Dependence of the quark-lepton complementarity on parametrizations of the Cabibbo-Kobayashi-Maskawa and Pontecorvo-Maki-Nakagawa-Sakata matrices

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The quark-lepton complementarity (QLC) is very suggestive in understanding possible relations between quark and lepton mixing matrices. We explore the QLC relations in all the possible angle-phase parametrizations and point out that they can approximately hold in five parametrizations. Furthermore, the vanishing of the smallest mixing angles in the Cabibbo-Kobayashi-Maskawa and Pontecorvo-Maki-Nakagawa-Sakata matrices can make sure that the QLC relations exactly hold in those five parametrizations. Finally, the sensitivity of the QLC relations to radiative corrections is also discussed.

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I. INTRODUCTION

The success of the standard model (SM) in describing the mass origin of elementary particles has satisfied many theoretical physicists, but it is now challenged by the existence of neutrino oscillations observed in the solar [1], atmospheric [2], reactor [3], and accelerator [4] neutrino experiments, which provide us with convincing evidence for neutrino masses and lepton flavor mixing. The underlying nature of neutrino mixings as compared with that of the quark mixings has inspired a large amount of speculation regarding symmetries in the quark-lepton world as well as other kinds of new physics beyond the SM [5].

In the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) [6] lepton mixing matrix, the most distinct feature is the existence of two large mixing angles, which is quite different from the pattern the Cabibbo-Kobayashi-Maskawa (CKM) [7] quark mixing matrix. To be specific, the PMNS matrix consists of a large and nearly maximal angle ϑ_{23} (atmospheric angle), a large but nonmaximal angle ϑ_{12} (solar angle), and a small angle ϑ_{13} (reactor angle) in the standard parametrization. An interesting phenomenological relation between the lepton and quark mixing angles, the so-called quark-lepton complementarity (QLC), has been noticed recently [8]. Namely, the sums of the mixing angles of quarks and leptons for the 1-2 and 2-3 mixings agree with 45°:

$$\theta_{12} + \vartheta_{12} \simeq 45^\circ, \qquad \theta_{23} + \vartheta_{23} \simeq 45^\circ, \tag{1}$$

where θ_{12} and θ_{23} are quark mixing angles. As for the 1-3 mixing angles of quarks and leptons, a similar relation $\theta_{13} + \vartheta_{13} \simeq 45^\circ$ does not hold because their sum is less than 10°.

Attempts to understand the deep meaning behind the QLC relations have been made. It has been interpreted as evidence for certain quark-lepton symmetry or quark-lepton unification, especially the possibility of the bimaximal and tri-bimaximal fermion mixing patterns and the deviation from them, which have been extensively discussed [9,10]. Some other aspects of the QLC relations have also been studied, such as their phenomenological implications [11] and renormalization group (RG) effects [12]. There are also the extended QLC relations proposed and discussed in the seesaw mechanisms [13]. Recent reviews about the QLC relations can be found in Ref. [14]. However, whether this relation is an accident or not remains an open question. In Ref. [15] Jarlskog points out that the QLC relations are parametrization variant and the specific models are far from being sufficiently pinned down to be useful for connecting quark and lepton mixing angles like this.

In this paper, we intend to analyze the parametrization dependence of the QLC relations by calculating the mixing angles of each possible parametrization. Among nine angle-phase parametrizations of the CKM and PMNS matrices, we find that five of them can have the approximate QLC relations. If the QLC relations are assumed to exactly hold in a certain parametrization such as the standard parametrization, we examine whether they are possible to exactly hold in other parametrizations. Furthermore, the stability of the QLC relations under the RG running is also studied in the Fritzsch-Xing (FX) parametrization [16].

The remaining part of this paper is organized as follows. In Sec. II, with the latest experimental data for the CKM and PMNS matrices, we calculate the mixing angles and sum them up in each parametrization to check the QLC relations. Section III is devoted to examining the relationships between different parametrizations, especially whether the QLC relations in one parametrization can also hold in other parametrizations, and what conditions should be satisfied. The RG running effects on the QLC relations are discussed in the FX parametrization both in the SM and the minimal supersymmetric standard model (MSSM) in Sec. IV.

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II. QLC RELATIONS IN DIFFERENT ANGLE-PHASE PARAMETRIZATIONS

In this section, we try to make numerical calculations of the mixing angles of quark and lepton flavor mixing matrices and examine the QLC relations for all the possible angle-phase parametrizations.

The 3×3 CKM quark mixing matrix can be expressed in terms of four independent parameters, which are usually taken as three rotation angles and one CP-violating phase angle. For a clear classification of this kind of angle-phase parametrizations, see [17]. It is pointed out that the CKM matrix V, if real and orthogonal, can in general be written as a product of three matrices R_{12}, R_{23}, and R_{31}, which describe simple rotations in the (1, 2), (2, 3), and (3, 1) planes.

$$R_{12}(\theta_{12}) = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$R_{23}(\theta_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \tag{2}$$
$$R_{31}(\theta_{13}) = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix},$$

where $s_{ij} \equiv \sin\theta_{ij}$, $c_{ij} \equiv \cos\theta_{ij}$, etc. After introducing the CP-violating phase ϕ, among all the 12 possible products only nine of them are structurally different, as the remaining three products are correlated with each other and lead essentially to the same form. And the specific forms of the nine possible angle-phase parametrizations are listed in the left column of Table I as P1 to P9, generally P1 corresponds to the standard parametrization [18] and P2 to the FX parametrization. To make it clear, we use θ and ϑ, respectively, to denote quark and lepton mixing angles. From P3 to P9 parametrization, ϑ_l, ϑ_ν, and ϑ in lepton flavor mixing correspond to θ_u, θ_d, and θ in the quark flavor mixing separately. For Majorana neutrinos, two additional parameters are needed in PMNS lepton mixing matrix, namely, two Majorana CP-violating phase angles, which do not affect oscillations [19].

In the calculation of quark mixing angles, we take the Wolfenstein parametrization with the accuracy of $O(\lambda^6)$ as proposed in [20], which is shown as below:

$$V_{\rm CKM} = \begin{pmatrix} 1-\frac{1}{2}\lambda^2-\frac{1}{8}\lambda^4 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda[1+\frac{1}{2}A^2\lambda^4(2\rho-1)+iA^2\lambda^4\eta] & 1-\frac{1}{2}\lambda^2-\frac{1}{8}(4A^2+1)\lambda^4 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2[1+\frac{1}{2}\lambda^2(2\rho-1)+i\lambda^2\eta] & 1-\frac{1}{2}A^2\lambda^4 \end{pmatrix}. \tag{3}$$

To calculate the moduli of the mixing matrix elements, we adopt the following inputs given by the Particle Data Group [18]:

$$\lambda = 0.2257^{+0.0009}_{-0.0010}, \qquad A = 0.814^{+0.021}_{-0.022}, \qquad \bar\rho = 0.135^{+0.031}_{-0.016}, \qquad \bar\eta = 0.349^{+0.015}_{-0.017}, \tag{4}$$

where $\bar\rho = \rho - \frac{1}{2}\rho\lambda^2 + (\frac{1}{2}A^2\rho - \frac{1}{8}\rho - A^2(\rho^2-\eta^2))\lambda^4 + O(\lambda^6)$ and $\bar\eta = \eta - \frac{1}{2}\eta\lambda^2 + (\frac{1}{2}A^2\eta - \frac{1}{8}\eta - 2A^2\rho\eta)\lambda^4 + O(\lambda^6)$. Then we obtain

$$|V_{\rm CKM}| = \begin{pmatrix} 0.974\,205^{-0.000\,21}_{+0.000\,23} & 0.225\,700^{+0.000\,90}_{-0.001\,00} & 0.003\,592^{+0.000\,40}_{-0.000\,34} \\ 0.225\,560^{+0.000\,90}_{-0.001\,00} & 0.973\,346^{-0.000\,27}_{+0.000\,29} & 0.041\,466^{+0.001\,41}_{-0.001\,48} \\ 0.008\,733^{+0.000\,11}_{-0.000\,27} & 0.040\,709^{+0.001\,44}_{-0.001\,48} & 0.999\,140^{-0.000\,06}_{+0.000\,06} \end{pmatrix}. \tag{5}$$

This result allows us to calculate the mixing angles in all the nine parametrizations according to the relations between angles and moduli.

For lepton mixing angles, the standard parametrization is expressed in terms of three mixing angles ϑ_{12}, ϑ_{13}, ϑ_{23} and one CP-violating phase angle φ. As shown below, the first row and third column have a pretty simple form.

$$V_{\rm PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -c_{12}s_{23}s_{13} - s_{12}c_{23}e^{-i\varphi} & -s_{12}s_{23}s_{13} + c_{12}c_{23}e^{-i\varphi} & s_{23}c_{13} \\ -c_{12}c_{23}s_{13} + s_{12}s_{23}e^{-i\varphi} & -s_{12}c_{23}s_{13} - c_{12}s_{23}e^{-i\varphi} & c_{23}c_{13} \end{pmatrix}, \tag{6}$$

where $s_{ij} = \sin\vartheta_{ij}$, $c_{ij} = \cos\vartheta_{ij}$ ($i, j = 1, 2, 3$). With the latest global fit of the experimental data given in [21], the three mixing angles read

$$\sin^2\vartheta_{12} = 0.312(1^{+0.128}_{-0.109})(2\sigma), \qquad \sin^2\vartheta_{23} = 0.466(1^{+0.292}_{-0.215})(2\sigma), \qquad \sin^2\vartheta_{13} = 0.016 \pm 0.010(1\sigma). \tag{7}$$

Because of the smallness of $\vartheta_{13} \simeq (7.27^{+2.012}_{-2.824})^\circ$, those terms including $\sin\vartheta_{13}$ could be neglected in the (2, 1), (2, 2), (3, 1),

TABLE I. Classification of different parametrizations for the flavor mixing matrix and the QLC relations.

Parametrization	Quark-Lepton Complementarity
P1: $V = R_{23}(\theta_{23})R_{31}(\theta_{13}, \phi)R_{12}(\theta_{12})$ $\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -c_{12}s_{23}s_{13} - s_{12}c_{23}e^{-i\phi} & -s_{12}s_{23}s_{13} + c_{12}c_{23}e^{-i\phi} & s_{23}c_{13} \\ -c_{12}c_{23}s_{13} + s_{12}s_{23}e^{-i\phi} & -s_{12}c_{23}s_{13} - c_{12}s_{23}e^{-i\phi} & c_{23}c_{13} \end{pmatrix}$	$\begin{array}{ccccc} \theta_{12}/\theta_{23}/\theta_{13} & & \vartheta_{12}/\vartheta_{23}/\vartheta_{13} & & \\ (13.04^{+0.053}_{-0.059})^\circ & + & (33.96^{+2.430}_{-2.137})^\circ & = & (47.00^{+2.483}_{-2.196})^\circ \\ (2.37^{+0.081}_{-0.085})^\circ & + & (43.05^{+7.839}_{-5.834})^\circ & = & (45.42^{+7.920}_{-5.919})^\circ \\ (0.20^{+0.023}_{-0.020})^\circ & + & (7.27^{+2.012}_{-2.824})^\circ & = & (7.47^{+2.035}_{-2.844})^\circ \end{array}$
P2: $V = R_{12}(\theta_u)R_{23}(\theta, \phi)R_{12}^{-1}(\theta_d)$ $\begin{pmatrix} s_u s_d c + c_u c_d e^{-i\phi} & s_u c_d c - c_u s_d e^{-i\phi} & s_u s \\ c_u s_d c - s_u c_d e^{-i\phi} & c_u c_d c + s_u s_d e^{-i\phi} & c_u s \\ -s_d s & -c_d s & c \end{pmatrix}$	$\begin{array}{ccccc} \theta_u/\theta_d/\theta & & \vartheta_l/\vartheta_\nu/\vartheta & & \\ (4.95^{+0.363}_{-0.305})^\circ & + & (10.58^{+1.310}_{-3.261})^\circ & = & (15.53^{+1.637}_{-3.566})^\circ \\ (12.11^{-0.262}_{+0.065})^\circ & + & (33.96^{+2.430}_{-2.137})^\circ & = & (46.67^{+2.168}_{-2.072})^\circ \\ (2.38^{+0.081}_{-0.085})^\circ & + & (43.54^{+7.956}_{-6.099})^\circ & = & (45.92^{+8.037}_{-6.184})^\circ \end{array}$
P3: $V = R_{23}(\theta_d)R_{12}(\theta, \phi)R_{23}^{-1}(\theta_u)$ $\begin{pmatrix} c_\theta & s_\theta c_u & -s_\theta s_u \\ -s_\theta c_d & c_\theta c_d c_u + s_d s_u e^{-i\phi} & -c_\theta c_d s_u + s_d c_u e^{-i\phi} \\ s_\theta s_d & -c_\theta s_d c_u + c_d s_u e^{-i\phi} & c_\theta s_d s_u + c_d c_u e^{-i\phi} \end{pmatrix}$	$\begin{array}{ccccc} \theta_u/\theta_d/\theta & & \vartheta_l/\vartheta_\nu/\vartheta & & \\ (0.91^{+0.096}_{-0.083})^\circ & + & (12.86^{+2.538}_{-4.477})^\circ & = & (13.77^{+2.634}_{-4.560})^\circ \\ (2.22^{+0.019}_{-0.059})^\circ & + & (43.05^{+7.839}_{-5.834})^\circ & = & (45.27^{+7.858}_{-5.893})^\circ \\ (13.04^{+0.053}_{-0.059})^\circ & + & (34.63^{+2.758}_{-2.538})^\circ & = & (47.67^{+2.811}_{-2.597})^\circ \end{array}$
P4: $V = R_{23}(\theta_d)R_{12}(\theta, \phi)R_{31}^{-1}(\theta_u)$ $\begin{pmatrix} c_\theta c_u & s_\theta & -c_\theta s_u \\ -s_\theta c_d c_u + s_d s_u e^{-i\phi} & c_\theta c_d & s_\theta c_d s_u + s_d c_u e^{-i\phi} \\ s_\theta s_d c_u + c_d s_u e^{-i\phi} & -c_\theta s_d & -s_\theta s_d s_u + c_d c_u e^{-i\phi} \end{pmatrix}$	$\begin{array}{ccccc} \theta_u/\theta_d/\theta & & \vartheta_l/\vartheta_\nu/\vartheta & & \\ (0.21^{+0.023}_{-0.020})^\circ & + & (8.74^{+2.733}_{-3.516})^\circ & = & (8.95^{+2.756}_{-3.536})^\circ \\ (2.39^{+0.086}_{-0.088})^\circ & + & (43.05^{+7.839}_{-5.834})^\circ & = & (45.44^{+7.925}_{-5.922})^\circ \\ (13.04^{+0.053}_{-0.059})^\circ & + & (33.65^{+2.189}_{-1.935})^\circ & = & (46.69^{+2.242}_{-1.994})^\circ \end{array}$
P5: $V = R_{31}(\theta_d)R_{23}(\theta_u, \phi)R_{12}^{-1}(\theta)$ $\begin{pmatrix} -s_\theta s_d s_u + c_\theta c_u e^{-i\phi} & -c_\theta s_d s_u - s_\theta c_u e^{-i\phi} & c_d s_u \\ s_\theta c_d & c_\theta c_d & s_d \\ -s_\theta s_d c_u - c_\theta s_u e^{-i\phi} & -c_\theta s_d c_u + s_\theta s_u e^{-i\phi} & c_d c_u \end{pmatrix}$	$\begin{array}{ccccc} \theta_u/\theta_d/\theta & & \vartheta_l/\vartheta_\nu/\vartheta & & \\ (0.21^{+0.023}_{-0.020})^\circ & + & (9.90^{+4.622}_{-4.326})^\circ & = & (10.11^{+4.645}_{-4.346})^\circ \\ (2.38^{+0.081}_{-0.085})^\circ & + & (42.62^{+7.354}_{-5.537})^\circ & = & (45.00^{+7.435}_{-5.622})^\circ \\ (13.05^{+0.054}_{-0.059})^\circ & + & (33.96^{+2.430}_{-2.137})^\circ & = & (47.01^{+2.484}_{-2.196})^\circ \end{array}$
P6: $V = R_{12}(\theta)R_{31}(\theta_u, \phi)R_{23}^{-1}(\theta_d)$ $\begin{pmatrix} c_\theta c_u & c_\theta s_d s_u + s_\theta c_d e^{-i\phi} & c_\theta c_d s_u - s_\theta s_d e^{-i\phi} \\ -s_\theta c_u & -s_\theta s_d s_u + c_\theta c_d e^{-i\phi} & -s_\theta c_d s_u - c_\theta s_d e^{-i\phi} \\ -s_u & s_d c_u & c_d c_u \end{pmatrix}$	$\begin{array}{ccccc} \theta_u/\theta_d/\theta & & \vartheta_l/\vartheta_\nu/\vartheta & & \\ (0.50^{+0.006}_{-0.015})^\circ & + & (22.41^{+4.993}_{-3.818}) & = & (22.91^{+4.999}_{-3.833})^\circ \\ (2.33^{+0.083}_{-0.085})^\circ & + & (37.99^{+7.102}_{-5.080})^\circ & = & (40.32^{+7.185}_{-5.165})^\circ \\ (13.04^{+0.053}_{-0.059})^\circ & + & (26.39^{-1.164}_{-0.021})^\circ & = & (39.43^{-1.111}_{-0.080})^\circ \end{array}$
P7: $V = R_{31}(\theta_u)R_{12}(\theta, \phi)R_{31}^{-1}(\theta_d)$ $\begin{pmatrix} c_\theta c_u c_d + s_u s_d e^{-i\phi} & s_\theta c_u & -c_\theta c_u s_d + s_u c_d e^{-i\phi} \\ -s_\theta c_d & c_\theta & s_\theta s_d \\ -c_\theta s_u c_d + c_u s_d e^{-i\phi} & -s_\theta s_u & c_\theta s_u s_d + c_u c_d e^{-i\phi} \end{pmatrix}$	$\begin{array}{ccccc} \theta_u/\theta_d/\theta & & \vartheta_l/\vartheta_\nu/\vartheta & & \\ (10.22^{+0.315}_{-0.320})^\circ & + & (45.62^{+1.233}_{-1.268})^\circ & = & (55.84^{+1.537}_{-1.577})^\circ \\ (10.42^{+0.304}_{-0.320})^\circ & + & (58.92^{+5.037}_{-3.769})^\circ & = & (69.34^{+5.341}_{-4.089})^\circ \\ (13.26^{+0.067}_{-0.073})^\circ & + & (52.69^{+6.791}_{-5.274})^\circ & = & (65.95^{+6.858}_{-5.347})^\circ \end{array}$
P8: $V = R_{12}(\theta)R_{23}(\theta_d, \phi)R_{31}(\theta_u)$ $\begin{pmatrix} -s_\theta s_d s_u + c_\theta c_u e^{-i\phi} & s_\theta c_d & s_\theta s_d c_u + c_\theta s_u e^{-i\phi} \\ -c_\theta s_d s_u - s_\theta c_u e^{-i\phi} & c_\theta c_d & c_\theta s_d c_u - s_\theta s_u e^{-i\phi} \\ -c_d s_u & -s_d & c_d c_u \end{pmatrix}$	$\begin{array}{ccccc} \theta_u/\theta_d/\theta & & \vartheta_l/\vartheta_\nu/\vartheta & & \\ (0.50^{+0.006}_{-0.016})^\circ & + & (27.75^{+8.734}_{-5.863})^\circ & = & (28.25^{+8.740}_{-5.879})^\circ \\ (2.33^{+0.083}_{-0.085})^\circ & + & (34.49^{+4.169}_{-3.562})^\circ & = & (36.82^{+4.252}_{-3.647})^\circ \\ (13.06^{+0.054}_{-0.060})^\circ & + & (42.43^{+6.631}_{-4.590})^\circ & = & (55.49^{+6.685}_{-4.650})^\circ \end{array}$
P9: $V = R_{31}(\theta_u)R_{12}(\theta, \phi)R_{23}(\theta_d)$ $\begin{pmatrix} c_\theta c_u & s_\theta c_d c_u - s_d s_u e^{-i\phi} & s_\theta s_d c_u + c_d s_u e^{-i\phi} \\ -s_\theta & c_\theta c_d & c_\theta s_d \\ -c_\theta s_u & -s_\theta c_d s_u - s_d c_u e^{-i\phi} & -s_\theta s_d s_u + c_d c_u e^{-i\phi} \end{pmatrix}$	$\begin{array}{ccccc} \theta_u/\theta_d/\theta & & \vartheta_l/\vartheta_\nu/\vartheta & & \\ (0.51^{+0.007}_{-0.016})^\circ & + & (24.86^{+5.223}_{-4.236})^\circ & = & (25.37^{+5.230}_{-4.252})^\circ \\ (2.43^{+0.084}_{-0.088})^\circ & + & (48.17^{+8.282}_{-6.463})^\circ & = & (50.60^{+8.366}_{-6.551})^\circ \\ (13.04^{+0.053}_{-0.059})^\circ & + & (24.09^{-2.114}_{+0.737})^\circ & = & (37.13^{-2.061}_{+0.678})^\circ \end{array}$

and $(3, 2)$ entries in the standard parametrization. Thus the moduli of the mixing matrix elements can be obtained,

$$|V_{\rm PMNS}| = \begin{pmatrix} 0.822\,795^{-0.0283}_{+0.0244} & 0.554\,083^{+0.0314}_{-0.0284} & 0.126\,491^{+0.0348}_{-0.0490} \\ 0.408\,176^{-0.0340}_{+0.0117} & 0.606\,129^{-0.0983}_{+0.0705} & 0.677\,159^{+0.0886}_{-0.0742} \\ 0.381\,303^{+0.0790}_{-0.0624} & 0.566\,223^{+0.0584}_{-0.0523} & 0.724\,883^{-0.1023}_{+0.1023} \end{pmatrix}. \quad (8)$$

With the relations between the moduli of mixing matrix elements and mixing angles in each of the parametrizations, we can get all the mixing angles. To be explicit, we explain how to get mixing angles in FX parametrization from the moduli of quark and lepton mixing angles. For the FX parametrization, $\theta_u = \arctan|V_{ub}/V_{cb}|$,

$\theta = \arccos|V_{tb}|$, and $\theta_d = \arctan|V_{td}/V_{ts}|$ hold. All the moduli of the CKM matrix elements have been illustrated in Eq. (5); accordingly θ_u, θ, and θ_d in FX parametrization can be obtained. So are the cases for the PMNS matrices.

The numerical results of quark and lepton mixing angles as well as their QLC relations are listed in the right column of Table I. It is obvious from the table that the QLC relations approximately hold in P1, P2, P3, P4, and P5 parametrizations but suffer from large deviation in the remaining four parametrizations. Thus the QLC relations are indeed parametrization dependent. Note that the parametrization proposed in Ref. [22] is equivalent to the standard parametrization when ignoring the CP-violating phase; hence the QLC relations will also hold in this parametrization. Furthermore, the distinct feature for those parametrizations accommodating the QLC relations is that they all have a simple form in their (1, 3) entries.

III. CONDITIONS FOR THE EXACT QLC RELATIONS TRANSFORMATION

Since the QLC relations depend on the forms of parametrizations, the exploration of those parametrizations that ensure this relation is necessary and pressing. Based on the hypothesis that the QLC relations hold in the standard parametrization,

$$\theta_{12} + \vartheta_{12} = 45°, \qquad \theta_{23} + \vartheta_{23} = 45°, \tag{9}$$

we aim to see under what conditions this is still the case for the corresponding angles in the other parametrizations. Taking the FX parametrization (i.e. P2 in Table I) as an example, we have

$$\begin{aligned} \tan\theta_d &= \frac{|V_{td}|}{|V_{ts}|} = \frac{|c_{\theta_{12}}c_{\theta_{23}}s_{\theta_{13}} - s_{\theta_{12}}s_{\theta_{23}}e^{-i\phi}|}{|s_{\theta_{12}}c_{\theta_{23}}s_{\theta_{13}} + c_{\theta_{12}}s_{\theta_{23}}e^{-i\phi}|}, \\ \cos\theta &= |V_{tb}| = |c_{\theta_{23}}c_{\theta_{13}}|. \end{aligned} \tag{10}$$

We first consider $\theta_d + \vartheta_\nu$, which is corresponding to the first relation in Eq. (9):

$$\tan(\theta_d + \vartheta_\nu) = \frac{\tan\theta_d + \tan\vartheta_\nu}{1 - \tan\theta_d\tan\vartheta_\nu} = \frac{\mathcal{C}\mathcal{A} + \mathcal{D}\mathcal{B}}{\mathcal{D}\mathcal{A} - \mathcal{C}\mathcal{B}}, \tag{11}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are defined as

$$\begin{aligned} \mathcal{A} &= |s_{\vartheta_{12}}c_{\vartheta_{23}}s_{\vartheta_{13}} + c_{\vartheta_{12}}s_{\vartheta_{23}}e^{-i\varphi}|, \\ \mathcal{B} &= |c_{\vartheta_{12}}c_{\vartheta_{23}}s_{\vartheta_{13}} + s_{\vartheta_{12}}s_{\vartheta_{23}}e^{-i\varphi}|, \\ \mathcal{C} &= |c_{\theta_{12}}c_{\theta_{23}}s_{\theta_{13}} - s_{\theta_{12}}s_{\theta_{23}}e^{-i\phi}|, \\ \mathcal{D} &= |s_{\theta_{12}}c_{\theta_{23}}s_{\theta_{13}} + c_{\theta_{12}}s_{\theta_{23}}e^{-i\phi}|. \end{aligned} \tag{12}$$

Using the QLC relation in Eq. (9), one can get $\mathcal{A}$ and $\mathcal{B}$ expressed in the form of the standard parametrization:

$$\begin{aligned} \mathcal{A} &= \tfrac{1}{2}|(c_{\theta_{12}} - s_{\theta_{12}})(c_{\theta_{23}} + s_{\theta_{23}})s_{\vartheta_{13}} \\ &\quad + (c_{\theta_{12}} + s_{\theta_{12}})(c_{\theta_{23}} - s_{\theta_{23}})e^{-i\varphi}|, \\ \mathcal{B} &= \tfrac{1}{2}|(c_{\theta_{12}} + s_{\theta_{12}})(c_{\theta_{23}} + s_{\theta_{23}})s_{\vartheta_{13}} \\ &\quad + (c_{\theta_{12}} - s_{\theta_{12}})(c_{\theta_{23}} - s_{\theta_{23}})e^{-i\varphi}|. \end{aligned} \tag{13}$$

After substituting the above expressions of $\mathcal{A}$ and $\mathcal{B}$ into Eq. (11), we find that it is hard to deduce any useful conclusion from it. Furthermore, we assume that the corresponding smallest angles in the standard parametrization for quark and lepton mixings are vanishing, i.e. $\theta_{13} = \vartheta_{13} = 0°$, and thus

$$\begin{aligned} \mathcal{A} &= \tfrac{1}{2}|(c_{\theta_{12}} + s_{\theta_{12}})(c_{\theta_{23}} - s_{\theta_{23}})|, \\ \mathcal{B} &= \tfrac{1}{2}|(c_{\theta_{12}} - s_{\theta_{12}})(c_{\theta_{23}} - s_{\theta_{23}})|, \\ \mathcal{C} &= |s_{\theta_{12}}s_{\theta_{23}}|, \\ \mathcal{D} &= |c_{\theta_{12}}s_{\theta_{23}}|, \end{aligned} \tag{14}$$

which leads to a remarkable result,

$$\tan(\theta_d + \vartheta_\nu) = 1, \tag{15}$$

from which the QLC relation in the FX parametrization $\theta_d + \vartheta_\nu = 45°$ exactly holds.

Now we turn to consider the second relation of Eq. (10). With the help of the QLC relation for θ_{23} and ϑ_{23} and the assumption of the vanishing smallest mixing angles $\theta_{13} = \vartheta_{13} = 0°$, one can obtain

$$\cos(\theta + \vartheta) = \frac{\sqrt{2}}{2}. \tag{16}$$

Again, the QLC relation holds for the FX parametrization in this situation. Namely,

$$\theta_d + \vartheta_\nu = 45° \quad \text{and} \quad \theta + \vartheta = 45°. \tag{17}$$

In fact, under the condition of $\theta_{13} = \vartheta_{13} = 0°$, the conclusion that the QLC relations hold in these parametrizatons can be exactly obtained. For example, from Eq. (10) we can easily get $\tan\theta_d = |\tan\theta_{12}|$ and $\cos\theta = |\cos\theta_{23}|$ in the FX parametrization if $\theta_{13} = \vartheta_{13} = 0°$. So is the case in the lepton sector, i.e. $\tan\vartheta_\nu = |\tan\vartheta_{12}|$ and $\cos\vartheta = |\cos\vartheta_{23}|$. Hence, the QLC relations hold in the FX parametrization. And the same conclusion can also be obtained for P3, P4, and P5 parametrizations in a similar procedure. The reason is simple that these five parametrizations are essentially equivalent to one another in the $\theta_{13} = \vartheta_{13} = 0°$ limit.

IV. ON THE STABILITY OF QLC RELATIONS RG RUNNING

As proposed in many papers, the quark-lepton symmetry implied by the QLC relations means that physics responsible for these relations should be realized at some scales, which might be the quark-lepton unification scale, Λ_{GUT}, or even higher scales, and the RG effects have been discussed in the framework of the standard parametrization [12,23]. Since there are specific advantages in the FX parametrization for the study of fermion mass matrices and B-meson physics [17], it is useful to examine the sensitivity of the QLC relations to the RG effects in this parametrization. And it has been shown that the RG equations of quark and lepton mixing angles have a particularly simple form in the FX parametrization [24,25]. Assume that QLC relations hold exactly at the scale M_Z in this parametrization:

$$\theta_d + \vartheta_\nu = 45°, \qquad \theta + \vartheta = 45°, \tag{18}$$

and thus

$$\dot{\theta}_d + \dot{\vartheta}_\nu = 0, \qquad \dot{\theta} + \dot{\vartheta} = 0, \tag{19}$$

where $\dot{\theta} = \frac{d\theta}{dt}$ with $t \equiv \ln(\mu/M_Z)$. We already have the RG equations of three quark mixing angles [24] and three Dirac neutrino mixing angles [25] in FX Parametrization:

$$\begin{aligned}
\dot{\theta}_u &= -\frac{1}{32\pi^2} C y_b^2 \sin 2\theta_u \sin^2\theta, \\
\dot{\theta}_d &= -\frac{1}{32\pi^2} C y_t^2 \sin 2\theta_d \sin^2\theta, \\
\dot{\theta} &= -\frac{1}{32\pi^2} C (y_b^2 + y_t^2) \sin 2\theta, \\
\dot{\vartheta}_l &= +\frac{C y_\tau^2}{16\pi^2} c_\nu s_\nu c_\vartheta c_\varphi (\xi_{13} - \xi_{23}), \\
\dot{\vartheta}_\nu &= +\frac{C y_\tau^2}{16\pi^2} c_\nu s_\nu [s_\vartheta^2 \xi_{12} + c_\vartheta^2 (\xi_{13} - \xi_{23})], \\
\dot{\vartheta} &= +\frac{C y_\tau^2}{16\pi^2} c_\vartheta s_\vartheta (s_\nu^2 \xi_{13} + c_\nu^2 \xi_{23}),
\end{aligned} \tag{20}$$

where $C = -1.5(+1)$ in the SM (MSSM), $\xi_{ij} \equiv (y_i^2 + y_j^2)/(y_i^2 - y_j^2)$, and y_α, y_a, and y_i ($\alpha = \tau$, $a = b$, t, and $i = 1, 2, 3$) stand, respectively, for the eigenvalues of the Yukawa coupling matrices of charged leptons, quarks, and neutrinos. In the case of the SM, the Yukawa couplings $y_i = \frac{m_i}{v} (i = 1, 2, 3)$, where the Higgs vacuum expection value (VEV) v is 174 GeV. In the MSSM, $m_\alpha = y_\alpha v \sin\beta$, $m_\gamma = y_\gamma v \cos\beta$ ($\alpha = u$, c, t, $\gamma = d$, s, b, e, μ, τ), where $\tan\beta$ is the ratio of two Higgs VEV's.

Some qualitative comments on the main features of Eq. (20) are in order.

(a) For the RG equations of quark flavor mixing angles in both the SM and MSSM, noticing that the value of θ is very small, we can safely claim that the RG running effects of θ_u, θ_d, and θ are highly suppressed. As a result, three quark mixing angles in FX parametrization will not change a lot under the RG running.

(b) In the lepton sector in the SM case, the derivatives of three mixing angles are proportional to $y_\tau^2 = (\frac{m_\tau}{v})^2 \simeq 10^{-4}$ [26]. Notice that $\xi_{ij} = -\frac{m_i^2 + m_j^2}{\Delta m_{ji}^2}$, with $\Delta m_{ji}^2 = m_j^2 - m_i^2$, $\Delta m_{21}^2 \simeq 7.7 \times 10^{-5}$ eV^2, and $|\Delta m_{32}^2| \approx |\Delta m_{31}^2| \approx 2.4 \times 10^{-3}$ eV^2 [21]. Thus the most sensitive angle to radiative corrections is ϑ_ν whose RG equation is the only one that consists of ξ_{12}. But we still cannot expect large running effects on θ_ν because the loop factor $1/16\pi^2$ makes the derivative even smaller. While in the MSSM case, where $y_\tau = \frac{m_\tau}{v \cos\beta}$, the RG running effects could be enhanced when $\tan\beta$ is significantly large.

As a result, if we sum up the derivatives of the corresponding mixing angles of quark and lepton sectors in Eq. (20), we can conclude that in the SM case the QLC relations are essentially stable under the RG running and in the MSSM case these relations might become unstable only when $\tan\beta$ is sufficiently large.

V. SUMMARY

To understand the deep meaning of the quark and lepton world, the quark-lepton symmetry topic has drawn a lot of attention in recent years. Among many of the aspects that imply the symmetry and unification in quark and lepton sectors, the QLC relations between the mixing angles of the CKM and PMNS matrices have been considered very interesting and suggestive. In this paper, we have calculated the QLC relations for each of the angle-phase parametrizations and find that these relations are parametrization dependent. Furthermore, the distinct feature of those parametrizations that can approximately accommodate the QLC relations is that they all have a simple form in the (1, 3) entries. Then based on the assumption that the QLC relations hold exactly in the standard parametrization, we make an exploration in the FX parametrization and get the conclusion that these relations can also hold as long as the smallest mixing angle θ_{13} is vanishing. Finally, we make clear that the QLC relations can essentially stay stable under the RG running effects in the SM and MSSM unless the value of $\tan\beta$ is sufficiently large.

Since we know from the above analysis that QLC relations are parametrization dependent, maybe this property of complementarity can be regarded as a criterion for picking up favorable parametrizations from this perspective. If so, the still unconfirmed lepton mixing pattern can also be deduced to some degree. For those parametrizations that accommodate QLC relations, they make the underlying symmetry between quark and lepton worlds more transparent. In fact, according to grand unified theories constraints for the fermion mixing matrices, QLC relations can be predicted in a good agreement with the experimental data, and the details have been explored in the previous reviews.

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