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Master formula approach to broken chiral $\mathrm{U}(3) \times \mathrm{U}(3)$ symmetry

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The master formula approach to chiral symmetry breaking proposed by Yamagishi and Zahed is extended to the $\mathrm{U}_R(3) \times \mathrm{U}_L(3)$ group in which effects of the $\mathrm{U}_A(1)$ anomaly and flavor symmetry breaking, $m_u \neq m_d \neq m_s$, are taken into account. New identities for the gluon topological susceptibility and π^0, η, $\eta' \to \gamma^{(*)}\gamma^{(*)}$ decays are derived, which embody the consequences of broken chiral symmetry in QCD without relying on any unphysical limits.

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I. INTRODUCTION

Phenomena of the η and η' mesons have been an attractive subject both theoretically and experimentally. Various properties of these mesons are closely related to the $\mathrm{U}_A(1)$ anomaly of QCD and the flavor mixing arising from the mass differences among u, d, s quarks. Since the $\mathrm{U}_A(1)$ anomaly is a realization of the nontrivial topology of the gluonic configurations, investigating reactions and decaying processes associated with the η and η' mesons will be one of the most feasible approaches to accessing the fundamental aspects of QCD.

Several theoretical approaches, such as the chiral perturbation theory [1,2] and the zero-momentum Ward identities with partially conserved axial-vector current hypothesis [3,4], have been applied for analyzing reaction processes of η and η' mesons and exploring the gluonic context of the low energy QCD. Most of these investigations start from certain unphysical limits such as chiral, soft pion, and large N_c.

The master formula approach to chiral symmetry breaking proposed by Yamagishi and Zahed is a powerful tool for analyzing hadronic processes which include ground state pseudoscalar mesons [5,6]. The approach is based on a set of master equations, which fully takes account of the consequences of broken chiral symmetry without relying on any unphysical limits or expansion schemes. Also, the master equations provide a systematic procedure, called the chiral reduction formula (χRF), to derive the chiral Ward identities satisfied by scattering amplitudes involving any number of pions with their physical masses. The advantage of this approach is that one can investigate the reaction dynamics, which cannot be determined by chiral symmetry, separately from the general structure required by the broken chiral symmetry. Once such a separation is made, any models and/or expansion schemes can be employed for describing the reaction dynamics without any contradiction with the constraints from broken chiral symmetry. A number of investigations based on the approach has been carried out for hadron reactions in the resonance region [6–11] and hadronic matter [12–18].

Up until now, the approach has been formulated within the two-flavor $\mathrm{SU}_R(2) \times \mathrm{SU}_L(2)$ [5] and three-flavor $\mathrm{SU}_R(3) \times \mathrm{SU}_L(3)$ [6] groups in the isospin symmetric limit. To analyze reaction processes involving η' and address the features of the low energy QCD mentioned above, however, we need to extend the approach to the $\mathrm{U}_R(3) \times \mathrm{U}_L(3)$ group incorporating the full flavor symmetry breaking due to $m_u \neq m_d \neq m_s$ and the $\mathrm{U}_A(1)$ anomaly. In this paper, we will describe how to make such an extension.

This paper is organized as follows. In Sec. II we first review the master formula approach for the isospin symmetric $\mathrm{SU}_R(2) \times \mathrm{SU}_L(2)$ group proposed in Ref. [5] and outline a derivation of the master equations with this simplest case. Then in Sec. III we describe the extension to the $\mathrm{U}_R(3) \times \mathrm{U}_L(3)$ group including finite quark masses and the $\mathrm{U}_A(1)$ anomaly. Several applications of the $\mathrm{U}_R(3) \times \mathrm{U}_L(3)$ master equations are presented in Sec. IV. The summary and outlook are given in Sec. V.

II. REVIEW OF ISOSPIN SYMMETRIC $\mathrm{SU}(2) \times \mathrm{SU}(2)$ CASE

A. The Veltman-Bell equations

Consider QCD with massive u and d quarks in the isospin symmetric limit: $m_u = m_d = \hat{m}$. The Lagrangian can be written as

$$\mathcal{L}^{\mathrm{SU}(2)}_{\mathrm{QCD}} = [\mathcal{L}^{\mathrm{SU}(2)}_{\mathrm{QCD}}]_{(0)} + \bar{q}\gamma^\mu(v^a_\mu + \gamma_5 a^a_\mu)\frac{\tau^a}{2}q(x) - \bar{q}(\hat{m} + s - ip^a\gamma_5\tau^a)q(x), \tag{1}$$

where $[\mathcal{L}^{\mathrm{SU}(2)}_{\mathrm{QCD}}]_{(0)}$ is the QCD Lagrangian in which the quark masses are set to zero; τ^a ($a = 1, 2, 3$) is the Pauli matrix for isospin; $q(x)$ is the isodoublet quark field $q = (u, d)^T$. The vector, axial-vector, scalar, and pseudoscalar external fields, $\phi = (v^a_\mu, a^a_\mu, s, p^a)$, are treated as sources to generate the corresponding currents and densities, $\mathcal{O} = (V^a_\mu, A^a_\mu, \Sigma, \Pi^a)$, which can be defined by $\mathcal{O} = \delta(\int d^4x \mathcal{L}^{\mathrm{SU}(2)}_{\mathrm{QCD}})/\delta\phi$.

A fundamental quantity in the theoretical framework developed in Ref. [5] is the extended S-matrix operator $\mathcal{S}[\phi]$, a functional of the external fields ϕ. This operator is unitary, $\mathcal{S}^\dagger\mathcal{S} = \mathcal{S}\mathcal{S}^\dagger = 1$, and is related to the vacuum-to-vacuum transition amplitude in the presence of the external fields ϕ: $Z[\phi] = \langle 0\,\text{out}|0\,\text{in}\rangle_\phi = \langle 0\,\text{in}|\mathcal{S}[\phi]|0\,\text{in}\rangle = \langle 0\,\text{out}|\mathcal{S}[\phi]|0\,\text{out}\rangle$. The Schwinger action principle allows one to express the quantum operators corresponding to $\mathcal{O} = (V^a_\mu, A^a_\mu, \Sigma, \Pi^a)$ as [5,19,20]

$$\mathcal{O}(x) = -i\mathcal{S}^\dagger \frac{\delta}{\delta\phi(x)}\mathcal{S}, \tag{2}$$

and their T^* product as

$$T^*[\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)] = (-i)^n\mathcal{S}^\dagger \frac{\delta}{\delta\phi(x_1)}\cdots\frac{\delta}{\delta\phi(x_n)}\mathcal{S}. \tag{3}$$

A more detailed description of the theoretical formulation based on the extended S-matrix can be found in the literature [19–21] and this will not be discussed here.

The system described by Eq. (1) and its effective theory have an approximate $\mathrm{SU}_R(2)\times\mathrm{SU}_L(2)$ chiral symmetry explicitly broken by the quark masses. It is known that such systems satisfy the divergence equations for the vector and axial currents, which, following Ref. [5], we call the Veltman-Bell (VB) equations [22,23]. The explicit forms are

$$\nabla^{\mu ac}V^c_\mu + \underline{a}^{\mu ac}A^c_\mu + \underline{p}^{ac}\Pi^c = 0, \tag{4}$$

$$\nabla^{\mu ac}A^c_\mu + \underline{a}^{\mu ac}V^c_\mu - (\hat{m}+s)\Pi^a + p^a\Sigma = 0, \tag{5}$$

where we have introduced the notation $\underline{X}^{ac} = \varepsilon^{abc}X^b$ (applicable to any quantity with one-isospin index X^b); $\nabla^{ac}_\mu = \delta^{ac}\partial_\mu + \underline{v}^{ac}_\mu$.

With Eq. (2), the VB equations (4) and (5) can be rewritten as a set of linear equations of the extended S-matrix:

$$\left[\nabla^{ac}_\mu \frac{\delta}{\delta v^c_\mu} + \underline{a}^{ac}_\mu \frac{\delta}{\delta a^c_\mu} + \underline{p}^{ac}\frac{\delta}{\delta p^c}\right]\mathcal{S} = 0, \tag{6}$$

$$\left[\nabla^{ac}_\mu \frac{\delta}{\delta a^c_\mu} + \underline{a}^{ac}_\mu \frac{\delta}{\delta v^c_\mu} - (s+\hat{m})\frac{\delta}{\delta p^a} + p^a\frac{\delta}{\delta s}\right]\mathcal{S} = 0. \tag{7}$$

Applying functional derivatives of ϕ to Eqs. (6) and (7) and using Eqs. (2) and (3), one can derive the vector and axial Ward identities satisfied by the operators $\mathcal{O} = (V^a_\mu, A^a_\mu, \Sigma, \Pi^a)$.

B. Master equations for the chiral symmetry breaking

The VB equations (4) and (5) [or equivalently Eqs. (6) and (7)] are satisfied by the systems in both the Wigner and Nambu-Goldstone phases. However, if the chiral symmetry in the system is spontaneously broken, the Nambu-Goldstone bosons (pions) appear and couple to the axial current and the isovector-pseudoscalar density:

$$\langle 0|A^a_\mu(x)|\pi^b(p)\rangle = f_\pi i p_\mu \delta^{ab}e^{-ipx}, \tag{8}$$

$$\langle 0|\Pi^a(x)|\pi^b(p)\rangle = G\delta^{ab}e^{-ipx}. \tag{9}$$

Here f_π is the pion decay constant, which remains finite in the chiral limit in contrast to those of the pion excitations [24]; G is the pseudoscalar coupling constant. By taking the matrix element of the axial VB equations (5) between the vacuum state $\langle 0|$ and the one-pion state $|\pi^a(p)\rangle$, and using Eqs. (8) and (9), we obtain the following mass relation:

$$f_\pi m^2_\pi = \hat{m}G, \tag{10}$$

which shows clearly the well-known result that the nonzero quark masses are responsible for the nonzero pion masses. The master equations proposed in Ref. [5] may be understood as the VB equations incorporating this information on chiral symmetry breaking.

It was shown in Ref. [5] that the information can be incorporated into the VB equations by making the following modifications:

(1) Introduce new pseudoscalar and scalar external fields, J^a and Y, defined by

$$J^a = Gp^a + f_\pi\nabla^{ac}_\mu a^{\mu c}, \tag{11}$$

$$Y = Gs, \tag{12}$$

and treat $\phi = (a^a_\mu, v^a_\mu, Y, J^a)$ as independent external fields.

(2) Introduce a new extended S-matrix $\hat{\mathcal{S}}$ as

$$\hat{\mathcal{S}} = \mathcal{S}\exp(-i\delta\mathcal{I}), \tag{13}$$

with

$$\delta\mathcal{I} = \int d^4x\left[Y(x)G^{-1}C + \frac{f^2_\pi}{2}a^{\mu a}(x)a^a_\mu(x)\right], \tag{14}$$

where a new constant C is introduced. (The physical meaning of C is explained below.)

With these modifications, the new current and density operators, $\hat{\mathcal{O}} = (j^a_{V\mu}, j^a_{A\mu}, \hat{\sigma}, \hat{\pi}^a)$, defined by $\hat{\mathcal{O}} = -i\hat{\mathcal{S}}^\dagger(\delta/\delta\phi)\hat{\mathcal{S}}$, are related with the original current and density operators, $\mathcal{O} = (V^a_\mu, A^a_\mu, \Sigma, \Pi^a)$, as follows:

$$V^a_\mu = j^a_{V\mu} + f_\pi\underline{a}^{ac}_\mu\hat{\pi}^c, \tag{15}$$

$$A^a_\mu = j^a_{A\mu} + f^2_\pi a^a_\mu - f_\pi\nabla^{ac}_\mu\hat{\pi}^c, \tag{16}$$

$$\Sigma = G\hat{\sigma} + C, \tag{17}$$

$$\Pi^a = G\hat{\pi}^a. \tag{18}$$

Here the new pseudoscalar density $\hat{\pi}^a$ satisfies $\langle 0|\hat{\pi}^a(x)|\pi^b(p)\rangle = \delta^{ab}e^{-ipx}$. This allows one to identify $\hat{\pi}^a$ with the normalized interpolating pion field. The change of the field variable $p^a \to J^a$ defined by Eq. (11) is responsible for the separation of the one-pion component from the axial current A^a_μ and thus is introduced to indicate that the system under consideration includes pions. The new axial current $j^a_{A\mu}$ has no one-pion component surviving on pion mass shell, $\langle 0|j^a_{A\mu}(x)|\pi^b(p)\rangle = 0$, in contrast to the original A^a_μ.

The second modification defined by Eqs. (13) and (14) is introduced to indicate that the action describing the original S-matrix operator, $\mathcal{S}$, should contain contact terms, as defined in Eq. (14), if the chiral symmetry is spontaneously broken. A justification of this statement has been provided in Ref. [5] making use of the gauged nonlinear sigma model. It is also noted that the difference between $\mathcal{S}$ and $\hat{\mathcal{S}}$ just comes from the contact terms of the c-number external fields and thus does not affect the physical observables.

The practical role of the second modification is twofold. The first term in the integrand of $\delta\mathcal{I}$ is introduced to take account of the quark-antiquark condensation. In fact, it explicitly introduces a shift in the scalar density [see Eq. (17)], amounting to the new constant, C. This constant carries part of the information on the condensation. (We allow $\langle\hat{\sigma}\rangle \neq 0$.) It turns out that C is expressed as the product of the pion decay constant and the pseudoscalar coupling constant:

$$C = f_\pi G. \tag{19}$$

This follows from the fact that $\hat{\pi}^a$ is the normalized interpolating pion field. (See Appendix A for the derivation.) The mass relation (10) can be rewritten as

$$f_\pi^2 m_\pi^2 = \hat{m}C. \tag{20}$$

If $\langle\hat{\sigma}\rangle = 0$, then $C = \langle\Sigma\rangle$ and Eq. (20) reduces to the Gell-Mann-Oakes-Renner (GMOR) relation. Therefore, $\langle\hat{\sigma}\rangle$ represents the deviation of the mass relation from the GMOR relation. In fact, the on-shell $(1/f_\pi)$ expansion scheme proposed in Ref. [5], which is the expansion in $1/f_\pi$ around the physical pion mass and is constructed so that the GMOR relation holds at the leading order, leads to $\langle\hat{\sigma}\rangle = 0 + \mathcal{O}(f_\pi^{-1})$ and $C = \langle\Sigma\rangle + \mathcal{O}(f_\pi^{-1})$. The second term in the integrand of $\delta\mathcal{I}$ results in the appearance of $f_\pi^2 a^a_\mu$ in Eq. (16). This ensures the existence of the contact term $\delta^{ab} f_\pi^2 g_{\mu\nu}$ in the two-point function of the axial current,

$$i\int d^4x e^{iqx}\langle T^*[A^a_\mu(x)A^b_\nu(0)]\rangle = \delta^{ab} f_\pi^2 g_{\mu\nu} - \delta^{ab} f_\pi^2 \frac{q_\mu q_\nu}{q^2 - m_\pi^2} + \cdots, \tag{21}$$

which must appear in the two-point function to have a correct chiral limit. (The symbol $\langle\rangle$ denotes the vacuum expectation value.)

Substituting Eqs. (11), (12), and (15)–(18) into the VB equations and using Eqs. (10) and (19), we have

$$[\nabla^{\mu ac} j^c_{V\mu} + \underline{a}^{\mu ac} j^c_{A\mu} + \underline{J}^{ac}\hat{\pi}^c] = 0, \tag{22}$$

$$\begin{aligned}[-\nabla^{\mu ae}\nabla^{ec}_\mu + \underline{a}^{\mu ae}\underline{a}^{ec}_\mu - m_\pi^2\delta^{ac} - Y f_\pi^{-1}\delta^{ac}]\hat{\pi}^c &= -J^a(x) - f_\pi^{-1}(\nabla^{\mu ac} j^c_{A\mu} + \underline{a}^{\mu ac} j^c_{V\mu}) \\ &\quad - [J^a - f_\pi \nabla^{\mu ac} a^c_\mu]\hat{\sigma}. \end{aligned}\tag{23}$$

These equations can be written in the functional derivative form as

$$\left(\nabla^{ac}_\mu \frac{\delta}{\delta v^c_\mu} + \underline{a}^{ac}_\mu \frac{\delta}{\delta a^c_\mu} + \underline{J}^{ac}\frac{\delta}{\delta J^c}\right)\hat{\mathcal{S}} = 0, \tag{24}$$

$$\left[-(\Box\delta^{ab} + m_\pi^2\delta^{ab} + K^{ab}_{\mathrm{SU(2)}})\frac{\delta}{\delta J^b} + iJ^a + f_\pi^{-1} t^a_A - (\nabla^{\mu ac} a^c_\mu - f_\pi^{-1}J^a)\frac{\delta}{\delta Y}\right]\hat{\mathcal{S}} = 0, \tag{25}$$

with

$$K^{ab}_{\mathrm{SU(2)}} = \nabla^{\mu ac}\nabla^{cb}_\mu - \underline{a}^{ac}_\mu \underline{a}^{\mu cb} + \delta^{ab}Y - \delta^{ab}\Box,$$

$$t^a_A = \nabla^{ac}_\mu \frac{\delta}{\delta a^c_\mu} + \underline{a}^{ac}_\mu \frac{\delta}{\delta v^c_\mu}.$$

By introducing the retarded and advanced Green functions satisfying

$$-[\Box\delta^{ab} + m_\pi^2\delta^{ab} + K^{ab}_{\mathrm{SU(2)}}(x)]G^{bc}_{R,A}(x,y) = \delta^{ac}\delta^{(4)}(x-y),$$

the axial VB equation (23) can be formally solved for the interpolating pion field $\hat{\pi}$. The functional derivative form of the solution is written as

$$\begin{aligned}\frac{\delta\hat{\mathcal{S}}}{\delta J^a(x)} &= i\hat{\mathcal{S}}\pi^a_{\mathrm{in}}(x) + i\hat{\mathcal{S}}\int d^4y G^{ab}_R(x,y)K^{bc}_{\mathrm{SU(2)}}(y)\pi^c_{\mathrm{in}}(y) \\ &\quad - \int d^4y G^{ab}_R(x,y)\bar{R}^b_{\mathrm{SU(2)}}(y)\hat{\mathcal{S}} \\ &= i\pi^a_{\mathrm{in}}(x)\hat{\mathcal{S}} + i\int d^4y G^{ab}_A(x,y)K^{bc}_{\mathrm{SU(2)}}(y)\pi^c_{\mathrm{in}}(y)\hat{\mathcal{S}} \\ &\quad - \int d^4y G^{ab}_A(x,y)\bar{R}^b_{\mathrm{SU(2)}}(y)\hat{\mathcal{S}}. \end{aligned}\tag{26}$$

Here π^a_{in} is the in-state asymptotic pion field, $\hat{\pi}^a \to \pi^a_{\mathrm{in}} + \cdots (t \to -\infty)$; $\bar{R}^a_{\mathrm{SU(2)}}(x)$ is defined by

$$\bar{R}^a_{\mathrm{SU(2)}}(x) = R^a_{\mathrm{SU(2)}}(x) + K^{ab}_{\mathrm{SU(2)}}(x)\frac{\delta}{\delta J^b(x)}, \tag{27}$$

with

$$R^a_{\mathrm{SU(2)}}(x) = \bigg[iJ^a + \frac{1}{f_\pi} t^a_A - K^{ab}_{\mathrm{SU(2)}} \frac{\delta}{\delta J^b} - \bigg(\nabla^{\mu ac} a^c_\mu - \frac{J^a}{f_\pi} \bigg) \frac{\delta}{\delta Y} \bigg](x). \tag{28}$$

Equations (24) and (26) constitute the master equations for the $\mathrm{SU}_R(2) \times \mathrm{SU}_L(2)$ chiral symmetry breaking in the isospin symmetric limit.

C. Chiral reduction formula

From the axial master equation (26), we can derive the commutation relations between the creation and annihilation operators of the pion and the extended S-matrix $\hat{\mathcal{S}}$,

$$[a^a_{\mathrm{in}}(k), \hat{\mathcal{S}}] = R^a_{\mathrm{SU(2)}}(k)\hat{\mathcal{S}}, \qquad [\hat{\mathcal{S}}, a^{a\dagger}_{\mathrm{in}}(k)] = R^a_{\mathrm{SU(2)}}(-k)\hat{\mathcal{S}}, \tag{29}$$

where $R^a_{\mathrm{SU(2)}}(k) = \int d^4x e^{ikx} R^a_{\mathrm{SU(2)}}(x)$. Note that we can rewrite the commutation relations in this form without using the asymptotic pion field. [Compare them with Eqs. (6.2) and (6.3) in Ref. [5].]

Iterative use of Eq. (29) results in the χRF for the on-shell scattering amplitudes involving any number of pions with their physical masses,

$$\begin{aligned} &\langle \alpha; k_1 a_1, \cdots, k_m a_m | \hat{\mathcal{S}} | \beta; l_1 b_1, \cdots, l_n b_n \rangle |_{\phi=0} \\ &\quad = [R^{a_1}_{\mathrm{SU(2)}}(k_1) \cdots R^{a_m}_{\mathrm{SU(2)}}(k_m) R^{b_1}_{\mathrm{SU(2)}}(-l_1) \cdots R^{b_n}_{\mathrm{SU(2)}}(-l_n)]_{\mathrm{S}} \\ &\quad\quad \times \langle \alpha | \hat{\mathcal{S}} | \beta \rangle |_{\phi=0}, \end{aligned} \tag{30}$$

where k_i (l_i) and a_i (b_i) are, respectively, the four momentum and isospin indices of the outgoing (incoming) pions; α and β stand for states of other particles. Here we consider the case that no two pions have equal momenta. The symbol $[\,]_{\mathrm{S}}$ represents normalized symmetric permutations of the functional derivative operators,

$$[\mathcal{D}_1 \cdots \mathcal{D}_n]_{\mathrm{S}} = \frac{1}{n!} \sum_{\mathrm{perms}} \mathcal{D}_1 \cdots \mathcal{D}_n. \tag{31}$$

This operation shows clearly the crossing symmetry in Eq. (30). By using Eq. (30) together with Eq. (3), scattering amplitudes are expressed in terms of Green's functions of the operators $\hat{\mathcal{O}} = (j^a_{V\mu}, j^a_{A\mu}, \hat{\sigma}, \hat{\pi}^a)$. The χRF takes the form of functional derivatives, and all constraints which stem from broken chiral symmetry are contained in $R^a_{\mathrm{SU(2)}}(k)$. The extension of the χRF to the off-shell pions has been discussed in detail in Ref. [25].

III. EXTENDING TO $\mathrm{U}_R(3) \times \mathrm{U}_L(3)$ WITH FLAVOR SYMMETRY BREAKING

In this section, we extend the master equations reviewed in Sec. II to the $\mathrm{U}_R(3) \times \mathrm{U}_L(3)$ group with $m_u \neq m_d \neq m_s$. In the remainder of this paper, the term "pion" ("π") is used for expressing the nonet ground pseudoscalar mesons generically, and the symbols $\pi^{\pm,0}$, $K^{+,0}$, $\bar{K}^{-,0}$, η, and η' are used for referring to the specific mesons.

A. The $\mathrm{U}_R(3) \times \mathrm{U}_L(3)$ VB equations

The Lagrangian now includes massive u, d, s quarks:

$$\begin{aligned} \mathcal{L}^{\mathrm{U(3)}}_{\mathrm{QCD}} &= [\mathcal{L}^{\mathrm{U(3)}}_{\mathrm{QCD}}]_{(0)} + \bar{q}\gamma^\mu (v^a_\mu + \gamma_5 a^a_\mu) \frac{\lambda^a}{2} q \\ &\quad - \bar{q}(m^a_q + s^a - ip^a\gamma_5)\lambda^a q - \theta\omega. \end{aligned} \tag{32}$$

Here $q^T = (u, d, s)$; $m^a_q \lambda^a = \mathrm{diag}(m_u, m_d, m_s)$; θ and ω are the vacuum angle and the gluon topological charge density, respectively. The term $[\mathcal{L}^{\mathrm{U(3)}}_{\mathrm{QCD}}]_{(0)}$ represents the QCD Lagrangian with the quark masses, the vacuum angle, and all external fields set to zero. The flavor matrix λ^a is taken to be one of the Gell-Mann matrices for $a = 1, \cdots, 8$ and $\lambda^0 = \sqrt{2/3}\mathbf{1}$ so that they satisfy $\mathrm{Tr}[\lambda^a \lambda^b] = 2\delta^{ab}$. The vector, axial-vector, scalar, and pseudoscalar external fields and the vacuum angle, $\phi = (v^a_\mu, a^a_\mu, s^a, p^a, -\theta)$, are sources to generate currents and densities, $\mathcal{O} = (V^a_\mu, A^a_\mu, \Sigma^a, \Pi^a, \omega)$, which are obtained from $\mathcal{O} = \delta(\int d^4x \mathcal{L}^{\mathrm{U(3)}}_{\mathrm{QCD}})/\delta\phi$. As in the SU(2) case, the quantum operators corresponding to $\mathcal{O}$ are defined by Eq. (2).

The VB equations for the $\mathrm{U}_R(3) \times \mathrm{U}_L(3)$ group can then be written as

$$\nabla^{\mu ac} V^c_\mu + \underline{a}^{\mu ac} A^c_\mu + \underline{p}^{ac} \Pi^c + (\underline{s} + \underline{m}_q)^{ac} \Sigma^c = 0, \tag{33}$$

$$\begin{aligned} &\nabla^{\mu ac} A^c_\mu + \underline{a}^{ab}_\mu V^b_\mu + \bar{p}^{ab} \Sigma^b \\ &\quad - (\bar{s} + \bar{m}_q)^{ac} \Pi^c - \mathrm{Tr}[\lambda^a]\omega = \Omega^a. \end{aligned} \tag{34}$$

Here we have introduced the notation[1] $\bar{X}^{ac} = d^{abc} X^b$ and $\underline{X}^{ac} = f^{abc} X^b$ (applicable to any quantity with one-flavor index X^b); $\nabla^{ac}_\mu = \partial_\mu \delta^{ac} + \underline{v}^{ac}_\mu$. It is well known that an additional nonconserving contribution in the axial VB equation, which amounts to $-\mathrm{Tr}[\lambda^a]\omega$, is attributable to the $\mathrm{U}_A(1)$ anomaly. Furthermore, for the $\mathrm{U}_R(3) \times \mathrm{U}_L(3)$ group, the non-Abelian anomaly Ω^a associated with the external gauge fields v^a_μ and a^a_μ also appears. Its explicit form is [26]

$$\begin{aligned} \Omega^a = \frac{N_c}{16\pi^2} \varepsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\bigg[\frac{\lambda^a}{2} \bigg(& F^V_{\mu\nu} F^V_{\rho\sigma} + \frac{1}{3} F^A_{\mu\nu} F^A_{\rho\sigma} \\ & + i\frac{8}{3}(a_\mu a_\nu F^V_{\rho\sigma} + a_\mu F^V_{\nu\rho} a_\sigma + F^V_{\mu\nu} a_\rho a_\sigma) \\ & - \frac{32}{3} a_\mu a_\nu a_\rho a_\sigma \bigg) \bigg], \end{aligned} \tag{35}$$

with $F^V_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - i[v_\mu, v_\nu] - i[a_\mu, a_\nu]$, $F^A_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - i[v_\mu, a_\nu] - i[a_\mu, v_\nu]$, $v_\mu =$

[1] The structure constants f^{abc} and d^{abc} are defined by $f^{abc} = -(i/4)\,\mathrm{Tr}[[\lambda^a, \lambda^b]\lambda^c]$ and $d^{abc} = (1/4)\,\mathrm{Tr}[\{\lambda^a, \lambda^b\}\lambda^c]$, respectively. With this definition, we have $f^{0bc} = 0$ and $d^{0bc} = \sqrt{2/3}\delta^{bc}$.

$v_\mu^a(\lambda^a/2)$, and $a_\mu = a_\mu^a(\lambda^a/2)$. We take a convention with $\varepsilon_{0123} = +1$.

The functional derivative form of the VB equations is given by

$$\left[\nabla_\mu^{ac}\frac{\delta}{\delta v_\mu^c} + \underline{a}_\mu^{ac}\frac{\delta}{\delta a_\mu^c} + \underline{p}^{ac}\frac{\delta}{\delta p^c} + (\underline{s} + \underline{m}_q)^{ac}\frac{\delta}{\delta s^c}\right]\mathcal{S} = 0, \tag{36}$$

$$\left[\nabla_\mu^{ac}\frac{\delta}{\delta a_\mu^c} + \underline{a}_\mu^{ac}\frac{\delta}{\delta v_\mu^c} + \bar{p}^{ac}\frac{\delta}{\delta s^c} - (\bar{s} + \bar{m}_q)^{ac}\frac{\delta}{\delta p^c} + \mathrm{Tr}[\lambda^a]\frac{\delta}{\delta\theta}\right]\mathcal{S} = i\Omega^a\mathcal{S}. \tag{37}$$

B. Flavor symmetry breaking and $\mathrm{U}_A(1)$ anomaly

Besides their key role in the singlet sector, the mass differences among the u, d, s quarks and the $\mathrm{U}_A(1)$ anomaly are responsible for the major complication in the construction of the $\mathrm{U}_R(3)\times\mathrm{U}_L(3)$ master equations. Therefore we first examine those effects carefully.

The mass differences of the quarks violate the flavor symmetry explicitly as well as the chiral symmetry. Thus the flavor U(3) basis cannot be the mass eigenstates of hadrons and mixing of the U(3) basis occurs. Necessary information on the flavor symmetry breaking here is summarized in the following matrix elements:

$$\langle 0|A_\mu^a(x)|P(p)\rangle = \tilde{f}_\pi^{aP} ip_\mu e^{-ipx}, \tag{38}$$

$$\langle 0|\Pi^a(x)|P(p)\rangle = G^{aP}e^{-ipx}, \tag{39}$$

$$\langle 0|\omega(x)|P(p)\rangle = \tilde{\mathcal{A}}^P e^{-ipx}, \tag{40}$$

where P denotes the physical pion states (the mass eigenstates): $P = (\pi^{\pm,0}, K^{\pm,0}, \bar{K}^0, \eta, \eta')$. (Capital indices are used to represent the mass eigenstates.) Because of the flavor symmetry breaking, the pion decay constant $\tilde{f}_\pi^{aP}$ and the pseudoscalar coupling constant G^{aP} now have two indices that specify the flavor U(3) basis and the mass eigenstates.[2] Also, it is noted that $\tilde{\mathcal{A}}^P$ has a nonzero value only for $P = \pi^0, \eta, \eta'$.

From the axial VB equation (34) and (38)–(40), we can derive the U(3) version of the mass relation:

$$\tilde{f}_\pi^{aQ}(m_\pi^2)^{QP} = \bar{m}_q^{ab}G^{bP} + \mathrm{Tr}[\lambda^a]\tilde{\mathcal{A}}^P, \tag{41}$$

where m_π^2 is the diagonal mass-squared matrix of the physical nonet pions. The appearance of $\tilde{\mathcal{A}}^P$ is attributable to ω in the axial VB equation and thus is nothing but a consequence of the $\mathrm{U}_A(1)$ anomaly.

As already mentioned above, the $\mathrm{U}_A(1)$ anomaly leads to the nonconserving term proportional to ω in the singlet axial VB equation. Because of this, the operators A_μ^0 and ω get renormalized under a change of the QCD scale [1]: $(A_\mu^0)_{\mathrm{ren}} = Z_A A_\mu^0$, $(\omega)_{\mathrm{ren}} = \omega + (Z_A - 1)(1/\mathrm{Tr}[\lambda^0])\partial^\mu A_\mu^0$, where Z_A is the renormalization factor. Accordingly, the singlet pion decay constant $\tilde{f}_\pi^{0P}$ and $\tilde{\mathcal{A}}^P$ also get renormalized as

$$(\tilde{f}_\pi^{0P})_{\mathrm{ren}} = Z_A\tilde{f}_\pi^{0P}, \tag{42}$$

$$(\tilde{\mathcal{A}}^P)_{\mathrm{ren}} = \tilde{\mathcal{A}}^P + (Z_A - 1)\frac{1}{\mathrm{Tr}[\lambda^0]}\tilde{f}_\pi^{0Q}(m_\pi^2)^{QP}. \tag{43}$$

These are thus scale dependent and not physical constants. This is in contrast to the octet pion decay constants, $\tilde{f}_\pi^{aP}$, with $a \neq 0$, which are scale independent. It is noted, however, that the combination $\tilde{f}_\pi^{0Q}(m_\pi^2)^{QP} - \mathrm{Tr}[\lambda^0]\tilde{\mathcal{A}}^P$ is invariant under a change of the QCD scale.

As pointed out by Shore and Veneziano (see, e.g., Refs. [4,27,28]), the renormalization group (RG) variant $\tilde{f}_\pi^{0P}$, which is defined as a coupling of the singlet axial current to the physical pions, does not satisfy GMOR type mass relations. They introduced a RG-invariant decay constant f_π^{0P} and showed that the use of this new constant provides a natural extension of the GMOR relation to the singlet sector.

Following Shore and Veneziano, let us introduce the RG invariant f_π^{0P}. Defining $\delta f_\pi^{0P} = \tilde{f}_\pi^{0P} - f_\pi^{0P}$, the combination $(\tilde{f}_\pi m_\pi^2)^{0P} - \mathrm{Tr}[\lambda^0]\tilde{\mathcal{A}}^P$ can be rewritten as

$$(\tilde{f}_\pi m_\pi^2)^{0P} - \mathrm{Tr}[\lambda^0]\tilde{\mathcal{A}}^P = (f_\pi m_\pi^2)^{0P} - \mathrm{Tr}[\lambda^0] \times[\tilde{\mathcal{A}}^P - (\delta f_\pi m_\pi^2)^{0P}]. \tag{44}$$

Because $(\tilde{f}_\pi m_\pi^2)^{0P} - \mathrm{Tr}[\lambda^0]\tilde{\mathcal{A}}^P$ and $(f_\pi m_\pi^2)^{0P}$ are individually RG invariant, $\mathcal{A}^P \equiv [\tilde{\mathcal{A}}^P - (\delta f_\pi m_\pi^2)^{0P}]$ is also RG invariant. The values of f_π^{0P} and $\mathcal{A}^P$ must be extracted from the experimental data. With this modification, the mass relation (41) becomes

$$\begin{aligned}(f_\pi m_\pi^2 f_\pi^T)^{ab} &= (\bar{m}_q G f_\pi^T)^{ab} + \mathrm{Tr}[\lambda^a](\mathcal{A}f_\pi^T)^b \\ &= (\bar{m}_q G f_\pi^T)^{ab} + A^{ab},\end{aligned} \tag{45}$$

where $f_\pi^{aP} \equiv \tilde{f}_\pi^{aP}$ for $a \neq 0$; $(f_\pi^T) = (f_\pi^T)^{Pa}$ is the transpose of f_π^{aP}; $A^{ab} \equiv \mathrm{Tr}[\lambda^a](\mathcal{A}f_\pi^T)^b$.

C. Master equations for the $\mathrm{U}_R(3)\times\mathrm{U}_L(3)$ symmetry breaking

We can incorporate the information on the symmetry breaking described in Sec. III B in a completely parallel way to the SU(2) case:

(1) Introduce a new pseudoscalar and scalar external fields J^P and Y^P defined by

[2] The breaking pattern of the flavor symmetry, i.e., $m_q^a\lambda^a = m_q^0\lambda^0 + m_q^3\lambda^3 + m_q^8\lambda^8$, implies that $\tilde{f}_\pi^{aP}$ and G^{aP} are block diagonal in $(1,2)\times(\pi^+,\pi^-)$, $(4,5)\times(K^+,K^-)$, $(6,7)\times(K^0,\bar{K}^0)$, and $(0,3,8)\times(\pi^0,\eta,\eta')$. In the isospin symmetric limit they become $(1,2,3)\times(\pi^+,\pi^-,\pi^0)$, $(4,5,6,7)\times(K^+,K^-,K^0,\bar{K}^0)$, and $(0,8)\times(\eta,\eta')$ and the first two blocks can be diagonalized.

$$J^P = p^a G^{aP} + (\tilde{f}_\pi^T)^{Pa}(\nabla^\mu \tilde{a}_\mu)^a - [(f_\pi^T)^{P0}\Box + (f_\pi^{-1})^{Pa}A^{a0}]\frac{1}{\mathrm{Tr}[\lambda^0]}\theta, \quad (46)$$

$$Y^P = s^a G^{aP}, \quad (47)$$

with $\tilde{a}_\mu^a = a_\mu^a + \delta^{a0}(1/\mathrm{Tr}[\lambda^0])\partial_\mu\theta$, and then consider $\phi = (a_\mu^a, v_\mu^a, Y^P, J^P, -\theta)$ as independent external field variables.

(2) Introduce a new extended S-matrix by

$$\hat{\mathcal{S}} = \mathcal{S}\exp[-i(\delta I + \delta I_\Omega)], \quad (48)$$

where

$$\delta I = \int d^4x\Big(Y^P(G^{-1})^{Pa}C^a + \frac{1}{2}\tilde{a}^{\mu a}(\tilde{f}_\pi\tilde{f}_\pi^T)^{ab}\tilde{a}_\mu^b - \frac{1}{2(\mathrm{Tr}[\lambda^0])^2}\theta(A + f_\pi f_\pi^T\Box)^{00}\theta - \frac{1}{\mathrm{Tr}[\lambda^0]}\partial^\mu\theta(f_\pi\tilde{f}_\pi^T)^{0b}\tilde{a}_\mu^b\Big), \quad (49)$$

and

$$\delta I_\Omega = \frac{N_c}{72\pi^2}\varepsilon^{\mu\nu\rho\sigma}\int d^4x\Big(a_\mu^a\nabla_\nu^{ac}a_\rho^c\sqrt{6}\tilde{a}_\sigma^0 - \frac{1}{2}a_\mu^0\partial_\nu a_\rho^0\sqrt{6}\tilde{a}_\sigma^0\Big). \quad (50)$$

Note that the new constant C^a, which corresponds to C in the SU(2) case, now has a flavor index. With these modifications, the new current and density operators $\hat{\mathcal{O}} = (j_{V\mu}^a, j_{A\mu}^a, \hat{\sigma}^P, \hat{\pi}^P, \hat{\omega})$ are defined by $\hat{\mathcal{O}} = -i\hat{\mathcal{S}}^\dagger\delta\hat{\mathcal{S}}/\delta\phi$ with $\phi = (v_\mu^a, a_\mu^a, Y^P, J^P, -\theta)$. The relations between the new and original operators are given by

$$V_\mu^a = j_{V\mu}^a + \underline{a}_\mu^{ab}(f_\pi\hat{\pi})^b + \frac{\delta(\delta I_\Omega)}{\delta v^{\mu a}}, \quad (51)$$

$$A_\mu^a = j_{A\mu}^a - \nabla_\mu^{ab}(\tilde{f}_\pi\hat{\pi})^b + (\tilde{f}_\pi\tilde{f}_\pi^T)^{ab}\tilde{a}_\mu^b - (\tilde{f}_\pi f_\pi^T)^{a0}\frac{1}{\mathrm{Tr}[\lambda^0]}\partial_\mu\theta + \frac{\delta(\delta I_\Omega)}{\delta a^{\mu a}}, \quad (52)$$

$$\Sigma^a = G^{aP}\hat{\sigma}^P + C^a, \quad (53)$$

$$\Pi^a = G^{aP}\hat{\pi}^P, \quad (54)$$

$$\omega = \hat{\omega} + \frac{1}{\mathrm{Tr}[\lambda^0]}\{A^{0a}[(f_\pi^{-1})^T]^{aP} - (\delta f_\pi)^{0P}\Box\}\hat{\pi}^P + \frac{1}{\mathrm{Tr}[\lambda^0]}(\delta f_\pi f_\pi^T)^{0a}\partial^\mu a_\mu^a + \frac{1}{(\mathrm{Tr}[\lambda^0])^2} \times [A^{00} + (\delta f_\pi\delta f_\pi^T)^{00}\Box]\theta + \frac{\delta(\delta I_\Omega)}{\delta(-\theta)}. \quad (55)$$

As in the SU(2) case, we can identify the new pseudoscalar density $\hat{\pi}^P$ with the (normalized) interpolating pion field satisfying $\langle 0|\hat{\pi}^A(x)|P(k)\rangle = \delta^{AP}e^{-ikx}$.

The modifications induced by Eqs. (46), (47), and (49) can be understood as a straightforward extension of, respectively, Eqs. (11), (12), and (14) in the SU(2) case. With the modification (46), the one-pion component is separated from A_μ^a and ω, and the resulting new operators $j_{A\mu}^a$ and $\hat{\omega}$ have no one-pion component surviving on pion mass shell, i.e., $\langle 0|j_{A\mu}^a|P(p)\rangle = \langle 0|\hat{\omega}|P(p)\rangle = 0$ at $p^2 = (m_\pi^2)^{PP}$. From Eq. (53), the new constant C^a again carries part of the quark-antiquark condensate: $\langle\Sigma^a\rangle = G^{aP}\langle\sigma^P\rangle + C^a$. The terms in Eq. (49), except for the first one, are introduced such that the two-point functions including A_μ^a and/or ω have the correct chiral limit in the presence of the pions.

A comment on the term δI_Ω is needed since it does not appear in the SU(2) case. As discussed by Kaiser and Leutwyler [1], the non-Abelian anomaly, Ω^a, which includes the external singlet axial gauge field a_μ^0, is not invariant under a change of the QCD scale because of the $U_A(1)$ anomaly. This leads to an inconsistency with the RG invariance of the VB equations.[3] The term δI_Ω is introduced to cure the inconsistency and corresponds to the sum of the two contact terms, P_1 and P_2, of Eq. (78) in Ref. [1]. With this additional term the non-Abelian anomaly Ω^a is replaced with the RG invariant Ω_0^a, which is of the same form as Ω^a, but a_μ^0 is replaced with $-(1/\mathrm{Tr}[\lambda^0])\partial_\mu\theta$.

Taking account of the modifications described above, the VB equations (33) and (34) can be rewritten as

$$\nabla^{\mu ab}j_{V\mu}^b + \underline{a}^{\mu ab}j_{A\mu}^b + \underline{(G^{-1}J)}^{ab}(G\hat{\pi})^b + [\underline{(G^{-1}Y)}^{ab} + \underline{m}_q^{ab}](G\hat{\sigma})^b + \chi_{V1}^{aP}\hat{\pi}^P + \chi_{V2}^a = 0, \quad (56)$$

and

$$[-\Box\delta^{PQ} - (m_\pi^2)^{PQ} - (f_\pi^{-1})^{Pa}K^{ab}f_\pi^{bQ}]\hat{\pi}^Q = -J^P - (f_\pi^{-1})^{Pa}(\nabla_\mu^{ab}j_a^{\mu b} + \underline{a}_\mu^{ab}j_V^{\mu b} - \mathrm{Tr}[\lambda^a]\hat{\omega}) - (f_\pi^{-1})^{Pa}\overline{(G^{-1}\mathcal{J})}^{ab}(G\hat{\sigma})^b + (f_\pi^{-1})^{Pa}(\Omega_0^a - \chi_A^a). \quad (57)$$

Here we have introduced

$$\chi_{V1}^{aP} = (\nabla^\mu\underline{\tilde{a}}_\mu)^{ab}\tilde{f}_\pi^{bP} - (\underline{\tilde{a}}^\mu\underline{v}_\mu)^{ab}\tilde{f}_\pi^{bP} - d^{abc}[(G^{-1})^T]^{bQ} \times (\tilde{f}_\pi^T)^{Qd}(\nabla^\mu\tilde{a}_\mu)^d G^{cP} + d^{abc}[(G^{-1})^T]^{bQ} \times [(f_\pi^T)^{Q0}\Box + (f_\pi^{-1})^{Qa}A^{a0}]\frac{1}{\mathrm{Tr}[\lambda^0]}\theta G^{cP}, \quad (58)$$

[3]The RG invariance of the VB equations arises from that of the S-matrix $\hat{\mathcal{S}}$.

$$\chi_{V2}^a = \underline{\tilde{a}}_\mu^{ab}(\tilde{f}_\pi \tilde{f}_\pi^T)^{bc}\tilde{a}^{\mu c} - \underline{\tilde{a}}_\mu^{ab}(\tilde{f}_\pi f_\pi^T)^{b0}\frac{1}{\mathrm{Tr}[\lambda^0]}\partial^\mu\theta + \underline{(G^{-1}Y)}^{ab}C^b, \tag{59}$$

$$\chi_A^a = \underline{v}_\mu^{ab}(f_\pi \tilde{f}_\pi^T)^{bc}\tilde{a}^{\mu c} - (f_\pi f_\pi^T)^{ab}\underline{v}_\mu^{bc}a^{\mu c} - \underline{v}_\mu^{ab}(f_\pi f_\pi^T)^{bc}\frac{1}{\mathrm{Tr}[\lambda^0]}\partial^\mu\theta, \tag{60}$$

$$K^{ab} = (\nabla^\mu\nabla_\mu)^{ab} - (\underline{a}^\mu\underline{a}_\mu)^{ab} + \overline{(G^{-1}Y)}^{ac}(Gf_\pi^{-1})^{cb} - \Box\delta^{ab}, \tag{61}$$

$$\mathcal{J}^P = J^P - (\tilde{f}_\pi^T)^{Pa}(\nabla_\mu\tilde{a}^\mu)^a + [(f_\pi^T)^{P0}\Box + (f_\pi^{-1})^{Pa}A^{a0}]\frac{1}{\mathrm{Tr}[\lambda^0]}\theta. \tag{62}$$

The terms χ_{V1}, χ_{V2}, and χ_A are attributable to the flavor symmetry breaking and thus they vanish in the flavor symmetric limit.

In deriving Eqs. (56) and (57), we have made use of a relation satisfied by the new constant C^a,

$$\bar{C}^{ab} = (Gf_\pi^T)^{ab}, \tag{63}$$

which can be regarded as the U(3) version of Eq. (19). This relation follows from the fact that $\hat{\pi}^P$ is the normalized interpolating pion field and can be derived by the same strategy as described for the SU(2) case in Appendix A. Then the mass relation (45) can be rewritten as

$$(f_\pi m_\pi^2 f_\pi^T)^{ab} = (\bar{m}_q\bar{C})^{ab} + A^{ab}. \tag{64}$$

From $(\bar{m}_q\bar{C})^T = (\bar{m}_q\bar{C})$ and $A^{ab} = 0$ for $a \neq 0$ [recall that $A^{ab} = \mathrm{Tr}[\lambda^a](\mathcal{A}f_\pi^T)^b$], one can fix A^{ab} up to one constant: $A^{ab} = \mathrm{Tr}[\lambda^a]\mathrm{Tr}[\lambda^b]A_\chi$. Within the $1/f_\pi$ expansion scheme [5], we have $\langle\hat{\sigma}^P\rangle = 0$ at the leading order, and therefore $C^0 = -(\sqrt{2/3})(\langle\bar{u}u\rangle + \langle\bar{d}d\rangle + \langle\bar{s}s\rangle)$, $C^3 = -(\langle\bar{u}u\rangle - \langle\bar{d}d\rangle)$, $C^8 = -(\sqrt{1/3})(\langle\bar{u}u\rangle + \langle\bar{d}d\rangle - 2\langle\bar{s}s\rangle)$, and $C^a = 0$ for the other flavor indices. In this case Eq. (64) reduces to the generalized GMOR relation given by Shore [4], which is derived by making use of the Ward identities in the zero-momentum (soft pion) limit. Because the leading order of the $1/f_\pi$ expansion scheme corresponds to taking the soft pion limit, we can identify the constant A_χ with the nonperturbative coefficient appearing in the gluon topological susceptibility in QCD [4]. The quantity $\langle\hat{\sigma}^P\rangle$ again represents the deviation of the mass relation (64) from the GMOR relation in the same way as for the SU(2) case.

The functional derivative form of Eqs. (56) and (57) is given by

$$T_V^a(x)\hat{\mathcal{S}} = 0, \tag{65}$$

$$T_A^a(x)\hat{\mathcal{S}} = 0, \tag{66}$$

with

$$T_V^a(x) = \nabla_\mu^{ab}\frac{\delta}{\delta v_\mu^b(x)} + \underline{a}_\mu^{ab}(x)\frac{\delta}{\delta a_\mu^b(x)} + \underline{(G^{-1}J)}^{ab}(x)G^{bP}\frac{\delta}{\delta J^P(x)} + [\underline{(G^{-1}Y)}^{ab}(x) + \underline{m}_q^{ab}]G^{bP}\frac{\delta}{\delta Y^P(x)} + \chi_{V1}^{aP}(x)\frac{\delta}{\delta J^P(x)} + i\chi_{V2}^a(x), \tag{67}$$

$$T_A^a(x) = f_\pi^{aP}[-\Box\delta^{PQ} - (m_\pi^2)^{PQ}]\frac{\delta}{\delta J^Q(x)} + f_\pi^{aP}R^P(x), \tag{68}$$

$$R^P(x) = -(f_\pi^{-1})^{Pa}K^{ab}(x)f_\pi^{bQ}\frac{\delta}{\delta J^Q(x)} + iJ^P(x) + (f_\pi^{-1})^{Pa}\times\left[\nabla_\mu^{ab}\frac{\delta}{\delta a_\mu^b(x)} + \underline{a}_\mu^{ab}\frac{\delta}{\delta v_\mu^b(x)} + \mathrm{Tr}[\lambda^a]\frac{\delta}{\delta\theta(x)}\right] + (f_\pi^{-1})^{Pa}\overline{(G^{-1}\mathcal{J})}^{ab}(x)G^{bQ}\frac{\delta}{\delta Y^Q(x)} - i(f_\pi^{-1})^{Pa}[\Omega_0^a(x) - \chi_A^a(x)]. \tag{69}$$

By introducing retarded and advanced Green's functions satisfying

$$[-\Box_x\delta^{PQ} - (m_\pi^2)^{PQ} - (f_\pi^{-1})^{Pa}K^{ab}(x)f_\pi^{bQ}]G_{R,A}^{QR}(x,y) = \delta^{PR}\delta^4(x-y),$$

one can formally solve Eq. (66) as

$$\frac{\delta}{\delta J^P(x)}\hat{\mathcal{S}} = i\hat{\mathcal{S}}\pi_{\mathrm{in}}^P(x) + i\hat{\mathcal{S}}\int d^4y G_R^{PQ}(x,y)(f_\pi^{-1})^{Qa}\times K^{ab}(y)f_\pi^{bR}\pi_{\mathrm{in}}^R(y) - \int d^4y G_R^{PQ}(x,y)\bar{R}^Q(y)\hat{\mathcal{S}} = i\pi_{\mathrm{in}}^P(x)\hat{\mathcal{S}} + i\int d^4y G_A^{PQ}(x,y)(f_\pi^{-1})^{Qa}\times K^{ab}(y)f_\pi^{bR}\pi_{\mathrm{in}}^R(y)\hat{\mathcal{S}} - \int d^4y G_A^{PQ}(x,y)\times\bar{R}^Q(y)\hat{\mathcal{S}}, \tag{70}$$

with

$$\bar{R}^P(x) = R^P(x) + (f_\pi^{-1})^{Pa}K^{ab}(x)f_\pi^{bQ}\frac{\delta}{\delta J^Q(x)}, \tag{71}$$

where π_{in}^P is the in-state asymptotic pion field. Equations (65) and (70) are the desired extension of the master equations to $\mathrm{U}_R(3)\times\mathrm{U}_L(3)$, incorporating the finite u, d, s quark masses and the $\mathrm{U}_A(1)$ anomaly. The major consequences of flavor symmetry breaking are the two-index character of the various constants and the appearance of $\chi_{V1}^{aP}\pi^P$, χ_{V2}^a, and χ_A^a terms. On the other hand, those arising from the $\mathrm{U}_A(1)$ anomaly are the mass relation (64) and the appearance of the operator $\hat{\omega}$.

D. Chiral reduction formula

The commutation relations of the pion creation and annihilation operator with the extended S-matrix $\hat{\mathcal{S}}$ are given by

$$[a^P_{\rm in}(k), \hat{\mathcal{S}}] = R^P(k)\hat{\mathcal{S}}, \qquad [\hat{\mathcal{S}}, a^{P\dagger}_{\rm in}(k)] = R^P(-k)\hat{\mathcal{S}}, \quad (72)$$

where $R^P(k) = \int dx e^{ikx} R^P(x)$. The $\mathrm{U}_R(3) \times \mathrm{U}_L(3)$ version of the χRF for on-shell pions can be expressed as

$$\begin{aligned}\langle \alpha; P_1(k_1), \cdots, P_m(k_m)|\hat{\mathcal{S}}|\beta; Q_1(l_1), \cdots, Q_n(l_n)\rangle|_{\phi=0} \\ = [R^{P_1}(k_1)\cdots R^{P_m}(k_m)R^{Q_1}(-l_1)\cdots R^{Q_n}(-l_n)]_{\rm S} \\ \times \langle \alpha|\hat{\mathcal{S}}|\beta\rangle|_{\phi=0},\end{aligned} \quad (73)$$

where k_i (l_i) is the four momentum of the outgoing (incoming) pion P_i (Q_i), and α and β label states of other particles. Here we again consider the case that no two pions have equal momenta. The symbol $[]_{\rm S}$ is defined in Eq. (31).

Before closing this section, we note that the singlet axial current $j^0_{A\mu}$ (or its functional derivative form) always appears as the RG-invariant combination $\partial^\mu j^0_{A\mu} - \mathrm{Tr}[\lambda^0]\hat{\omega}$ in the axial master equation (70) and the χRF (73). [In the absence of the external fields, $j^0_{A\mu}$ and $\hat{\omega}^0$ are renormalized as $(j^0_{A\mu})_{\rm ren} = Z_A j^0_{A\mu}$ and $(\hat{\omega})_{\rm ren} = \hat{\omega} + (Z_A - 1)(1/\mathrm{Tr}[\lambda^0]) \times \partial^\mu j^0_{A\mu}$, respectively.] Therefore, the existence of $\hat{\omega}$, which originates from the $\mathrm{U}(1)_A$ anomaly, is crucial for ensuring the RG invariance of the master equations and the χRF.

IV. APPLICATIONS

As an illustration, we will present several applications of the $\mathrm{U}_R(3) \times \mathrm{U}_L(3)$ master equations and the χRF.

A. Gluon topological susceptibility

By using Eqs. (3) and (55), we can derive the chiral Ward identity for the gluon topological susceptibility χ:

$$\begin{aligned}\chi &= -i \int d^4x \langle T^*[\omega(x)\omega(0)]\rangle \\ &= A_\chi - i\int d^4x \langle T^*[\hat{\omega}(x)\hat{\omega}(0)]\rangle - i6A_\chi^2 (f_\pi^{-1})^{P0}(f_\pi^{-1})^{Q0} \\ &\quad \times \int d^4x \langle T^*[\hat{\pi}^P(x)\hat{\pi}^Q(0)]\rangle - i2\sqrt{6}A_\chi (f_\pi^{-1})^{P0} \\ &\quad \times \int d^4x \langle T^*[\hat{\omega}(x)\hat{\pi}^P(0)]\rangle.\end{aligned} \quad (74)$$

This shows clearly how the pion poles contribute to the gluon topological susceptibility. The constant A_χ of the first term corresponds to the leading contribution in the large N_c limit of QCD with massive quarks, $\chi = A_\chi + \mathcal{O}(1/N_c)$ [28]. The appearance of A_χ is ensured by the modification introduced by the third term of δI [Eq. (49)]. The RG transformation property of $\hat{\omega}$ implies the RG invariance of the zero-momentum projected two-point functions $\int d^4x \langle T^*[\hat{\omega}(x)\hat{\pi}^P(0)]\rangle$ and $\int d^4x \langle T^*[\hat{\omega}(x)\hat{\omega}(0)]\rangle$, and therefore Eq. (74).

Making use of the chiral Ward identities of $\langle T^*[\hat{\omega}(x)\hat{\pi}^P(0)]\rangle$ and $\langle T^*[\hat{\pi}^P(x)\hat{\pi}^Q(0)]\rangle$ derived from the axial master equation (70) (see Appendix B for the results), the above identity can be further written as

$$\begin{aligned}\chi &= A_\chi - 6A_\chi^2[(f_\pi m_\pi^2 f_\pi^T)^{-1}]^{00} - i(1 \\ &\quad - 6A_\chi[(f_\pi m_\pi^2 f_\pi^T)^{-1}]^{00})^2 \int d^4x \langle T^*[\hat{\omega}(x)\hat{\omega}(0)]\rangle \\ &\quad + 6A_\chi^2[(f_\pi m_\pi^2 f_\pi^T)^{-1}]^{0a}\bar{m}_q^{ac}\overline{(G\langle\hat{\sigma}\rangle)}^{cb}[(f_\pi m_\pi^2 f_\pi^T)^{-1}]^{b0}.\end{aligned} \quad (75)$$

Here the second term comes from the pion pole in $\langle T^*[\hat{\pi}^P(x)\hat{\pi}^Q(0)]\rangle$. This is the most general expression constrained only by the broken chiral symmetry.

We observe that our result consistently reduces to those obtained in previous works by taking appropriate limits. With the mass relation (64), we obtain

$$A_\chi - 6A_\chi^2[(f_\pi m_\pi^2 f_\pi^T)^{-1}]^{00} = A_\chi\left(1 + A_\chi \sum_{q=u,d,s}\frac{1}{m_q C_q}\right)^{-1}, \quad (76)$$

where $C_u = \tilde{C} + C_3$, $C_d = \tilde{C} - C_3$, and $C_s = (C_0 - \sqrt{2}C_8)/\sqrt{6}$ with $\tilde{C} = (2C_0 + C_8)/\sqrt{3}$. The on-shell expansion scheme [5] gives $C_q = -\langle\bar{q}q\rangle$ ($q = u, d, s$) at the leading order. In this case, the first and second terms of our general expression (75) (i.e., the contributions from the leading term in the large N_c limit and the pion pole term) reproduce the classic result [4,29]. Also, our result approaches zero in the chiral limit, $\chi \to 0 (m_q \to 0)$, showing the noncommutative character of the $N_c \to \infty$ and $m_q \to 0$ limits [30].

The third and forth terms in Eq. (75) are new. The third term shows how the higher meson states X, including hybrids states and glueballs, contribute to the gluon topological susceptibility through $\langle 0|\hat{\omega}|X\rangle \neq 0$. The forth term is proportional to $\langle\hat{\sigma}^P\rangle$, which characterizes the deviation of the mass relation (64) from the GMOR relation.

B. Two-photon decay of π^0, η, η' mesons

Next let us consider the two-photon decay of the π^0, η, and η' mesons: $P(p) \to \gamma^{(*)}(q_1)\gamma^{(*)}(q_2)$ with $P = (\pi^0, \eta, \eta')$. For this purpose, we need to evaluate the amplitude $\int dx_{q_1} dy_{q_2} \langle 0|\hat{\mathcal{S}}T^*[j^a_{V\mu}(x)j^b_{V\nu}(y)]|\pi^P(p)\rangle|_{\phi=0}$, where $\int dx_q \equiv \int d^4x \exp(iqx)$. The χRF gives

$$\begin{aligned}&\int dx_{q_1} dy_{q_2} \langle 0|\hat{\mathcal{S}}T^*[j^{a\mu}_V(x)j^{b\nu}_V(y)]|\pi^P(p)\rangle|_{\phi=0} \\ &= \int dx_{q_1} dy_{q_2} (-i)^2 \frac{\delta}{\delta v^a_\mu(x)}\frac{\delta}{\delta v^b_\nu(y)} R^P(p)\langle\hat{\mathcal{S}}\rangle|_{\phi=0} \\ &= \int dx_{q_1} dy_{q_2} dz_{-p} (f_\pi^{-1})^{Pc}\Big[i\frac{\delta}{\delta v^a_\mu(x)}\frac{\delta}{\delta v^b_\nu(y)}\Omega_0^c(z) \\ &\quad + i\langle T^*[j^{a\mu}_V(x)j^{b\nu}_V(y)W^c(z)]\rangle\Big],\end{aligned} \quad (77)$$

where $W^c = \partial^\mu j^c_{A\mu} - \mathrm{Tr}[\lambda^c]\hat{\omega}$. With expressions for the electromagnetic current, $j^\mu_{\rm em} = (j^3_V)^\mu + (1/\sqrt{3})(j^8_V)^\mu$, and the external electromagnetic field, $-eA_\mu = v^3_\mu = \sqrt{3}v^8_\mu$, we can derive a general expression for the $P \to \gamma^{(*)}\gamma^{(*)}$ decay amplitude:

$$g_{\gamma\gamma P}(q_1^2, q_2^2; p^2) = (f_\pi^{-1})^{Pc}\Big[c^c_{\rm em}\frac{e^2N_c}{8\pi^2} - \delta^{c0}\sqrt{6}F_{\gamma\gamma\hat{\omega}} \times (q_1^2, q_2^2; p^2) + F^c_{\gamma\gamma A1}(q_1^2, q_2^2; p^2) - F^c_{\gamma\gamma A2}(q_1^2, q_2^2; p^2)\Big], \quad (78)$$

with $(c^3_{\rm em}, c^8_{\rm em}, c^0_{\rm em}) = (2/3, 2\sqrt{3}/9, 4\sqrt{6}/9)$. Here we have defined

$$\int dx_{q_1}dy_{q_2}\langle 0|\hat{S}T^*[j^\mu_{\rm em}(x)j^\nu_{\rm em}(y)]|\pi^P(p)\rangle|_{\phi=0} = i\epsilon^{\mu\nu\alpha\beta}q_{1\alpha}q_{2\beta}g_{\gamma\gamma P}(q_1^2, q_2^2; p^2), \quad (79)$$

$$\int dx_{q_1}dy_{q_2}dz_{-p}\langle T^*[j^\mu_{\rm em}(x)j^\nu_{\rm em}(y)\hat{\omega}(z)]\rangle = \epsilon^{\mu\nu\alpha\beta}q_{1\alpha}q_{2\beta}F_{\gamma\gamma\hat{\omega}}(q_1^2, q_2^2; p^2), \quad (80)$$

and

$$\int dx_{q_1}dy_{q_2}dz_{-p}\langle T^*[j^\mu_{\rm em}(x)j^\nu_{\rm em}(y)\partial_\lambda j^{c\lambda}_A(z)]\rangle = \epsilon^{\mu\nu\alpha\beta}q_{1\alpha}q_{2\beta}[F^c_{\gamma\gamma A1}(q_1^2, q_2^2; p^2) - F^c_{\gamma\gamma A2}(q_1^2, q_2^2; p^2)], \quad (81)$$

modulo $(2\pi)^4\delta^4(p - q_1 - q_2)$. In Eq. (81) we have used the general expression of the vector-vector-axial correlation function [31,32] [denoting $F^c_{\gamma\gamma Ai}(q_1^2, q_2^2; p^2)$ as $F^c_{\gamma\gamma Ai}$]:

$$\frac{1}{i}\int dx_{q_1}dy_{q_2}\langle T^*[j^\mu_{\rm em}(x)j^\nu_{\rm em}(y)j^{c\lambda}_A(0)]\rangle = \sum_{i=1}^{6}F^c_{\gamma\gamma Ai}I_i^{\lambda\mu\nu},$$

with

$$I_1^{\lambda\mu\nu} = \epsilon^{\lambda\mu\nu\alpha}q_{1\alpha}, \qquad I_2^{\lambda\mu\nu} = \epsilon^{\lambda\mu\nu\alpha}q_{2\alpha},$$
$$I_3^{\lambda\mu\nu} = \epsilon^{\lambda\mu\alpha\beta}q_{1\alpha}q_{2\beta}q_2^\nu, \qquad I_4^{\lambda\mu\nu} = \epsilon^{\lambda\nu\alpha\beta}q_{1\alpha}q_{2\beta}q_1^\mu,$$
$$I_5^{\lambda\mu\nu} = \epsilon^{\lambda\mu\alpha\beta}q_{1\alpha}q_{2\beta}q_1^\nu, \qquad I_6^{\lambda\mu\nu} = \epsilon^{\lambda\nu\alpha\beta}q_{1\alpha}q_{2\beta}q_2^\mu.$$

The broken chiral symmetry relates the $P \to \gamma^{(*)}\gamma^{(*)}$ decay amplitude to the correlation functions of the current and density operators (80) and (81). The correlation functions themselves, however, cannot be fixed from symmetry requirements and thus some dynamical inputs are needed to evaluate them.

Equation (78) is applicable both for on- and off-shell pions. Using the partially conserved axial-vector current hypothesis $g_{\gamma\gamma P}(q_1^2, q_2^2; m_P^2) \sim g_{\gamma\gamma P}(q_1^2, q_2^2; 0)$ and setting q_1 and q_2 to the photon point, $q_1^2 = q_2^2 = 0$, our general expression (78) consistently reduces to the two-photon decay formula of π^0, η, η' mesons derived by Shore [4,28],

$$g_{\gamma\gamma P}(0, 0; m_P^2) \sim g_{\gamma\gamma P}(0, 0; 0) = (f_\pi^{-1})^{Pc}\Big[c^c_{\rm em}\frac{e^2N_c}{8\pi^2} - \sqrt{6}F_{\gamma\gamma\hat{\omega}}(0, 0; 0)\Big], \quad (82)$$

if we identify $F_{\gamma\gamma\hat{\omega}}(0, 0; 0)$ as $Ag_{G\gamma\gamma}$. Here we have used $F^c_{\gamma\gamma A1}(0, 0; 0) = F^c_{\gamma\gamma A2}(0, 0; 0) = 0$ [32]. The RG invariance of the operator $W^c = \partial^\mu j^c_{A\mu} - \mathrm{Tr}[\lambda^c]\hat{\omega}$ implies that the combination $\sqrt{6}F_{\gamma\gamma\hat{\omega}} - F^0_{\gamma\gamma A1} + F^0_{\gamma\gamma A2}$ is also RG invariant. Therefore $F_{\gamma\gamma\hat{\omega}}(0, 0; 0)$ is by itself RG invariant.

V. SUMMARY AND OUTLOOK

We have derived an extension of the master equations for chiral symmetry breaking proposed in Refs. [5,6] to the $\mathrm{U}_R(3) \times \mathrm{U}_L(3)$ chiral group, carefully taking into account the $\mathrm{U}_A(1)$ anomaly and full flavor symmetry breaking $m_u \neq m_d \neq m_s$. With the master equations and the χRF, new chiral Ward identities for the gluon topological susceptibility χ and $P \to \gamma^{(*)}\gamma^{(*)}$ decay amplitude have been derived, showing how the constraints from broken chiral symmetry enter into those quantities without relying on any unphysical limits. Then we have seen that our general results consistently reduce to those obtained in previous studies by taking appropriate limits.

The χRF is applicable to any reaction processes which include ground state pseudoscalar mesons, e.g., π, K, $\bar{K}$, η, η' production reactions on a baryon target and heavy meson decays such as $J/\Psi \to 3P$, $\gamma 2P$, $\gamma 3P$, $\cdots$ with $P = \pi, K, \bar{K}, \eta, \eta'$. The heavy meson decays are interesting in relation to new meson resonance states appearing in the decay processes [33,34]. A careful treatment of the final state interactions in the decay processes will be vital for exploring properties of such new meson states. The χRF enables one to separate details of each reaction mechanism from the general framework required by broken chiral symmetry, and thus will provide a useful theoretical basis for the analysis of such processes. Investigations in this direction will be discussed elsewhere.

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APPENDIX A: DERIVATION OF EQ. (19)

In this Appendix we describe the derivation of Eq. (19), which relates the new constant C to the pion decay constant

f_π and the pseudoscalar coupling constant G. The strategy used here can be straightforwardly applied to the U(3) case [Eq. (63)]; in this case the algebraic operations just become more complicated.

Originally the operator $R^a_{\mathrm{SU(2)}}(k)$ has the following form:

$$R^a_{\mathrm{SU(2)}}(k) = \int d^4x e^{+ikx}\Big[i\frac{f_\pi}{GC}J^a + \frac{1}{f_\pi}t^a_A - K^{ab}_{\mathrm{SU(2)}}\frac{\delta}{\delta J^b} - \left(\nabla^{\mu ac}a^c_\mu - \frac{J^a}{f_\pi}\right)\frac{\delta}{\delta Y} + i\left(1-\frac{f_\pi}{GC}\right) \times f_\pi \nabla^{\mu ab}a^b_\mu\Big](x). \quad \text{(A1)}$$

Equation (19) is obtained by making use of the fact that the $\hat{\pi}^a$ is the *normalized* interpolating pion field satisfying $\langle 0|\hat{\pi}^a(x)|\pi^b(p)\rangle = \delta^{ab}e^{-ipx}$. By applying the χRF to the matrix element $\langle 0|\hat{\pi}^a(x)|\pi^b(p)\rangle$, we have (note that $\hat{\mathcal{S}}|0\rangle = \hat{\mathcal{S}}^\dagger|0\rangle = |0\rangle$ following from the stability of the vacuum state),

$$\delta^{ab}e^{-ipx} = \langle 0|\hat{\pi}^a(x)|\pi^b(p)\rangle = -i\langle 0|\hat{\mathcal{S}}^\dagger \frac{\delta}{\delta J^a(x)}R^b_{\mathrm{SU(2)}}(-p)\hat{\mathcal{S}}|0\rangle|_{\phi=0}, \quad \text{(A2)}$$

with

$$\begin{aligned} -i\hat{\mathcal{S}}^\dagger \frac{\delta}{\delta J^a(x)}R^b_{\mathrm{SU(2)}}(-p)\hat{\mathcal{S}} &= +\frac{f_\pi}{GC}\delta^{ab}e^{-ipx} + \frac{\hat{\sigma}(x)}{f_\pi}\delta^{ab}e^{-ipx} + \frac{i}{f_\pi}\int d^4y e^{-ipy}(ip)^\mu T^*[\hat{\pi}^a(x)j^b_{A\mu}(y)] + \mathcal{O}(\phi) \\ &= +\frac{f_\pi}{GC}\delta^{ab}e^{ipx} + \frac{\delta^{ab}}{f_\pi}(p^2-m_\pi^2)\int d^4y e^{-ipy}\Delta_R(x-y)\hat{\sigma}(y) - \frac{1}{f_\pi}\int d^4y e^{-ipy}p^\mu j^b_{A\mu}(y)\pi^a_{\mathrm{in}}(x) \\ &\quad + \frac{1}{f_\pi^2}\varepsilon^{abc}\int d^4y e^{-ipy}\Delta_R(x-y)[-2ip^\mu + (\partial_y)^\mu]j^c_{V\mu}(y) \\ &\quad - \frac{1}{f_\pi}\int d^4y \Delta_R(x-y)(\partial_y)^\mu \hat{\mathcal{S}}^\dagger[\hat{\mathcal{S}}j^a_{A\mu}(y), a^{b\dagger}_{\mathrm{in}}(p)] + \mathcal{O}(\phi), \end{aligned} \quad \text{(A3)}$$

where $(\partial_y)^\mu = [\partial/(\partial y_\mu)]$. In the last step of Eq. (A3), we have used Eqs. (26) and (27) with $R^a_{\mathrm{SU(2)}}$ replaced by Eq. (A1), and the relation

$$\begin{aligned} \hat{\mathcal{S}}^\dagger[\hat{\mathcal{S}}j^a_{A\mu}(y), a^{b\dagger}_{\mathrm{in}}(p)] &= -i\frac{\delta}{\delta a^{\mu a}(y)}R^b_{\mathrm{SU(2)}}(-p)\hat{\mathcal{S}} \\ &= \delta^{ab}(\partial_y)_\mu[e^{-ipy}\hat{\sigma}(y)] \\ &\quad + \varepsilon^{abc}\frac{1}{f_\pi}e^{-ipy}j^c_{V\mu}(y) \\ &\quad - \frac{1}{f_\pi}p^\nu\int d^4z e^{-ipz} \\ &\quad \times T^*[j^a_{A\mu}(y)j^b_{A\nu}(z)] + \mathcal{O}(\phi). \end{aligned} \quad \text{(A4)}$$

Noticing that $\langle 0|j^a_{A\mu}|\pi^b\rangle = 0$, $\langle 0|j^a_{V\mu}|0\rangle = 0$, and $p^2 = m_\pi^2$, Eq. (A2) gives

$$1 = \frac{f_\pi}{GC}. \quad \text{(A5)}$$

APPENDIX B: COMMUTATION RELATIONS AND CHIRAL WARD IDENTITIES

1. Commutation relations

The functional derivative operators defined in Eqs. (67) and (68) satisfy the following commutation relations [defining $T^a_V(k) = \int d^4x \exp(ikx)T^a_V(x)$ and $T^a_A(k) = \int d^4x \exp(ikx)T^a_A(x)$]:

$$[T^a_V(k), T^b_V(k')] = -f^{abc}T^c_V(k+k'), \quad \text{(B1)}$$

$$[T^a_V(k), T^b_A(k')] = -f^{abc}T^c_A(k+k'), \quad \text{(B2)}$$

$$[T^a_A(k), T^b_A(k')] = -f^{abc}T^c_V(k+k'). \quad \text{(B3)}$$

With Eqs. (68) and (B3) we further obtain

$$\begin{aligned}[T^a_A(k), R^P(k')] &= -(f_\pi^{-1})^{Pb}f^{abc}T^c_V(k+k') \\ &\quad + [k'^2\delta^{PQ} - (m_\pi^2)^{PQ}]\int d^4x e^{i(k+k')x} \\ &\quad \times \left[if^{aQ}_\pi + \hat{d}^{aQC}\frac{\delta}{\delta Y^C(x)}\right], \end{aligned} \quad \text{(B4)}$$

where $\hat{d}^{aBC} = d^{abc}(G^{-1})^{Bb}G^{cC}$. If k'_μ is the on-shell pion momentum satisfying $(k')^2 = (m_\pi^2)^{PP}$, we have $[T^a_A(k), R^P(k')]\hat{\mathcal{S}} = 0$. Therefore, we can make use of the same prescription as proposed in Ref. [25] to derive the off-shell extension of the U(3) χRF (73).

2. Chiral Ward identities for the two-point functions

Here we summarize several chiral Ward identities used for obtaining the results in Sec. IVA. Making use of Eqs. (55), (2), and (3), the two-point function of ω can be expressed as

$$\int d^4xe^{-ip(x-y)}\langle T^*[\omega(x)\omega(y)]\rangle = \int d^4xe^{-ip(x-y)}\langle T^*[\hat{\omega}(x)\hat{\omega}(y)]\rangle + i\Big[A_\chi + \frac{1}{6}(\delta\tilde{f}_\pi\delta\tilde{f}_\pi^T)^{00}(-p^2)\Big] + \mathcal{N}^P(p^2) \times \int d^4xe^{-ip(x-y)}\langle T^*[\hat{\omega}(x)\hat{\pi}^P(y)]\rangle + \mathcal{N}^P(p^2)\int d^4xe^{-ip(x-y)}\langle T^*[\hat{\omega}(y)\hat{\pi}^P(x)]\rangle + \mathcal{N}^P(p^2)\mathcal{N}^Q(p^2)\int d^4xe^{-ip(x-y)}\langle T^*[\hat{\pi}^P(x)\hat{\pi}^Q(y)]\rangle, \tag{B5}$$

with $\mathcal{N}^P(p^2) = \sqrt{6}A_\chi(f_\pi^{-1})^{P0} + (p^2/\sqrt{6})(\delta\tilde{f}_\pi)^{0P}$. This identity reduces to that of the gluon topological susceptibility [Eq. (74)] by setting $p_\mu = 0$ and $y = 0$.

Similarly, with the axial master equation (70), the two-point function of the interpolating pion field can be written as

$$\begin{aligned}\int d^4xe^{ip(x-y)}\langle T^*[\hat{\pi}^A(x)\hat{\pi}^B(y)]\rangle &= \int d^4xe^{ip(x-y)}\langle\hat{\pi}^B(y)\pi_{\rm in}^A(x)\rangle + i\int d^4xe^{ip(x-y)}\Delta_R^{AB}(x-y) + i\int d^4xe^{ip(x-y)}\Delta_R^{AC}(x-y)\\&\quad\times(f_\pi^{-1})^{Cc}\hat{d}^{cBD}\langle\hat{\sigma}^D\rangle - \int d^4xe^{ip(x-y)}\int dz\Delta_R^{AC}(x-z)(f_\pi^{-1})^{Cc}\langle T^*[W^c(z)\hat{\pi}^B(y)]\rangle\\&= i\int d^4xe^{ip(x-y)}\Delta_F^{AB}(x-y) + i\int d^4xe^{ip(x-y)}\Delta_R^{AC}(x-y)(f_\pi^{-1})^{Cc}\hat{d}^{cBD}\langle\hat{\sigma}^D\rangle\\&\quad+ i\int d^4xe^{ip(x-y)}\Delta_R^{BC}(y-x)(f_\pi^{-1})^{Cc}\hat{d}^{cAD}\langle\hat{\sigma}^D\rangle + i\int d^4xe^{ip(x-y)}\int dz\Delta_R^{AC}(x-z)\\&\quad\times\Delta_R^{BF}(y-z)(f_\pi^{-1})^{Fd}[(m_\pi^2)^{CD} - 6A_\chi(f_\pi^{-1})^{C0}(f_\pi^{-1})^{D0}]\hat{d}^{dDE}\langle\hat{\sigma}^E\rangle + \int d^4xe^{ip(x-y)}\\&\quad\times\int dzdz'\Delta_R^{AC}(x-z)\Delta_R^{BD}(y-z')(f_\pi^{-1})^{Cc}(f_\pi^{-1})^{Dd}\langle T^*[W^c(z)W^d(z')]\rangle.\end{aligned} \tag{B6}$$

Here we have used $W^a = \partial^\mu j_{A\mu}^a - {\rm Tr}[\lambda^a]\hat{\omega}$; $\Delta_R^{AB}(x-y)$ and $\Delta_F^{AB}(x-y)$ are, respectively, the retarded and Feynman propagators satisfying $[-\Box_x\delta^{PA} - (m_\pi^2)^{PA}]\Delta_{R(F)}^{AB}(x-y) = \delta^{PB}\delta^4(x-y)$. In the last step, we have made use of Eq. (70) and the following Ward identities:

$$\int d^4xe^{ip(x-y)}\langle T^*[j_{A\mu}^a(x)\hat{\pi}^A(y)]\rangle = \int d^4xe^{ip(x-y)}\langle j_{A\mu}^a(x)\pi_{\rm in}^A(y)\rangle + \int d^4xe^{ip(x-y)}p_\mu\Delta_R^{AC}(y-x)(f_\pi^{-1})^{Cc}(\tilde{f}_\pi{}^T)^{Ba}\hat{d}^{cBD}\langle\hat{\sigma}^D\rangle - \int d^4xe^{ip(x-y)}\int dz\Delta_R^{AC}(y-z)(f_\pi^{-1})^{Cc}\langle T^*[j_{A\mu}^a(x)W^c(z)]\rangle, \tag{B7}$$

$$\begin{aligned}\int d^4xe^{ip(x-y)}\langle T^*[\hat{\omega}(x)\hat{\pi}^A(y)]\rangle &= \int d^4xe^{ip(x-y)}\langle\hat{\omega}(x)\pi_{\rm in}^A(y)\rangle - i\int d^4xe^{ip(x-y)}\sqrt{6}A_\chi\Delta_R^{AC}(y-x)(f_\pi^{-1})^{Cc}(f_\pi^{-1})^{B0}\hat{d}^{cBD}\langle\hat{\sigma}^D\rangle\\&\quad- i\int d^4xe^{ip(x-y)}\frac{p^2}{\sqrt{6}}\Delta_R^{AC}(y-x)(f_\pi^{-1})^{Cc}[\delta f_\pi^T]^{Ba}\hat{d}^{cBD}\langle\hat{\sigma}^D\rangle\\&\quad- \int d^4xe^{ip(x-y)}\int dz\Delta_R^{AC}(y-z)(f_\pi^{-1})^{Cc}\langle T^*[\hat{\omega}(x)W^c(z)]\rangle.\end{aligned} \tag{B8}$$

Setting $p^2 = 0$ and $y = 0$, Eqs. (B6) and (B8) become

$$\begin{aligned}(f_\pi^{-1})^{A0}(f_\pi^{-1})^{B0}\int d^4x\langle T^*[\hat{\pi}^A(x)\hat{\pi}^B(0)]\rangle &= -i[(f_\pi m_\pi^2 f_\pi^T)^{-1}]^{00} - i2[(f_\pi m_\pi^2 f_\pi^T)^{-1}]^{0c}(f_\pi^{-1})^{C0}\hat{d}^{cCD}\langle\hat{\sigma}^D\rangle\\&\quad+ i[(f_\pi m_\pi^2 f_\pi^T)^{-1}]^{0a}[(f_\pi m_\pi^2 f_\pi^T)^{-1}]^{0c}[(f_\pi m_\pi^2 f_\pi^T)^{ab} - A^{ab}](f_\pi^{-1})^{Cb}\hat{d}^{cCD}\langle\hat{\sigma}^D\rangle\\&\quad+ 6[(f_\pi m_\pi^2 f_\pi^T)^{-1}]^{00}[(f_\pi m_\pi^2 f_\pi^T)^{-1}]^{00}\int d^4x\langle T^*[\hat{\omega}(x)\hat{\omega}(0)]\rangle,\end{aligned} \tag{B9}$$

and

$$(f_\pi^{-1})^{A0}\int d^4x\langle T^*[\hat{\omega}(x)\hat{\pi}^A(0)]\rangle = i\sqrt{6}A_\chi[(f_\pi m_\pi^2 f_\pi^T)^{-1}]^{0c}(f_\pi^{-1})^{B0}\hat{d}^{cBD}\langle\hat{\sigma}^D\rangle - \sqrt{6}[(f_\pi m_\pi^2 f_\pi^T)^{-1}]^{00}\int d^4x\langle T^*[\hat{\omega}(x)\hat{\omega}(0)]\rangle, \tag{B10}$$

respectively. Substituting these equations into Eq. (74), we obtain Eq. (75).

[1] R. Kaiser and H. Leutwyler, Eur. Phys. J. C **17**, 623 (2000).
[2] B. Borasoy and S. Wetzel, Phys. Rev. D **63**, 074019 (2001).
[3] G. M. Shore, Nucl. Phys. **B569**, 107 (2000).
[4] G. M. Shore, Nucl. Phys. **B744**, 34 (2006).
[5] H. Yamagishi and I. Zahed, Ann. Phys. (N.Y.) **247**, 292 (1996).
[6] C.-H. Lee, H. Yamagishi, and I. Zahed, Nucl. Phys. **A653**, 185 (1999).
[7] J. V. Steele, H. Yamagishi, and I. Zahed, Nucl. Phys. **A615**, 305 (1997).
[8] J. V. Steele, H. Yamagishi, and I. Zahed, Phys. Rev. D **57**, 1703 (1998).
[9] H. Kamano and M. Arima, Phys. Rev. C **69**, 025206 (2004).
[10] H. Kamano, M. Morishita, and M. Arima, Phys. Rev. C **71**, 045201 (2005).
[11] H. Kamano and M. Arima, Phys. Rev. C **73**, 055203 (2006).
[12] J. V. Steele, H. Yamagishi, and I. Zahed, Phys. Lett. B **384**, 255 (1996).
[13] J. V. Steele, H. Yamagishi, and I. Zahed, Phys. Rev. D **56**, 5605 (1997).
[14] C.-H. Lee, H. Yamagishi, and I. Zahed, Phys. Rev. C **58**, 2899 (1998).
[15] J. V. Steele and I. Zahed, Phys. Rev. D **60**, 037502 (1999).
[16] K. Dusling, D. Teaney, and I. Zahed, Phys. Rev. C **75**, 024908 (2007).
[17] K. Dusling and I. Zahed, Phys. Rev. C **80**, 014902 (2009).
[18] K. Dusling and I. Zahed, Nucl. Phys. **A825**, 212 (2009).
[19] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Wiley, New York, 1980).
[20] H. Yamagishi, Int. J. Mod. Phys. A **4**, 2591 (1989).
[21] H. Yamagishi, Prog. Theor. Phys. **78**, 886 (1987).
[22] M. Veltman, Phys. Rev. Lett. **17**, 553 (1966).
[23] J. S. Bell, Nuovo Cimento A **50**, 129 (1967).
[24] A. Höll, A. Krassnigg, and C. D. Roberts, Phys. Rev. C **70**, 042203(R) (2004).
[25] H. Kamano, Prog. Theor. Phys. **116**, 839 (2006).
[26] W. A. Bardeen, Phys. Rev. **184**, 1848 (1969).
[27] G. M. Shore and G. Veneziano, Nucl. Phys. **B381**, 3 (1992).
[28] G. M. Shore, in *String Theory and Fundamental Interactions*, edited by M. Gasperini and J. Maharana, Lecture Notes in Physics (Springer, New York, 2007).
[29] P. D. Vecchia and G. Veneziano, Nucl. Phys. **B171**, 253 (1980).
[30] E. Witten, Nucl. Phys. **B156**, 269 (1979).
[31] S. L. Adler, Phys. Rev. **177**, 2426 (1969).
[32] G. M. Shore and G. Veneziano, Mod. Phys. Lett. A **8**, 373 (1993).
[33] M. Ablikim *et al.*, Phys. Rev. Lett. **95**, 262001 (2005).
[34] M. Ablikim *et al.*, Phys. Rev. Lett. **97**, 142002 (2006).