Delta\tau) \times \tilde{\Delta}^{LM}_{\ell_3\ell_4}(k,\tau_s) \mathcal{C}^{\ell_3 0}_{\ell 0 \ell_1 0} \mathcal{C}^{\ell_2 0}_{\ell 0 \ell_4 0} \mathcal{C}^{\ell_3(\ell_1-\ell)}_{\ell_4(\ell_2-\ell)L(\ell_1-\ell_2)}. \tag{36}$$

Equation (36) depicts how power in SI violating terms at small SH multipoles ℓ_3, ℓ_4 at the last scattering, free-stream to higher SH multipoles ℓ_1, ℓ_2 at the present epoch. Note the structural similarity to Eq. (10) for the statistical isotropic case.

It is useful to provide explicit expressions for Eq. (36) in two particular cases, when the asymptotic moments of the CMB brightness fluctuation contain only diagonal terms and off-diagonal terms, respectively.

The evolution equation which involves only the diagonal terms of the moments of the brightness fluctuation at last scattering are

$$\tilde{\Delta}^{LM}_{\ell_1,\ell_1}(k,\tau) = \sum_{\ell} C_1(L,\ell,\ell_1) j_\ell(k\Delta\tau) \tilde{\Delta}^{LM}_{\ell_1-\ell,\ell_1-\ell}(k,\tau_s), \tag{37}$$

with

$$C_1(L,\ell,\ell_1) = \frac{\Pi_{\ell\ell\ell_1\ell_3}}{\sqrt{2\ell}} \left[\frac{\ell_1!}{\ell!\ell_3!}\right]^2 \frac{(2\ell)!(2\ell_3+1)!}{(2\ell_1+1)!} \times \delta_{\ell_3,\ell_1-\ell} \sqrt{\frac{(2\ell_3)!}{(2\ell_3-L)!} \frac{(2\ell_3)!}{(2\ell_3+L+1)!}}, \tag{38}$$

where $\ell, \ell_1 \gg |\ell_1 - \ell|$. The details are given in Appendix C. The term under the square root captures the L dependence of the free streaming of $L > 0$ terms.

The evolution equation which involves only the off-diagonal terms of the moments of the brightness fluctuation at last scattering are

$$\tilde{\Delta}^{LM}_{\ell_1,\ell_2}(k,\tau) = \sum_{\ell} C_2(\ell,\ell_1,\ell_2) j_\ell(k\Delta\tau) \tilde{\Delta}^{LM}_{\ell_1-\ell,\ell_2-\ell}(k,\tau_s) \times \delta_{L,\ell_1-\ell_2}, \tag{39}$$

with

$$C_2(\ell,\ell_1,\ell_2) = \frac{(-1)^{(\ell_1+\ell_2)/2}}{\sqrt{2\ell}} \frac{\Pi_{\ell\ell\ell_1\ell_2}}{\Pi_{\ell_1-\ell}} \sqrt{\frac{\ell_1+\ell_2-2\ell+1}{\ell_1+\ell_2+1}} \times \frac{(2\ell)!\ell_1!\ell_2!}{(\ell!)^2(\ell_1-\ell)!(\ell_2-\ell)!} \times \left[\frac{(2\ell_1-2\ell+1)!(2\ell_2-2\ell+1)!}{(2\ell_1+1)!(2\ell_2+1)!}\right]^{1/2}, \tag{40}$$

where $\ell, \ell_n \gg |\ell_n - \ell|, L$ with $n = 1, 2$. The details are given in Appendix C. As in Eq. (38), the term under the square root captures the L dependence of the free-streaming of $L > 0$ terms.

C. SI violating physical effects at last scattering

The patterns of the CMB temperature field, i.e. the angular correlations observed today, are traced back to inhomogeneities at the last scattering surface. In the tight coupling regime of the baryon-photon fluid, one expects power only at small SH multipoles of the $\tilde{\Delta}^{LM}_{\ell_1\ell_2}(k,\tau_s)$. The generalized evolution equation of the CMB brightness fluctuations (29) free-streams this power at small SH multipoles in both SI and non-SI moments with the same bipolar moment L, retaining $|\ell_1 - \ell_2|$. Any observed violation of SI today is easier to interpret as generalized moments arising due to simple physics just beyond the fluid approximation regime. In this section, we illustrate this point explicitly for SI violation in the CMB anisotropy in the presence of a homogeneous magnetic field at last scattering.

The SH coefficients $a_{\ell m}$ can be expressed in terms of the CMB brightness fluctuations at last scattering using Eq. (13) and the generalized evolution equation (31) as

$$a_{\ell m} = 4\pi \int \frac{d^3k}{(2\pi)^3} \phi(\vec{k}) \sum_{\ell_1 m_1 LM} \beta_{\ell_1 \ell} C^{LM}_{\ell_1 m_1 \ell m} Y_{\ell_1 m_1}(\hat{k}) \times \sum_{\ell_2 \ell_3 \ell_4} C(L, \ell_2, \ell_1, \ell) j_{\ell_2}(k\Delta\tau) \tilde{\Delta}^{LM}_{\ell_3 \ell_4}(k, \tau_s), \quad (41)$$

where $C(L, \ell_2, \ell_1, \ell)$ is defined in Eq. (32).

Hence, in general $\langle a_{\ell m} a_{\ell' m'} \rangle$ correlations measured at present are related to $\langle \tilde{\Delta}^{LM}_{\ell_1 \ell_2}(k, \tau_s) \tilde{\Delta}^{L'M'}_{\ell_3 \ell_4}(k, \tau_s) \rangle$ correlation between the generalized Boltzmann fluctuations at the last scattering surface.

In particular, SI violation encoded in the off-diagonal correlation $\langle a_{\ell m} a_{\ell' m'} \rangle$ (and nonzero BipoSH $A^{LM}_{\ell \ell'}$, $L > 0$) is related as

$$A^{JN}_{\ell \ell'} \sim \{\tilde{\Delta}^{L}_{\ell_1 \ell}(k, \tau_s) \otimes \tilde{\Delta}^{L'}_{\ell_2 \ell'}(k, \tau_s)\}_{JN} \quad (42)$$

as in Eqs. (24)–(26) or more generally for different bipolar coefficients J' and N' as in Eq. (22) when the power spectrum is also anisotropic.

Using Eq. (11), correlations of the CMB brightness fluctuations are

$$\langle \Delta(\vec{k}, \hat{n}, \tau_s) \Delta(\vec{k}, \hat{n}', \tau_s) \rangle = (4\pi)^2 \sum_{\ell_1 \ell_2 LM} \sum_{\ell_3 \ell_4 L'M'} \beta_{\ell_1 \ell_2} \beta_{\ell_3 \ell_4} \tilde{\Delta}^{LM}_{\ell_1 \ell_2}(k, \tau_s) \times \tilde{\Delta}^{L'M'}_{\ell_3 \ell_4}(k, \tau_s) \{Y_{\ell_1}(\hat{k}) \otimes Y_{\ell_2}(\hat{n})\}_{LM} \{Y_{\ell_3}(\hat{k}) \otimes Y_{\ell_4}(\hat{n}')\}_{L'M'}. \quad (43)$$

We illustrate the generality and power of our formalism using the case for a uniform magnetic field. We show the correlations of the CMB brightness fluctuations in this particular case are sourced by the bipolar dipole ($L = 1$) terms of Eq. (11) with $\ell_1 = \ell_2 = 1$, where

$$\Delta(\vec{k}, \hat{n}, \tau_s) = 4\pi i \sqrt{3} \sum_M \tilde{\Delta}^{1M}_{11}(k, \tau_s) \{Y_1(\hat{k}) \otimes Y_1(\hat{n})\}_{1M} = 3i \sqrt{\frac{3}{2}} \sum_M \tilde{\Delta}^{1M}_{11}(k, \tau_s) (\hat{k} \times \hat{n})_M, \quad (44)$$

with $M = \{-1, 0, +1\}$. Here $(\hat{k} \times \hat{n})_M$ is the usual cross product written as irreducible products of the rotation group [51].

Using standard vector identity [51] and Eq. (8),

$$(\hat{n} \cdot \hat{n}')(\hat{k} \cdot \hat{k}) - (\hat{n} \cdot \hat{k})(\hat{n}' \cdot \hat{k}) = (\hat{n} \times \hat{k}) \cdot (\hat{n}' \times \hat{k}), \quad (45)$$

the temperature correlations in the presence of a uniform magnetic field as discussed in [30] [see Eq. (A3)] are given by

$$\langle \Delta(\vec{k}, \hat{n}, \tau_s) \Delta(\vec{k}, \hat{n}', \tau_s) \rangle \propto \tilde{\Delta}^{1M}_{11}(k, \tau_s) \tilde{\Delta}^{1M'}_{11}(k, \tau_s) \times (\hat{n} \times \hat{k}) \cdot (\hat{n}' \times \hat{k}). \quad (46)$$

Here the bipolar dipole terms of the CMB brightness fluctuation $\tilde{\Delta}^{1M}_{11}(k, \tau_s) \tilde{\Delta}^{1M'}_{11}(k, \tau_s)$ in Eq. (44) encapsulates the source term due to the presence of a uniform magnetic field [30].

In Eq. (41), the bipolar dipole terms $\tilde{\Delta}^{1M}_{11}(k, \tau_s)$ give rise to SH coefficients:

$$a_{\ell m} = 4\pi \int \frac{d^3k}{(2\pi)^3} \phi(\vec{k}) \sum_{m_1 M} \tilde{\Delta}^{1M}_{11}(k, \tau_s) \times \delta_{M, m+m_1} [\{(\cdots) Y_{\ell - 2 m_1}(\hat{k}) + (\cdots) Y_{\ell m_1}(\hat{k})\} j_{\ell-1}(k\Delta\tau) + \{(\cdots) Y_{\ell m_1}(\hat{k}) + (\cdots) Y_{\ell + 2 m_1}(\hat{k})\} j_{\ell+1}(k\Delta\tau)]. \quad (47)$$

The angular correlations in this case turn out to be

$$\langle a_{\ell m} a^*_{\ell' m'} \rangle = (\cdots) \delta_{\ell' \ell} \delta_{mm'} + (\cdots) \delta_{\ell' \ell \pm 2} \delta_{mm'}. \quad (48)$$

It is interesting to note that the known diagonal ($\ell' = \ell$) and off-diagonal ($\ell' = \ell \pm 2$) correlations in the presence of a homogeneous magnetic field [25,30,31], can be easily recovered in our approach. Work is in progress to relate other cases of SI violation originating in the physics at the last scattering surface using this formalism.

V. CONCLUSIONS

The search for subtle statistical isotropy breakdown in the universe is highly motivated by numerous theoretical scenarios. The fluctuations in the cosmic microwave background are arguably the most promising observational probe of the SI of the universe. The violation of SI could have its origin not only in the anisotropic primordial power spectrum, but also in the SI violation in the fluctuations of the baryon-photon fluid at last scattering. SI deviations generated by a general form of anisotropic primordial power spectrum for isotropic Boltzmann functions have been studied in the recent literature [43]. This paper includes this equally important possibility of a general SI breakdown in the CMB photon distribution function. We study the generalized case of SI violation in terms of bipolar spherical harmonic (BipoSH) brightness fluctuations, substantially extending the scope of origin of SI violation solely from the anisotropic primordial power spectrum.

The breakdown of SI in the CMB brightness fluctuation results in off-diagonal terms in the SH space angular correlations $\langle a_{\ell m} a_{\ell' m'} \rangle$, or, equivalently, in the coefficients of the bipolar spherical harmonic (BipoSH) representation [39,40]. We relate the measurable BipoSH coefficients to SI deviations in the baryon-photon fluid as well as the primordial power spectrum. The observable BipoSH coefficients can be compactly expressed in terms of BipoSH brightness fluctuations terms through products of standard Clebsch-Gordan coefficients and a Wigner-9j function. We also present the expression for the simpler case of an isotropic primordial power spectrum, where the BipoSH coefficients turn out to be given through a compact combination of a Wigner-6j symbol and a Clebsch-Gordan coefficient. We also provide the large SH multipole limit for these coefficients for the terms encoding deviations from SI at low BipoSH multipoles.

We obtain the generalized free-stream evolution equation for the SI violation encoded in terms of the BipoSH brightness fluctuations introduced in our work. We demonstrate that different modes of BipoSH brightness fluctuations at the present epoch have to evolve from same bipolar modes at the last scattering. The moments of the CMB brightness fluctuations $\tilde{\Delta}^{LM}_{\ell_3\ell_4}(\hat{k}, \tau_s)$ at last scattering are expected to be nonzero at small values of SH multipoles ℓ_3 and ℓ_4 due to tight coupling. However, our results show that the power in these SI violating terms at low SH multipoles would be transferred during free-stream evolution to higher multipoles ℓ_1 and ℓ_2 in $\tilde{\Delta}^{LM}_{\ell_1\ell_2}(\hat{k}, \tau)$ at the present epoch, retaining LM and $|\ell_3 - \ell_4|$. This is akin to the well-known free-streaming evolution of power in the SI brightness fluctuation at low SH multipole power at last scattering to large SH multipole at present. For clearly highlighting the structural similarity, we present the evolution of BipoSH brightness fluctuations in the asymptotic case of large values of the final SH multipoles today relative to the initial SH multipoles at last scattering.

While much of the claimed observational evidence of SI breakdown, such as the "axis of evil," "north-south asymmetry," etc., pertains to relatively small values of the SH multipoles, the significance is largely obscured by dominance of cosmic variance. However, SI violation at small SH multipoles in the baryon-photon plasma at last scattering would free-stream to large SH multipoles at present and, consequently, would be easier to establish from CMB observations. A program of study to relate the BipoSH brightness fluctuations in the baryon-photon fluid for different physical scenarios is currently underway. We have used our formalism to represent and match the well-known case for SI violation in the presence of a homogeneous magnetic field. We illustrate how the angular correlations in such a case could be seeded by the dipole term of the generalized CMB brightness fluctuation and would have diagonal ($\ell' = \ell$) and off-diagonal ($\ell' = \ell \pm 2$) terms.

In summary, our work strongly motivates closer study of all possible SI violating phenomena and scenarios in the simple baryon-photon plasma, since these could potentially provide more readily observable signature of SI violation in the universe. It is also encouraging that it may have observational implications in light of the recent WMAP-7 discovery [4]. Since, the quadrupolar anisotropy anomaly with nonzero BipoSH coefficients related as $A^{2M}_{\ell\ell} \sim -2A^{2M}_{\ell-2\ell}$ rules out an origin in anisotropic power spectrum, this may well be related to SI violations in the CMB brightness fluctuations. This possibility is also bolstered by the fact that the non-SI effect peaks at acoustic $l \sim 200$ scales pointing to some nontrivial physics at the last scattering surface. Extension of this formalism to CMB polarization should be readily possible. The formalism and the initial conclusions are important and timely in light of higher precision and resolution CMB anisotropy and polarization data expected in the near future, in particular, from the ongoing Planck Surveyor CMB mission.

APPENDIX A: BIPOLAR COEFFICIENTS FOR SI DEVIATIONS

The BipoSH coefficients are defined in Eq. (19). For a directional dependent primordial power spectrum as in Eq. (16), these bipolar coefficients can be evaluated using the angular correlations in Eq. (17) in the following way:

$$A^{JN}_{\ell\ell'} = (A^{JN}_{\ell\ell'})_{\mathfrak{l}=0} + \sqrt{4\pi}\int \frac{k^2 dk}{2\pi^2} P_0(k) \sum_{\mathfrak{l}\ell_1\ell_2 LL'} \frac{\Pi_{\ell_1\ell_3}}{\Pi_{\ell_2}} \times \beta_{\ell_1\ell}\beta^*_{\ell_2\ell'}\mathcal{C}^{\ell_2 0}_{\ell_1 0\mathfrak{l}0}\sum_{\mathfrak{m}} g_{\mathfrak{l}\mathfrak{m}} \sum_{MM'} \tilde{\Delta}^{LM}_{\ell_1\ell}(k,\tau)[\tilde{\Delta}^{L'M'}_{\ell_2\ell'}(k,\tau)]^* \times \sum_{m_1m_2mm'} (-1)^{m'} \mathcal{C}^{LM}_{\ell_1m_1\ell m}\mathcal{C}^{L'M'}_{\ell_2m_2\ell'm'}\mathcal{C}^{JN}_{\ell m\ell'-m'}\mathcal{C}^{\ell_2m_2}_{\ell_1m_1\mathfrak{l}\mathfrak{m}}. \quad \text{(A1)}$$

We use the following symmetry properties of the Clebsch-Gordan coefficients in Eq. (8.4.10) from [51],

$$\mathcal{C}^{\ell_2m_2}_{\ell_1m_1\mathfrak{l}\mathfrak{m}} = (-1)^{\mathfrak{l}+\ell_1-\ell_2}\mathcal{C}^{\ell_2-m_2}_{\ell_1-m_1\mathfrak{l}-\mathfrak{m}} = (-1)^{\ell_1-\ell_2}\frac{\Pi_{\ell_2}}{\Pi_{\ell_1}}(-1)^{\mathfrak{m}}\mathcal{C}^{\ell_1m_1}_{\ell_2m_2\mathfrak{l}-\mathfrak{m}} \quad \text{(A2)}$$

and

$$\mathcal{C}^{JN}_{\ell m\ell'-m'} = (-1)^{\ell'-m'}\frac{\Pi_J}{\Pi_\ell}\mathcal{C}^{\ell m}_{\ell'm'JN}, \quad \text{(A3)}$$

where $\Pi_{\ell_1\ell_2\ldots\ell_n} = [(2\ell_1+1)(2\ell_2+1)\ldots(2\ell_n+1)]^{1/2}$ has been defined for convenient notational simplicity. We use the formula for summation of the product of four Clebsch-Gordan coefficients given in Eq. (8.7.26) from [51]:

$$\sum_{m_1m_2mm'} \mathcal{C}^{LM}_{\ell_1m_1\ell m}\mathcal{C}^{L'M'}_{\ell_2m_2\ell'm'}\mathcal{C}^{\ell m}_{\ell'm'JN}\mathcal{C}^{\ell_1m_1}_{\ell_2m_2\mathfrak{l}-\mathfrak{m}} = \sum_{\ell_3m_3}\Pi_{\ell\ell_1L'\ell_3}\mathcal{C}^{\ell_3m_3}_{L'M'JN}\mathcal{C}^{LM}_{L'M'\ell_3m_3}\begin{Bmatrix} L & \ell_1 & \ell \\ L' & \ell_2 & \ell' \\ \ell_3 & \mathfrak{l} & J \end{Bmatrix}. \quad \text{(A4)}$$

The bipolar coefficients in Eq. (A1) simplify to

$$A^{JN}_{\ell\ell'} = (A^{JN}_{\ell\ell'})_{\mathfrak{l}=0} + \sqrt{4\pi}\int \frac{k^2 dk}{2\pi^2} P_0(k) \sum_{\mathfrak{l}\ell_1\ell_2 LL'} \Pi_{\mathfrak{l}\ell_1 LL'}(-1)^{\mathfrak{l}+L'+\ell'+\ell_1-\ell_2} \beta_{\ell_1\ell}\beta^*_{\ell_2\ell'}\mathcal{C}^{\ell_2 0}_{\ell_1 0\mathfrak{l}0}\sum_{\mathfrak{l}} g_{\mathfrak{l}\mathfrak{m}} \sum_{\ell_3 m_3}\begin{Bmatrix} J & \ell_3 & \mathfrak{l} \\ \ell & L & \ell_1 \\ \ell' & L' & \ell_2 \end{Bmatrix}$$
$$\times\, \mathcal{C}^{JN}_{\ell_3 m_3 \mathfrak{l}\mathfrak{m}}\{\tilde{\Delta}^{L}_{\ell_1\ell}\otimes\tilde{\Delta}^{L'}_{\ell_2\ell'}\}_{\ell_3 m_3}. \qquad \text{(A5)}$$

$\{\tilde{\Delta}^{L}_{\ell_1\ell}\otimes\tilde{\Delta}^{L'}_{\ell_1\ell'}\}_{JN}$ are the bipolar products in L and L' as defined in Eq. (21).

For statistically isotropic primordial perturbations, the angular correlation in Eq. (23) can be written in terms of bipolar coefficients as

$$(A^{JN}_{\ell\ell'})_{\mathfrak{l}=0} = 4\pi\int\frac{k^2dk}{2\pi^2}P_0(k)\sum_{\ell_1 LL'}\beta_{\ell_1\ell}\beta^*_{\ell_1\ell'}\sum_{MM'}\tilde{\Delta}^{LM}_{\ell_1\ell}(k,\tau)[\tilde{\Delta}^{L'M'}_{\ell_1\ell'}(k,\tau)]^*\sum_{m_1mm'}(-1)^{m'}\mathcal{C}^{JN}_{\ell m\ell'-m'}\mathcal{C}^{LM}_{\ell_1m_1\ell m}\mathcal{C}^{L'M'}_{\ell_1m_1\ell'm'}. \qquad \text{(A6)}$$

The summation in the above equation can be simplified using Eq. (8.7.17) from [51] as follows:

$$\begin{aligned}\sum_{m_1mm'}(-1)^{m'}\mathcal{C}^{JN}_{\ell m\ell'-m'}\mathcal{C}^{LM}_{\ell_1m_1\ell m}\mathcal{C}^{L'M'}_{\ell_1m_1\ell'm'} &= (-1)^{\ell_1+L+J}\Pi_{L'J}\mathcal{C}^{LM}_{L'M'JN}\begin{Bmatrix}\ell' & \ell_1 & L'\\ L & J & \ell\end{Bmatrix}\\ &= (-1)^{\ell_1+L+J}\Pi_{L'J}(-1)^{L'-M'}\frac{\Pi_L}{\Pi_J}\mathcal{C}^{JN}_{LML'-M'}\begin{Bmatrix}L & J & L'\\ \ell' & \ell_1 & \ell\end{Bmatrix}\\ &= (-1)^{\ell_1+L+L'+J}\Pi_{LL'}(-1)^{M'}\mathcal{C}^{JN}_{LML'-M'}\begin{Bmatrix}L & J & L'\\ \ell' & \ell_1 & \ell\end{Bmatrix}.\end{aligned} \qquad \text{(A7)}$$

Thus the BipoSH coefficient can be simplified to

$$\begin{aligned}(A^{JN}_{\ell\ell'})_{\mathfrak{l}=0} &= 4\pi\int\frac{k^2dk}{2\pi^2}P_0(k)\sum_{\ell_1LL'}(-1)^{\ell_1+L+L'+J}\Pi_{LL'}\beta_{\ell_1\ell}\beta^*_{\ell_1\ell'}\begin{Bmatrix}L & J & L'\\ \ell' & \ell_1 & \ell\end{Bmatrix}\sum_{MM'}\tilde{\Delta}^{LM}_{\ell_1\ell}(k,\tau)[\tilde{\Delta}^{L'M'}_{\ell_1\ell'}(k,\tau)]^*(-1)^{M'}\mathcal{C}^{JN}_{LML'-M'}\\ &= 4\pi\int\frac{k^2dk}{2\pi^2}P_0(k)\sum_{\ell_1LL'}(-1)^{\ell_1+L+L'+J}\Pi_{LL'}\beta_{\ell_1\ell}\beta^*_{\ell_1\ell'}\begin{Bmatrix}L & J & L'\\ \ell' & \ell_1 & \ell\end{Bmatrix}\{\tilde{\Delta}^{L}_{\ell_1\ell}\otimes\tilde{\Delta}^{L'}_{\ell_1\ell'}\}_{JN},\end{aligned} \qquad \text{(A8)}$$

where symmetry properties of the Clebsch-Gordan coefficients, Eqs. (8.4.10) in [51] has been used. To consider low bipolar deviations from statistical isotropy, i.e. $(L, L', J \ll \ell, \ell', \ell_1)$, we use the asymptotic relation for the Wigner-6j function as given in Eq. (9.9.1) from [51]:

$$\begin{Bmatrix}L & J & L'\\ \ell' & \ell_1 & \ell\end{Bmatrix} \approx \frac{(-1)^{L+J+\ell'-\ell_1}}{\sqrt{2\ell_1}\Pi_{L'}}\mathcal{C}^{L'(\ell'-\ell_1)}_{L(\ell-\ell_1)J(\ell'-\ell)} \approx \frac{(-1)^{J+\ell+\ell'}}{\sqrt{2\ell_1}\Pi_J}\mathcal{C}^{J(\ell-\ell')}_{L(\ell-\ell_1)L'(\ell_1-\ell')}. \qquad \text{(A9)}$$

Using the above relation, the asymptotic limit to the BipoSH coefficients turn out to be

$$(A^{JN}_{\ell\ell'})_{\mathfrak{l}=0} \approx 4\pi\int\frac{k^2dk}{2\pi^2}P_0(k)\sum_{\ell_1LL'}(-1)^{\ell+\ell'+\ell_1+L+L'}\frac{\Pi_{LL'}}{\sqrt{2\ell_1}\Pi_J}\beta_{\ell_1\ell}\beta^*_{\ell_1\ell'}\mathcal{C}^{J(\ell-\ell')}_{L(\ell-\ell_1)L'(\ell_1-\ell')}\{\tilde{\Delta}^{L}_{\ell_1\ell}\otimes\tilde{\Delta}^{L'}_{\ell_1\ell'}\}_{JN}. \qquad \text{(A10)}$$

APPENDIX B: GENERALIZED EVOLUTION EQUATION FOR STATISTICAL ISOTROPY BREAKDOWN

Starting with Eq. (29), expanding $P_\ell(\hat{k}\cdot\hat{n}) = \frac{4\pi}{2\ell+1}\sum_m Y^*_{\ell m}(\hat{k})Y_{\ell m}(\hat{n})$ and evaluating the double integral using the equation

$$\int d\Omega_{\hat{n}}Y_{\ell m}(\hat{n})Y_{\ell_1m_1}(\hat{n})Y_{\ell_2m_2}(\hat{n})^* = \frac{1}{\sqrt{4\pi}}\frac{\Pi_{\ell\ell_1}}{\Pi_{\ell_2}}\mathcal{C}^{\ell_2 0}_{\ell 0\ell_1 0}\mathcal{C}^{\ell_2m_2}_{\ell m\ell_1m_1}, \qquad \text{(B1)}$$

the most general evolution equation becomes

$$\tilde{\Delta}^{\,LM}_{\ell_1\ell_2}(k,\tau) = \frac{1}{\beta_{\ell_1\ell_2}}\sum_{\ell}(-i)^{\ell}\Pi^2_{\ell}j_{\ell}(k\Delta\tau)\sum_{\ell_3\ell_4L'M'}\beta_{\ell_3\ell_4}\tilde{\Delta}^{L'M'}_{\ell_3\ell_4}(k,\tau_s)\frac{\Pi_{\ell_1\ell_4}}{\Pi_{\ell_2\ell_3}}\mathcal{C}_g, \qquad \text{(B2)}$$

where

$$\mathcal{C}_g = \mathcal{C}^{\ell_3 0}_{\ell 0 \ell_1 0}\mathcal{C}^{\ell_2 0}_{\ell 0 \ell_4 0}\sum_{m_1 m_2}\mathcal{C}^{LM}_{\ell_1 m_1 \ell_2 m_2}\sum_{m_3 m_4}\mathcal{C}^{L'M'}_{\ell_3 m_3 \ell_4 m_4}\sum_{m}\mathcal{C}^{\ell_3 m_3}_{\ell m \ell_1 m_1}\mathcal{C}^{\ell_2 m_2}_{\ell m \ell_4 m_4}. \tag{B3}$$

Using the summation formula for four Clebsch-Gordan coefficients given by Eq. (9.1.8) from [51],

$$\sum_{m m_1 m_2 m_3 m_4}\mathcal{C}^{L'M'}_{\ell_3 m_3 \ell_4 m_4}\mathcal{C}^{\ell_3 m_3}_{\ell_1 m_1 \ell m}\mathcal{C}^{LM}_{\ell_1 m_1 \ell_2 m_2}\mathcal{C}^{\ell_2 m_2}_{\ell m \ell_4 m_4} = \delta_{L'L}\delta_{M'M}(-1)^{\ell_1+\ell+\ell_4+L'}\Pi_{\ell_2\ell_3}\begin{Bmatrix}\ell_1 & \ell & \ell_3\\ \ell_4 & L' & \ell_2\end{Bmatrix}, \tag{B4}$$

Equation (B2) simplifies to

$$\begin{aligned}\tilde{\Delta}^{LM}_{\ell_1\ell_2}(k,\tau) &= \sum_{\ell\ell_3\ell_4}(-i)^{\ell}\Pi_{\ell\ell\ell_1\ell_4}\frac{\beta_{\ell_3\ell_4}}{\beta_{\ell_1\ell_2}}(-1)^{\ell_3+\ell_4+L}j_\ell(k\Delta\tau)\tilde{\Delta}^{LM}_{\ell_3\ell_4}(k,\tau_s)\mathcal{C}^{\ell_3 0}_{\ell 0\ell_1 0}\mathcal{C}^{\ell_2 0}_{\ell 0 \ell_4 0}\begin{Bmatrix}\ell_1 & \ell & \ell_3\\ \ell_4 & L & \ell_2\end{Bmatrix}\\ &= \sum_{\ell\ell_3\ell_4}(-i)^{\ell}\Pi_{\ell\ell\ell_1\ell_4}\frac{\beta_{\ell_3\ell_4}}{\beta_{\ell_1\ell_2}}(-1)^{\ell_3+\ell_4+L}j_\ell(k\Delta\tau)\tilde{\Delta}^{LM}_{\ell_3\ell_4}(k,\tau_s)\mathcal{C}^{\ell_3 0}_{\ell 0\ell_1 0}\mathcal{C}^{\ell_2 0}_{\ell 0 \ell_4 0}\begin{Bmatrix}\ell_1 & L & \ell_2\\ \ell_4 & \ell & \ell_3\end{Bmatrix}.\end{aligned} \tag{B5}$$

Substituting the multipole dependent coefficients as

$$\begin{aligned}C(L,\ell,\ell_1,\ell_2) &= (-i)^{\ell}\Pi_{\ell\ell\ell_1\ell_4}\frac{\beta_{\ell_3\ell_4}}{\beta_{\ell_1\ell_2}}(-1)^{\ell_3+\ell_4+L}\\ &\quad\times\mathcal{C}^{\ell_3 0}_{\ell 0\ell_1 0}\mathcal{C}^{\ell_2 0}_{\ell 0 \ell_4 0}\begin{Bmatrix}\ell_1 & L & \ell_2\\ \ell_4 & \ell & \ell_3\end{Bmatrix},\end{aligned} \tag{B6}$$

the generalized evolution equation for deviations of statistical isotropy in the CMB brightness fluctuation reduces to

$$\tilde{\Delta}^{LM}_{\ell_1\ell_2}(k,\tau) = \sum_{\ell\ell_3\ell_4}C(L,\ell,\ell_1,\ell_2)j_\ell(k\Delta\tau)\tilde{\Delta}^{LM}_{\ell_3\ell_4}(k,\tau_s). \tag{B7}$$

APPENDIX C: DIAGONAL TERMS OF THE ASYMPTOTIC MOMENTS OF CMB BRIGHTNESS FLUCTUATION

In Eq. (34), taking $e = 0$, i.e. $\ell = R$ and putting $\ell_1 = R + \ell_4$ and $\ell_2 = R + \ell_3$, we can write Eq. (35) as

$$\begin{Bmatrix}\ell_4 & L & \ell_3\\ \ell+\ell_4 & \ell & \ell+\ell_3\end{Bmatrix} \approx \frac{(-1)^L}{\sqrt{2\ell}\,\Pi_{\ell_3}}\mathcal{C}^{\ell_3\ell_4}_{\ell_4\ell_3 L(\ell_4-\ell_3)}. \tag{C1}$$

From the properties of the Clebsch-Gordan coefficient, we get $\ell_3 = \ell_4$ and hence $\ell_1 = \ell_2$. Thus in Eq. (31) substituting $\ell_3 = \ell_1 - \ell$, we get the factor

$$\begin{aligned}C_1(L,\ell,\ell_1) &= (-i)^{\ell}\Pi^2_{\ell}\Pi_{\ell_1}\Pi_{\ell_1-\ell}\frac{\beta_{\ell_1-\ell,\ell_1-\ell}}{\beta_{\ell_1,\ell_1}}(-1)^L\\ &\quad\times\mathcal{C}^{(\ell_1-\ell)0}_{\ell 0\ell_1 0}\mathcal{C}^{\ell_1 0}_{\ell 0(\ell_1-\ell)0}\frac{(-1)^L}{\sqrt{2\ell}\,\Pi_{\ell_1-\ell}}\\ &\quad\times\mathcal{C}^{(\ell_1-\ell)(\ell_1-\ell)}_{(\ell_1-\ell)(\ell_1-\ell)L0}.\end{aligned} \tag{C2}$$

Now using Eq. (8.4.10) from [51],

$$\begin{aligned}\mathcal{C}^{(\ell_1-\ell)0}_{\ell 0\ell_1 0} &= (-1)^{\ell+\ell_1-(\ell_1-\ell)}\mathcal{C}^{(\ell_1-\ell)0}_{\ell_1 0\ell 0} = \mathcal{C}^{(\ell_1-\ell)0}_{\ell_1 0\ell 0},\\ \mathcal{C}^{\ell_1 0}_{\ell 0(\ell_1-\ell)0} &= (-1)^{\ell}\frac{\Pi_{\ell_1}}{\Pi_{\ell_1-\ell}}\mathcal{C}^{(\ell_1-\ell)0}_{\ell_1 0\ell 0}.\end{aligned} \tag{C3}$$

Using Eqs. (8.5.34) and (8.5.42) from [51], we get

$$[\mathcal{C}^{(\ell_1-\ell)0}_{\ell_1 0\ell 0}]^2 = \left[\frac{\ell_1!}{\ell!(\ell_1-\ell)!}\right]^2\frac{(2\ell)!(2\ell_1-2\ell+1)!}{(2\ell_1+1)!}, \tag{C4}$$

and

$$\mathcal{C}^{(\ell_1-\ell)(\ell_1-\ell)}_{(\ell_1-\ell)(\ell_1-\ell)L0} = (2\ell_1-2\ell)!\sqrt{\frac{2\ell_1-2\ell+1}{(2\ell_1-2\ell-L)!(2\ell_1-2\ell+L+1)!}}. \tag{C5}$$

Thus

$$\begin{aligned}C_1(L,\ell,\ell_1) &= \frac{\Pi^2_{\ell}\Pi_{\ell_1}\Pi_{\ell_3}}{\sqrt{2\ell}}\left[\frac{\ell_1!}{\ell!\ell_3!}\right]^2\frac{(2\ell)!(2\ell_3+1)!}{(2\ell_1+1)!}\\ &\quad\times\sqrt{\frac{(2\ell_3)!}{(2\ell_3-L)!}\frac{(2\ell_3)!}{(2\ell_3+L+1)!}}\delta_{\ell_3,\ell_1-\ell}.\end{aligned} \tag{C6}$$

Substituting this in Eq. (31), the evolution equation becomes

$$\tilde{\Delta}^{LM}_{\ell_1,\ell_1}(k,\tau) = \sum_{\ell}C_1(L,\ell,\ell_1)j_\ell(k\Delta\tau)\tilde{\Delta}^{LM}_{\ell_1-\ell,\ell_1-\ell}(k,\tau_s), \tag{C7}$$

where $\ell, \ell_1 \gg |\ell_1 - \ell|$.

APPENDIX D: OFF-DIAGONAL TERMS OF THE ASYMPTOTIC MOMENTS OF CMB BRIGHTNESS FLUCTUATION

In Eq. (34), with $e = 0$, i.e. $\ell = R$,

$$\begin{Bmatrix} \ell_4 & L & \ell_3 \\ \ell_1 & \ell & \ell_2 \end{Bmatrix} \approx \frac{(-1)^{\ell_4+L+\ell_1-\ell}}{\sqrt{2\ell}\Pi_{\ell_3}} \mathcal{C}^{\ell_3(\ell_1-\ell)}_{\ell_4(\ell_2-\ell)L(\ell_1-\ell_2)}. \tag{D1}$$

From conditions of Clebsch-Gordan coefficients

$$|\ell_1 - \ell| \leq \ell_3 \qquad |\ell_2 - \ell| \leq \ell_4 \qquad |\ell_1 - \ell_2| \leq L.$$

For off-diagonal terms of $\tilde{\Delta}^{LM}_{\ell_1,\ell_2}$, we consider the minimum nonzero values of ℓ_3, ℓ_4 and L:

$$\ell_1 - \ell = \ell_3 \qquad \ell_2 - \ell = \ell_4 \qquad \ell_1 - \ell_2 = L.$$

Thus in Eq. (31) we would get the factor

$$C_2(\ell, \ell_1, \ell_2) = (-i)^\ell \Pi^2_\ell \Pi_{\ell_1} \Pi_{\ell_2-\ell} \frac{\beta_{\ell_1-\ell,\ell_2-\ell}}{\beta_{\ell_1\ell_2}} \frac{(-1)^{L+\ell_1+\ell_2}}{\sqrt{2\ell}\Pi_{\ell_1-\ell}} \times \mathcal{C}^{(\ell_1-\ell)0}_{\ell 0 \ell_1 0} \mathcal{C}^{\ell_2 0}_{\ell 0 (\ell_2-\ell)0} \mathcal{C}^{(\ell_1-\ell)(\ell_1-\ell)}_{(\ell_2-\ell)(\ell_2-\ell)LL}. \tag{D2}$$

Now using the symmetries of Clebsch-Gordan coefficients given by Eq. (8.5.34) from [51], we get

$$\mathcal{C}^{(\ell_1-\ell)0}_{\ell 0 \ell_1 0} = \mathcal{C}^{(\ell_1-\ell)0}_{\ell_1 0 \ell 0} = \frac{(-1)^\ell \ell_1!}{\ell!(\ell_1-\ell)!} \left[\frac{(2\ell)!(2\ell_1 - 2\ell + 1)!}{(2\ell_1+1)!}\right]^{1/2}. \tag{D3}$$

Similarly,

$$\mathcal{C}^{\ell_2 0}_{\ell 0 (\ell_2-\ell)0} = (-1)^\ell \frac{\Pi_{\ell_2}}{\Pi_{\ell_2-\ell}} \mathcal{C}^{(\ell_2-\ell)0}_{\ell_2 0 \ell 0} = \frac{\Pi_{\ell_2}}{\Pi_{\ell_2-\ell}} \frac{\ell_2!}{\ell!(\ell_2-\ell)!} \times \left[\frac{(2\ell)!(2\ell_2 - 2\ell + 1)!}{(2\ell_2+1)!}\right]^{1/2}. \tag{D4}$$

Using Eq. (8.5.37) from [51] we get

$$\mathcal{C}^{(\ell_1-\ell)(\ell_1-\ell)}_{(\ell_2-\ell)(\ell_2-\ell)LL} = \delta_{\ell_2-\ell+L,\ell_1-\ell} = \delta_{\ell_2+L,\ell_1}. \tag{D5}$$

Thus,

$$C_2(\ell, \ell_1, \ell_2) = \frac{(-1)^{(\ell_1+\ell_2)/2}}{\sqrt{2\ell}} \frac{(2\ell)!\ell_1!\ell_2!}{(\ell!)^2(\ell_1-\ell)!(\ell_2-\ell)!} \times \left[\frac{(2\ell_1 - 2\ell + 1)!(2\ell_2 - 2\ell + 1)!}{(2\ell_1+1)!(2\ell_2+1)!}\right]^{1/2} \times \frac{\Pi_{\ell\ell\ell_1\ell_2}}{\Pi_{\ell_1-\ell}} \sqrt{\frac{\ell_1 + \ell_2 - 2\ell + 1}{\ell_1 + \ell_2 + 1}}. \tag{D6}$$

Substituting this in Eq. (31), the evolution equation becomes

$$\tilde{\Delta}^{LM}_{\ell_1,\ell_2}(k, \tau) = \sum_\ell C_2(\ell, \ell_1, \ell_2) j_\ell(k\Delta\tau) \tilde{\Delta}^{LM}_{\ell_1-\ell,\ell_2-\ell}(k, \tau_s) \times \delta_{L,\ell_1-\ell_2}, \tag{D7}$$

where ℓ, $\ell_n \gg |\ell_n - \ell|$, L with $n = 1, 2$.

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