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Small shear viscosity in the semiquark gluon plasma

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At nonzero temperature in QCD, about the deconfining phase transition there is a semiquark gluon plasma (semi-QGP), where the expectation value of the (renormalized) Polyakov loop is less than one. This can be modeled by a semiclassical expansion about a constant field for the vector potential, A_0, which is diagonal in color. We compute the shear viscosity in the semi-QGP by using the Boltzmann equation in the presence of this background field. To leading, logarithmic order in weak coupling, the dominant diagrams are given by the usual scattering processes of $2 \to 2$ particles. For simplicity we also assume that both the number of colors and flavors are large. Near the critical temperature T_c, where the expectation value of the Polyakov loop is small, the overall density of colored fields decreases according to their color representation, with the density of quarks vanishes linearly with the loop, and that of gluons, quadratically. This decrease in the overall density dominates changes in the transport cross section. As a result, relative to that in the perturbative QGP, near T_c the shear viscosity in the semi-QGP is suppressed by two powers of the Polyakov loop. In a semiclassical expansion, the suppression of colored fields depends only upon which color representation they lie in, and not upon their mass. That light and heavy quarks are suppressed in a common manner may help to explain the behavior of charm quarks at RHIC.

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I. INTRODUCTION

The collisions of heavy ions at ultrarelativistic energies have exhibited a multitude of surprising results [1–4]. Experiments at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven have shown that in peripheral collisions, the anisotropy with respect to the reaction plane, or elliptic flow, can described by nearly ideal hydrodynamics, with a small value for the shear viscosity. In kinetic theory, the shear viscosity is inversely proportional to the squared scattering amplitude [5], so a small shear viscosity could be due to strong coupling. This suggests that is fruitful to consider the analogy to $\mathcal{N} = 4$ supersymmetric gauge theories, where one can compute at infinite coupling for an infinite number of colors. For $\mathcal{N} = 4$ supersymmetry, the ratio of the shear viscosity to the entropy density is $1/4\pi$, which is conjectured to be a universal lower bound [6,7]. The shear viscosity has also been computed from numerical simulations on the lattice [8], in the hadronic phase [9], and in the perturbative quark gluon plasma (QGP) [10–19].

In this paper we adopt an alternate approach from $\mathcal{N} = 4$ supersymmetry. Instead of trying to work down from infinite coupling, we work up from small coupling. While perturbation theory at nonzero temperature is badly behaved, resummed perturbation theory works down to much lower temperatures, to a few times T_c, where T_c is the critical temperature for deconfinement [20–24]. This suggests that nonperturbative effects dominate the region near T_c, which we have termed the semi-QGP [25–29]. In this paper, the fourth in a series [30–32], we compute the shear viscosity in a simple approximation for the semi-QGP.

The basis of this approach are measurements of the order parameters for deconfinement. Consider a straight Wilson line in the direction of imaginary time,

$$L = P \exp\left(i \int_0^{1/T} d\tau A_\tau \right), \tag{1}$$

where P denotes path ordering, T is the temperature, τ the imaginary time, and A_τ is the timelike component of the gauge field, in the fundamental representation. The Wilson line is a matrix in color space, and so is not gauge invariant, but its eigenvalues are. The simplest measure of the eigenvalues of the Wilson line is its trace, which is the Polyakov loop,

$$\ell = \frac{1}{N_c} \mathrm{tr} L. \tag{2}$$

This Polyakov loop is directly related to the propagator of an infinitely heavy quark, which for Eq. (2) is in the fundamental representation. Heuristically, one can view the Polyakov loop as measuring the excess free energy f_q which arises from adding a colored, heavy quark to a thermal bath, $\langle \ell \rangle \sim \exp(-f_q/T)$.

This Polyakov loop is a bare quantity, but a renormalized quantity can be extracted from simulations on the lattice [32–36]. These indicate that the expectation value of the Polyakov loop in Eq. (2) is approximately constant and near one for temperatures greater than $\sim 3T_c$. Below $\sim 3T_c$, however, the Polyakov loop is significantly less than one. In a pure gauge theory, the Polyakov loop is a strict order parameter for deconfinement, so below T_c the excess free energy f_q is infinite, and the expectation value of the Polyakov loop vanishes identically. With dynamical

quarks, the Polyakov loop is only an approximate order parameter: because of screening by quark antiquark pairs, below T_c the excess free energy f_q is finite, and $\langle \ell \rangle$ is nonzero. In a gauge theory with three colors and three light flavors, as in QCD, numerical simulations find that the Polyakov loop is small at T_c, $\langle \ell \rangle \approx 0.2$, and essentially vanishes below $\approx 0.8T_c$; see, e.g., Fig. (13) of Bazavov *et al.*, [34,35]. We dub the region in which the Polyakov loop deviates from one as the semi-QGP [30–32].

Similar considerations have also motivated what is known at the Polyakov–Nambu–Jona-Lasino model [29]. In these models, the only dynamical variable related to deconfinement is the simplest Polyakov loop, that in the fundamental representation, Eq. (2). For an $SU(N_c)$ gauge theory, however, the thermal Wilson line has $N_c - 1$ independent eigenvalues. These are characterized by traces of higher powers of L, $\mathrm{tr}L^2$ through $\mathrm{tr}L^{N_c-1}$. These higher moments of the thermal Wilson line are not directly accessible on the lattice, since loops in different representations have distinct renormalization constants. These loops have a simple physical interpretation, as the propagator of a heavy quark in that representation.

In the end, we parametrize our results in terms of the Polyakov loop in the fundamental representation. Even so, in intermediate steps, we could not represent the computations by a simplified model in which only the simplest Polyakov loop enters: the matrix structure is essential. We discuss this further in the Conclusions, Sec. V.

Our approach to the semi-QGP is the following: To the classical Lagrangian, we add terms which drive confinement, such as [26]

$$\mathcal{L}_{\mathrm{eff}} = b_{\mathrm{fuzzy}} T^2 T_c^2 |\mathrm{tr}L|^2; \tag{3}$$

there are many other possible terms [25–27]. To date, a simple form for the effective Lagrangian for the semi-QGP has not been obtained; this would allow one to compute both the pressure and the renormalized loop from the same effective Lagrangian [37]. Clearly analysis from numerical simulations, especially in effective models, is essential to gaining this understanding [28].

We shall see that knowing the full form for the effective Lagrangian is irrelevant to the question we address in this paper: near T_c, where the expectation value of the Polyakov loop is small, is the shear viscosity suppressed, or enhanced? Naive expectation, based upon kinetic theory, indicates that as the cross section decreases, the shear viscosity increases. In contrast, we find that the shear viscosity in the semi-QGP is smaller than in the perturbative QGP.

We note that our approach to the semi-QGP breaks strong resemblance to "double-trace" deformations of the vacuum theory [38]. In these models, a term such as Eq. (3) is added to drive the theory into a confining phase.

This paper is organized as follows: In Sec. II, we consider a semiclassical expansion of the semi-QGP. For technical reasons we assume that both N_c and N_f are large. In Sec. III, we discuss how to compute transport coefficients in the semi-QGP using kinetic theory. This is a standard approach, except that kinetic theory is in the presence of a background field, A_τ in Eq. (2). This background field is important near the phase transition, and suppresses the shear viscosity. Section IV gives numerical results. In Sec. V we discuss possible phenomenological implications of our results. We find that in a semiclassical expansion of the semi-QGP, near T_c all colored particles are suppressed in a universal manner. This is natural, as a type of "bleaching" of color as $T \to T_c^+$. In particular, this color bleaching is independent of the mass of the field. We suggest that this might help to explain the otherwise puzzling results on the behavior of charm quarks at RHIC.

II. EFFECTIVE THEORY IN THE SEMI-QGP

A semi-QGP is characterized by the Polyakov loop, Eq. (2). Static quantities, such as the pressure, are determined by an effective Lagrangian of Polyakov loops [25–29]. What we require is an effective theory in real time, which can be used to compute transport coefficients. Start with the partition function for QCD,

$$Z(T) = \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\Big[i \int_C d^4x \mathcal{L} \Big], \tag{4}$$

where $\mathcal{L}$ is the Lagrangian,

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{tr}F_{\mu\nu}^2 + \bar{\psi}(i\not{D} - m)\psi, \tag{5}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$; ψ and $\bar{\psi}$ is the quark field, $\not{D} = \gamma^\mu(\partial_\mu - igA_\mu)$, and m is the quark mass. C denotes a path in complex time, Fig. 1. We work in Minkowski spacetime with a metric, $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The timelike gauge field on the imaginary time in Eq. (1), A_τ, corresponds to iA_0 in this notation. We defer a discussion of a fully self-consistent approach, where terms such as Eq. (3) are added, to later analysis. Including such a term would not alter our results at leading order in g, because to leading order the shear viscosity is dominated by the scattering of hard particles, with momenta $\sim T$, while terms in Eq. (3) affect fields at soft momenta.

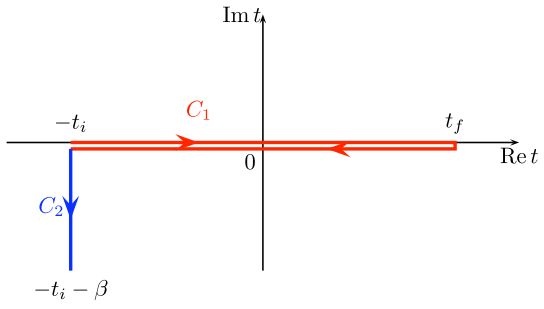

FIG. 1 (color online). Complex-time path.

For the generators of $SU(N_c)$, we use what may be called 't Hooft's double line basis [31,39]:

$$[t^{ab}]_{cd} = \frac{1}{\sqrt{2}}\left(\delta^a_c\delta^b_d - \frac{1}{N_c}\,\delta^{ab}\delta_{cd}\right). \tag{6}$$

Somewhat unconventionally, we denote indices in the fundamental representation as $a, b, c, d, \ldots = 1 \ldots N_c$. In the double line basis, generators are denoted by a pair of indices in the fundamental representation, ab, so that in all there are N_c^2 generators. This is one more generator than in an orthonormal basis. As an overcomplete basis, the trace of two generators is not a delta function, but a projection operator:

$$\mathrm{tr}\, t^{ab}t^{cd} = \frac{1}{2}P^{ab,cd}, \qquad P^{ab,cd} = \delta^{ad}\delta^{bc} - \frac{1}{N_c}\delta^{ab}\delta^{cd}. \tag{7}$$

The commutator of two generators is

$$[t^{ab}, t^{cd}] = i\sum_{e,f=1}^{N_c} f^{(ab,cd,ef)}t^{fe}, \tag{8}$$

where $f^{(ab,cd,ef)}$ is the structure constant,

$$f^{(ab,cd,ef)} = \frac{i}{\sqrt{2}}(\delta^{ad}\delta^{cf}\delta^{eb} - \delta^{af}\delta^{cb}\delta^{ed}). \tag{9}$$

The double line basis is obviously convenient at large N_c, as then the terms $\sim 1/N_c$ in Eqs. (6) and (7) can be dropped. It is also of use at finite N_c, however, since then each line in a generator represents a flow of color charge, and charge conservation is obvious; this is especially true in the presence of a background field [31].

We denote four momenta as $P^\mu = (p^0, \boldsymbol{p})$, where p_0 is the timelike component of the momenta, and $\boldsymbol{p}$ the spatial momentum. In thermal equilibrium at a temperature T, $p_0 = i\omega_n$, where ω_n is a Matsubara frequency: $\omega_n = 2n\pi T$ for bosons, and $= (2n+1)\pi T$ for fermions, where n is an integer. In imaginary time, a nonzero value of the Polyakov loop is modeled by taking a constant field for A_0,

$$[A_0^{\mathrm{cl}}]_{ab} = i\delta_{ab}\frac{Q^a}{g}. \tag{10}$$

We expand about the classical field, A_0^{cl}, in fluctuations, B_μ:

$$A_\mu = \delta_{\mu 0}A_0^{\mathrm{cl}} + B_\mu. \tag{11}$$

The covariant derivatives in this particular background field are especially simple:

$$[i(\partial_0 - igA_0^{\mathrm{cl}})\psi]_a \to (p_0 + iQ^a)\psi_a;$$
$$[i(\partial_0 - ig[A_0^{\mathrm{cl}}, .])B_\mu]_{ab} \to (p_0 + iQ^{ab})[B_\mu]_{ab};$$
$$Q^{ab} = Q^a - Q^b. \tag{12}$$

Note that the adjoint covariant derivative involves the quantity Q^{ab}, which is the difference of two color charges. The corresponding statistical distribution functions are

$$n_a(E) = \frac{1}{e^{|E-iQ^a|_R/T}+1} \quad \text{for quarks,}$$
$$n_{ab}(E) = \frac{1}{e^{|E-iQ^{ab}|_R/T}-1} \quad \text{for gluons,} \tag{13}$$

where $|\cdots|_R$ is defined as

$$|z|_R = \begin{cases} +z & \mathrm{Re}\, z > 0 \\ -z & \mathrm{Re}\, z < 0 \end{cases}. \tag{14}$$

From the statistical distribution functions we see that the background field acts like an imaginary chemical potential for color. As such, in a given field Q^a these distribution functions are complex, and so unphysical. The only physically meaningful quantities are integrals over distributions of Q, and these give results which are sensible.

To see how this comes about, consider summing a quark propagator over its color indices. This enters, for example, in the computation of the pressure at leading order. The sum is

$$\frac{1}{N_c}\sum_a \frac{1}{e^{(E-iQ^a)/T}+1} = \sum_{n=1}^{\infty}(-)^{n+1}e^{-nE/T}\frac{1}{N_c}\,\mathrm{tr}L^n. \tag{15}$$

We can always assume that the vacuum expectation value of $\mathrm{tr}L$ is real. In the pure glue theory, this can be enforced by a global $Z(N_c)$ rotation; with dynamical quarks, this is automatic. In Eq. (15) there is an infinite series in powers of $\mathrm{tr}L^n$, the expectation values of which are real. The computation of the shear viscosity in this paper provides another example where integrals over Q's give physically sensible results.

We wish to compute correlation functions near thermal equilibrium, and so need to continue from imaginary to real time. We adopt the usual path, illustrated in Fig. 1, along $C = C_1 \cup C_2$. This includes integration in real time, $\mathrm{Re}t$, along C_1 from an initial time t_i, to a final time, t_f, and back (in practice, both times are assumed to be infinite). Then one integrates in imaginary time, along C_2 from $\mathrm{Im}t$: $0 \to -1/T = -\beta$.

It is natural to take the background field only for the part of the path in imaginary time. Consider an ordinary chemical potential, introduced as a Lagrange multiplier for the number operator. The chemical potential does not affect the Hamiltonian, nor the evolution in real time: it enters only to change the statistical distribution functions in thermal equilibrium. Thus we do the same for an imaginary chemical potential for color: along Fig. 1 we take the background field to be nonzero along C_2, and to vanish along C_1.

While we will not use the real time formalism, it helps to understand the choice of background field in time. The propagator is a two by two matrix,

$$[G_{\mu\nu}(K)]_{ab,cd} = -g_{\mu\nu}P_{ab,cd}D_{ab}(K), \tag{16}$$

where

$$D_{ab}(K) = \begin{pmatrix} D_{ab}^{11}(K) & D_{ab}^{12}(K) \\ D_{ab}^{21}(K) & D_{ab}^{22}(K) \end{pmatrix} \tag{17}$$

with

$$\begin{aligned} D_{ab}^{11}(K) &= \frac{i}{K^2 + i\epsilon} + n_{ab}(k_0) 2\pi\delta(K^2), \\ D_{ab}^{12}(K) &= (n_{ab}(k_0) + \theta(-k_0)) 2\pi\delta(K^2), \\ D_{ab}^{21}(K) &= (n_{ab}(k_0) + \theta(k_0)) 2\pi\delta(K^2), \\ D_{ab}^{22}(K) &= \frac{-i}{K^2 - i\epsilon} + n_{ab}(k_0) 2\pi\delta(K^2), \end{aligned} \tag{18}$$

in Feynman gauge [40], where $\theta(k^0)$ is the Heaviside step function. This form shows that the zero temperature parts of the propagator, $1/(K^2 \pm i\epsilon)$, are independent of the Q's. The color charges only enter through the statistical distribution functions, $n_{ab}(k_0)$.

Instead of using the real time formalism, we compute in imaginary time, and then analytically continue. The color dependent frequency ω_n becomes

$$i\omega_n + iQ^{ab} \to \omega \pm i\epsilon, \tag{19}$$

where $\pm$ corresponds to retarded and advanced propagation.

After computing diagrams, we need to sum over distributions of the Q's. To avoid having to deal with trace terms at finite N_c, we work in the limit of large N_c. The pressure is of order $\sim N_c^0$ in the hadronic phase, and $\sim N_c^2$ in the deconfined phase. Thus a mean field approximation, as in Eq. (10), should be a reasonable approximation in the deconfined phase at large N_c. At large N_c, color sums can be replaced by integrals:

$$\frac{1}{N_c}\sum_a \to \int da = \int_{-\pi}^{\pi} d\varphi \rho(\varphi), \tag{20}$$

where we introduce the eigenvalue density $\rho(\varphi) = da/d\varphi$, with $\varphi = Q/T$. Moments of the Polyakov loops are Fourier transforms of the eigenvalue density,

$$\ell_n = \frac{1}{N_c}\,\mathrm{tr}L^n = \int_{-\pi}^{\pi} d\varphi \rho(\varphi) e^{i\varphi n}. \tag{21}$$

In particular, we write the the Polyakov loop in the fundamental representation, ℓ_1, simply as ℓ. In the confined phase, all moments of the Polyakov loop vanish, $\ell_n = 0$, and the eigenvalue density is constant. In the perturbative QGP, the moments of all Polyakov loops are close to 1, $\ell_n = 1$, and the eigenvalue density is close to a delta function.

Since we do not now have a complete effective theory of the semi-QGP, we do not know the exact eigenvalue density as a function of temperature. We do know the expectation value of the simplest Polyakov loop, ℓ, as a function of temperature, and so we consider two representative forms for the eigenvalue density. One ansatz is to take a step function,

$$\rho_{\mathrm{step}}(\varphi, \lambda) = \frac{1}{2\lambda}\theta(\lambda - |\varphi|), \tag{22}$$

where λ is a parameter characterizing the eigenvalue density. With this ansatz, the expectation value of the Polyakov loops are

$$\ell_n \equiv \frac{1}{N_c}\,\mathrm{tr}L^n = \frac{\sin n\lambda}{n\lambda}. \tag{23}$$

The other ansatz we take from a soluble model in two dimensions, that of Gross and Witten [41]:

$$\rho_{\mathrm{GW}}(\varphi, \lambda) = \begin{cases} \frac{1}{2\pi}(1 + \lambda\cos\varphi) & \lambda \le 1 \\ \frac{\sqrt{\lambda}}{\pi}\cos\frac{\varphi}{2}\left(1 - \lambda\sin^2\frac{\varphi}{2}\right)^{1/2} & \lambda > 1, \end{cases} \tag{24}$$

where $|\varphi| < 2\sin^{-1}(1/\lambda)^{1/2}$ is satisfied for $\lambda > 1$. For $\lambda \le 1$, $\ell_{\pm 1} = 1/(2\lambda)$, while all others vanish. With this ansatz, there is no simple expression for ℓ_n when $\lambda > 1$.

Rather unexpectedly, we shall see that our results for the shear viscosity are almost independent of the choice of these eigenvalue densities. We do not know if this is a property special to the shear viscosity, or if it is generic.

III. TRANSPORT COEFFICIENTS

In this section we consider transport coefficients in a semi-QGP. Hydrodynamics is described by evolution in time of locally conserved currents, such as the energy-momentum tensor, which satisfy $\partial_\mu T^{\mu\nu} = 0$. For simplicity, we assume that the only conserved charges are those for energy and momentum. In hydrodynamics the energy-momentum tensor $T^{\mu\nu}$ can be expanded as [42]

$$T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + \Pi^{\mu\nu}, \tag{25}$$

where u^μ is the local velocity, $u_\mu u^\mu = 1$; $\mathcal{E} = T^{\mu\nu}u_\mu u_\nu$ is the energy density and $\mathcal{P}$ is the local pressure, both of which are Lorentz scalars; lastly, $\Delta^{\mu\nu}$ is a spatial projection operator,

$$\Delta^{\mu\nu} = u^\mu u^\nu - g^{\mu\nu}. \tag{26}$$

For the metric we take $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

$\Pi^{\mu\nu}$ measures the deviation from thermal equilibrium. Near thermal equilibrium it can be expanded in a gradient expansion,

$$\Pi^{\mu\nu} = \eta\sigma^{\mu\nu} + \zeta\Delta^{\mu\nu}\Delta^{\alpha\beta}\partial_\alpha u_\beta + \cdots, \tag{27}$$

where

$$\sigma^{\mu\nu} = \Delta^{\mu\rho}\Delta^{\nu\sigma}(\partial_\sigma u_\rho + \partial_\rho u_\sigma - \tfrac{2}{3}g_{\sigma\rho}\Delta^{\alpha\beta}\partial_\alpha u_\beta). \tag{28}$$

The first order transport coefficients ζ and η are, respectively, the bulk and shear viscosities. The transport coefficients reflect microscopic properties of the theory, and are an input into a hydrodynamical analysis.

These transport coefficients can be computed from the Kubo formula, which relates them to the correlation functions [43]:

$$\eta = \frac{1}{20}\lim_{\omega\to 0}\frac{1}{\omega}\int d^4x e^{i\omega t}\theta(t)\langle[\pi_{ij}(x),\pi^{ij}(0)]\rangle, \tag{29}$$

$$\zeta = \frac{1}{2}\lim_{\omega\to 0}\frac{1}{\omega}\int d^4x e^{i\omega t}\theta(t)\langle[\mathcal{P}(x),\mathcal{P}(0)]\rangle, \tag{30}$$

where $\mathcal{P}(x) = -T^i{}_i(x)/3$ and $\pi_{ij}(x) = T_{ij}(x) + g_{ij}\mathcal{P}(x)$. These correlation functions are not simple to compute in perturbation theory, though. They are sensitive to pinch singularities, and it is necessary to resum an infinite series of diagrams.

Another way of computing these transport coefficients is to use kinetic theory and the Boltzmann equation. To leading order, this is equivalent to a resummation of the relevant diagrams in the Kubo formula [13,14]. We will use the Boltzmann equation to compute the shear viscosity in a semi-QGP in Sec. III B. Our analysis is a straightforward extension of the computation of Arnold, Moore, and Yaffe [16], of the shear viscosity in the perturbative QGP, to the semi-QGP.

A. Shear viscosity in the semi-QGP

Before going to the technical details of the computation, we give a qualitative discussion of how the shear viscosity arises in kinetic theory in the QGP, and how it differs in the semi-QGP.

For the transport theory of a relativistic gas [44], the shear viscosity η is given as

$$\eta \approx \tfrac{4}{15} n\bar{p}\lambda_{\rm mfp}, \tag{31}$$

where n is the number density, $\bar{p}$ is the mean momentum, and $\lambda_{\rm mfp}$ is the mean free path. For light particles, where the mass is much less than the temperature, the mean momentum $\bar{p}\sim T$, and the number density is $n\sim T^3$. The mean free path is $\lambda_{\rm mfp}\sim 1/(n\sigma)$, where σ is the transport cross section, so that the shear viscosity is $\eta\sim T/\sigma$.

In QCD, the transport cross section is $\sigma\sim g^4\ln(T/m_{\rm Debye})/T^2$, where $m_{\rm Debye}\sim\sqrt{g^2n/T}\sim gT$ is the Debye mass. There is a Coulomb logarithm, $\ln(T/m_{\rm Debye})$, due an infrared singularity from forward scattering [45], so in all,

$$\eta\sim\frac{T^3}{g^4\ln(1/g)}. \tag{32}$$

The Boltzmann equation gives the same powers of g and $\ln(1/g)$, but a strictly perturbative analysis gives the wrong overall coefficient. To get the correct result perturbatively requires the resummation of an infinite set of diagrams [11,16].

In weak coupling, σ is small and the shear viscosity is large. This sounds counter intuitive, but is not: when perturbed, it takes a long time for a weakly interacting gas to move back towards thermal equiblibrium.

Conversely, the most natural way to obtain a small shear viscosity is if the cross section, and so the coupling constant, are large. The best illustration of this is a $\mathcal{N}=4$ supersymmetric gauge theory, where in the limit of infinite coupling and an infinite number of colors, the ratio of the shear viscosity to the entropy density, s, is $\eta/s = 1/4\pi$ [6]. This is conjectured to be a universal lower bound on η/s.

The analysis in a semi-QGP is distinct from ordinary kinetic theory. It is characterized by the partial ionization of color [30–32], so that both the density of colored particles, and the interaction cross section, depend nontrivially upon the Polyakov loop.

Consider first the density. From Eq. (13), a background Q field acts like an imaginary chemical potential for color. For ordinary statistical distribution functions, the only way to obtain a small density of particles is if they are heavy, and so Boltzmann suppressed. A nonzero Q field provides another. When $Q\neq 0$, the statistical distribution function for a given particle is complex valued. Thus while the distribution functions are of order one in magnitude, because of cancelling phases, they can vanish after averaging over the Q's. This happens in the confined phase at $N_c=\infty$, where the expectation value of $\mathrm{tr}L^n$ vanishes for all n. For a quark, which lies in the fundamental representation, the trace of the statistical distribution function is given in Eq. (15), and vanishes term by term. Just above T_c, where ℓ is nonzero but small, the distribution function for such a field vanishes as a single power of the loop,

$$\left\langle\sum_a n_a\right\rangle\sim N_cT^3\ell; \tag{33}$$

the bracket $\langle\cdots\rangle$ denotes the average over the Q distribution, and an integral over the particles three momentum.

Gluons in the adjoint representation carry two Q charges, so the Boltzmann expansion of the statistical distribution function involves $\exp(i(Q^a-Q^b))$. Summing over the color indices, $\sum_a e^{iQ^a}\sum_b e^{-iQ^b} = \mathrm{tr}L\,\mathrm{tr}L^\dagger$, so the density of gluons then vanishes as the square of the loop,

$$\left\langle\sum_{ab} n_{ab}\right\rangle\sim N_c^2T^3\ell^2; \tag{34}$$

we assume that the expectation of ℓ is real. This suppression of the density has no analogy in the perturbative regime, where the density of massless fields in necessarily a pure number times T^3.

The average mean free path is the ratio of the density, divided by a transport cross section. For the scattering of two gluons,

$$\lambda^{gl}_{\rm mfp} \sim \frac{\langle \sum_{ab} n_{ab} \rangle}{\langle \sum_{ab,cd} n_{ab} n_{cd} \sigma_{ab;cd} \rangle}; \tag{35}$$

$\sigma_{ab;cd}$ is the transport cross section for $2 \to 2$ scattering between gluons with color indices ab and cd.

For the sake of argument, assume first that there are no correlations between different colors. If true, then the cross section would factorize,

$$\left\langle \sum_{ab,cd} n_{ab} n_{cd} \sigma_{ab;cd} \right\rangle = \left\langle \sum_{ab} n_{ab} \right\rangle^2 \langle \sigma_{ab;cd} \rangle \sim N_c^4 \sigma \ell^4. \tag{36}$$

Here σ is a typical perturbative cross section, $\sim g^4 \log(1/g)$. The mean free path would then diverge at small ℓ, $\lambda^{gl}_{\rm mfp} \sim 1/(\sigma \ell^2)$. This factor of $1/\ell^2$ would cancel a similar factor in the density, so that the shear viscosity would be as in ordinary perturbation theory, $\eta \sim T/\sigma$, (approximately) independent of ℓ at small ℓ.

As we shall see in the next section, however, there are strong correlations between different colors, so that the cross section does not factorize as in Eq. (36). Instead, it vanishes only quadratically with the loop,

$$\left\langle \sum_{ab,cd} n_{ab} n_{cd} \sigma_{ab;cd} \right\rangle \sim N_c^4 \sigma \ell^2, \tag{37}$$

At first sight this dependence on ℓ seems counter intuitive: the scattering process involves the scattering of two gluons, with color indices ab and cd. Expanding the statistical distribution function for one gluon brings in powers of $\exp(i(Q^a - Q^b))$, while expanding that for the other gluon brings in powers of $\exp(i(Q^c - Q^d))$. Summing over all color indices, the naive expectation is that the cross section is proportional to the fourth power of the loop, one from each factor of $\exp(iQ)$. The scattering processes, however, include a planar diagram in which two of the color indices are equal, say $b = c$. For this term, the factor of $\exp(-iQ^b)$ from one gluon cancels, *identically*, against the factor of $\exp(+iQ^c)$ from the other gluon. This explains why Eq. (37) is proportional to ℓ^2, and not to ℓ^4, as in Eq. (36).

In the pure glue theory, then, from Eq. (35) the mean free path is approximately independent of the expectation value of the loop, $\lambda^{gl}_{\rm mfp} \sim 1$. Even in the confined phase, the mass dimension of Q is set by the temperature; thus the typical momentum remains $\sim T$, and $\bar{p} \sim T$. Consequently, from Eq. (31) the shear viscosity is proportional to the overall gluon density, and so vanishes quadratically with the loop, $\eta \sim \ell^2$.

With dynamical quarks the counting differs, although somewhat unexpectedly, the result is similar. The cross section for the scattering between a quark, with color a, and an antiquark, with color b, is

$$\left\langle \sum_{a,b} n_a n_b \sigma_{a;b} \right\rangle. \tag{38}$$

If we could factorize the cross section, as in Eq. (36), then this would be proportional to the density of quarks, squared, or $\sim \ell^2$. There is such a contribution when the colors are unequal, $a \neq b$. In addition, though, there is a planar diagram from when the colors are equal, $a = b$. In this case the chemical potential for the quark, $\exp(iQ^a)$, cancels identically against that from the antiquark, $\exp(-iQ^b)$, so that Eq. (38) is not $\sim \ell^2$, but constant. The mean free path for quarks,

$$\lambda^{qk}_{\rm mfp} \sim \frac{\langle \sum_{a} n_a \rangle}{\langle \sum_{a,b} n_a n_b \sigma_{a;b} \rangle}; \tag{39}$$

is then proportional to the quark density, $\sim \ell$. The average momentum remains $\sim T$, so that the shear viscosity is the product of the quark density, $\sim \ell$, times the mean free path. Thus like gluons, in all the quark contribution to the shear viscosity is $\eta^{qk} \sim \ell^2$.

Putting in the factors of the coupling constant, we find that either in the pure glue theory, or in the theory with dynamical quarks, that the shear viscosity vanishes quadratically in the loop,

$$\eta \sim \frac{T^3}{g^4 \ln 1/g} \ell^2, \tag{40}$$

relative to the result in the perturbative QGP.

We show later that the results for quarks is in fact numerically close to that of the pure glue theory. While we consider a large N_c limit in which N_f is also large, this similarity between the shear viscosity in the pure glue theory, and with dynamical quarks, is rather unexpected.

B. Derivation of the shear viscosity in the semi-QGP

In this section, we use the Boltzmann equation to compute the shear viscosity in the semi-QGP [10–12,16–19]. Of course we assume that the coupling constant is small enough so that a quasiparticle picture is applicable, and kinetic theory is valid. The computation is precisely like that in the ordinary QGP, except that we need to compute in the presence of a background field for A_0. In this section we consider a purely gluonic theory, considering the case with dynamical quarks in the next section.

If the statistical distribution function for a gluon with color ab is f_{ab}, then in kinetic theory the energy-momentum tensor is

$$T^{\mu\nu} = 2 \int d\Gamma P^\mu P^\nu f_{ab}. \tag{41}$$

The factor of 2 in Eq. (41) comes from the following relativistic normalization of the phase space including color and spin:

$$\int d\Gamma = \sum_{ab,s} \int \frac{d^3p}{(2\pi)^3 2E}, \tag{42}$$

where s denotes the spin of the gluon, and $E = |\boldsymbol{p}|$ is the energy of the gluon. We are interested in a system slightly away from local thermal equilibrium, so we write the statistical distribution function as

$$f_{ab} = f^{(0)}_{ab} + \delta f_{ab}, \tag{43}$$

where $f^{(0)}_{ab}$ is the gluon distribution function in local thermal equilibrium,

$$f^{(0)}_{ab} = \frac{1}{e^{\beta(u\cdot P^{ab})} - 1}, \tag{44}$$

with $P^{\mu;ab} = P^\mu - i\delta^{\mu 0} Q^{ab}$, and δf_{ab} measures the deviation from equilibrium.

The constant background field $A^{\rm cl}_0$ induces an imaginary color charge. We assume that after averaging over the Q's, that the net color charge vanishes. A spatially dependent color charge arises in related problems, such as for 't Hooft loops, when there is a background color electric field [31,46–48].

As discussed in Sec. II, the background field vanishes along the part of the contour in real time in Fig. 1. Thus the time derivative in the Boltzmann equation does not involve the background field,

$$2P^\mu \partial_\mu f_{a_1 b_1}(\boldsymbol{x}, \boldsymbol{p}, t) = -C_{a_1 b_1}[f]. \tag{45}$$

We do not consider a force term, which may be dynamically generated in an expanding plasma [17]. In Eq. (45), $C[f]$ is the collision term for the scattering of $2 \to 2$ gluons,

$$\begin{aligned} C_{a_1 b_1}[f] = \prod_{i=2}^{4} \int d\Gamma_i (2\pi)^4 \delta^4(P_1 + P_2 - P_3 - P_4) |\mathcal{M}|^2 \\ \times (f_{a_1 b_1} f_{a_2 b_2}(1 + f_{a_3 b_3})(1 + f_{a_4 b_4}) \\ - (1 + f_{a_1 b_1})(1 + f_{a_2 b_2}) f_{a_3 b_3} f_{a_4 b_4}), \end{aligned} \tag{46}$$

where $\mathcal{M}$ is the scattering amplitude, which depends upon the background field. To leading order, it is necessary to include not only the scattering of $2 \to 2$ particles, but also processes involving splitting, of $1 \to 2$ particles, and joining, for $2 \to 1$ particles [16,49,50]. The processes of soft multiple splitting and joining are known as the Landau-Pomeranchuk-Migdal (LPM) effect [51]. In this paper we compute to leading order in both the coupling, g^2, and logarithms of g. At leading logarithmic order, the result is dominated by the scattering of $2 \to 2$ particles in the t-channel, and the LPM effect can be neglected.

In the pure glue theory, the corresponding diagram in the t-channel is shown in Fig. 2. In the double line basis, there are four diagrams. The infrared singularity is cutoff by using a dressed propagator in the hard thermal loop (HTL) approximation, in the presence of the background field [31,52],

$$D_{\mu\nu;ab,cd}(K) = P^L_{\mu\nu} \frac{\boldsymbol{k}^2}{K^2} D^L_{ab,cd}(K) + P^T_{\mu\nu} D^T_{ab,cd}(K), \tag{47}$$

D^L and D^T are the propagators for the longitudinal and transverse gluons in the HTL approximation, respectively. For the amplitude at tree level, gauge dependent terms can be ignored. Projection operators for the transverse and longitudinal directions are

$$\begin{aligned} P^T_{\mu\nu} &= g_{\mu i}\left(-g^{ij} - \frac{k^i k^j}{\boldsymbol{k}^2}\right) g_{i\nu}, \\ P^L_{\mu\nu} &= -g_{\mu\nu} + \frac{K_\mu K_\nu}{K^2} - P^T_{\mu\nu}. \end{aligned} \tag{48}$$

while the transverse and longitudinal propagators are

$$\begin{aligned} D^T_{ab,cd}(K) &= \left[\frac{i}{K^2 - m^2[x^2 + \frac{x(1-x^2)}{2} \ln\frac{x+1}{x-1}]}\right]_{ab,cd}, \\ D^L_{ab,cd}(K) &= \left[\frac{i}{k^2 + 2m^2(1 - \frac{x}{2}\ln\frac{x+1}{x-1})}\right]_{ab,cd}, \end{aligned} \tag{49}$$

where $x = k_0/|\boldsymbol{k}|$. The thermal mass, $[m^2]_{ab,cd}$, depends upon the details of the color distribution [31]. In general, it is not diagonal, and different components of $[m^2]_{aa,bb}$, with no summation over the indices a and b, mix with one another. In the limit of large N_c, $[m^2]_{ab,cd}$ becomes diagonal, and mixing is suppressed by $1/N_c$. In the following, we work only to leading order of N_c, in which the propagators are diagonal:

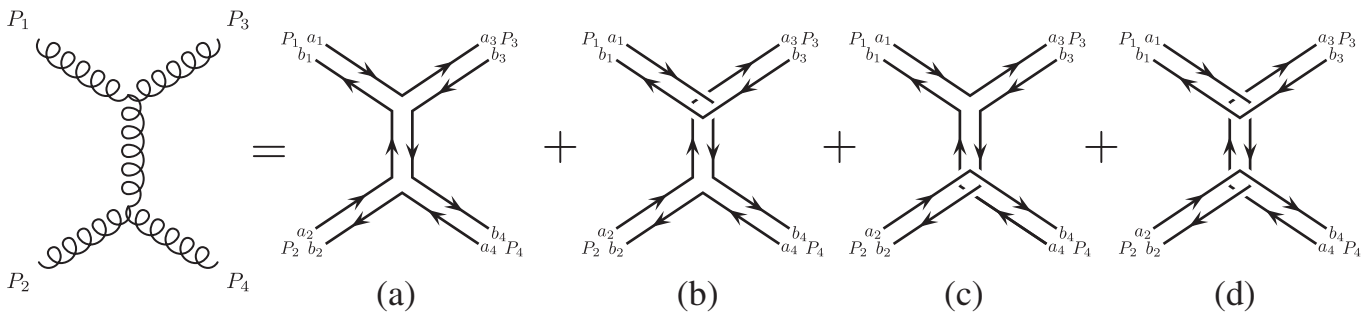

FIG. 2. Feynman diagram of the gluon scattering with the normal and double line notations in the t-channel.

$$D^T_{ab,cd}(K) = \delta_{ad}\delta_{bc}D^T_{ab}(K) = \delta_{ad}\delta_{bc}\left[\frac{i}{K^2 - [m^2]_{ab}[x^2 + \frac{x(1-x^2)}{2}\ln\frac{x+1}{x-1}]}\right],$$
$$D^L_{ab,cd}(K) = \delta_{ad}\delta_{bc}D^L_{ab}(K) = \delta_{ad}\delta_{bc}\left[\frac{i}{k^2 + 2[m^2]_{ab}(1 - \frac{x}{2}\ln\frac{x+1}{x-1})}\right], \quad (50)$$

where

$$[m^2]_{ab} = \frac{g^2 N_c T^2}{6}\frac{1}{2N_c}\sum_{e=1}^{N_c}(\mathcal{A}(Q^{ae}) + \mathcal{A}(Q^{be})), \quad (51)$$

and

$$\mathcal{A}(Q) = 1 - 6q(1-q), \quad (52)$$

where $q = Q/(2\pi T)$, and q is defined modulo 1 [31].

We note that at large N_c, the combination g^2N_c, and so the Debye mass, are of order one. In absence of the background field, $Q = 0$, $[m^2]_{ab}$ coincides with the ordinary thermal mass, $[m^2]_{ab} = g^2N_cT^2/6$.

For the shear viscosity, we find that the result is proportional to that in zero field,

$$\eta = \eta_{\rm pert}\mathcal{R}(\ell), \quad (53)$$

where $\eta_{\rm pert}$ is the usual result in perturbation theory [16],

$$\eta_{\rm pert} = \frac{2N_c^2 c_\eta T^3}{(g^2N_c)^2\ln[1/(g^2N_c)]}; \quad (54)$$

c_η is a constant of order one, which depends upon N_c and N_f.

Note that at large N_c, the free energy and so the entropy are large, of order N_c^2. The shear viscosity is of the same order, so that the ratio of the shear viscosity, to the entropy, is of order one. Our task is now to derive $\mathcal{R}(\ell)$.

1. Kinematics and Scattering amplitude

Before solving the Boltzmann equation we review the kinematics of scattering processes for $2 \to 2$ particles, involving the exchange of a soft gluon [11]. In terms of the Mandelstam variables,

$$s \equiv (P_1 + P_2)^2, \qquad t \equiv (P_1 - P_3)^2, \qquad u \equiv (P_1 - P_4)^2. \quad (55)$$

The dominant kinematic region in Fig. 2 is $|t| \ll |s| \simeq |u|$, as then the amplitude, squared, is enhanced by $\sim 1/t^2$. We introduce the quantities:

$$P_1 = P + \frac{K}{2}, \qquad P_2 = P' - \frac{K}{2}, \qquad P_3 = P - \frac{K}{2}, \qquad P_4 = P' + \frac{K}{2}, \quad (56)$$

in which the Mandelstam variables become

$$s = (P + P')^2, \qquad t = K^2, \qquad u = (P - P')^2. \quad (57)$$

There are three angles which enter, as shown in Fig. 3:

$$\cos\theta = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}, \qquad \cos\theta' = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}', \qquad \cos\varphi = \frac{(\hat{\boldsymbol{p}}\times\hat{\boldsymbol{k}})\cdot(\hat{\boldsymbol{p}}'\times\hat{\boldsymbol{k}})}{|\hat{\boldsymbol{p}}\times\hat{\boldsymbol{k}}||\hat{\boldsymbol{p}}'\times\hat{\boldsymbol{k}}|}, \quad (58)$$

where unit vector are written as $\hat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|$, *etc.* For forward scattering, θ is close to θ', and

$$\cos\varphi \simeq \frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}' - x^2}{1 - x^2}, \quad (59)$$

using $\cos\theta' \simeq x = k_0/|\boldsymbol{k}|$. The Mandelstam variables become

$$s \simeq 2pp'(1-x^2)(1-\cos\varphi), \qquad u \simeq -2pp'(1-x^2)(1-\cos\varphi), \qquad t \simeq -k^2(1-x^2), \quad (60)$$

with $p = |\boldsymbol{p}|$, $p' = |\boldsymbol{p}'|$, and $k = |\boldsymbol{k}|$. The amplitude in the t-channel is

$$i\mathcal{M} = J^{\mu;ab}_{s_1,s_3}(P_1, P_3)D_{\mu\nu;ab,cd}(K)J^{\nu;cd}_{s_2,s_4}(P_2, P_4), \quad (61)$$

where $J^{\mu;ab}_{s,s'}$ is the color current, and $D_{\mu\nu;ab,cd}$ is the gluon propagator in Eq. (47). For forward scattering, $P_1 \simeq P_3 \simeq P$, and the form of the current is dictated by Lorentz symmetry,

$$J^{\mu;ab}_{s_1,s_3}(P_1, P_3) \simeq 2igP^\mu t^{ab}_r\delta_{s_1,s_3}, \quad (62)$$

where t^{ab}_r is a generator of $SU(N)$ in the representation r. For the adjoint representation,

$$[t^{ab}_{\rm adj}]^{cd,ef} = if^{(cd,ab,ef)}. \quad (63)$$

The spin or helicity is denoted by s_1 and s_3. The amplitude for gluon scattering becomes

$$i\mathcal{M} = -4g^2\delta_{s_1,s_3}\delta_{s_2,s_4}\sum_{ab,cd}t^{ab}_{13}t^{cd}_{24}P^\mu P^\nu D_{\mu\nu;ab,cd}(K) = -4g^2\delta_{s_1,s_3}\delta_{s_2,s_4}\sum_{ab}t^{ab}_{13}t^{ba}_{24}pp'[D^{ab}_L(K) + (1-x^2)\cos\varphi D^{ab}_T(K)], \quad (64)$$

using the shorthand notation $t^{ab}_{ij} = [t^{ab}_{\rm adj}]^{a_ib_i,a_jb_j}$. The square

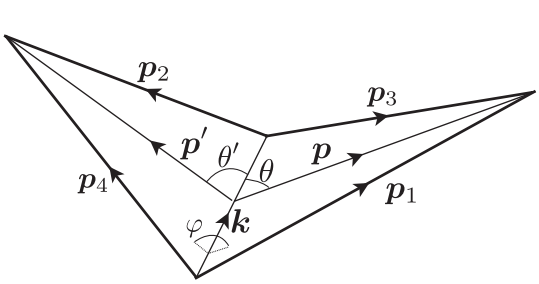

FIG. 3. Kinematics of the two body scattering.

of the amplitude is

$$|\mathcal{M}|^2 = \sum_{ab,cd} 16g^4 p^2 p'^2 t_{13}^{ab} t_{24}^{ba} t_{13}^{cd} t_{24}^{dc} \delta_{s_1,s_3}\delta_{s_2,s_4} \times [D_L^{ab}(K) + (1-x^2)\cos\varphi D_T^{ab}(K)] \times [D_L^{cd}(K) + (1-x^2)\cos\varphi D_T^{cd}(K)]^*. \quad (65)$$

In bare perturbation theory, the thermal mass vanishes, and this becomes

$$|\mathcal{M}|^2 = 4g^4 \sum_{ab,cd} t_{13}^{ab} t_{24}^{ba} t_{13}^{cd} t_{24}^{dc} \delta_{s_1,s_3}\delta_{s_2,s_4} \frac{s^2}{t^2}. \quad (66)$$

It is rather useful to write phase space in terms of the variables k, p, p', x and φ, instead of $\boldsymbol{p}_1$ to $\boldsymbol{p}_4$. By energy-momentum conservation,

$$\delta(E_1 + E_2 - E_3 - E_4) \simeq \delta(k(\cos\theta - \cos\theta')) = \frac{1}{k}\delta(\cos\theta - \cos\theta'), \quad (67)$$

the integral over phase space becomes

$$\prod_{i=2}^{4} \int d\Gamma_i (2\pi)^4 \delta^{(4)}(P_1 + P_2 - P_3 - P_4) \simeq \frac{1}{2^2(2\pi)^6} \prod_{i=2}^{4} \sum_{a_i b_i, s_i} \int dk k dp p dp' dx d\varphi (2\pi)^3 \times \delta^{(3)}\left(\boldsymbol{p}_1 - \boldsymbol{p} - \frac{\boldsymbol{k}}{2}\right). \quad (68)$$

2. Chapman-Enskog method

Because the Boltzmann equation is nonlinear, it is not easy to solve exactly. Instead, we use the Chapman-Enskog method to estimate the transport coefficients [53]. Start with a solution of the Boltzmann equation local thermal equilibrium $f_{ab}^{(0)}$, Eq. (44). By the conservation of energy-momentum and charge, the collision term vanishes, and gives the following relation:

$$\frac{f_{a_1b_1}^{(0)}}{1+f_{a_1b_2}^{(0)}} \frac{f_{a_2b_2}^{(0)}}{1+f_{a_2b_2}^{(0)}} = \frac{f_{a_3b_3}^{(0)}}{1+f_{a_3b_3}^{(0)}} \frac{f_{a_4b_4}^{(0)}}{1+f_{a_4b_4}^{(0)}}. \quad (69)$$

We are interested in a nonequilibrium system which is close to local thermal equilibrium, and so expand the distribution function around $f_{ab}^{(0)}$,

$$f_{ab} = f_{ab}^{(0)} + \epsilon f_{ab}^{(1)} + \epsilon^2 f_{ab}^{(2)} + \cdots. \quad (70)$$

This corresponds to expanding in spatial derivatives, or equivalently, the Knudsen number. The parameter ϵ keeps track of the order of derivatives, and is taken to one after computation. $f^{(n)}$ gives the transport coefficients of n-th order, e.g., $f^{(1)}$ gives the shear and bulk viscosities. The collision term is expanded in powers of ϵ as

$$C[f] = \sum_{n=0}^{\infty} \epsilon^n C^{(n)}, \quad (71)$$

where

$$C^{(n)} = \frac{1}{n!}\frac{d^n}{d\varepsilon^n} C[f]|_{\epsilon=0}. \quad (72)$$

The first term, $C^{(0)}$, obviously vanishes, because $f_{ab}^{(0)}$ is the solution in thermal equilibrium. In the left hand side of the Boltzmann equation, the spatial derivative, ∂_i, is taken to be of order ϵ. In the Chapman-Enskog method, the time derivative is expanded in powers of ϵ,

$$\partial_t = \sum_{n=1}^{\infty} \epsilon^n \partial_t^{(n)}. \quad (73)$$

We show in the following that $\partial_t^{(n)}$ is determined by energy-momentum conservation and local thermal equilibrium, Eqs. (81) and (82). $f_{ab}^{(n)}$ is parametrized as

$$f_{ab}^{(n)} = f_{ab}^{(0)}(1 + f_{ab}^{(0)})\Phi^{(n)} \quad (74)$$

for useful. To ϵ^n, the Boltzmann equation is written as

$$2P^\mu \partial_\mu^{(n)} f_{a_1b_1} = -\mathcal{L}\Phi_{a_1b_1}^{(n)} + K_{a_1b_1}^{(n)}, \quad (75)$$

where the partial derivative, to order ϵ^n, is

$$P^\mu \partial_\mu^{(n)} f_{ab} = p^0 \sum_{k=1}^{n} \partial_t^{(k)} f_{ab}^{(n-k)} + p^i \partial_i f_{ab}^{(n-1)}, \quad (76)$$

and

$$K_{a_1b_1}^{(n)} = -C_{a_1b_1}^{(n)} + \mathcal{L}\Phi_{a_1b_1}^{(n)}. \quad (77)$$

We introduce the linearized operator $\mathcal{L}$,

$$\mathcal{L}\Phi_{a_1b_1}^{(n)} = \prod_{i=2}^{4} \int d\Gamma_i (2\pi)^4 \delta^{(4)}(P_1 + P_2 - P_3 - P_4) \times |\mathcal{M}|^2 f_{a_1b_1}^{(0)} f_{a_2b_2}^{(0)} (1 + f_{a_3b_3}^{(0)})(1 + f_{a_4b_4}^{(0)}) \times (\Phi_1^{(n)} + \Phi_2^{(n)} - \Phi_3^{(n)} - \Phi_4^{(n)}). \quad (78)$$

In vanishing background field, the operator $\mathcal{L}$ is self-adjoint and positive semidefinite:

$$\int d\Gamma \Phi^\dagger(\mathcal{L}\Phi) = \int d\Gamma (\mathcal{L}\Phi)^\dagger \Phi \geq 0 \quad \text{for} \quad \Phi \neq 0. \quad (79)$$

While positivity is not automatic in a nonzero background field, we assume it is, after averaging over color. Our analysis shows no signs of such unphysical behavior.

Equation (75) is a linear equation for $\Phi^{(n)}$. Since K_n and $\partial_\mu^{(n)} f_{ab}$ are as functionals of $f_{ab}^{(k)}$ for $0 \leq k \leq n-1$, and does not include $f_{ab}^{(n)}$, $\Phi^{(n)}$ can be obtained order by order.

There is an ambiguity, however. $\mathcal{L}$ contains zero eigenvalues, which correspond to conserved currents, $\Phi = P^\mu$

and 1. To remove this ambiguity, we require that at each order in ϵ, the deviation from thermal equilibrium is orthogonal to the zero modes,

$$\int d\Gamma P^0 P^\mu f_{ab}^{(n)} = 0 \tag{80}$$

for $n \geq 1$. Then the conserved charge density only depends upon $f_{ab}^{(0)}$,

$$T^{0\mu} = 2\int d\Gamma P^0 P^\mu f_{ab}^{(0)}. \tag{81}$$

We also require that energy and momentum are conserved order by order in ϵ,

$$2\partial_t^{(n)} \int d\Gamma P^0 P^\nu f_{ab}^{(0)} + 2\partial_i \int d\Gamma P^i P^\nu f_{ab}^{(n-1)} = 0, \tag{82}$$

which defines the time derivative to ϵ^n.

The viscosities are determined at leading order,

$$2P^\mu \partial_\mu^{(1)} f_{a_1 b_1} = -\mathcal{L}\Phi_{a_1 b_1}^{(1)}, \tag{83}$$

after using $K_1 = 0$. The left hand side is

$$2P^\mu \partial_\mu^{(1)} f_{ab} = -2P^{ab;\nu} P^\mu f_{ab}^{(0)}(1 + f_{ab}^{(0)})(u_\nu \partial_\mu^{(1)} \beta + \beta \partial_\mu^{(1)} u_\nu). \tag{84}$$

Working in the local rest frame, where $u_0 = 1$, $u_i = 0$, $\partial_\mu^{(1)} u_0 = 0$ and $\partial_\mu^{(1)} u_i \neq 0$,

$$2P^\mu \partial_\mu^{(1)} f_{ab} = -f_{ab}^{(0)}(1 + f_{ab}^{(0)})\Big[2E^2\Big(\partial_t^{(1)}\beta + \frac{\beta}{3}\partial_i u_i\Big) + 2Ep^i(\partial_i \beta + \beta \partial_t^{(1)} u_i) + \beta p^i p^j \sigma_{ij} - iQ^{ab}(E\partial_t^{(1)}\beta + p^i \partial_i \beta)\Big]. \tag{85}$$

By assumption, the last term, proportional to Q^{ab}, vanishes after averaging over color, and so we drop it in computing the shear viscosity. This equation contains $\partial_t^{(1)}$, which is yet to be determined. From Eq. (82), to leading order

$$\int d\Gamma P^\nu P^\mu \partial_\mu^{(1)} f_{ab}^{(0)} = 0, \tag{86}$$

which reads

$$\partial_t^{(1)}\beta + \frac{\beta}{3}\partial_i u_i = 0, \qquad \beta \partial_t^{(1)} u_i + \partial_i \beta = 0. \tag{87}$$

Using Eq. (87) in Eq. (85), only last term on the left hand side survives,

$$2P^\mu \partial_\mu^{(1)} f = -\beta f_{ab}^{(0)}(1 + f_{ab}^{(0)}) p^i p^j \sigma_{ij}. \tag{88}$$

The trace of this equation vanishes, so that the bulk viscosity vanishes in this calculation. At linear order, the Boltzmann equation becomes

$$\beta f_{a_1 b_1}^{(0)}(1 + f_{a_1 b_1}^{(0)}) p^i p^j \sigma_{ij} = \mathcal{L}\Phi_{a_1 b_1}^{(1)}. \tag{89}$$

Acting with the inverse of the collision operator, $\mathcal{L}^{-1}$, on both sides of Eq. (89), we obtain the formal solution,

$$\Phi^{(1)} = \mathcal{L}^{-1}\beta f_{a_1 b_1}^{(0)}(1 + f_{a_1 b_1}^{(0)}) p^i p^j \sigma_{ij}. \tag{90}$$

Note that the condition of orthogonality, Eq. (81), ensures that zero modes are projected out, so that the inverse operator $\mathcal{L}^{-1}$ is well defined. We parametrize $\Phi^{(1)}$ as

$$\Phi^{(1)} = \Big(p^i p^j - \frac{p^2}{3}\delta^{ij}\Big)\sigma_{ij}\chi(p) = \sigma_{ij}\chi^{ij}(p). \tag{91}$$

Remember that σ_{ij} is traceless, so that here we can add $p^2\delta^{ij}\sigma_{ij}/3$ on the left hand side, and take χ^{ij} to be traceless. The transport coefficients are independent of one another, so we write

$$\beta f_{a_1 b_1}^{(0)}(1 + f_{a_1 b_1}^{(0)})(p^i p^j - \tfrac{1}{3}\delta^{ij} p^2) = \mathcal{L}\chi_{a_1 b_1}^{ij}. \tag{92}$$

Using the solution of the Boltzmann equation, we rewrite Π^{ij} in Eq. (25) to obtain

$$\Pi^{(1)ij} = 2\int d\Gamma p^i p^j f_{ab}^{(1)} = 2\int d\Gamma p^i p^j f_{ab}^{(0)}(1 + f_{ab}^{(0)})\Phi^{(1)}, \tag{93}$$

where $\Pi^{(1)ij}$ denotes ϵ^1. Using Eq. (91), and performing the integral over angles gives

$$\int \frac{d\Omega}{4\pi} p^i p^j p^k p^l = \frac{1}{15} p^4(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}), \tag{94}$$

we find

$$\Pi^{(1)ij} = \frac{2}{5}\sigma^{ij}\int d\Gamma\Big(p^k p^l - \frac{1}{3}\delta^{kl}p^2\Big) f^{(0)}(1 + f^{(0)})\chi^{kl}. \tag{95}$$

The rank two tensor in the integrand is nothing but $\mathcal{L}\chi^{kl}$ in Eq. (92), so that

$$\Pi^{(1)ij} = \sigma^{ij}\frac{2T}{5}\int d\Gamma \chi^{kl}\mathcal{L}\chi^{kl}. \tag{96}$$

The shear viscosity is then

$$\eta = \frac{2T}{5}\int d\Gamma(\chi^{ij}\mathcal{L}\chi^{ij}). \tag{97}$$

In order to determine χ_{ij}, we need to know the inverse of a linear operator, with an infinite number of matrix elements.

We first find a formal solution, expanding χ in orthogonal polynomials, $\chi = \sum c_n \chi_n(p)$, and

$$\beta^6 \int d\Gamma f_{ab}(1 + f_{ab}) p^4 \chi_n(p)\chi_m(p) = d_n \delta_{nm}, \tag{98}$$

where d_n is a normalization constant. We choose $\chi_n = \sum_{m=0}^n b_{nm}(p/T)^m$ to be a monic polynomial, i.e., the leading coefficient is unity, $b_{nn} = 1$. This polynomial can be

systematically obtained by Gram-Schmidt orthogonalization. The first three terms are

$$\chi_0 = 1, \tag{99}$$

$$\chi_1 = \left(\frac{p}{T}\right) + b_{10}, \tag{100}$$

$$\chi_2 = \left(\frac{p}{T}\right)^2 + b_{21}\left(\frac{p}{T}\right) + b_{20}, \tag{101}$$

where the b_{nm} are dimensionless constants:

$$b_{10} = -\frac{h_1}{h_0}, \tag{102}$$

$$b_{20} = \frac{h_1 h_3 - h_2^2}{h_0 h_2 - h_1^2}, \tag{103}$$

$$b_{21} = \frac{h_1 h_2 - h_0 h_3}{h_0 h_2 - h_1^2}, \tag{104}$$

where

$$h_n = \beta^{6-n} \int d\Gamma f_{ab}(1 + f_{ab}) p^{4+n} = \frac{\Gamma(6+n)}{2\pi^2} \sum_{k=1}^{\infty} \frac{|\mathrm{tr} L^k|^2}{k^{5+n}}. \tag{105}$$

The normalization constants are obtained from Eq. (98)–(101) as

$$d_0 = h_0, \tag{106}$$

$$d_1 = h_2 + h_1 b_{10}, \tag{107}$$

$$d_2 = h_4 + h_3 b_{21} + h_2 b_{20}. \tag{108}$$

Taking $\int d\Gamma \chi_n^{ij}$ to Eq. (92), the Boltzmann equation reduces to a matrix form,

$$S_n = \sum_m \mathcal{L}_{nm} c_m, \tag{109}$$

where

$$S_n = \beta \int d\Gamma \chi_n^{ij} f_{ab}^{(0)}(1 + f_{ab}^{(0)}) \left(p^i p^j - \frac{p^2}{3}\delta^{ij}\right), \tag{110}$$

with

$$\mathcal{L}_{nm} = \int d\Gamma (\chi_n^{ij} \mathcal{L} \chi_m^{ij}). \tag{111}$$

Formally, the solution can be written as

$$\chi^{ij} = \sum_{n,m} \chi_n^{ij} [\mathcal{L}^{-1}]_{nm} S_m. \tag{112}$$

Inserting this into Eq. (97), we obtain

$$\eta = \frac{2T}{5} \sum_{n,m} S_n [\mathcal{L}^{-1}]_{nm} S_m. \tag{113}$$

From Eqs. (94) and (98),

$$S_n = \frac{2T^5}{3} d_0 \delta_{n,0}, \tag{114}$$

from which the shear viscosity is

$$\eta = \frac{8T^{11}}{45} d_0^2 [\mathcal{L}^{-1}]_{00}. \tag{115}$$

This involves the zero-zero component of the inverse of $\mathcal{L}$. In general, finding the inverse of $\mathcal{L}$ is not easy, as it involves an infinite number of matrix elements. It is thus necessary to truncate the number of matrix elements. This gives a lower bound on the solution, because Eq. (113) has a quadratic form, and by Eq. (79), $\mathcal{L}$ is positive semidefinite. We will see the solution converges quickly as the size of the matrix for $\mathcal{L}$ increases, with the solution for one matrix element within 0.6% of the exact solution [see Fig. 4 and related discussion following Eq. (137)]. Using just one matrix element, we obtain an approximate form for the shear viscosity,

$$\eta \approx \frac{8T^{11}}{45} \frac{d_0^2}{\mathcal{L}_{00}}. \tag{116}$$

We now turn to estimating the matrix element,

$$\mathcal{L}_{nm} = \frac{1}{4} \prod_{i=1}^{4} \int d\Gamma_i (2\pi)^4 \delta^{(4)}(P_1 + P_2 - P_3 - P_4) |\mathcal{M}|^2 f_{a_1 b_1}^{(0)} f_{a_2 b_2}^{(0)} (1 + f_{a_3 b_3}^{(0)})(1 + f_{a_4 b_4}^{(0)}) \times (\chi_n^{ij}(p_1) + \chi_n^{ij}(p_2) - \chi_n^{ij}(p_3) - \chi_n^{ij}(p_4)) \times (\chi_m^{ij}(p_1) + \chi_m^{ij}(p_2) - \chi_m^{ij}(p_3) - \chi_m^{ij}(p_4)). \tag{117}$$

The coefficient 1/4 is caused by symmetrizing the colli-

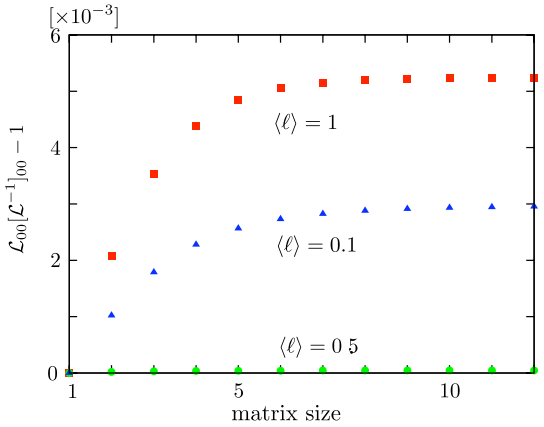

FIG. 4 (color online). Matrix size dependence of $\mathcal{L}_{00}[\mathcal{L}^{-1}]_{00} - 1$.

sion term. We are interested only in the forward scattering amplitude, where the exchanged momentum, k, is small. In the integrand of Eq. (117), to leading order in k,

$$\begin{aligned}\chi_n^{ij}(p_1) - \chi_n^{ij}(p_3) &= \chi_n^{ij}\Big(p + \frac{k}{2}\Big) - \chi_n^{ij}\Big(p - \frac{k}{2}\Big) \\ &= \Big(p^i k^j + k^i p^j - 2(\boldsymbol{p}\cdot\boldsymbol{k})\frac{p^i p^j}{p^2}\Big)\chi_n(p) \\ &\quad + \Big(p^i p^j - \frac{p^2\delta^{ij}}{3}\Big)(\boldsymbol{p}\cdot\boldsymbol{k}) \\ &\quad \times \frac{1}{p^3}(p^2\chi_n(p))' + \mathcal{O}(k^2). \end{aligned} \tag{118}$$

Further,

$$\begin{aligned}&(\chi_n^{ij}(p_1) + \chi_n^{ij}(p_2) - \chi_n^{ij}(p_3) - \chi_n^{ij}(p_4))(\chi_m^{ij}(p_1) \\ &\quad + \chi_m^{ij}(p_2) - \chi_m^{ij}(p_3) - \chi_m^{ij}(p_4)) \\ &\simeq 2(p^2k^2 - (\boldsymbol{p}\cdot\boldsymbol{k})^2)\chi_n(p)\chi_m(p) \\ &\quad + \frac{2}{3}(\boldsymbol{p}\cdot\boldsymbol{k})^2\frac{1}{p^2}(p^2\chi_n(p))'(p^2\chi_m(p))' + (p\leftrightarrow p') \\ &= k^2(2(1-x^2)p^2\chi_n(p)\chi_m(p) \\ &\quad + \frac{2x^2}{3}(p^2\chi_n(p))'(p^2\chi_m(p))') + (p\leftrightarrow p') \\ &\equiv k^2 F_{nm}(p,p',x). \end{aligned} \tag{119}$$

In the second line, we dropped the cross term $(\chi_n^{ij}(p_1) - \chi_n^{ij}(p_3))(\chi_m^{ij}(p_2) - \chi_m^{ij}(p_4))$ that vanishes in the integral of θ and ϕ. The function $F_{nm}(p, p', x)$ is independent of k; the matrix element becomes

$$\begin{aligned}\mathcal{L}_{nm} \simeq \frac{1}{4}\prod_{i=1}^{4}\int d\Gamma_i (2\pi)^4\delta^{(4)}(P_1 + P_2 - P_3 - P_4) \\ \times |\mathcal{M}|^2 f^{(0)}_{a_1b_1} f^{(0)}_{a_2b_2}(1 + f^{(0)}_{a_3b_3}) \\ \times (1 + f^{(0)}_{a_4b_4})k^2 F_{nm}(p,p',x). \end{aligned} \tag{120}$$

Expanding the statistical distribution functions at small k_0,

$$\begin{aligned}&f^{(0)}_{a_1b_1} f^{(0)}_{a_2b_2}(1 + f^{(0)}_{a_3b_3})(1 + f^{(0)}_{a_4b_4}) \\ &\quad = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\exp\Big[-\beta\sum_{j=1}^{4} n_j(E_j - iQ^{a_jb_j})\Big] \\ &\quad \simeq \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\mathcal{G}(Q,n) \\ &\qquad \times \mathrm{e}^{-\beta(n_1(p+(k_0/2))+n_2(p-(k_0/2))+n_3(p'+(k_0/2))+n_4(p'-(k_0/2)))}, \end{aligned} \tag{121}$$

where

$$\mathcal{G}(Q,n) = \exp\Big[i\beta\sum_{j=1}^{4} n_j Q^{a_jb_j}\Big]. \tag{122}$$

Note $\mathcal{G}(0, n) = 1$. Equation (120) has no singularity at $k_0 = 0$, so taking $k_0 \to 0$,

$$\begin{aligned}&f^{(0)}_{a_1b_1} f^{(0)}_{a_2b_2}(1 + f^{(0)}_{a_3b_3})(1 + f^{(0)}_{a_4b_4}) \\ &\quad \simeq \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\mathcal{G}(Q,n)e^{-\beta((n_1+n_3)p+(n_2+n_4)p')}. \end{aligned} \tag{123}$$

Writing the collision term using this, and using the explicit form of the squared amplitude in Eq. (65) as

$$\begin{aligned}\mathcal{L}_{nm} &= 2\frac{g^4}{(2\pi)^5}\prod_{i=1}^{4}\sum_{a_ib_i}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\mathcal{G}(Q,n)t_{13}^{ab}t_{24}^{ba}t_{13}^{cd}t_{24}^{dc} \\ &\quad \times \int_0^{\infty} dp\, p^2 e^{-\beta p(n_1+n_3)}\int_0^{\infty} dp'\, p'^2 e^{-\beta p'(n_2+n_4)} \\ &\quad \times \int_{-1}^{1} dx\int_{-\pi}^{\pi}\frac{d\varphi}{2\pi}\int_0^{\infty} dk k^3 F_{nm}(p,p';x) \\ &\quad \times [D_L^{ab}(K) + (1-x^2)\cos\varphi D_T^{ab}(K)] \\ &\quad \times [D_L^{cd}(K) + (1-x^2)\cos\varphi D_T^{cd}(K)]^*. \end{aligned} \tag{124}$$

The infrared singularity for the longitudinal component is cut off by the Debye mass, but that for the transverse component is regularized by the dynamical screening mass, $\sim xm^2$. The is infrared safe for $x \neq 0$, and is finite after integration over x. Consider the following integral:

$$\int_0^T dk\frac{k^3}{(k^2+m_1^2)(k^2+m_2^2)} \approx \frac{1}{4}\ln\Big(\frac{T^4}{m_1^2m_2^2}\Big) \approx \frac{1}{2}\ln\frac{1}{g^2N_c}, \tag{125}$$

where we use m_1^2 and $m_2^2 \sim g^2N_cT^2$, and we keep only the logarithmic term. This expression is valid as long as $\mathrm{tr}L \neq 0$. Under these approximations, the integral over k in Eq. (124) can be done, giving

$$\begin{aligned}&\int_0^{\infty} dk k^3 |D_L + (1-x^2)\cos\varphi D_T|^2 \\ &\quad \approx \frac{1}{2}(1-\cos\varphi)^2\ln\frac{1}{g^2N_c}. \end{aligned} \tag{126}$$

We can then perform the angular integral,

$$\begin{aligned}&\int_{-1}^{1} dx\int_{-\pi}^{\pi}\frac{d\varphi}{2\pi}F_{nm}(p,p',x)(1-\cos\varphi)^2 \\ &\quad = 4p^2\chi_n(p)\chi_m(p) + \frac{2}{3}(p^2\chi_n(p))'(p^2\chi_m(p))' \\ &\qquad + (p\leftrightarrow p') = \frac{20}{3}p^2\sum_{l=0}^{n+m}C_{nm,l}\Big[\Big(\frac{p}{T}\Big)^l + \Big(\frac{p'}{T}\Big)^l\Big], \end{aligned} \tag{127}$$

where

$$C_{nm,l} = \sum_{s=0}^{n}\sum_{t=0}^{m} b_{n,s}b_{m,t}\Big(1 + \frac{1}{10}(st+2l)\Big)\delta_{s+t,l}. \tag{128}$$

This integral is symmetric under exchange of p and p', so we can replace $p^2 + p'^2$ by $2p'^2$. Using

$$\int_0^\infty dp p^2 e^{-\beta p(n_1+n_3)} \int_0^\infty dp' p'^{4+l} \beta^l e^{-\beta p'(n_2+n_4)} = \frac{48T^8\Gamma(5+l)}{(n_1+n_3)^3(n_2+n_4)^{5+l}\Gamma(5)}, \tag{129}$$

the matrix element $\mathcal{L}_{nm}$ becomes

$$\mathcal{L}_{nm} = \frac{20T^8 g^4}{\pi^5} \ln\left(\frac{1}{g^2 N_c}\right) \prod_{i=1}^{4} \sum_{a_i,b_i} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \times \mathcal{G}(Q,n) t_{13}^{ab} t_{24}^{ba} t_{13}^{cd} t_{24}^{dc} \frac{1}{(n_1+n_3)^3} \times \sum_{l=0}^{n+m} \frac{C_{nm,l}}{(n_2+n_4)^{5+l}} \frac{\Gamma(5+l)}{\Gamma(5)}. \tag{130}$$

This matrix element is $\sim T^8 g^4 \ln(1/g^2 N_c)$, so that $\eta \sim T^3/(g^4 \ln 1/(g^2 N_c))$, as expected. Thus in the semi-QGP, the factors of temperature and the coupling constant are the same as in the perturbative QGP; the dependence upon the background field appears only through $\mathcal{G}(Q,n)$.

The next step is to sum over color. There are 16 diagrams contributing to the viscosity. Interference terms are not planar diagrams, and so can be dropped at large N_c. In Fig (2), the contribution of diagrams (a) and (b) coincides with that of (d) and (c). The color structure of diagrams (a) and (b) are

$$\begin{aligned} \mathcal{G}_{(a)}(Q,n) &= \exp[i\beta(Q^a(n_1+n_3) + Q^b(n_2-n_1) \\ &\quad + Q^c(n_4-n_3) - Q^d(n_2+n_4))], \\ \mathcal{G}_{(b)}(Q,n) &= \exp[i\beta(Q^a(n_1+n_4) + Q^b(n_2+n_3) \\ &\quad - Q^c(n_1+n_3) - Q^d(n_2+n_4))], \end{aligned} \tag{131}$$

respectively. Introducing

$$\begin{aligned} m_1 &= n_1 + n_3, \qquad m_2 = n_2 + n_4, \\ m_3 &= n_3, \qquad m_4 = n_4, \end{aligned} \tag{132}$$

the matrix element becomes

$$\mathcal{L}_{nm} = \frac{10T^8(g^2N_c)^2N_c^2}{\pi^5} \ln\left(\frac{1}{g^2N_c}\right) \times \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{m_1} \sum_{m_4=1}^{m_2} \sum_{l=0}^{n+m} \frac{\ell_{m_1}}{m_1^3} \frac{\ell_{m_2}}{m_1^{5+l}} \times \frac{\Gamma(5+l)}{\Gamma(5)} C_{nm,l} (\ell_{m_4-m_3} \ell_{m_2+m_3-m_1-m_4} + \ell_{m_1-m_3+m_4} \ell_{m_2-m_4+m_3}). \tag{133}$$

This is the final form for $\mathcal{L}_{nm}$. If the background vanishes,

$$\mathcal{L}_{nm} = \frac{20T^8(g^2N_c)^2N_c^2}{\pi^5} \zeta(2)\zeta(4) \ln\left(\frac{1}{g^2N_c}\right) \times \sum_{l=0}^{n+m} \frac{\zeta(4+l)}{\zeta(4)} \frac{\Gamma(5+l)}{\Gamma(5)} C_{nm,l}, \tag{134}$$

where $\zeta(z)$ is the Riemann zeta-function. In particular, for the inverse of $\mathcal{L}$, the zero-zero component dominates,

$$\mathcal{L}_{00} = \frac{20T^8(g^2N_c)^2N_c^2}{\pi^5} \zeta(2)\zeta(4) \ln\left(\frac{1}{g^2N_c}\right), \tag{135}$$

where $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. At finite N_c, N_c^4 is replaced by $N_c^2(N_c^2-1)$. In zero background field,

$$d_0(\ell=1) = \frac{60N_c^2}{\pi^2}\zeta(5), \tag{136}$$

where N_c^2 is replaced by (N_c^2-1) at finite N_c.

In zero background field, at large N_c the shear viscosity is

$$\eta_{\rm pert} = \frac{8T^{11}}{45} \frac{d_0^2}{\mathcal{L}_{00}(\ell=1)} = 540\zeta^2(5)\left(\frac{2}{\pi}\right)^5 \frac{T^3N_c^2}{(g^2N_c)^2 \ln 1/(g^2N_c)}. \tag{137}$$

This result was first obtained by Baym *et al.* [11]. Our result is valid at large N_c, with the correct result at finite N_c is obtained by replacing the overall factor of N_c^2 in the numerator by $N_c^2 - 1$.

Comparing with Eq. (54), the coefficient $c_\eta = 26.98$. This result is very close to the exact result, $c_\eta = 27.13$, which was obtained numerically by Arnold, Moore, and Yaffe [16]. We confirmed their results with several large matrices. In Fig (4) we show the dependence of $[\mathcal{L}^{-1}]_{00} \times (\ell=0)$ upon the size of the matrix. The solution converges quickly as the size of the matrix increases, with the solution for one matrix element within $\sim$0.6% of the exact result. Therefore, in the following we limit ourselves to the approximation of one matrix element.

In order to compare the viscosity in the semi-QGP to $\eta_{\rm pert}$, we calculate the ratio $\mathcal{R}(\ell)$ in Eq. (53),

$$\mathcal{R}(\ell) = \frac{\eta}{\eta_{\rm pert}} = \frac{d_0^2(\ell=1)[\mathcal{L}^{-1}(\ell=1)]_{00}}{d_0^2(\ell)[\mathcal{L}^{-1}(\ell)]_{00}}. \tag{138}$$

Inserting the explicit form, we obtain

$$\mathcal{R}(\ell) = \left[\frac{1}{\zeta(5)} \sum_{n=1}^{\infty} \frac{|\ell_n|^2}{n^5}\right]^2 \left[\frac{1}{2\zeta(4)\zeta(2)} \times \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{m_1} \sum_{m_4=1}^{m_2} \frac{\ell_{m_1}}{m_1^3} \frac{\ell_{m_2}}{m_2^5} \times (\ell_{m_4-m_3} \ell_{m_2+m_3-m_1-m_4} + \ell_{m_1-m_3+m_4} \ell_{m_2-m_4+m_3})\right]^{-1}. \tag{139}$$

This is our main result. If the background field vanishes, i.e., all $\ell_n = 1$, this ratio is unity, $\mathcal{R}(\ell) = 1$. Conversely, if all $\ell_n = 0$, this ratio vanishes, $\mathcal{R}(\ell) = 0$. Numerically we will see how $\mathcal{R}(\ell)$ changes as a function of ℓ in the next section. Before going to the numerics, we estimate $\mathcal{L}$ near T_c, where the Polyakov loop is small. Neglecting higher powers of ℓ, and higher moments, ℓ_n for $n \geq 2$,

$$\mathcal{R}(\ell) = \frac{\pi^6}{270\zeta^2(5)} \frac{\ell^2}{1+\ell^2} \simeq 3.31 \frac{\ell^2}{1+\ell^2}. \tag{140}$$

Thus the shear viscosity is $\sim \ell^2$ at small ℓ.

Let us see why the shear viscosity becomes small. Comparing Eqs. (31) and (35) to Eq. (116), one identifies

$$\sum_{ab} n_{ab} \sim T^3 d_0(\ell); \qquad \sum_{ab,cd} n_{ab} n_{cd} \sigma_{ab;cd} \sim \frac{\mathcal{L}_{00}(\ell)}{T^4}. \tag{141}$$

As mentioned in Sec. III A, both the gluon number $d_0 \sim N_c^2 \ell^2$, and the transport cross section multiplied by density squared, $\sum_{ab,cd} n_{ab} n_{cd} \sigma_{ab;cd} \sim N_c^4 \ell^2$, are small when ℓ is.

This suggests two important points. First, gluons (and other colored fields) are suppressed as $T \to T_c^+$. This "bleaching" of colored fields is most natural in the confined phase; it is universal, and depends only upon the color representation, independent of the fields spin, flavor, or mass. Secondly, even if the coupling constant is moderate, the correlation between gluons in the semi-QGP effectively becomes large; this is why the viscosity is small when ℓ is. Conversely, the interaction between small color representations becomes large: e.g., quark and antiquark scattering includes two channels, an adjoint an a singlet. For small ℓ, the singlet channel dominates that in the adjoint. This reflects that as one approaches the confined phase, interactions are dominated by the formation of color singlet states, such as glueballs and mesons.

IV. NUMERICAL RESULTS

A. The shear viscosity in the semi-QGP

In this section we compute numerically the ratio $\mathcal{R}(\ell)$ in Eq. (139). This is the ratio of the shear viscosity in the semi-QGP, when $\langle \ell_n \rangle \neq 1$, to that in the perturbative QGP, when all $\langle \ell_n \rangle = 1$. The results are plotted in Fig. 5 as a function of $\langle \ell \rangle$. Of course the ratio $\mathcal{R}(\ell)$ is a function not only of the first moment of the eigenvalue distribution, $\langle \ell \rangle$, but of higher moments, $\langle \ell_n \rangle$. The problem is that while $\langle \ell \rangle$ is easily extracted from numerical simulations on the lattice, the full eigenvalue distribution is not.

To illustrate the range of possible results, as mentioned in Sec. II we consider two different distributions, which we hope are representative. The first is a step function, Eq. (22); the second is that of a Gross-Witten model, Eq. (24). The different results for $\mathcal{R}(\ell)$ is illustrated in Fig. 5: there the result for a step function is shown by a dashed green line, and that for a Gross-Witten model, a solid red line. The difference between the two distributions are at most a few percent, over the entire range of ℓ. In fact, up to $\ell = 0.6$, an approximate solution without higher moments in Eq. (140), denoted by a blue dotted line shown in Fig. 5, is close to the other two distributions. This suggests that at least for the shear viscosity, the results are not very sensitive to higher moments of the eigenvalue distribution. We do not know if this is generic, or special to the shear viscosity, computed at leading logarithmic order

We find suppression of the shear viscosity at small ℓ, $\mathcal{R}(\ell) \sim \ell^2$ as $\ell \to 0$. Conversely, we find that $\mathcal{R}(\ell)$ is enhanced for $\ell \sim 1.0$. For $\ell \simeq 0.9$; $\mathcal{R}(\ell) \sim 1.25$; near $\ell = 1$, an approximate form is $\sim 1 + 1.47\sqrt{1-\ell}$. We do not have a simple explanation for this enhancement; if it is a peculiarity of working at leading logarithmic order, *etc.*

We also computed the shear viscosity with dynamical quarks; details will be presented elsewhere [54]. As in the pure glue theory, the dominant processes to leading logarithmic order are those involving the exchange of a soft particle in the t-channel. The diagrams which contribute are illustrated in Fig. 6. There are contributions from the scattering between a quark and an antiquark, or scattering between two quarks, Fig. 6(a); scattering between a gluon and a quark, Fig. 6(b); and Compton scattering between quarks and gluons, Fig. 6(c) and Fig. 6(d).

However, the physical processes which dominate are very different from the pure glue theory. At small ℓ, the dominant scattering processes are not between two gluons, but between a quark and an antiquark, as shown in Fig. 6(a) . Contributions to the shear viscosity arise from two parts, d_0 and $\mathcal{L}_{00}$, shown in Eq. (116). d_0 is proportional to the number density; for gluons, this is of order ℓ^2, while with quarks, it is $\sim \ell$. In $\mathcal{L}_{00}$, gluon-gluon scattering gives $\mathcal{L}_{00} \sim \ell^2$; the scattering of a quark and an antiquark includes a channel, Fig. 6(a), which is color singlet, so that $\mathcal{L}_{00} \sim \ell^0$. Therefore, for a purely gluonic theory, the shear viscosity is $\eta \sim d_0^2/\mathcal{L}_{00} \sim (\ell^2)^2/\ell^2 \sim \ell^2$; with dynamical quarks, it is also $\eta \sim d_0^2/\mathcal{L}_{00} \sim (\ell)^2/\ell^0 \sim \ell^2$. The similar

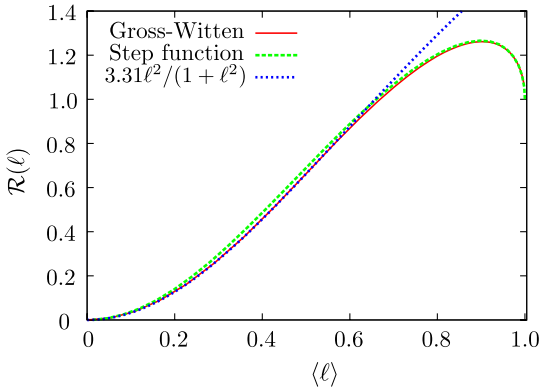

FIG. 5 (color online). Numerical results for $\mathcal{R}(\ell)$, Eq. (139). The two ansatzes for the eigenvalues are a step function (dashed line) and Gross-Witten (solid line). The dotted line denotes a simple analytic approximation at small ℓ, Eq. (140).

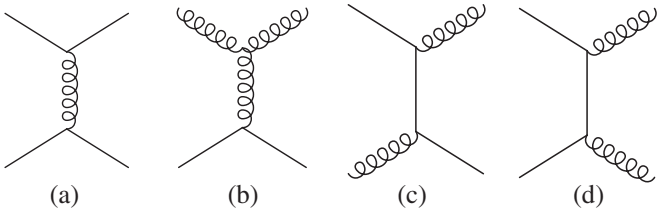

FIG. 6. Feynman diagram of quark contributions to the shear viscosity in the leading order. The straight line denotes quarks or antiquarks, and the curly line denotes gluons.

behavior of η at small ℓ must presumably reflect some more fundamental physics, although we do not know yet what it is.

In the numerical results, we took the ratio of the number of flavors to color to be fixed at several values, equal to $N_f/N_c = 0$ (the pure glue theory), 1/3, 2/3 and 1. The results are illustrated in Fig. 7 [30]. There is a weak, but non-negligible, dependence upon N_f/N_c.

B. The ratio of the shear viscosity to the entropy

In applications to hydrodynamics, what matters is not the shear viscosity by itself, but the ratio of the shear viscosity to the entropy density, η/s. For $\mathcal{N} = 4$ supersymmetric gauge theories, the gauge/gravity duality gives $\eta/s = 1/4\pi$ [6,7], which is conjectured to be a universal lower bound.

In this subsection we compute the behavior of η/s in the semi-QGP. In principle, with a complete theory of the semi-QGP the entropy density would follow from the eigenvalue density. Since such an eigenvalue density is not presently known, we use results directly from the lattice, $s_{\rm lat}$ [35]. This is not a significant limitation, since any complete theory of the semi-QGP would necessarily give an entropy density in agreement with $s_{\rm lat}$.

We also need to know the temperature dependence of the coupling constant in QCD. We use the formula for the running of the coupling to two loop order,

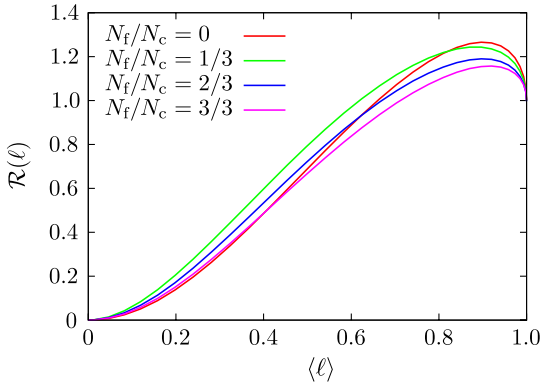

FIG. 7 (color online). Flavor dependence of the viscosity.

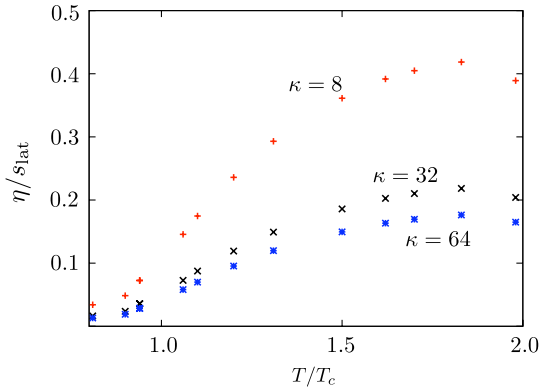

FIG. 8 (color online). Ratio of the shear viscosity to the lattice entropy.

$$N_c\alpha_s = \frac{N_c g^2(T)}{4\pi} \approx \frac{4\pi}{\beta_0(x)\ln\frac{T^2}{\Lambda^2} + \frac{\beta_1(x)}{\beta_0(x)}\ln\ln\frac{T^2}{\Lambda^2}}, \quad (142)$$

where the coefficients of the beta function at large N_c are

$$\beta_0 = \tfrac{1}{3}(11 - 2x), \quad (143)$$

$$\beta_1 = \tfrac{1}{3}(34 - 13x), \quad (144)$$

with $x = N_f/N_c$. We determine the renormalization mass scale, Λ, such that $\alpha_s(T_c) \approx 0.3$ [24]. This value of the coupling is moderate in strength, so that our computation, to leading logarithmic order, may not be reliable. To parametrize this uncertainty, we introduce a phenomenological parameter κ to represent effects at higher order in the shear viscosity,

$$\eta = \frac{2N_c^2 c_\eta \mathcal{R}(\ell) T^3}{(N_c g^2(T))^2 \ln(\kappa/(g^2(T)N_c))}. \quad (145)$$

In Fig. 8 we plot $\eta/s_{\rm lat}$, when $N_c = N_f$, for three values of κ, $\kappa = 8$, 32, and 64. As the temperature decreases, the coupling increases, and the shear viscosity, $\eta \sim 1/g^4(T)$, decreases. For larger values of κ, the dependence on the temperature is weaker.

The shear viscosity $\eta \sim N_c^2/(g^2N_c)^2$; with g^2N_c fixed at large N_c, $\eta \sim N_c^2$. Our computation is valid to leading logarithmic order, and includes the effects of quarks and gluons. Hadrons, such as mesons and glueballs, are subleading, and contribute $\sim$1 to η. Thus when at small $\ell \sim 1/N_c$, our approximations break down. We expect that this is the reason why our results for $\eta/s_{\rm lat}$ violates the conjectured lower bound, near and below T_c.

V. BLEACHING COLOR IN THE SEMI QUARK GLUON PLASMA

We conclude with some general comments about our results, and some speculative remarks.

The simplest way of viewing deconfinement in a non-Abelian gauge theory is in analogy to ionization in an

Abelian plasma [30,45]. Define the lower limit of the semi-QGP as $T^-_{\rm semi}$, and the upper limit, as $T^+_{\rm semi}$. Recall that in QCD, lattice simulations indicate that the semi-QGP exists between $T^-_{\rm semi} \approx 0.8T_c$ and $T^+_{\rm semi} \approx 3 - 4T_c$, as discussed in the Introduction and Refs. [30–32].

There is no ionization of color below $T^-_{\rm semi}$, so the only states are colorless hadrons. The ionization of color is partial in the semi-QGP, between $T^-_{\rm semi}$ and $T^+_{\rm semi}$, and total above $T^+_{\rm semi}$, in the perturbative QGP. A simple corollary of this picture is that scattering process involving colored states disappears as $T \to T^-_{\rm semi}$ from above. One may say that the scattering of colored particles is ''bleached'' by the vanishing of the Polyakov loop. In this paper we computed how the shear viscosity is bleached, an example especially relevant for experiment [4].

One result which followed from our analysis, but which we did not stress, is the following. The Polyakov loop is related to the propagator of an infinitely heavy, test quark. Yet we find that the bleaching of dynamical, light colored fields, such as gluons and massless quarks, is identical to that of heavy quarks. That is, at least within the approximations in which we work, the bleaching of color is universal, *independent* of the mass of the dynamical fields.

Thus it is necessary to discuss what assumptions are implicit in our analysis; these are equivalent to the statement that the semi-QGP exists only in a relatively narrow window about T_c.

That $T^-_{\rm semi}$ is not too far below T_c implies that it is reasonable to speak of confinement in a theory with dynamical quarks. It is not difficult to think of gauge theories for which this is not true. Consider a gauge theory with three colors and eight massless flavors, where lattice simulations appear to indicate that chiral symmetry is spontaneously broken in the vacuum [55]. If so, then there is a chiral phase transition at which this symmetry is restored at a finite, nonzero temperature. Nevertheless, in this theory the chiral phase transition bears little relation to the deconfining phase transition of the pure gauge theory. This follows simply by counting the number of degrees of freedom: the hadronic pressure, from an ideal gas of 63 Goldstone bosons, is large, $\approx 2/3$ that of the ideal QGP. Concomitantly, presumably the expectation value of the (renormalized, fundamental representation) Polyakov loop is also significant at temperatures well below that for the restoration of chiral symmetry. In this case, one might say that the semi-QGP begins at a temperature far below T_c, but the physics is really just eight flavors of quarks thoroughly wash out the $Z(3)$ global symmetry of the pure glue $SU(3)$ theory. The physics is dominated by chiral symmetry, and its restoration, with deconfinement a minor perturbation on that.

The other assumption is that $T^+_{\rm semi}$ is not much greater than T_c. Assume that the contrary were true, $T^+_{\rm semi} \gg T_c$: then we expect that both the ratio of the pressure, to the ideal gas pressure, and also the (renormalized, fundamental representation) Polyakov loop would deviate from unity, from $T^+_{\rm semi}$ all of the way down to T_c. While straight Polyakov loops presumably dominate near $T^+_{\rm semi}$, it is unreasonable to expect that they dominate all of the way down to T_c, if $T^+_{\rm semi} \gg T_c$. Instead, we suggest that in this case it is necessary to include thermal Wilson lines, and the corresponding Polyakov loops, which oscillate an even number of times in τ: $0 \to 1/T$. (The number of oscillations must be even because a Wilson loop, as a quantity formed from bosonic fields, must be periodic in τ.) As one goes to temperatures much lower than $T^+_{\rm semi}$, the effects of oscillatory Polyakov loops become more significant. An effective theory of Polyakov loops can still be constructed; it is just that a new type of Polyakov loops have to be folded in.

If effects from oscillatory Polyakov loops are important, then it is clear that the propagation of light fields *would* differ from that of heavy fields. A heavy field propagates in a straight line in imaginary time, but a light field performs a random walk; the lighter the field, the more the dominant paths in the path integral include those which fluctuate from a straight path. Thus if $T^+_{\rm semi}$ were much larger than T_c, the light fields would feel the effects of oscillatory Wilson lines, but the heavier fields would not.

In fact, even in a pure gauge theory, where the semi-QGP exists only in a narrow region in temperature, there is no reason why oscillatory Polyakov loops could not be constructed and measured. On the lattice, it will be awkward to discretize them, but they could be measured, on a sufficiently fine lattice. Our principal assumption is then that they are not needed when the semi-QGP exists in a narrow region in temperature.

We comment that there is a soluble limit in which one can show that only straight Polyakov loops control deconfinement. For a theory on a sphere in the three spatial dimensions, the radius R of the sphere can be made of femtometer dimensions, so that the non-Abelian coupling constant runs to small values. This theory is soluble, and has a true phase transition at infinite N_c [56]. Near T_c, the deconfining transition is controlled by the simplest Polyakov loop, a straight lop in the fundamental representation. At zero coupling, deconfinement occurs at a temperature T_c, which is equal to a pure number times $1/R$; at this point, the mass term for this Polyakov loop, alone, changes sign, jumping to a value of $1/2$. Although the full behavior of this model has not yet been computed, since the only dimensional scale in the problem is the radius of the sphere, it is most likely the semi-QGP only persists up to temperatures a few times $1/R$; i.e., $T^+_{\rm semi}/T_c$ is a number of order one.

There is a heuristic explanation as to why lattice simulations appear to find that $T^+_{\rm semi}$ is just a few times T_c. If $T^+_{\rm semi} \gg T_c$, then given how the coupling constant in QCD runs with temperature [24], there would have to be a perturbative mechanism for generating the eigenvalue re-

pulsion necessary in the semi-QGP [26]. This would indicate a perturbative instability for QCD in a thermal bath, which seems unnatural. Instead, lattice simulations find that the deviations from conformality, apart from the usual perturbative corrections from the conformal anomaly, are nonperturbative, due to corrections $\sim 1/T^2$ [26,34–36]. If so, in a narrow regime in temperature it is most natural that there are only a few operators which contribute, and that these few only involve straight Polyakov loops.

Thus our conclusions about the universality of the bleaching of color are special to a semi-QGP which exists only in a narrow region of temperature. We also do not imply that all properties of fields in the semi-QGP are independent of their mass: only the bleaching of color. This result may be of significance for experiment. One of the real puzzles of the experimental data from the Relativistic Heavy Ion Collider is that the behavior of heavy quarks, such as charm, appears to be rather similar to that of light quarks: see Refs. [1,2], and especially Ref. [3]. This is very difficult to understand if the behavior arises from energy loss, which is very different for light quarks than for heavy. Our analysis suggests a completely different mechanism may be responsible: the bleaching of color. Of course a more careful analysis is necessary in order to confirm this suggestion.

If RHIC is in a conformally invariant regime, as suggested by $\mathcal{N} = 4$ supersymmetric gauge theories, then results for heavy ions at the LHC should be similar to RHIC: η/s remains small, essentially unchanged with temperature, and the behavior of heavy quarks will remain like that of light quarks.

In contrast, if the semi-QGP is valid, then there will be dramatic differences between RHIC and the LHC. If experiments at RHIC probe a region near T_c, then those at the LHC should probe temperatures significantly, perhaps a factor of 2, higher. At the LHC, the ratio of the shear viscosity, to the entropy density, will be large. Also, in a perturbative QGP, the behavior of heavy quarks should differ significantly from that of light quarks. The difficulty is that these differences are only for initial times and temperatures; inevitably, a system at the LHC, even if it starts in the perturbative QGP, cools through the semi-QGP.

In the end, we eagerly await the experimental results for heavy ions from the LHC, which will decide which theory is correct.

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