

Violation of quark-hadron duality and spectral chiral moments in QCDMartín González-Alonso,¹ Antonio Pich,¹ and Joaquim Prades²¹*Departament de Física Teòrica and IFIC, Universitat de València-CSIC, Apt. Correus 22085, E-46071 València, Spain*²*CAFPE and Departamento de Física Teórica y del Cosmos, Universidad de Granada,**Campus de Fuente Nueva, E-18002 Granada, Spain*

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We analyze the spectral moments of the $V - A$ two-point correlation function. Using all known short-distance constraints and the most recent experimental data from tau decays, we determine the lowest spectral moments, trying to assess the uncertainties associated with the so-called violations of quark-hadron duality. We have generated a large number of acceptable spectral functions, satisfying all conditions, and have used them to extract the wanted hadronic parameters through a careful statistical analysis. We obtain accurate values for the χ PT couplings L_{10} and C_{87} , and a realistic determination of the dimension six and eight contributions in the operator product expansion, $\mathcal{O}_6 = (-5.4_{-1.6}^{+3.6}) \cdot 10^{-3} \text{ GeV}^6$ and $\mathcal{O}_8 = (-8.9_{-7.4}^{+12.6}) \cdot 10^{-3} \text{ GeV}^8$, showing that the duality-violation effects have been underestimated in previous literature.

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I. INTRODUCTION

QCD sum rules (QCDSRs) [1,2] have been widely used during the last 30 years to study many important aspects of QCD. They constitute a very useful tool, enabling us with a powerful connection between QCD parameters and physical observables.

The basic assumption behind the QCDSR techniques is that the quark and hadron degrees of freedom provide two dual descriptions of the same strong interaction dynamics. This quark-hadron duality is a consequence of the assumed confinement of QCD. In more technical terms, a QCDSR is a dispersion relation relating the value of a given two-point correlation function at some Euclidean value of Q^2 with an integral over the corresponding spectral function in the Minkowskian domain. Quark-hadron duality allows us to calculate this Minkowskian integral in terms of hadrons, using the available experimental data. Ideally, the resulting QCDSR is an exact mathematical relation arising from analyticity and confinement (duality). In practice, however, a series of approximations need unavoidably to be adopted in its specific numerical implementation. In the Euclidean region the correlator is approximated by its short-distance operator product expansion (OPE) [3], truncated to a given finite order. In the Minkowskian region, since experimental data is only available at low energies, the integral over the physical spectral function is usually cut at a certain finite invariant-mass s_0 ; from s_0 up to ∞ , one then adopts the short-distance information provided by the OPE.

The uncertainties associated with all these approximations are usually known as violations of quark-hadron duality. They are difficult to estimate, because of our inability to make reliable QCD calculations at low and intermediate energies. The normal way to assess the theoretical uncertainties of QCDSRs consists in estimating the OPE truncation error and testing the stability of the results

with variations of s_0 . However, this method is too naive and can underestimate the effects not included in the OPE, i.e. the difference between the physical correlator and its OPE approximation.

Violations of QCD quark-hadron duality [4] have been relatively poorly studied and often disregarded. Its importance in finite energy sum rules (FESRs) has attracted some attention recently [5–8], owing to the phenomenological need for higher accuracies. To estimate the size of these effects is of course of maximal importance, if we want to master the strong interaction at all energies and be able to perform precision QCD calculations. This importance extends to all particle physics when one realizes that those calculations are often necessary to disentangle new physics from the standard model. Moreover, duality violations will also be present in new-physics scenarios characterized by a strongly interacting dynamics. A better knowledge of duality violations in QCD would help to understand their role in more exotic theories.

In the following, we present a detailed analysis of the possible numerical impact of duality violations in the description of the two-point correlation function of a left- and a right-handed vector currents. This is a very good laboratory to test the problem because this correlator is an order parameter of chiral-symmetry breaking: in the massless quark limit it vanishes to all orders in perturbation theory; its operator product expansion only contains power-suppressed contributions, starting with dimension six. In the absence of any theory of duality violations, we will use a generic, but theoretically motivated, model [4,9] to assess the phenomenological relevance of these effects.

The theoretical ingredients of our analysis are presented in the next section. Section III contains a detailed discussion of the behavior of the physical spectral function at high energies. Using the most recent experimental data, we generate a large number of “acceptable” spectral functions

which satisfy all known QCD constraints. Our numerical results, obtained through a careful statistical analysis of the whole set of possible spectral functions, are given in Sec. IV. Section V summarizes our findings.

II. THEORETICAL FRAMEWORK

The basic objects of the theoretical analysis are the two-point correlation functions of the vector and axial-vector quark currents $\mathcal{J}_{ij}^\nu(x)$, defined as follows:

$$\begin{aligned} \Pi_{ij,\mathcal{J}}^{\mu\nu}(q) &\equiv i \int d^4x e^{iqx} \langle 0 | T(\mathcal{J}_{ij}^\mu(x) \mathcal{J}_{ij}^\nu(0)^\dagger) | 0 \rangle \\ &= (-g^{\mu\nu} q^2 + q^\mu q^\nu) \Pi_{ij,\mathcal{J}}^{(1)}(q^2) \\ &\quad + q^\mu q^\nu \Pi_{ij,\mathcal{J}}^{(0)}(q^2). \end{aligned} \quad (1)$$

Although our analysis can be applied to any correlation function, we will only study here the nonstrange correlators and therefore $\mathcal{J}_{ij}^\mu(x)$ will denote the Cabibbo-allowed vector or axial-vector currents, $V_{ud}^\mu(x) = \bar{u} \gamma^\mu d$ and $A_{ud}^\mu = \bar{u} \gamma^\mu \gamma_5 d$. Moreover, we will concentrate on the $J = 0 + 1$ part of the $V - A$ difference, that is nothing but the correlation function of the left- and right-handed currents, $L_{ud}^\mu(x) \equiv V_{ud}^\mu(x) - A_{ud}^\mu(x)$ and $R_{ud}^\mu(x) \equiv V_{ud}^\mu(x) + A_{ud}^\mu(x)$, that is

$$\Pi(s) \equiv \Pi_{ud,V}^{(0+1)}(s) - \Pi_{ud,A}^{(0+1)}(s) \equiv \frac{2f_\pi^2}{s - m_\pi^2} + \bar{\Pi}(s), \quad (2)$$

where we have made explicit the contribution of the pion pole to the longitudinal axial-vector two-point function. We will work in the isospin limit, $m_u = m_d$, where $\Pi_{ud,V}^{(0)}(q^2) = 0$.

The correlator $\bar{\Pi}(s)$ is analytic in the entire complex s plane, except for a cut on the positive real axis which starts at the threshold $s_{th} = 4m_\pi^2$. Applying Cauchy's theorem in the circuit in Fig. 1 to the function $w(s)\Pi(s)$, one gets the exact relation:

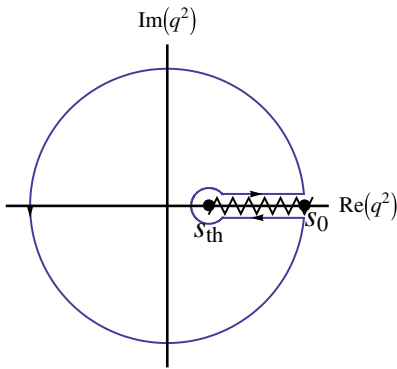


FIG. 1 (color online). Analytic structure of $\bar{\Pi}(s)$.

$$\begin{aligned} \int_{s_{th}}^{s_0} ds w(s) \rho(s) + \frac{1}{2\pi i} \oint_{|s|=s_0} ds w(s) \Pi(s) \\ = 2f_\pi^2 w(m_\pi^2) + \text{Res}_{s=0}[w(s)\Pi(s)], \end{aligned} \quad (3)$$

where $\rho(s) \equiv \frac{1}{\pi} \text{Im} \bar{\Pi}(s)$, $w(s)$ is a general analytic weight function except maybe at the origin where it can have poles, and $\text{Res}_{s=0}[F(s)]$ is the residue of $F(s)$ at $s = 0$. Integrals of the chiral spectral function $\rho(s)$ times $w(s)$ from threshold s_{th} up to s_0 are usually called spectral chiral moments $M_{w(s)}(s_0)$; when $s_0 \rightarrow \infty$ we will denote them $M_{w(s)}$ for brevity.

In order to evaluate the contour integral of (3), one approximates $\Pi(s)$ with its OPE expression

$$\Pi^{\text{OPE}}(s) = \sum_{k=3} C_{2k}(\nu) \frac{\langle O_{2k} \rangle(\nu)}{(-s)^k} \equiv \sum_{k=3} \frac{\mathcal{O}_{2k}}{(-s)^k}, \quad (4)$$

where $\langle O_{2k} \rangle(\nu)$ are vacuum expectation values of operators with dimension $d = 2k$; their associated Wilson coefficients $C_{2k}(\nu)$ contain logarithmic dependences with $-s$. Notice that both are ν -dependent quantities, but this dependence cancels in their product \mathcal{O}_{2k} . The use of the OPE introduces a systematic error in the relation (3), which is expected to be dominated by the region close to the positive real axis where the OPE approximation does not apply (except at $s_0 = \infty$ [10]).

Let us rewrite Eq. (3) in the form:

$$\begin{aligned} \int_{s_{th}}^{s_0} ds w(s) \rho(s) + \frac{1}{2\pi i} \oint_{|s|=s_0} ds w(s) \Pi^{\text{OPE}}(s) \\ + \text{DV}[w(s), s_0] \\ = 2f_\pi^2 w(m_\pi^2) + \text{Res}_{s=0}[w(s)\Pi(s)], \end{aligned} \quad (5)$$

where

$$\text{DV}[w(s), s_0] \equiv \frac{1}{2\pi i} \oint_{|s|=s_0} ds w(s) (\Pi(s) - \Pi^{\text{OPE}}(s)) \quad (6)$$

parametrizes the violation of quark-hadron duality that we are interested in. Notice that $\text{DV}[w(s), s_0]$ depends on the weight function $w(s)$ and on the circuit radius s_0 . The relation (5) contains all the elements of a standard sum rule. The first term is the hadronic part, that in our case is nothing but an integral of the $V - A$ nonstrange spectral function that has been measured in τ decays (for $s < m_\tau^2$) [11–15], while the second term is the OPE contribution to the contour integral at $|s| = s_0$. The second line contains the pion-pole contribution and the residue at the origin for negative power weight functions, $1/s^n$, which is calculable with chiral perturbation theory (χ PT) [16].

Sum rules of this type have been applied countless times in the last 30 years in order to extract theoretical parameters like quark masses [17,18], the strong coupling constant [19], QCD condensates [13,20–30] or χ PT couplings

[20,21,30–32]. Or, used in the other way around, to make predictions of hadronic observables.

In the chiral limit ($m_u = m_d = 0$) the correlator $\Pi(s)$ vanishes identically to all orders in perturbation theory and therefore its OPE contains only power-suppressed contributions from dimension $d = 2k$ operators, starting at $d = 6$, as we have already indicated in (4). The nonzero up and down quark masses induce tiny corrections with dimensions two and four, which are negligible at high values of s . This makes this correlator a very interesting object in the study of nonperturbative QCD.

In order to analyze duality-violation (DV) effects in different sum rules, we will use the weights $w(s) = s^n$, with $n = -2, -1, 2, 3$, that generate the following four FESRs¹:

$$M_{-2}(s_0) \equiv \int_{s_{\text{th}}}^{s_0} ds \frac{1}{s^2} \rho(s) = 16C_{87}^{\text{eff}} - \text{DV}[1/s^2, s_0], \quad (7)$$

$$M_{-1}(s_0) \equiv \int_{s_{\text{th}}}^{s_0} ds \frac{1}{s} \rho(s) = -8L_{10}^{\text{eff}} - \text{DV}[1/s, s_0], \quad (8)$$

$$M_2(s_0) \equiv \int_{s_{\text{th}}}^{s_0} ds s^2 \rho(s) = 2f_\pi^2 m_\pi^4 + \mathcal{O}_6 - \text{DV}[s^2, s_0], \quad (9)$$

$$M_3(s_0) \equiv \int_{s_{\text{th}}}^{s_0} ds s^3 \rho(s) = 2f_\pi^2 m_\pi^6 - \mathcal{O}_8 - \text{DV}[s^3, s_0], \quad (10)$$

where $L_{10}^{\text{eff}} \equiv -\frac{1}{8} \bar{\Pi}(0)$ and $C_{87}^{\text{eff}} \equiv \frac{1}{16} \bar{\Pi}'(0)$ are quantities that can be written in terms of low-energy χ PT constants [31], while $\mathcal{O}_{6,8}$ are defined in Eq. (4). These four sum rules have been used in the past [13,20–32] to extract the values of either the χ PT couplings L_{10} and C_{87} , or the vacuum expectation values of the dimension six and eight operators appearing in the OPE. In those works the DV effects were just inferred from the s_0 -stability (if not just neglected), that as we will see can be a misleading method. Here we want to analyze the effect of DV on these four observables using a different approach that will be introduced in the following sections.

For the computation of the hadronic integral representation of the moments $M_n(s_0)$ we will use the 2005 ALEPH data on semileptonic τ decays [11], shown in Fig. 2, which provide the most recent and precise measurement of the $V - A$ spectral function $\rho(s)$.

¹Here we neglect the logarithmic corrections to the Wilson coefficients in the OPE. The error associated to this approximation is expected to be smaller than the other errors involved in the analysis, as was found, e.g. in Refs. [24,33].

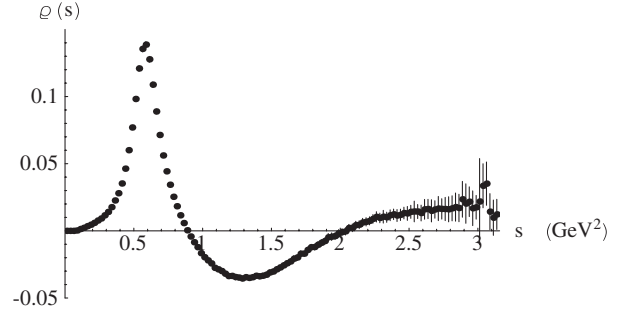


FIG. 2. Nonstrange $V - A$ spectral function $\rho(s) = \frac{1}{\pi} \text{Im} \Pi_{ud, V-A}^{(0+1)}(s)$ measured from hadronic τ decays by ALEPH [11].

A. Theoretically-known spectral moments

In the four sum rules introduced in the previous section, we use the experimental data to extract theoretical information, namely, the value of the corresponding parameters or, equivalently, the value of the spectral moments for $s_0 \rightarrow \infty$, M_n . There exist a few additional sum rules where we know theoretically the value of the spectral moments when $s_0 \rightarrow \infty$. These sum rules will play a special role in our analysis because they give us very valuable information on the spectral function $\rho(s)$ for $s \geq s_0$. The three sum rules that we will use are

$$M_0 = \int_{s_{\text{th}}}^{\infty} ds \rho(s) = 2f_\pi^2, \quad (11)$$

$$M_1 = \int_{s_{\text{th}}}^{\infty} ds s \rho(s) = 2f_\pi^2 m_\pi^2, \quad (12)$$

$$\int_{s_{\text{th}}}^{\infty} ds s \log\left(\frac{s}{\lambda^2}\right) \rho(s) \Big|_{m_q=0} = (m_{\pi^0}^2 - m_{\pi^+}^2)_{\text{EM}} \frac{8\pi}{3\alpha} f_0^2. \quad (13)$$

The relations (11) and (12) are the well-known first and second Weinberg sum rules (WSRs) [34], while the third identity is the pion sum rule (π SR) giving the electromagnetic pion mass splitting in the chiral limit [35]. In the second WSR there are contributions of the form $\mathcal{O}(m_q^2 \alpha s_0)$ [36], where s_0 is the upper limit of the integral, but they are negligible for the values of s_0 that we are considering.

B. Duality violation

To get vanishing DV in sum rules like (5) and (7)–(10) one could consider working with an infinite Cauchy radius s_0 , but this is clearly not an option because the spectral function $\rho(s)$ is only known up to $s_{\text{max}} = m_\tau^2$. We can predict the value of $\rho(s)$ at high-enough energies using perturbative QCD, but there is an intermediate region above s_{max} where perturbation theory is still not reliable. Therefore we have to deal with this DV unavoidably, and it

is important to keep in mind that at $s_0 \sim 3 \text{ GeV}^2$ it can represent a sizable contribution to the sum rules, as the WSRs show clearly (see e.g. Fig. 1 in Ref. [22]).

Since the solution to QCD is not known yet, DV is almost by definition a noncalculable quantity and that is the reason why it has been taken to be negligible very often. But in order to make precise and reliable predictions one must worry about the size of this effect. As it is commonly done, we have defined the DV in Eq. (6) as the uncertainty associated with the use of the OPE. The usual strategy to estimate the size of the DV has been to look at the stability of the spectral moments with variations of s_0 . This stability can be improved adopting the so-called “pinched weights” [24], polynomial weight functions with a zero at $s = s_0$ which suppresses the contribution to the integral (6) from the region close to the positive real axis. As we will see, the stability with s_0 obtained with these weights can be misleading in some situations.

Taking into account that $DV[w(s), s_0]$ vanishes for $s_0 \rightarrow \infty$, one can easily rewrite Eq. (5) in the form [5–7,9]

$$DV[w(s), s_0] = \int_{s_0}^{\infty} ds w(s) \rho(s), \quad (14)$$

expressing the DV effect as an hadronic integral that can be analyzed phenomenologically.

We know from QCD that the spectral function $\rho(s)$ has to vanish at high values of s and, consequently, we expect the region right above s_0 to be the most relevant in (14). This makes the pinched weights an interesting tool to minimize the DV. However, in (14) we can see something that is hidden in (6), namely, that one has to worry also about the possible enhancement of the contribution from the high-energy part of the integral ($s \gg s_0$) produced by the pinched weights. And thus, we see that the use of these weights can worsen the situation. Another direct consequence from (14), unless accidental cancellations occur, is that by weighting less the high-energy part of the spectral integral one can get smaller DV. In particular, for our spectral moments $M_n(s_0)$, one expects the DV effects to increase with increasing values of n . Thus, the size of the DV will be smaller in the determination of L_{10}^{eff} than in the determination of the chiral moment M_2 .

To quantify the DV uncertainties of a given sum rule we must then estimate the possible behavior of the spectral function beyond s_0 . The DV is an estimate of the freedom in the behavior of the spectral function above s_0 , once all the theoretical and phenomenological knowledge on that spectral function and on its moments has been taken into account. For instance, QCD tells us that $\rho(s)$ must go *quickly* enough to zero when $s \rightarrow \infty$. This is valuable information, but one can still imagine infinite possible shapes for the spectral function and, therefore, the limits imposed on DV effects are poor and not good enough for most phenomenological analyses.

Some theoretically motivated models for the DV were advocated in Ref. [4]. We will adopt a simple parametrization of the spectral function at high energies, based in the resonance model proposed in [4] and similar to the one used in Refs. [7,9]. Following the discussion above, we add more physical constraints to the behavior of $\rho(s)$ and require that it satisfies the WSRs and the π SR [6]. Our goal is to generate a bunch of physically acceptable spectral functions and translate this information into DV limits.

Similar work has been done in [7,9] to estimate the DV uncertainties associated with the determination of α_s from hadronic τ decay data. An important difference of our present study with those works is that they make separate analyses for the vector and axial-vector channels, without imposing the constraints from the WSRs and π SR. In fact, one can easily check that those sum rules are not satisfied for the vast majority of the generated spectral functions used in [7,9] (as can be seen in Fig. 2 of Ref. [8]). So the results found there cannot be applied to the $V - A$ channel that we want to study here.

III. ACCEPTABLE $V - A$ SPECTRAL FUNCTIONS

A. Spectral-function parametrization

We split the integral of the spectral function $\rho(s)$ in two parts. For the low-energy part of the integral we will use the ALEPH data, whereas in the rest of the integration range we will work under the assumption that the spectral function is well described by the following parametrization:

$$\rho(s \geq s_z) = \kappa e^{-\gamma s} \sin(\beta(s - s_z)), \quad (15)$$

that has κ , γ , β , and s_z as free parameters. From the ALEPH data we know that the $V - A$ spectral function $\rho(s)$ has a second zero around 2 GeV^2 (see Fig. 2), which is represented in our parametrization through the s_z parameter. We will take this zero as the separation point between the use of the data and the use of the model.

At high values of s this parametrization appears naturally in the equidistant resonance-based model with finite widths introduced in [4]. It has also been used for the vector and axial-vector correlators in Ref. [9], based on the expected exponential falloff associated with the intrinsic error of an asymptotic expansion; the sine function reflects the periodicity of the daughter trajectories in the spectrum of the Regge theory.

In the region $2.0 \text{ GeV}^2 \leq s \leq 3.3 \text{ GeV}^2$ the proposed parametrization is compatible with the ALEPH data; the corresponding χ^2 fit gives the result²

$$\begin{aligned} \chi_{\text{min}}^2(\kappa, \gamma, \beta, s_z) &= \chi^2(1.00, 1.05, 0.40, 2.03) = 4.4 \\ &\ll \text{d.o.f.} = 43. \end{aligned} \quad (16)$$

In fact the compatibility appears to be too good, in the

²Hereafter, unless otherwise stated, we include all correlations among the points.

sense that the minimum χ^2 is much smaller than the number of degrees of freedom (d.o.f.): $43 = 45$ points—2 parameters. This low value of χ_{\min}^2 was also found in Refs. [9,37].

B. Imposing constraints

As we have already said, the WSRs and the π SR in (11)–(13) are an important source of information on $\rho(s)$, for s values beyond the range of the τ data. In the literature, the use of this information has been mostly limited to define the so-called “duality points,” values of s_0 for which the WSRs are satisfied, i.e. $DV[s^n, s^0] = 0$ ($n = 0, 1$). These duality points are frequently used to evaluate the other FESRs, but this introduces an unknown systematic error and several ambiguities, like which duality point is the best option.

We will fully use that information by imposing that the spectral function $\rho(s)$, given by the latest ALEPH data below $s_z \sim 2 \text{ GeV}^2$ and Eq. (15) for $s > s_z$, fulfils the two WSRs and the π SR within uncertainties. This requirement constrains the regions in the parameter space of model (15) that are compatible with both QCD and the data. We will find all possible tuples³ $(\kappa, \gamma, \beta, s_z)$ which are compatible with such constraints by fitting the model. In this way, we analyze how much freedom is left for the shape of the spectral function after imposing all we know on $\rho(s)$ from data plus QCD. We will also require the compatibility between models and data in the region⁴ $1.7 \text{ GeV}^2 \leq s \leq 3.15 \text{ GeV}^2$.

The four imposed conditions can be written quantitatively in the following form⁵:

$$\int_0^{s_z} \rho(s)^{\text{ALEPH}} ds + \int_{s_z}^{\infty} \rho(s; \kappa, \gamma, \beta, s_z) ds = 2f_{\pi}^2 = (17.1 \pm 0.4) \times 10^{-3} \text{ GeV}^2, \quad (17)$$

³We will talk about “tuple” referring to a set of values $(\kappa, \gamma, \beta, s_z)$.

⁴Although we are assuming that the model describes correctly the spectral function beyond $s_z \sim 2 \text{ GeV}^2$, we impose the compatibility with the data from 1.7 GeV^2 to ensure the continuity of the spectral function in the matching region between the data and the model.

⁵The quoted errors in Eqs. (17) and (18) are just data errors, whereas in (19) the main uncertainty comes from the fact that quark masses do not vanish in nature and we are using real data (not chiral-limit data). We estimate this uncertainty taking for the pion decay constant the value $f_0 = 87 \pm 5 \text{ MeV}$, that covers a range that includes the physical value and the different estimates of the chiral-limit value [38]. We also include a small uncertainty coming from the residual scale dependence of the logarithm, which is proportional to the second WSR. We consider $\lambda \sim 1 \text{ GeV}$ a good choice of scale because higher values would suppress the high-energy part of the integral (the information that we want to use), while smaller values would generate larger τ -data errors in (19), losing also information about the high-energy region.

$$\int_0^{s_z} \rho(s)^{\text{ALEPH}} s ds + \int_{s_z}^{\infty} \rho(s; \kappa, \gamma, \beta, s_z) s ds = 2f_{\pi}^2 m_{\pi}^2 = (0.3 \pm 0.8) \times 10^{-3} \text{ GeV}^4, \quad (18)$$

$$\begin{aligned} & \int_0^{s_z} \rho(s)^{\text{ALEPH}} s \log\left(\frac{s}{1 \text{ GeV}^2}\right) ds \\ & + \int_{s_z}^{\infty} \rho(s; \kappa, \gamma, \beta, s_z) s \log\left(\frac{s}{1 \text{ GeV}^2}\right) ds \\ & = (m_{\pi^0}^2 - m_{\pi^+}^2)_{\text{EM}} \frac{8\pi}{3\alpha} f_0^2 \\ & = -(10.9 \pm 1.5) \times 10^{-3} \text{ GeV}^4, \end{aligned} \quad (19)$$

$$\chi^2(\kappa, \gamma, \beta, s_z) < \chi_{\text{crit}}^2 = \text{d.o.f.} = 54. \quad (20)$$

C. Selection process of acceptable models

After defining the minimal conditions that a tuple has to satisfy in order to be accepted, we perform a scanning over the 4-dimensional parameter space, looking for physically acceptable tuples. We emphasize the importance of taking properly into account the data correlations. For instance, if one analyzes the compatibility of a null spectral function with the ALEPH data in the region $(2, 3.15) \text{ GeV}^2$, the resulting minimum χ^2 is very sensitive to these correlations:

$$\chi^2(0.0, \gamma, \beta, s_z)/\text{d.o.f.} = 0.99 \text{ (correlations included)}, \quad (21)$$

$$\chi^2(0.0, \gamma, \beta, s_z)/\text{d.o.f.} = 4.58 \text{ (correlations included)}. \quad (22)$$

To perform the parameter-space scanning process, we adopt the following procedure. First, we define a rectangular region such that it contains the four-dimensional ellipsoid defined by $\chi^2(\kappa, \gamma, \beta, s_z) = \text{d.o.f.}$, and we create a lattice with $20^4 = 16 \times 10^4$ points, that is, 16×10^4 tuples (or functions). We find that 1789 of them satisfy our set of minimal conditions; i.e., 1789 of them represent possible shapes of the physical spectral function beyond 2 GeV^2 . Figure 3 shows the statistical distribution of the parameters of our model after the selection process. In Fig. 4 we show the distribution of the quantity $\chi^2(\kappa, \gamma, \beta, s_z)$ for those tuples that have passed the selection process. We find that all accepted tuples generate values of χ^2 larger than 10.0, i.e., tuples following the central values of the experimental points do not pass the selection process; neither do the tuples that go above the central values. Thus, our model indicates clearly that the third bump of the spectral function should be smaller than what the ALEPH data suggest (see Fig. 2). The size of this third bump is an important issue that future high-quality τ decay data could clarify. For illustrative purposes, Fig. 5

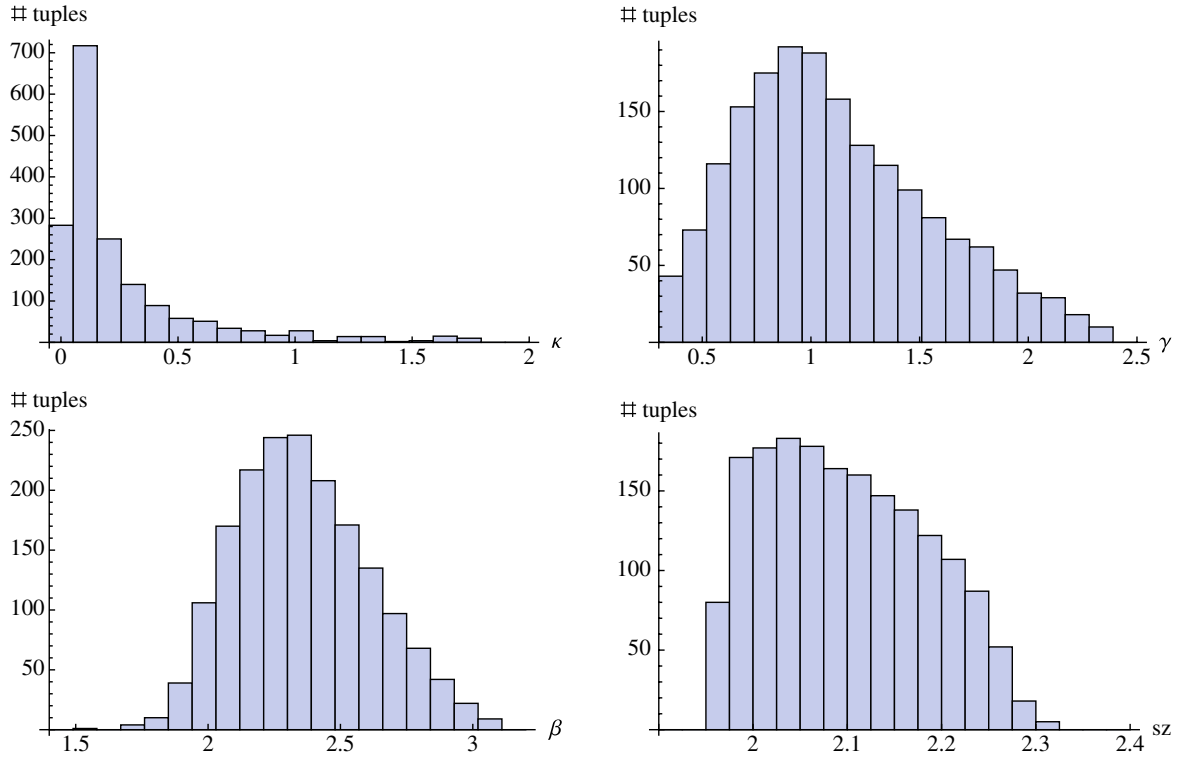


FIG. 3 (color online). Statistical distribution of acceptable models in the parameter space κ (upper-left), γ (upper-right), β (lower-left), and s_z (lower-right).

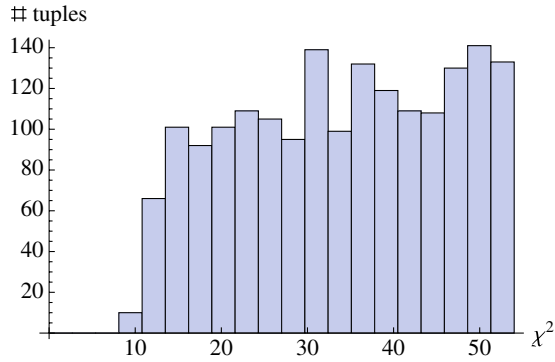


FIG. 4 (color online). Distribution of $\chi^2(\kappa, \gamma, \beta, s_z)$ values for acceptable tuples.

shows one of the hundreds of functions that satisfy our set of conditions.

IV. NUMERICAL RESULTS

For each one of the hundreds of functions that have passed our selection process, we can calculate the associated values of C_{87}^{eff} , L_{10}^{eff} , \mathcal{O}_6 , and \mathcal{O}_8 , simply carrying out the integrals of Eqs. (7)–(10) with $s_0 \rightarrow \infty$. The results of this analysis are summarized in Fig. 6, which shows the statistical distribution of the calculated parameters. From these distributions, one gets the final numbers:

$$\begin{aligned} C_{87}^{\text{eff}} &= (8.167_{-0.002}^{+0.007} \pm 0.12) \times 10^{-3} \text{ GeV}^{-2} \\ &= (8.17 \pm 0.12) \times 10^{-3} \text{ GeV}^{-2}, \end{aligned} \quad (23)$$

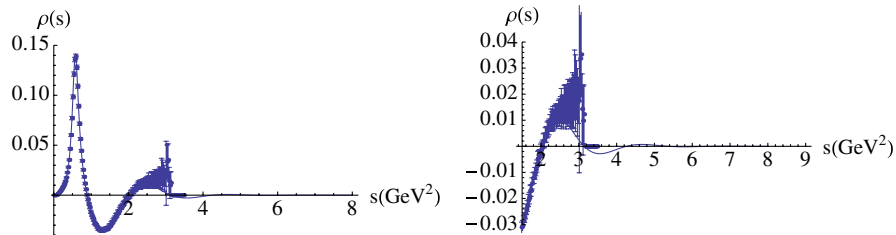


FIG. 5 (color online). Spectral function $\rho(s)$ generated with $(\kappa, \gamma, \beta, s_z) = (0.24, 1.23, 2.82, 2.03)$, together with the experimental ALEPH data [11]. $\chi^2 = 38.7$ for this tuple.

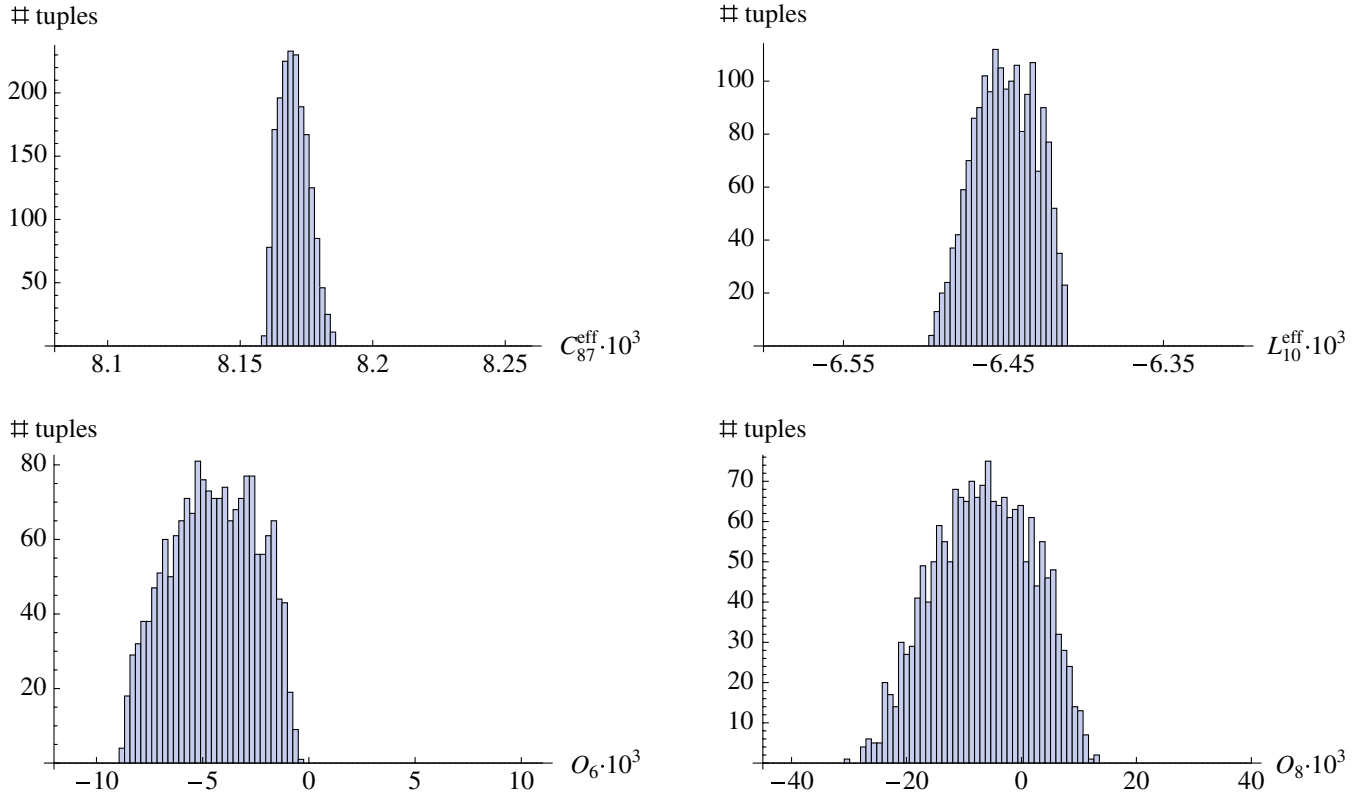


FIG. 6 (color online). Statistical distribution of values of C_{87}^{eff} (upper-left), L_{10}^{eff} (upper-right), \mathcal{O}_6 (lower-left), and \mathcal{O}_8 (lower-right) for acceptable models.

$$\begin{aligned} L_{10}^{\text{eff}} &= (-6.46_{-0.01}^{+0.03} \pm 0.07) \times 10^{-3} \\ &= (-6.46_{-0.07}^{+0.08}) \times 10^{-3}, \end{aligned} \quad (24)$$

$$\begin{aligned} \mathcal{O}_6 &= (-5.4_{-1.0}^{+3.4} \pm 1.2) \times 10^{-3} \text{ GeV}^6 \\ &= (-5.4_{-1.6}^{+3.6}) \times 10^{-3} \text{ GeV}^6, \end{aligned} \quad (25)$$

$$\begin{aligned} \mathcal{O}_8 &= (-8.9_{-7.1}^{+12.4} \pm 2.1) \times 10^{-3} \text{ GeV}^8 \\ &= (-8.9_{-7.4}^{+12.6}) \times 10^{-3} \text{ GeV}^8, \end{aligned} \quad (26)$$

where the first error is that associated to the high-energy region (integral from s_z to infinity), that we compute from the dispersion of the histograms of Fig. 6, and the second error is that associated to the low-energy region (integral from zero to s_z), that we compute in a standard way from the ALEPH data. This results correspond to the 68% probability region (one sigma). Since the first error is not Gaussian we show also now the 95% probability results (95% of the acceptable spectral functions give a result within the quoted interval):

$$\begin{aligned} C_{87}^{\text{eff}} &= (8.167_{-0.007}^{+0.011} \pm 0.24) \times 10^{-3} \text{ GeV}^{-2} \\ &= (8.17 \pm 0.24) \times 10^{-3} \text{ GeV}^{-2}, \end{aligned} \quad (27)$$

$$\begin{aligned} L_{10}^{\text{eff}} &= (-6.46_{-0.03}^{+0.04} \pm 0.14) \times 10^{-3} \\ &= (-6.46_{-0.14}^{+0.15}) \times 10^{-3}, \end{aligned} \quad (28)$$

$$\begin{aligned} \mathcal{O}_6 &= (-5.4_{-2.7}^{+4.2} \pm 2.4) \times 10^{-3} \text{ GeV}^6 \\ &= (-5.4_{-3.6}^{+4.8}) \times 10^{-3} \text{ GeV}^6, \end{aligned} \quad (29)$$

$$\begin{aligned} \mathcal{O}_8 &= (-8.9_{-15.1}^{+16.9} \pm 4.2) \times 10^{-3} \text{ GeV}^8 \\ &= (-8.9_{-15.7}^{+17.4}) \times 10^{-3} \text{ GeV}^8. \end{aligned} \quad (30)$$

Our calculations have been done with a very simple, but physically motivated, parametrization of DV [4,9]. Most likely this parametrization does not represent the actual shape of the $V - A$ spectral function, but it accounts for the possible freedom of the function $\rho(s)$ beyond 2 GeV^2 and its consequences on the observables. Our statistical analysis translates the present ignorance on the high-energy behavior of $\rho(s)$ into a clear quantitative assessment on the uncertainties of the phenomenologically extracted parameters.

As expected, the DV effects have very little impact on the values of C_{87}^{eff} and L_{10}^{eff} , because the corresponding FESRs (7) and (8) are dominated by the low-energy region where the available data sits. Our results are in excellent agreement with the most recent determination of these parameters, using the same ALEPH τ data, performed in

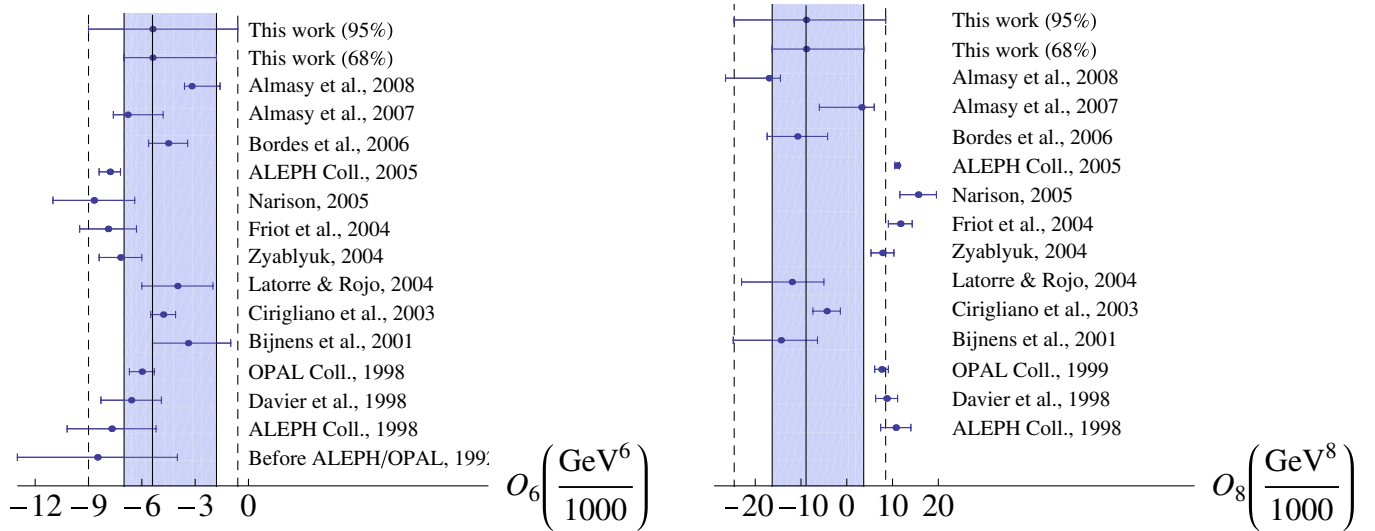


FIG. 7 (color online). Comparison of our results for \mathcal{O}_6 (left) and \mathcal{O}_8 (right) with previous determinations [11–13,20,22,24,27–30,40–42] (we show for every method the most recent determination). The (blue) filled band shows our results at 65% C.L., while the 95% probability regions are indicated by the dotted lines.

Ref. [31]: $C_{87}^{\text{eff}} = (8.18 \pm 0.14) \times 10^{-3} \text{ GeV}^{-2}$ and $L_{10}^{\text{eff}} = -(6.48 \pm 0.06) \cdot 10^{-3}$.

The situation is not so good for the moments M_2 and M_3 (or equivalently \mathcal{O}_6 and \mathcal{O}_8), which are sensitive to the high-energy behavior of the spectral function. The present ALEPH data, together with the constraints from the WSRs and the π SR, are not good enough to determine the sign of \mathcal{O}_8 ; the DV uncertainties turn out to be too large in this case. Our results are slightly better for \mathcal{O}_6 , where there is no doubt in the sign, but again the effects of DV imply larger uncertainties than what was estimated in previous works. Our results are compared in Fig. 7 with previous determinations of \mathcal{O}_6 and \mathcal{O}_8 . One recognizes in the figure the existence of two groups of results that disagree between them. For \mathcal{O}_6 there is a small tension between a bigger or smaller value, whereas in the case of \mathcal{O}_8 the disagreement affects to the sign and is more sizeable. In some cases the discrepancy appears to be related with a two-fold ambiguity in the adopted choice of duality points. Our analysis indicates that the DV error was grossly underestimated in most of the previous works based on FESRs (7)–(10). Only Refs. [22,28] quote uncertainties similar to ours, although our error bands are slightly shifted in such a way that the tension with the other estimates is reduced.

V. SUMMARY

The phenomenological requirement for increasing precisions in the determinations of hadronic parameters makes it necessary to assess the size of small effects which previously could be considered negligible. In particular, a substantial improvement of QCDSR results, needed to determine many hadronic observables both in the standard model and in models beyond it, could only be possible with a better control of DV.

Violations of quark-hadron duality are difficult to estimate because those effects are unknown by definition. They originate in the uncertainties associated with the use of the OPE to approximate the exact physical correlator. As defined in Eq. (6), DV effects correspond to an OPE approximation performed in the complex plane, outside the Minkowskian region, which deteriorates in the vicinity of the real axis. Using analyticity, the size of DV can be related with an integral of the hadronic spectral function from s_0 up to ∞ , given in Eq. (14), which allows us to perform a phenomenological analysis.

We have studied the possible role of DV in the two-point correlation function $\Pi(s)$. This $V - A$ nonstrange correlator is very well suited for this analysis because: (i) it is a purely nonperturbative quantity in the chiral-limit, (ii) there are well-known theoretical constraints, and (iii) there exist good available data from τ decays. Moreover, different moments of its spectral function provide hadronic parameters of high phenomenological relevance.

We have assumed a generic, but theoretically motivated, behavior of the spectral function at high energies, where data are not available, with four free parameters. This allows us to study how much freedom in $\rho(s)$ could be tolerated, beyond the requirement that all known QCD constraints are satisfied. Performing a numerical scanning over the four-dimensional parameter space, we have generated a large number of acceptable spectral functions, satisfying all conditions, and have used them to extract the wanted hadronic parameters through a careful statistical analysis. The dispersion of the numerical results provides then a good quantitative assessment of the actual uncertainties.

We have determined four hadronic parameters of special interest: C_{87}^{eff} , L_{10}^{eff} , \mathcal{O}_6 , and \mathcal{O}_8 . Our final numerical results

are given in Eqs. (23)–(26) for the one sigma results and (27)–(30) for the 95% probability results. The parameters C_{87}^{eff} and L_{10}^{eff} are in excellent agreement with the most recent determination using FESRs and the same ALEPH data [31]. The vacuum condensate \mathcal{O}_6 is an important input for the calculation of the CP -violating kaon parameter ε_K' ; it dominates the $\Delta I = 3/2$ contribution to ε_K' [22,23]. The determination of this contribution is an important goal of lattice QCD calculations and independent information is required to test the reliability of those results. We will study the consequences of our results for ε_K' in a forthcoming publication [39].

Our analysis indicates that the DV error was grossly underestimated in most of the previous QCDSR determinations of \mathcal{O}_6 and \mathcal{O}_8 based on the FESRs (7)–(10). The present $V - A$ nonstrange tau data between 2 GeV² and 3 GeV² [11] is not good enough to constrain the spectral function with the needed accuracy. Good data in that

energy region with much smaller experimental uncertainties is clearly required. Future high-statistics τ -decay data samples could allow a substantial improvement of our results, helping to clarify the actual high-energy behavior of the $V - A$ spectral function.

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