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Null cosmological singularities and free strings

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We continue exploring free strings in the background of null Kasner-like cosmological singularities, following K. Narayan, arXiv:0904.4532. We study the free string Schrodinger wave functional along the lines of K. Narayan, arXiv:0807.1517. We find the wave functional to be nonsingular in the vicinity of singularities whose Kasner exponents satisfy certain relations. We compare this with the description in other variables. We then study certain regulated versions of these singularities where the singular region is replaced by a substringy but nonsingular region and study the string spectra in these backgrounds. The string modes can again be solved for exactly, giving some insight into how string oscillator states get excited near the singularity.

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I. INTRODUCTION

In this article, we continue exploring free string propagation in the background of null cosmological singularities, following [1], and motivated by [2–5], as well as some earlier investigations, e.g. [6–26].

Time-dependent generalizations of AdS/CFT where the bulk contains null or spacelike cosmological singularities were studied in [2–4], with a nontrivial dilaton e^{Φ} that vanishes at the location of the cosmological singularity, the curvatures behaving as $R_{MN} \sim \partial_M\Phi\partial_N\Phi$. The gauge theory duals are $\mathcal{N} = 4$ Super Yang-Mills theories with a time-dependent gauge coupling $g_{\rm YM}^2 = e^{\Phi}$, and [2–4] describe aspects of the dual descriptions of the bulk cosmological singularities. From the bulk point of view, supergravity breaks down and possible resolutions of the cosmological singularity stem from stringy effects. Indeed noting $\alpha' \sim \frac{1}{\sqrt{g_s N}} = \frac{1}{g_{\rm YM}\sqrt{N}}$ from the usual AdS/CFT dictionary and extrapolating naively to these time-dependent cases with a nontrivial dilaton, we have $\alpha' \sim \frac{1}{e^{\Phi/2}\sqrt{N}}$ indicating vanishing effective tension for stringy excitations, when $e^{\Phi} \to 0$ near the singularity. In general, we expect that stringy effects are becoming important near the bulk singularity, corresponding to possible gauge coupling effects in the dual gauge theory. It is therefore interesting to understand world sheet string effects in the vicinity of the singularity. Owing to the technical difficulties with string quantization in an AdS background with RR flux, we would like to look for simpler, purely gravitational backgrounds as toy models whose singularity structure shares some essential features with the backgrounds in the AdS/CFT investigations.

In [1], we described purely gravitational spacetimes, with no other background fields turned on, containing null Kasner-like cosmological singularities where tidal forces diverge. The Kasner exponents satisfy certain algebraic relations stemming from the supergravity equations of motion. These are related by a coordinate transformation to anisotropic plane waves with singularities. We then studied the free string spectra in these backgrounds, aided by the fact that the string mode functions can be exactly solved for in these backgrounds. Using the mode asymptotics in the near singularity region, the world sheet Hamiltonian in light cone gauge can be simplified enabling a detailed analysis of the string spectrum.

The Schrodinger wave functional was found to be a useful diagnostic for the response of gauge theories to time-dependent coupling sources [4]: these theories are dual to AdS cosmologies with spacelike singularities, in particular, with dilaton profiles vanishing as $e^{\Phi} \sim t^p$, $p > 0$, with t being a timelike time coordinate. Among other things, [4] found that the quantum mechanical wave function of the system, describing its response to the external time dependence, in general acquires a time-dependent phase factor. For $p \geq 1$, this phase becomes wildly oscillating and diverges as $t \to 0^-$. As a result, the wave function of the system (in the Schrodinger picture) does not have a well-defined limit as $t \to 0^-$. In contrast for $p < 1$, the phase factor does not diverge and the wave function has a well-defined limit as $t \to 0^-$. The energy diverges in both cases. By contrast, null singularities with $e^{\Phi} \sim (x^+)^p$ appear to be better defined [2], with a well-defined "near singularity" wave function and no divergences[1]. These findings could acquire possible modifications due to gauge coupling renormalization effects, as discussed in [4].

Motivated by this, in the present paper, we continue investigating null singularities and the string world sheet description, following [1]. Along the lines of the Schrodinger wave functional analysis [4] described above, we study the Schrodinger equation for the string wave

[1]Indeed, in terms of the redefined gauge fields $\tilde{A}_\mu = e^{-\Phi/2}A_\mu$, the gauge theory interaction terms become unimportant near the singularity [2] and the light cone Hamiltonian containing simply the kinetic terms gives rise to a free light cone Schrodinger equation for the gauge theory wave functional. Operators involving $\tilde{A}_\mu$ are likely to not have local bulk duals as argued in [2], again suggesting stringy effects.

functions using the free string Hamiltonian to study the response of strings to null Kasner-like cosmological singularities, thereby gaining insights into string propagation across these null singularities. We find that for singularities whose Kasner exponents satisfy certain relations, the wavefunction has a well-defined limit near the singularity, as $x^+ \to 0^-$. We compare this with the corresponding description in other variables. We then describe certain regulated versions of these spacetimes where the singular region is excised by a substringy but nonsingular region (albeit by a nonanalytic regulator), and study the string spectra in these regulated regions. The string modes can luckily be again solved for exactly, giving some insight into how string oscillator states turn on near the singularity. In particular, comparing the (instantaneous) masses of string oscillator states with the local energy (curvature) scales in the regulated near-singularity region, we find that a finite number of oscillator states are light for a finite regulator.

In Sec. II, we review some key points of [1], and then discuss the Schrodinger wave functional in Sec. III. In Sec. IV, we describe various dimensional properties of these spacetimes that make this description consistent with the no-scale property of plane wave spacetimes (that these are related to by a coordinate transformation). Finally in Sec. V, we discuss some regulated versions of these singularities and string propagation in them, closing with a discussion in Sec. VI.

II. REVIEWING NULL SINGULARITIES AND STRINGS

We are interested in purely gravitational spacetime backgrounds (for simplicity) that have a big-crunch (-bang) type of singularity at some value of the lightlike time coordinate x^+. Thus consider

$$ds^2 = e^{f(x^+)}(-2dx^+dx^- + dx^idx^i) + e^{h(x^+)}dx^mdx^m, \quad (1)$$

with two null scale factors (and all other backgrounds fields vanishing). It is straightforward to generalize this to multiple scale factors $e^{h_m(x^+)}$. Simple classes of null Kasner-like cosmological singularities arise in the vicinity of $x^+ = 0$ with

$$ds^2 = (x^+)^a(-2dx^+dx^- + dx^idx^i) + (x^+)^b dx^mdx^m, \quad a > 0, \quad (2)$$

where $i = 1, 2$, $m = 3, \ldots, D-2$. A solution with $a < 0$ can be transformed to one with $a > 0$ by redefining $y^+ = \frac{1}{x^+}$. These are Ricci-flat solutions to the Einstein equations if $R_{++} = 0$, i.e.

$$\frac{1}{2}(f')^2 - f'' + \frac{D-4}{4}(-2h'' - (h')^2 + 2f'h') = 0$$
$$\Rightarrow a^2 + 2a + \frac{D-4}{2}(-b^2 + 2b + 2ab) = 0. \quad (3)$$

The first equation, in terms of the scale factors e^f, e^h, shows that the curvature for the 4D scale factor e^f is sourced by those for the "internal space" scale factor e^h: indeed the h (and more generally h_m for multiple scale factors) are the analogs of the dilaton scalar in the AdS/CFT cosmological context [2–4] where the corresponding equation was $R^{(4)}_{++} = \frac{1}{2}(\partial_+\Phi)^2$. That is, the kinetic terms $(\partial_+ h_m)^2$ (and related cross-terms) play the role of the dilaton in driving the singular behavior of the 4D part of the spacetime.

For any $b \neq a$, these give solutions $2a = -2 - (D-4)b \pm \sqrt{4 + (D-4)(D-2)b^2}$. Requiring $a > 0$ means we take the positive radical. Requiring unambiguous analytic continuation from $x^+ < 0$ to $x^+ > 0$ across the singularity means a, b are even integers: this is more restrictive but such solutions do exist.[2] While no curvature invariants diverge in these null backgrounds, the Riemann components R_{+I+I}, $I = i, m$, are nonvanishing giving diverging tidal forces: analyzing the deviation of null geodesic congruences, we find the accelerations a^i, $a^m \sim \frac{1}{(x^+)^{2a+2}}$. We refer to [1] for details on the various properties of these spacetimes and the Kasner exponents.

We would like to study string propagation in these backgrounds. Starting with the closed string world sheet action $S = -\int \frac{d\tau d\sigma}{4\pi\alpha'}\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X)$, we use light cone gauge $x^+ = \tau$ and set $h_{\tau\sigma} = 0$, with $E(\tau, \sigma) = \sqrt{-\frac{h_{\sigma\sigma}}{h_{\tau\tau}}}$, as in [27] (see also [28]), obtaining

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\Big(-Eg_{IJ}\partial_\tau X^I\partial_\tau X^J + \frac{1}{E}g_{IJ}\partial_\sigma X^I\partial_\sigma X^J - 2Eg_{+-}\partial_\tau X^-\Big). \quad (4)$$

Then setting the momentum conjugate to X^- to a τ-independent constant $p_- = \frac{Eg_{+-}}{2\pi\alpha'} = -\frac{1}{2\pi\alpha'}$ by a τ-independent σ-reparametrization invariance (not fixed by the gauge fixing above), we obtain $E = -\frac{1}{g_{+-}}$, giving

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma((\partial_\tau X^i)^2 - e^{2f(\tau)}(\partial_\sigma X^i)^2 + e^{h(\tau)-f(\tau)}(\partial_\tau X^m)^2 - e^{h(\tau)+f(\tau)}(\partial_\sigma X^m)^2), \quad (5)$$

containing only the physical transverse degrees of freedom $X^I \equiv X^i$, X^m, of the string. We can now calculate the Hamiltonian $H[X^-, p_-, X^I, \Pi^I]$ [using $E = \frac{2\pi\alpha' p_-}{g_{+-}}$ in (4)], and solve for X^- using $\partial_\tau X^- = \frac{\partial H}{\partial p_-}$.

The light cone gauge quantization of strings in these backgrounds is aided by the fact that the classical string

[2]From eq. (10) of [1], we have $(a, b) = (0, 2), (44, -2), (44, 92), (2068, -92)\ldots$, for $D = 26$ (bosonic string), and $(a, b) = (0, 2), (12, -2), (12, 28), (180, -28), (180, 390), \ldots$, for $D = 10$ (superstring).

modes here can be exactly solved for from the world sheet equations of motion following from (5): we have the mode functions

$$f_n^I(\tau) = \sqrt{n\tau^{d_I}}\left(c_{n1}^I J_{(d_I/2a+2)}\left(\frac{n\tau^{a+1}}{a+1}\right) + c_{n2}^I Y_{(d_I/2a+2)}\left(\frac{n\tau^{a+1}}{a+1}\right)\right), \quad (6)$$

where $d_I = 1,\ 2\nu$, for $I = i,\ m$, respectively, $\nu = \frac{a+1-b}{2}$ and $c_{n1}^I,\ c_{n2}^I$ are constants. Using the basis modes $f_n^I(\tau)e^{in\sigma}$, we can then mode expand the world sheet coordinate fields $X^I(\tau,\sigma)$. Then by the usual procedure to simplify the Hamiltonian using the mode expansion, we obtain

$$\begin{aligned} H = {} & \frac{1}{2\alpha'}((\dot{X}_0^i)^2 + \tau^{b-a}(\dot{X}_0^m)^2) + \frac{1}{2\alpha'}\sum_n |k_n^i|^2((\{a_n^i, a_{-n}^i\} \\ & + \{\tilde{a}_n^i, \tilde{a}_{-n}^i\})(|\dot{f}_n^i|^2 + n^2\tau^{2a}|f_n^i|^2) - \{a_n^i, \tilde{a}_n^i\}((\dot{f}_n^i)^2 \\ & + n^2\tau^{2a}(f_n^i)^2) - \{a_{-n}^i, \tilde{a}_{-n}^i\}((\dot{f}_n^{i*})^2 + n^2\tau^{2a}(f_n^{i*})^2)) \\ & + \frac{1}{2\alpha'}\sum_n |k_n^m|^2((\{a_n^m, a_{-n}^m\} + \{\tilde{a}_n^m, \tilde{a}_{-n}^m\})(\tau^{b-a}|\dot{f}_n^m|^2 \\ & + n^2\tau^{b+a}|f_n^m|^2) - \{a_n^m, \tilde{a}_n^m\}(\tau^{b-a}(\dot{f}_n^m)^2 + n^2\tau^{b+a}(f_n^m)^2) \\ & - \{a_{-n}^m, \tilde{a}_{-n}^m\}(\tau^{b-a}(\dot{f}_n^{m*})^2 + n^2\tau^{b+a}(f_n^{m*})^2)). \end{aligned} \quad (7)$$

We now introduce a cutoff null surface at $x^+ \equiv \tau = \tau_c$, akin to a stretched horizon outside a black hole horizon, and will evaluate the various mode asymptotics on this constant null-time surface. The mode function asymptotics show distinct behavior for the low lying (small oscillation number $n \lesssim n_c \sim \frac{1}{\tau_c^{a+1}}$) and highly stringy (large $n \gg n_c$) modes in the near singularity region. From the mode functions, we find the asymptotics as $\tau = \tau_c \to 0$,

$$f_n^I \to \lambda_{n0}^I + \lambda_{n\tau}^I \tau_c^{d_I}, \qquad \frac{n\tau_c^{a+1}}{(a+1)} \lesssim 1,$$

$$f_n^i \sim \frac{1}{\tau_c^{a/2}} e^{-in\tau_c^{a+1}/l(a+1)}, \qquad f_n^m \sim \frac{1}{\tau_c^{b/2}} e^{-in\tau_c^{a+1}/(a+1)}, \qquad \frac{n\tau_c^{a+1}}{(a+1)} \gg 1, \quad (8)$$

where the $\lambda_{n0}^I,\ \lambda_{n\tau}^I$ are constant coefficients arising from the Bessel function expansions and involving c_{n1}^I, c_{n2}^I. The highly stringy modes (positive frequency in the second line, with $c_{n1}^I = 1,\ c_{n2}^I = -i$) are essentially ultrashort wavelength relative to the cutoff length scale τ_c. Note that this implies that for $\tau_c^{a+1} \gtrsim 1$, the $n = 1$ oscillator state is already "highly stringy". Note that for any nonzero if infinitesimal regularization τ_c, there exist such highly stringy modes, of sufficiently high n that are oscillatory. Details on the near singularity string spectrum can be found in [1]. Later (Sec. IV) we will revisit this being explicit about the length scales involved, to gain insight into how oscillator states turn on near the singularity.

III. WAVE FUNCTIONS AND THE SCHRODINGER PICTURE

Here we will use the Schrodinger equation to describe wave functions for near singularity string states and gain insights into free string propagation across the singularity.[3] Our analysis of the general Schrodinger wave functional has parallels with that of the Schrodinger wave functionals for gauge theory duals of AdS cosmologies with spacelike singularities in [4].

A. Wave functions, oscillator states and the Schrodinger equation

From [1] (Sec. III), the string world sheet Hamiltonian in terms of the oscillator operators for the low-lying and highly stringy modes is

$$\begin{aligned} H_< = {} & \pi\alpha'((p_{i0})^2 + \tau^{a-b}(p_{m0})^2) + \sum_n \frac{\pi}{2(a+1)n^2} \\ & \times\left(\frac{1}{|c_{n0}^i|}(b_{n\tau}^{i\dagger}b_{n\tau}^i + n^2\tau^{2a}b_{n0}^{i\dagger}b_{n0}^i\right) \\ & + \frac{1}{|c_{n0}^m|}((2\nu)^2\tau^{a-b}b_{n\tau}^{m\dagger}b_{n\tau}^m + n^2\tau^{b+a}b_{n0}^{m\dagger}b_{n0}^m)), \\ H_> \sim {} & \tau^a \sum_{I;n\gg n_c} \frac{1}{a+1}(a_{-n}^I a_n^I + \tilde{a}_{-n}^I\tilde{a}_n^I + n), \end{aligned} \quad (9)$$

with the b_n^I-oscillators defined as linear combinations $b_{n0}^I = \lambda_{n0}^I a_n^I - \lambda_{n0}^{I*}\tilde{a}_{-n}^I$, $b_{n\tau}^I = \lambda_{n\tau}^I a_n^I - \lambda_{n\tau}^{I*}\tilde{a}_{-n}^I$, of the creation-annihilation operators $a_n^I,\ \tilde{a}_n^I$, for the low-lying modes. This can be seen by analyzing the Hamiltonian (7) and using the mode asymptotics (8).

We see that the Hamiltonian does not mix the various operators for different I-directions, and also for different oscillator levels n. Thus we can write $H = \sum_I H_I$, where

$$\begin{aligned} H_i^< = {} & \pi\alpha'(p_{i0})^2 + \sum_{n\lesssim n_c} \frac{\pi}{2(a+1)|c_{n0}^i|n^2} \\ & \times (b_{n\tau}^{i\dagger}b_{n\tau}^i + n^2\tau^{2a}b_{n0}^{i\dagger}b_{n0}^i) \\ H_m^< = {} & \pi\alpha'\tau^{a-b}(p_{m0})^2 + \sum_{n\lesssim n_c} \frac{\pi}{2(a+1)|c_{n0}^m|n^2} \\ & \times ((2\nu)^2\tau^{a-b}b_{n\tau}^{m\dagger}b_{n\tau}^m + n^2\tau^{b+a}b_{n0}^{m\dagger}b_{n0}^m), \\ H_I^> \sim {} & \tau^a \sum_{I;n\gg n_c} \frac{1}{a+1}(a_{-n}^I a_n^I + \tilde{a}_{-n}^I\tilde{a}_n^I + n). \end{aligned} \quad (10)$$

Each H_I decouples into a contribution $H_I^<$ from the low-

[3] Our analysis is essentially from the bosonic parts of the string world sheet theory. Since the world sheet fermion terms are quadratic (with covariant derivatives) for these purely gravitational backgrounds, we expect that including them will not qualitatively change our results here.

lying modes ($n \lesssim n_c \sim \frac{1}{\tau_c^{a+1}}$) and another $H_I^>$ from the highly stringy modes ($n \gg n_c \sim \frac{1}{\tau_c^{a+1}}$).

In this free string limit, the general state $|\Psi\rangle$ then factorizes into a product of states $\prod_{I,<,>}|\Psi_I^<\rangle|\Psi_I^>\rangle$, decoupled both in the various I-directions as well as between the low-lying and highly stringy modes. The Schrodinger equation $i\frac{\partial}{\partial\tau}|\Psi\rangle = H|\Psi\rangle$ for the general state then factorizes into a set of equations for each I-direction as

$$i\frac{\partial}{\partial\tau}|\Psi_I\rangle = H_I|\Psi_I\rangle \tag{11}$$

In the near singularity limit $\tau \to 0$, we see that these equations simplify and become (keeping only the leading terms)

$$\begin{aligned} i\frac{\partial}{\partial\tau}|\Psi_i^<\rangle &= H_i^<|\Psi_i^<\rangle \sim \left(\pi\alpha'(p_{i0})^2 + \sum_{n\lesssim n_c}\frac{\pi}{2(a+1)|c_{n0}^i|n^2}b_{n\tau}^{i\dagger}b_{n\tau}^i\right)|\Psi_i^<\rangle, \\ i\frac{\partial}{\partial\tau}|\Psi_m^<\rangle &= H_m^<|\Psi_m^<\rangle \sim \tau^{a-b}\left(\pi\alpha'(p_{m0})^2 + \sum_{n\lesssim n_c}\frac{(2\nu)^2\pi}{2(a+1)|c_{n0}^m|n^2}b_{n\tau}^{m\dagger}b_{n\tau}^m\right)|\Psi_m^<\rangle, \qquad b>0, \\ &\sim \tau^{a+b}\left(\sum_{n\lesssim n_c}\frac{\pi}{2(a+1)|c_{n0}^m|}b_{n0}^{m\dagger}b_{n0}^m\right)|\Psi_m^<\rangle, \qquad b<0, \\ i\frac{\partial}{\partial\tau}|\Psi_I^>\rangle &= H_I^<|\Psi_I^<\rangle \sim \tau^a\left(\sum_{I;n\gg n_c}\frac{1}{a+1}(a_{-n}^I a_n^I + \tilde{a}_{-n}^I\tilde{a}_n^I + n)\right)|\Psi_I^>\rangle. \end{aligned} \tag{12}$$

These equations can be recast as Schrodinger equations with time-independent Hamiltonians $i\partial_{\lambda_I}|\Psi_I\rangle = H_I'|\Psi\rangle$ (with $\partial_{\lambda_I}H_I = 0$) in terms of some time parameter λ_I depending on the particular modes in question. For instance, $\lambda_I^> = \int \tau^a d\tau = \frac{\tau^{a+1}}{a+1}$ for the highly stringy modes in the last line above. Thus we see that $|\Psi_i^<\rangle$ is well defined across $\tau \to 0$ as long this time-independent Schrodinger equation has a well-defined time parameter λ_I. From the form of these equations, we see that for both $b>0$, $b<0$, the states $|\Psi_m^<\rangle$ are well-defined as long as $\tau^{a+1-b} = \tau^{2\nu}$ is well-defined as $\tau \to 0$, i.e. if $2\nu \geq 0$.

The highly stringy modes are very high frequency and essentially do not see the time-dependence of the background at all, giving a free Schrodinger equation (in flat space effectively) in the last line of (12). Thus the states $|\Psi_I^>\rangle$ are well defined as $\tau \to 0$.

Let us now make a comment on a point particle propagating in these backgrounds, with action $S = \int d\tau \frac{1}{2}\xi(\tau)g_{IJ}\dot{x}^I\dot{x}^J$, where $\xi(\tau)$ is the worldline metric. Fixing lightcone gauge $x^+ = \tau$, the lightcone momentum $p_- = \frac{\xi(\tau)g_{+-}}{2}$ is conserved. This gives the conjugate momenta $p_I = \xi(\tau)g_{II}\dot{x}^I$ and the Hamiltonian $H = \frac{1}{2\xi(\tau)g_{II}}p_I^2 = \frac{1}{2p_-}(p_i^2 + \tau^{a-b}p_m^2)$. We can also solve for x^- using $\partial_\tau x^- = \frac{\partial H}{\partial p_-}$, using $\xi(\tau) = \frac{2p_-}{g_{+-}}$. The fact that this point particle Hamiltonian appears well-defined dovetails with the fact that the low-lying oscillator modes of the string have asymptotics similar to the zero mode (point particle). Note however that the spacetimes in question are singular only due to diverging tidal forces arising in congruences of null geodesics. This suggests that such a singularity might reflect in wave propagation of field modes in the near singularity region, rather than in single particle propagation. It would be interesting to explore this further.

B. The Schrodinger wave functional

In [1] (and reviewed earlier), we described a "nuts-and-bolts" approach to string propagation near these null Kasner-like singularities, by solving for the world sheet string mode functions, constructing the Hamiltonian and then the near singularity string spectrum. This then leads to the Schrodinger equation description of the string wave function in the previous subsection. Here we will describe a more general Schrodinger wave functional for string states. This has parallels with the analysis of [4]. We will see how this dovetails with the earlier analysis.

The world sheet string Hamiltonian following from the action (5) is

$$\begin{aligned} H = \frac{1}{4\pi\alpha'}\int d\sigma[(2\pi\alpha')^2(\Pi^i)^2 + \tau^{2a}(\partial_\sigma X^i)^2 \\ + (2\pi\alpha')^2\tau^{a-b}(\Pi^m)^2 + \tau^{a+b}(\partial_\sigma X^m)^2]. \end{aligned} \tag{13}$$

As we will elaborate on in Sec. IV (incorporating length scales in this system), the range of $\int d\sigma$ is $\int_0^{2\pi|p_-|\alpha'} d\sigma$, involving the light cone momentum p_-. This Hamiltonian is the physical Hamiltonian $H = -p_+$ satisfying the physical state condition $m^2 = -2g^{+-}p_+p_- - g^{II}(p_{I0})^2$. Let us denote by $\Psi[X^I(\sigma), \tau]$ the wave functional for string fields $X^I(\sigma) \equiv X^i(\sigma), X^m(\sigma)$. Then the Schrodinger equation for the wave functional and the functional momentum operator is

$$\begin{aligned} i\partial_\tau\Psi[X^I, \tau] &= H[X^I, \tau]\Psi[X^I, \tau], \\ \Pi^I[\sigma] &= -i\frac{\delta}{\delta X^I[\sigma]}. \end{aligned} \tag{14}$$

In light cone gauge $x^+ = \tau$, this is essentially the Schrodinger equation for the evolution in spacetime $i\frac{\partial}{\partial x^+}\Psi$ of the string wavefunctional. Since spatial translations are symmetries on the world sheet, spatial momenta are conserved. Now we see that the Hamiltonian is simply the sum $H = \sum_I H_I[X^I]$ of decoupled contributions from each of the string coordinate fields X^I. Therefore in the free string limit, it is consistent to assume that the string wave functional also factorizes into decoupled pieces as

$$\Psi[X^I] = \prod_I \Psi_I[X^I] = \prod_{i,m} \Psi^i[X^i]\Psi^m[X^m]. \tag{15}$$

Then the Schrodinger equation becomes

$$\sum_I \frac{i\partial_\tau \Psi_I}{\Psi_I} = \frac{\sum_I H_I[X^I]\Psi}{\Psi} = \sum_I \frac{H_I \Psi_I}{\Psi_I}. \tag{16}$$

Thus it is consistent to assume that this equation is separable into decoupled equations for each string coordinate field as

$$i\partial_\tau \Psi_I[X^I, \tau] = H_I[X^I]\Psi^I[X^I, \tau]. \tag{17}$$

For the X^i, this becomes (since $a > 0$)

$$\begin{aligned} i\partial_\tau \Psi_i[X^i, \tau] &= \int \frac{d\sigma}{4\pi\alpha'}[(2\pi\alpha')^2(\Pi^i)^2 \\ &\quad + \tau^{2a}(\partial_\sigma X^i)^2]\Psi^i[X^i, \tau] \xrightarrow{\tau\to 0} \\ &\quad - \pi\alpha' \int d\sigma \frac{\delta^2}{\delta X^i[\sigma]^2}\Psi^i[X^i, \tau], \end{aligned} \tag{18}$$

This is the Schrodinger equation for free propagation in a time-independent background, as in flat space: thus we conclude that free string propagation in the x^i-directions is nonsingular.

For the X^m, the Schrodinger equation becomes

$$\begin{aligned} i\partial_\tau \Psi_m[X^m, \tau] = \frac{1}{4\pi\alpha'}\int d\sigma\bigg[&-(2\pi\alpha')^2 \tau^{a-b}\frac{\delta^2}{\delta X^m[\sigma]^2} \\ &+ \tau^{a+b}(\partial_\sigma X^m)^2\bigg]\Psi^m[X^m, \tau]. \end{aligned} \tag{19}$$

This has different behavior depending on the Kasner exponents a, b. For $b > 0$, the kinetic term dominates as $\tau \to 0$ and we have

$$i\partial_\tau \Psi_m[X^m, \tau] = -\pi\alpha'\tau^{a-b}\int d\sigma \frac{\delta^2}{\delta X^m[\sigma]^2}\Psi^m[X^m, \tau], \tag{20}$$

which can be recast as the free Schrodinger equation

$$\begin{aligned} i\partial_\lambda \Psi_m[X^m, \lambda] &= -\pi\alpha' \int d\sigma \frac{\delta^2}{\delta X^m[\sigma]^2}\Psi^m[X^m, \lambda], \\ \lambda &= \int d\tau \tau^{a-b} = \frac{\tau^{2\nu}}{2\nu}. \end{aligned} \tag{21}$$

This is well defined for $2\nu = a + 1 - b \geq 0$. Alternatively, we can solve for the time dependence of (20) to obtain

$$\Psi[X^m, \tau] = e^{i\pi\alpha'(\tau^{2\nu}/2\nu)\int d\sigma(\delta^2/\delta X^m[\sigma]^2)}\Psi[X^m]. \tag{22}$$

The phase in the functional operator is well-defined if $2\nu \geq 0$: else we obtain a "wildly" oscillating phase as $\tau \to 0$.

For $b < 0$, the potential term dominates and the Schrodinger equation becomes

$$i\partial_\tau \Psi_m[X^m, \tau] = \frac{\tau^{a+b}}{4\pi\alpha'}\int d\sigma(\partial_\sigma X^m)^2\Psi^m[X^m, \tau], \quad [b < 0], \tag{23}$$

$$\begin{aligned} \Rightarrow i\partial_\lambda \Psi_m[X^m, \lambda] &= \frac{1}{4\pi\alpha'}\int d\sigma(\partial_\sigma X^m)^2\Psi^m[X^m, \lambda], \\ \lambda &= \frac{\tau^{a+b+1}}{a+b+1} = \frac{\tau^{2\nu}}{2\nu}, \end{aligned} \tag{24}$$

which is again well defined if $2\nu = a + 1 - |b| \geq 0$. Alternatively, we can solve for the time-dependent phase of the wave function as $\Psi[X^m, \tau] \sim e^{-i(\tau^{2\nu}/8\pi\nu\alpha')\int d\sigma(\partial_\sigma X^m)^2}\Psi[X^m]$, with a well-defined phase if $2\nu \geq 0$.

Now to see the highly stringy modes, we write the Hamiltonian (13) as

$$\begin{aligned} H^>[X^I] &= \frac{\tau^a}{4\pi\alpha'}\int d\sigma\bigg(\frac{(2\pi\alpha')^2(\Pi^I)^2}{g_{II}} + g_{II}(\partial_\sigma X^I)^2\bigg) \\ &= \frac{\tau^a}{4\pi\alpha'}\int d\sigma(a^I_\sigma a^{I\dagger}_\sigma + a^{I\dagger}_\sigma a^I_\sigma), \end{aligned} \tag{25}$$

in terms of the instantaneous creation-annihilation operators $a^I_\sigma = \frac{1}{\sqrt{2}}(\sqrt{g_{II}}(\partial_\sigma X^I) + \frac{i(2\pi\alpha')\Pi^I}{\sqrt{g_{II}}})$. This rewriting of the Hamiltonian above is only valid for modes that are sufficiently high frequency that the background time dependence appears frozen to them: indeed apart from the τ^a prefactor, this is essentially the flat space string Hamiltonian as it should be for such modes. Recasting in terms of mode expansions, we see that the Hamiltonian above is essentially the same as that for the highly stringy modes [i.e. $H^>$, last line of (9)]. The corresponding Schrodinger equation then is

$$\begin{aligned} i\partial_\tau \Psi^>[X^I, \tau] &= H^>[X^I]\Psi^>[X^I, \tau] \\ \Rightarrow i\partial_{\tau^{a+1}/(a+1)}\Psi^>[X^I, \tau] &= H^>_{\text{flat}}\Psi^>[X^I, \tau]. \end{aligned} \tag{26}$$

The form of Eqs. (18), (20), (23), and (26) is similar to (12).

Thus the general Schrodinger picture wave functional $\Psi[X^I(\sigma)]$ description recovers the earlier "nuts-and-bolts" description and is consistent with various detailed aspects of the spectrum and wave functions described in [1] and in the previous subsection. This suggests that string propagation is well-defined across null singularities with $2\nu = a + 1 - b \geq 0$. These are in fact the singularities where the classical near-singularity string mode functions do not diverge [1].

C. Other variables and the Schrodinger wave functional

Here we compare the above with the corresponding description in other variables that arise if we use other spacetime coordinates.

In terms of the affine parameter $\lambda = \frac{(x^+)^{a+1}}{a+1}$, we can write these spacetimes as

$$ds^2 = -2d\lambda dx^- + \lambda^{A_I}(dx^I)^2, \qquad A_I = \frac{a_I}{a+1}. \quad (27)$$

Now $g_{+-} = -1$ and fixing light cone gauge $\tau = \lambda$, the string world sheet Hamiltonian becomes

$$H = \frac{1}{4\pi\alpha'}\int d\sigma\left((2\pi\alpha')^2\frac{(\Pi^I)^2}{g_{II}} + g_{II}(\partial_\sigma X^I)^2\right) = \frac{1}{4\pi\alpha'}\int d\sigma\left((2\pi\alpha')^2\frac{(\Pi^I)^2}{\tau^{A_I}} + \tau^{A_I}(\partial_\sigma X^I)^2\right), \quad (28)$$

Now consider $A_I > 0$. Then as $\tau \to 0$, the kinetic terms dominate and the Schrodinger equation becomes

$$i\partial_\tau\Psi[X^I,\tau] = -\pi\alpha'\tau^{-A_I}\int d\sigma\frac{\delta^2}{\delta X^{I2}}\Psi[X^I,\tau] \quad (29)$$

giving for the time-dependence

$$\Psi[X^I,\tau] \sim e^{-i\pi\alpha'(\tau^{1-A_I}/1-A_I)\int d\sigma(\delta^2/\delta X^{I2})}\Psi[X^I]. \quad (30)$$

The phase in the functional operator is well-defined if $A_I < 1$. Alternatively, we can recast (29) as a free Schrodinger equation in terms of the time parameter τ^{1-A_I} which is well-defined if $A_I < 1$.

For spacetimes with $A_I < 0$, the potential terms dominate and we have

$$i\partial_\tau\Psi[X^I,\tau] = \frac{\tau^{-|A_I|}}{4\pi\alpha'}\int d\sigma(\partial_\sigma X^I)^2\Psi[X^I,\tau], \quad (31)$$

giving for the time-dependence

$$\Psi[X^I,\tau] \sim e^{-i((\tau^{1-|A_I|})/(4\pi\alpha'(1-|A_I|)))\int d\sigma(\partial_\sigma X^I)^2}\Psi[X^I], \quad (32)$$

which is well defined for $|A_I| < 1$.

The condition $|A_I| < 1$ is equivalent to $2\nu \geq 0$ (e.g. $\frac{b}{a+1} \leq 1$) stated earlier. These spacetimes are in a sense the analogs of the cases $p < 1$ with a well-defined wave functional phase arising in the gauge theories dual to AdS cosmologies with spacelike singularities [4]. There is no manifest time-dependent divergence in the Hamiltonian here, unlike the dilaton prefactor there.

Now we discuss Brinkman coordinates. From [1], we know that the coordinate transformation $x^I = (x^+)^{-a_I/2}y^I$, where $a_I \equiv a, b$, recasts the spacetimes (2) in manifest plane-wave form (this is also valid for singularities with multiple Kasner exponents)

$$ds^2 = -2(x^+)^a dx^+ dy^- + \left[\sum_I\left(\frac{a_I^2}{4} - \frac{a_I(a+1)}{2}\right)(y^I)^2\right] \times \frac{(dx^+)^2}{(x^+)^2} + (dy^I)^2. \quad (33)$$

Here we have redefined $y^- = x^- + (\frac{\sum_I a_I(y^I)^2}{4(x^+)^{a+1}})$. For $a_I = a, b$ distinct, these are in general anisotropic plane-waves with singularities. After further redefining to the affine parameter $\lambda = \frac{(x^+)^{a+1}}{a+1}$, we obtain the metric

$$ds^2 = -2d\lambda dy^- + \sum_I \chi_I(y^I)^2\frac{d\lambda^2}{\lambda^2} + (dy^I)^2, \qquad \chi_I = \frac{A_I}{4}(A_I - 2). \quad (34)$$

The string world sheet Hamiltonian in light cone gauge $\tau = \lambda$ is

$$H = \frac{1}{4\pi\alpha'}\int d\sigma\left((2\pi\alpha')^2(\Pi_y^I)^2 + (\partial_\sigma y^I)^2 - \sum_I\frac{\chi_I}{\tau^2}(y^I)^2\right). \quad (35)$$

We see that the Hamiltonian in these variables y^I contains a mass-term which diverges as $\tau \to 0$. The wave functional then acquires a wildly oscillating phase as $\tau \to 0$

$$\Psi[y^I,\tau] \sim e^{-(i/\tau)\sum_I\chi_I(y^I)^2}\Psi[y^I]. \quad (36)$$

This renders a well-defined Schrodinger wave functional interpretation near the singularity difficult in these Brinkman coordinates.

In the Rosen-like coordinates with the variables X^I, as we have seen, there is no such divergent mass-term and the Schrodinger wave functional shows smooth behavior across the singularity for spacetimes with $|A_I| < 1$. In some sense, this is akin to the difference between the X and $\tilde{X}$ dual gauge theory variables discussed in [4]. Of course physical observables defined appropriately presumably are well-defined independent of the choice of variables, although they might be more transparent in some variables.

IV. NULL SINGULARITIES, STRINGS, AND LENGTH SCALES

We now describe some relevant length scales that arise in string propagation in the vicinity of null cosmological singularities, essentially drawing various results from [1] but being explicit about length scales. Our goal is to gain insights into how string oscillator states get excited in the near singularity region. In the next section, we will study regulated versions of the singularity which will render further support to this picture.

The no-scale nature of these spacetimes, i.e. requiring no explicit length scale, is manifest in the plane-wave (Brinkman) form (33) and (34). In the Rosen coordinates (2), where the null cosmology interpretation is manifest, the dimensions of various coordinates are nontrivial, as they should be to maintain the no-scale property. In particular, the affine parameter $\lambda = \frac{(x^+)^{a+1}}{a+1}$ is of dimension length (L), so that $\dim x^+ \equiv L^{1/(a+1)}$. This implies that $\dim x^i \equiv L^{1-a/(2(a+1))}$ and $\dim x^m \equiv L^{1-b/(2(a+1))}$. The length scale characterizing the near singularity region is set by the tidal forces, or the acceleration: this gives the scale of curvature as $a^i \equiv M_c = \frac{1}{(x^+)^{2a+2}}$ of dimension $\frac{1}{L^2}$.

The light cone gauge condition fixes $\dim\tau^{a+1} = \dim\sigma = L$. We now introduce the coordinate length l of the string, so that $\int d\sigma \equiv \int_0^{2\pi l} d\sigma$. From the world sheet Lagrangian, we then see that the momentum conjugate to X^- is $p_- = -\frac{l}{2\pi\alpha'}$, so that the coordinate length l is related to the light cone momentum of the string as

$$l = 2\pi|p_-|\alpha' \equiv 2\pi p_-\alpha'. \tag{37}$$

We see that $p_- \leq 0$ and will denote $|p_-|$ by p_- for convenience in what follows (note that $p^+ = -g^{+-}p_-$ is positive but time-dependent). Our conventions agree with those of [29] for flat space. The corresponding Hamiltonian, $-p_+$, reexpressing the momenta Π^I in terms of $\partial_\tau X^I$, is

$$H = \frac{1}{4\pi\alpha'}\int d\sigma[(\partial_\tau X^i)^2 + e^{2f(\tau)}(\partial_\sigma X^i)^2 + e^{h(\tau)-f(\tau)}(\partial_\tau X^m)^2 + e^{h(\tau)+f(\tau)}(\partial_\sigma X^m)^2] \tag{38}$$

then shows that the dimensions of each term are consistent with $\dim X^i = L^{1-a/(2(a+1))}$, $\dim X^m = L^{1-b/(2(a+1))}$. The Hamiltonian then has $\dim H = \dim\frac{1}{\tau} = L^{-1/(a+1)}$.

The mode function asymptotics for the low-lying and highly stringy modes in the near singularity region on a cutoff null surface $x^+ = \tau = \tau_c$ are

$$f_n^i \to \lambda_{n0}^i + \lambda_{n\tau}^i\frac{\tau_c}{l^{1/(a+1)}}, \qquad f_n^m \to \lambda_{n0}^m + \lambda_{n\tau}^m\frac{\tau_c^{2\nu}}{l^{2\nu/(a+1)}},$$
$$\frac{n\tau_c^{a+1}}{l(a+1)} \ll 1, \qquad f_n^i \sim \frac{l^{a/(2(a+1))}}{\tau_c^{a/2}}e^{-in\tau_c^{a+1}/l(a+1)},$$
$$f_n^m \sim \frac{l^{b/(2(a+1))}}{\tau_c^{b/2}}e^{-in\tau_c^{a+1}/l(a+1)}, \qquad \frac{n\tau_c^{a+1}}{l(a+1)} \gg 1, \tag{39}$$

where the λ_{n0}^I, $\lambda_{n\tau}^I$ are constant coefficients arising from the Bessel function expansions (and in the second line, we have chosen positive frequency modes, $c_{n1}^I = 1$, $c_{n2}^I = -i$). A mode is highly stringy on this cutoff surface if

$$n \gg \frac{l}{\tau_c^{a+1}} \sim \frac{p_-\alpha'}{\tau_c^{a+1}}, \tag{40}$$

the characteristic scale being a combination of the light cone momentum and the string scale. Thus this implies that for $\tau_c^{a+1} \gtrsim p_-\alpha'$, the $n = 1$ oscillator state is already "highly stringy."

The masses of the highly stringy states is $m^2 \sim \frac{1}{\alpha'} \times (a_n^{i\dagger}a_n^i + \ldots)$. Thus a single oscillator excitation has mass $m^2 = \frac{n}{\alpha'}$. Comparing with the typical curvature scale (set by the tidal forces), we find

$$\frac{m^2}{M_c} \equiv \frac{m^2}{a^i} \sim \frac{n}{\alpha'(\frac{1}{\tau_c^{2a+2}})}. \tag{41}$$

Thus oscillator states satisfying

$$\frac{p_-\alpha'}{\tau_c^{a+1}} \ll n \ll \frac{\alpha'}{\tau_c^{2a+2}} \tag{42}$$

are light relative to the typical energy scales in the near singularity region. The first inequality is from our definition of highly stringy modes [second line of (39)]. This also implicitly requires that $p_-\tau_c^{a+1} \ll 1$, as $\tau \to 0$. These are in a sense the "instantaneous" masses of string states on the surface $\tau = \tau_c$. This picture suggests that the typical tidal forces in the near singularity region are sufficiently high that they excite several highly stringy states. However for a given cutoff surface $\tau = \tau_c$ away from the singularity, only oscillators with $n \lesssim \frac{\alpha'}{\tau_c^{2a+2}}$ are light. As $\tau_c \to 0$, this upper cutoff on the oscillator number also increases. Thus as we approach the singularity, all oscillator states become light and get excited.

In what follows, we will study certain regulated versions of these singularities and string propagation in them, which vindicates the picture above.

V. REGULATING THE SINGULARITY

We now describe some regulated versions of such null singularities. These will in fact require an explicit length scale at which the singularity is regulated, so that they are not "no-scale" anymore. Some other regularized versions of plane-wave singularities have been discussed in e.g. [25] (see also [9,30] for interesting related discussions in other kinds of null singularities).

We see that some natural analytic regulators appear to violate some energy conditions so that they are not allowed regularizations. For instance, consider modifying (2) with a metric ansatz of the form (1) where the 4D scale factor is now $e^f = L^a[(\frac{x^+}{L})^2 + \epsilon^2]^{a/2}$. Thus the scale factor departs from the earlier one at a length scale given by L (of $\dim\tau$), within which the spacetime is not singular. ϵ is a small regulating parameter. This gives the Ricci curvature for the 4D part of the spacetime as

$$e^f = L^a\left(\left(\frac{x^+}{L}\right)^2 + \epsilon^2\right)^{a/2} \Rightarrow R_{++}^{(4)} = \frac{1}{2}(f')^2 - f'' \stackrel{x^+\to 0}{\longrightarrow} -\frac{a}{(L\epsilon)^2} < 0. \tag{43}$$

Since this is essentially the 4D local energy density T_{++}, such a regulator violates energy positivity in the regulated region. Similar observations also hold for an analytic regulator of the form $e^f = L^a((\frac{x^+}{L})^a + \epsilon^2)$.

In terms of the D-dim system, we find no solution to $R_{++}^{(D)} = 0$ whose 4D scale factor e^f is of the above form:

the additional terms in $R^{(D)}_{++}$ in the regulated region ($x^+ \ll L\epsilon$) are of the form $\frac{-(2a+(D-4)b)}{(L\epsilon)^2}$, which is negative definite.[4]

This kind of a regulator can be thought of as a universal near singularity $x^+ \to 0$ limit of $e^f = L^a[1-(1-\epsilon)e^{-(x^+/L))^2}]^{a/2} \sim L^a[\epsilon + (1-\epsilon)(\frac{x^+}{L})^2]^{a/2}$, or other regulators, and so such an energy condition violation is a fairly basic problem of low energy regulators of the singularity. A similar feature also occurs in the AdS dilatonic null cosmologies discussed in [2].

An alternative regularization, although not analytic, is

$$e^f = L^a\left(\frac{|x^+|}{L} + \epsilon\right)^a, \qquad e^h = L^b\left(\frac{|x^+|}{L} + \epsilon\right)^b. \quad (44)$$

This does not have the problem above: we find

$$\begin{aligned} R^{(4)}_{++} &= \frac{a(a+2)}{2(|x^+| + L\epsilon)^2}, \\ R^{(D)}_{++} &= \frac{1}{2}(f')^2 - f'' + \frac{D-4}{4}(-2h'' - (h')^2 + 2f'h') \\ &= \frac{a^2 + 2a + \frac{D-4}{2}(-b^2 + 2b + 2ab)}{(|x^+| + L\epsilon)^2}. \end{aligned} \quad (45)$$

Thus this regulated system is automatically a solution to $R^{(D)}_{++} = 0$ since the expression in the numerator vanishes for the original singular solution. The Riemann curvature components and the tidal forces, given by the geodesic deviation, are

$$\begin{aligned} R_{+i+i} &= \frac{a(a+2)}{4}(|x^+| + L\epsilon)^{a-2}, \\ R_{+m+m} &= \frac{b(2a+2-b)}{4}(|x^+| + L\epsilon)^{b-2}, \\ a^i, a^m &\sim \frac{1}{L^{2a+2}(\frac{|x^+|}{L} + \epsilon)^{2a+2}}, \end{aligned} \quad (46)$$

so that in the regulated region $|x^+| \ll L\epsilon$, the curvature scale is $\frac{1}{(L\epsilon)^{a+1}}$, large but finite, and so are the tidal forces. Null geodesics propagating solely along x^+ (at constant x^-, x^i, x^m) with cross section along the x^i or x^m directions have an affine parameter

$$\lambda = \text{const}\int dx^+(|x^+| + L\epsilon)^a = \text{const}\frac{(|x^+| + L\epsilon)^{a+1}}{a+1}. \quad (47)$$

It is worth mentioning that although the regulated spacetime appears nonanalytic, the geodesics, affine parameter, and curvature are continuous in the regulated region as we cross $x^+ = 0$.

[4]From eq. (9) of [1], we have $2a + (D-4)b = -2 \pm \sqrt{2 + (D-4)(D-2)b^2}$. Requiring $a > 0$ means we take the positive radical. It can then be shown that $2a + (D-4)b > 0$ if $a, b \neq 0$.

This regulated spacetime can also be recast as a plane wave

$$ds^2 = -2(|x^+| + \epsilon)^a dx^+ dy^- + \left[\sum_I\left(\frac{a_I^2}{4} - \frac{a_I(a+1)}{2}\right) \times (y^I)^2\right]\frac{(dx^+)^2}{(|x^+| + \epsilon)^2} + (dy^I)^2, \quad (48)$$

whose singularity is now regulated.

There is of course nothing sacrosanct in such a regularization of the singularity. Our purpose here is to simply use the regularization (44) as a crutch to gain insights into string oscillator states turning on. It would be interesting to explore these further with perhaps an analytic regulator, possibly with other fields (e.g. the dilaton) turned on.

In the next subsection, we will find that the string spectrum can be solved exactly in these regulated backgrounds too.

A. Strings near the regulated singularity

We are primarily interested in the approach from early times to the almost-singular region to see how string oscillator states turn on, so the nonanalyticity in the metric across $x^+ = 0$ will not concern us. Let us therefore study the spacetime (1) with the scale factors (44) for $\tau = x^+ < 0$. For simplicity, we will abuse notation and use $\tau = x^+$ rather than $-\tau = -x^+$. Also in what follows, we will denote p_- to mean $|p_-|$ as before, for convenience.

The world sheet action is given in (4), which we would like to quantize using lightcone gauge, as in [1]. Keeping the string length factors explicit, we set the momentum conjugate to X^- to a τ-independent constant $p_- = \frac{Eg_{+-}l}{2\pi\alpha'} = -\frac{l}{2\pi\alpha'}$ by a τ-independent reparametrization invariance, thus obtaining $E = -\frac{1}{g_{+-}}$. This then gives the reduced action in the second line of (4), containing only the physical transverse oscillation modes $X^I \equiv X^i, X^m$, of the string.

The string world sheet equations of motion in the regulated near singularity region are

$$\begin{aligned} \partial_\tau^2 X^i - L^{2a}\left(\frac{\tau}{L} + \epsilon\right)^{2a}\partial_\sigma^2 X^i &= 0, \\ \partial_\tau^2 X^m + \frac{b-a}{L(\frac{\tau}{L} + \epsilon)}\partial_\tau X^m - L^{2a}\left(\frac{\tau}{L} + \epsilon\right)^{2a}\partial_\sigma^2 X^m &= 0. \end{aligned} \quad (49)$$

Defining a new (dimensionless) variable $\tau' = \frac{\tau}{L} + \epsilon$, these can be recast as the equations of motion in the singular spacetime [1] in terms of the variable τ'. Then we can read off the solutions for the mode functions (with $\nu = \frac{a+1-b}{2}$),

$$\begin{aligned} f_n^I(\tau) = \sqrt{\frac{nL^{d_I}}{l^{d_I/(a+1)}}\left(\frac{\tau}{L} + \epsilon\right)^{d_I}}\Bigg[& c_{n1}^I J_{(d_I/2a+2)} \times \left(\frac{nL^{a+1}(\frac{\tau}{L} + \epsilon)^{a+1}}{l(a+1)}\right) \\ & + c_{n2}^I Y_{(d_I/2a+2)}\left(\frac{nL^{a+1}(\frac{\tau}{L} + \epsilon)^{a+1}}{l(a+1)}\right)\Bigg], \end{aligned} \quad (50)$$

where we have introduced factors of l to make f_n^I dimensionless. It is straightforward to see that removing the regulator as $\epsilon \to 0$ reduces these mode functions to the ones in (6): in particular, the scale L disappears as the no-scale singular spacetime is recovered for $\epsilon \to 0$.

The mode expansion for the string world sheet fields is

$$X^I(\tau,\sigma) = X_0^I(\tau) + \sum_{n=1}^{\infty}(k_n^I f_n^I(\tau)(a_n^I e^{in\sigma/l} + \tilde{a}_n^I e^{-in\sigma/l}) + k_n^{I*} f_n^{I*}(\tau)(a_{-n}^I e^{-in\sigma/l} + \tilde{a}_{-n}^I e^{in\sigma/l})). \quad (51)$$

Working out the momenta and commutation relations, it can be shown that $k_n^i = \frac{i}{n}\sqrt{\frac{\pi\alpha' l^{-1+1/(a+1)}}{2|c_{n0}^i|(a+1)}}$, $k_n^m = \frac{i}{n}\times\sqrt{\frac{\pi\alpha' l^{-1+2\nu/(a+1)}}{2|c_{n0}^m|(a+1)}}$. The α', l dependences can be also fixed by dimensional analysis. The oscillator algebras are $[a_n^I, a_{-m}^J] = [\tilde{a}_n^I, \tilde{a}_{-m}^J] = n\delta^{IJ}\delta_{nm}$. The Hamiltonian (38) in this case simplifies to

$$H = \frac{l}{2\alpha'}((\dot{X}_0^i)^2 + \tau^{b-a}(\dot{X}_0^m)^2) + \frac{l}{2\alpha'}\sum_n |k_n^i|^2\bigg[(\{a_n^i, a_{-n}^i\} + \{\tilde{a}_n^i, \tilde{a}_{-n}^i\})\bigg(|\dot{f}_n^i|^2 + \frac{n^2L^{2a}}{l^2}\bigg(\frac{|\tau|}{L}+\epsilon\bigg)^{2a}|f_n^i|^2\bigg) - \{a_n^i, \tilde{a}_n^i\}\bigg((\dot{f}_n^i)^2 + \frac{n^2L^{2a}}{l^2}\bigg(\frac{|\tau|}{L}+\epsilon\bigg)^{2a}(f_n^i)^2\bigg) - \{a_{-n}^i, \tilde{a}_{-n}^i\}\bigg((\dot{f}_n^{i*})^2 + \frac{n^2L^{2a}}{l^2}\bigg(\frac{|\tau|}{L}+\epsilon\bigg)^{2a}(f_n^{i*})^2\bigg)\bigg] + \frac{l}{2\alpha'}\sum_n |k_n^m|^2\bigg[(\{a_n^m, a_{-n}^m\} + \{\tilde{a}_n^m, \tilde{a}_{-n}^m\})\bigg(\tau^{b-a}|\dot{f}_n^m|^2 + \frac{n^2L^{b+a}}{l^2}\bigg(\frac{|\tau|}{L}+\epsilon\bigg)^{b+a}|f_n^m|^2\bigg) - \{a_n^m, \tilde{a}_n^m\}\bigg(\tau^{b-a}(\dot{f}_n^m)^2 + \frac{n^2L^{b+a}}{l^2}\bigg(\frac{|\tau|}{L}+\epsilon\bigg)^{b+a}(f_n^m)^2\bigg) - \{a_{-n}^m, \tilde{a}_{-n}^m\}\bigg(\tau^{b-a}(\dot{f}_n^{m*})^2 + \frac{n^2L^{b+a}}{l^2}\bigg(\frac{|\tau|}{L}+\epsilon\bigg)^{b+a}(f_n^{m*})^2\bigg)\bigg]. \quad (52)$$

We can now study the asymptotics of these mode functions and then of stringy states in the regulated (but highly curved) near singularity region. In particular, focussing on the regulated region $\tau \ll L$, the mode functions above become

$$f_n^I(\tau) \sim \sqrt{\frac{n(L\epsilon)^{d_I}}{l^{d_I/(a+1)}}}\bigg[c_{n1}^I J_{(d_I/2a+2)}\bigg(\frac{n(L\epsilon)^{a+1}}{l(a+1)}\bigg) + c_{n2}^I Y_{(d_I/2a+2)}\bigg(\frac{n(L\epsilon)^{a+1}}{l(a+1)}\bigg)\bigg], \quad (53)$$

Now the mode asymptotics change depending on the cutoff length scale $L_c = (L\epsilon)^{a+1}$.

The low-lying string modes (small oscillation number $n \ll \frac{l}{L_c}$) have mode function asymptotics as $\tau \to 0$

$$f_n^i \to \lambda_{n0}^i + \lambda_{n\tau}^i \frac{\tau + L\epsilon}{l^{1/(a+1)}}, \qquad f_n^m \to \lambda_{n0}^m + \lambda_{n\tau}^m \frac{(\tau+L\epsilon)^{2\nu}}{l^{2\nu/(a+1)}}, \quad (54)$$

so that[5] (for $2\nu \geq 0$)

$$f_n^i \to \lambda_{n0}^i, \qquad \dot{f}_n^i \to \frac{\lambda_{n\tau}^i}{l^{1/(a+1)}}, \qquad f_n^m \to \lambda_{n0}^m, \qquad \dot{f}_n^m \to \frac{\lambda_{n\tau}^m}{l^{2\nu/(a+1)}}(2\nu)(\tau+L\epsilon)^{2\nu-1}. \quad (55)$$

Then the Hamiltonian (52) for these low-lying modes simplifies and can be rewritten as

$$H_< = \frac{\pi\alpha'}{l}((p_{i0})^2 + \tau^{a-b}(p_{m0})^2) + \sum_n \frac{\pi}{2(a+1)n^2}\times\bigg(\frac{1}{|c_{n0}^i|}\bigg(\frac{1}{l^{1/(a+1)}}b_{n\tau}^{i\dagger}b_{n\tau}^i + n^2\frac{(\tau+L\epsilon)^{2a}}{l^{2-1/(a+1)}}b_{n0}^{i\dagger}b_{n0}^i\bigg) + \frac{1}{|c_{n0}^m|}\bigg((2\nu)^2\frac{(\tau+L\epsilon)^{a-b}}{l^{2\nu/(a+1)}}b_{n\tau}^{m\dagger}b_{n\tau}^m + n^2\frac{(\tau+L\epsilon)^{b+a}}{l^{(2\nu+2b)/(a+1)}}b_{n0}^{m\dagger}b_{n0}^m\bigg)\bigg), \quad (56)$$

where we have defined new oscillator modes (and their Hermitian conjugates)

$$b_{n0}^I = \lambda_{n0}^I a_n^I - \lambda_{n0}^{I*}\tilde{a}_{-n}^I, \qquad b_{n\tau}^I = \lambda_{n\tau}^I a_n^I - \lambda_{n\tau}^{I*}\tilde{a}_{-n}^I, \qquad I = i, m. \quad (57)$$

The algebra and other properties of these b^I operators are as discussed in [1]. The string oscillator masses $m^2 = -2g^{+-}p_+p_- - g^{II}(p_{I0})^2$ recalling that $p_- = -\frac{l}{2\pi\alpha'}$, $-p_+ = H$, then work out in the regulated region to (for $2\nu \geq 0$)

[5]The constant coefficients λ^I, from the Bessel expansions, are

$$\lambda_{n\tau}^i = \sqrt{n}\bigg(\frac{n}{2a+2}\bigg)^{(1/2a+2)}\frac{c_{n1}^i + c_{n2}^i\cot\frac{\pi}{2a+2}}{\Gamma(\frac{2a+3}{2a+2})},$$
$$\lambda_{n0}^i = -c_{n2}^i\sqrt{n}\bigg(\frac{n}{2a+2}\bigg)^{-(1/2a+2)}\frac{\operatorname{cosec}\frac{\pi}{2a+2}}{\Gamma(\frac{2a+1}{2a+2})},$$
$$\lambda_{n\tau}^m = \sqrt{n}\bigg(\frac{n}{2a+2}\bigg)^{(\nu/a+1)}\frac{c_{n1}^m + c_{n2}^m\cot\frac{\nu\pi}{a+1}}{\Gamma(\frac{a+\nu+1}{a+1})},$$
$$\lambda_{n0}^m = -c_{n2}^m\sqrt{n}\bigg(\frac{n}{2a+2}\bigg)^{-(\nu/a+1)}\frac{\operatorname{cosec}\frac{\nu\pi}{a+1}}{\Gamma(\frac{a+1-\nu}{a+1})}.$$

$$m^2(\tau) \to \frac{1}{2\alpha'(a+1)} \sum_{i,m;n\lesssim\frac{l}{L_c}} \bigg(\frac{l^{a/(a+1)}}{(L\epsilon)^a} \frac{N^i_{n\tau}}{n^2|c^i_{n0}|} + \frac{(L\epsilon)^a}{l^{a/(a+1)}} \frac{N^i_{n0}}{|c^i_{n0}|} + \frac{(2\nu)^2 l^{b/(a+1)}}{(L\epsilon)^b} \frac{N^m_{n\tau}}{n^2|c^m_{n0}|} + \frac{(L\epsilon)^b}{l^{b/(a+1)}} \frac{N^m_{n0}}{|c^m_{n0}|} \bigg), \quad (58)$$

defining

$$N^i_{n\tau} = b^{i\dagger}_{n\tau} b^i_{n\tau}, \qquad N^i_{n0} = b^{i\dagger}_{n0} b^i_{n0}, \qquad N^m_{n\tau} = b^{m\dagger}_{n\tau} b^m_{n\tau}, \qquad N^m_{n0} = b^{m\dagger}_{n0} b^m_{n0}. \quad (59)$$

The time-dependence in the masses shows that single excitations are light relative to the local curvature scale in the regulated region if (from the $N^i_{n\tau}$ prefactor)

$$\frac{l^{a/(a+1)}/(\alpha'(L\epsilon)^a)}{1/(L\epsilon)^{2a+2}} \ll 1 \Rightarrow (L\epsilon)^{a+2} \ll \frac{\alpha'^{1/(a+1)}}{p_-^{a/(a+1)}}. \quad (60)$$

To obtain some intuition for this, note that for $a \sim 0$ (almost flat space), we have $L_c^2 = (L\epsilon)^2 \ll \frac{\alpha'}{p_-^a} \sim \alpha'$, i.e. the regulating length scale L_c is substringy. Similar expressions can be obtained from the $N^I_{n0,\tau}$ prefactors.

For the case $2\nu < 0$, the mode function asymptotics are different: in particular, the mode functions $f^I_n(\tau)$ grow large but are still finite due to the regulator. Then we can again calculate the Hamiltonian and the oscillator masses for this case.

Now we turn to the other asymptotic region of the modes: the modes are oscillatory for

$$n \gg \frac{l}{L_c} = \frac{p_-\alpha'}{L_c}. \quad (61)$$

The mode function asymptotics for (these oscillatory) highly stringy modes in the regulated region are (for positive frequency modes with $c_1 = 1$, $c_2 = -i$)

$$f^i_n \sim \frac{l^{a/(2(a+1))}}{(L\epsilon)^{a/2}} e^{-in(L\epsilon)^{a+1}/l(a+1)}, \qquad f^m_n \sim \frac{l^{b/(2(a+1))}}{(L\epsilon)^{b/2}} e^{-in(L\epsilon)^{a+1}/l(a+1)}, \quad (62)$$

and their derivatives are

$$\dot{f}^i_n \sim \bigg(-\frac{in(L\epsilon)^a}{l} - \frac{a}{2L\epsilon} \bigg) \frac{e^{-in(L\epsilon)^{a+1}/(a+1)}}{(L\epsilon)^{a/2}}, \qquad \dot{f}^m_n \sim \bigg(-\frac{in(L\epsilon)^a}{l} - \frac{b}{2L\epsilon} \bigg) \frac{e^{-in(L\epsilon)^{a+1}/(a+1)}}{(L\epsilon)^{b/2}}. \quad (63)$$

The Hamiltonian (7) for highly stringy modes then simplifies to

$$H_> \sim \frac{(L\epsilon)^a}{l} (a^I_{-n} a^I_n + \tilde{a}^I_{-n} \tilde{a}^I_n + n), \quad (64)$$

where the constant prefactor arises as $\frac{l}{2\alpha'} l^{-1+(1/a+1)} \alpha' \frac{(L\epsilon)^a}{l^2} l^{(a/a+1)}$. Thus the (instantaneous) masses of the highly stringy states in the regulated region are

$$m^2 \sim -g^{+-} H p_- \sim \frac{1}{(L\epsilon)^a} \frac{(L\epsilon)^a}{l} \frac{l}{\alpha'} (N^I_n + \tilde{N}^I_n + n) \sim \frac{1}{\alpha'} (N^I_n + \tilde{N}^I_n + n). \quad (65)$$

Relative to the local curvature scale given by $\frac{1}{L_c^2}$, these modes are light for oscillator states satisfying

$$\frac{p_-\alpha'}{L_c} \ll n \ll \frac{\alpha'}{L_c^2}. \quad (66)$$

This implicitly requires $p_- \ll \frac{1}{L_c}$. The number of such oscillator levels from is $\frac{\alpha'}{L_c^2}(1 - p_- L_c)$. Thus for any finite $p_- \ll \frac{1}{L_c}$, only a finite set of the highly stringy oscillator states are excited in the regulated near singularity region, as expected. In the singular limit $L_c \to 0$, all oscillator states are light and the number of excited oscillator states diverges. Conversely in the sector $p_- \sim \frac{1}{L_c}$, the window of light highly stringy states pinches off.

As the light cone momentum p_- increases, the lower cutoff in (66) increases and the oscillator states that are highly stringy must have higher n. Conversely, in the $p_- = 0$ zero mode sector, essentially all oscillator states are highly stringy.

With the singularity regulated at the string scale $L_c \sim l_s$, we see that no string oscillators are turned on in the regulated region, i.e. $n \sim 1$ is already not a light state, from (66). If instead the regulator is the Planck length $L_c \sim l_p$, then the oscillator state of highest level turned on is $n \sim (\frac{l_s}{l_p})^2 \sim \frac{1}{g_s^{2/(D-2)}}$, using the naive relation for the Newton constant $G_D = l_p^{D-2} = g_s^2 l_s^{D-2}$. This implicitly requires $p_- \ll M_p$. Thus in the weakly coupled (or free) string limit $g_s \to 0$, we have $n \gg 1$ in the regulated region with a large number of highly stringy oscillator states excited.

In a reduced quantum mechanics of the oscillator modes, the wave function of the nth highly stringy oscillator state again has an oscillating phase (using sec. 3.3 of [1]) but one that is nondivergent now, due to the regulator. The overall damping of the real Gaussian part is also finite. We recall from [1] that the wave functions for the low-lying oscillator states are regular even in spacetimes with singularities for $2\nu \geq 0$. In the regulated spacetime, the wave functions are also well behaved for the cases $2\nu < 0$.

VI. DISCUSSION

We have described the Schrodinger picture wave functional for string propagation across null singularities, reconciling this with the description in terms of the conventional Hamiltonian of oscillator states. The nonsingular behavior of the wave functional suggests that free string propagation is well defined across null cosmological singularities satisfying certain relations among their Kasner exponents ($2\nu \geq 0$). These are in fact the singularities for which the classical near-singularity string mode functions are nondivergent. While most of our discussion has been for two scale factors (or Kasner exponents), it is straightforward to generalize this to multiple Kasner exponents. In other variables, such as those arising in Brinkman coordinates, the presence of a wildly oscillating phase makes such an interpretation of the Schrodinger wave functional difficult. We then discussed the role of length scales and also studied string propagation in spacetimes of this sort where the singularity has been regulated at some scale (with a certain regulator). This gives a slightly clearer picture of how string oscillator modes get excited near the singularity. In particular for a finite substringy regulator, there is a finite (but large) number of string oscillator states excited near the singularity. If the regularization occurs at the Planck scale, the highest such oscillator state turned on is of level $n \sim (\frac{l_s}{l_p})^2$. Thus although the Schrodinger wave functional suggests well-defined free string propagation for some of these singularities, it is conceivable that the total production of light string states is divergent, as already suggested in [6]. An important related issue involves understanding the backreaction on the background of such light string modes. Also note that our discussion of the Schrodinger wave functional is essentially at the level of first quantized single strings. The fact that there is a proliferation of light string oscillator states suggests that perhaps a second quantized framework, e.g. string field theory, incorporating string interactions, might be useful for a more complete understanding of string propagation across such singularities. A simple, if trite, possibility is simply that strings get highly excited in the near singularity region but pass through without significant interaction, then smoothly get deexcited as we evolve past the singularity towards late times. It would be interesting to explore these further, and also understand the apparently ill-defined singularities (with $2\nu < 0$).

We have essentially used the scale factors $h_m(x^+)$ in our solutions here to simulate the role of the dilaton there in that the internal $h_m(x^+)$ scale factors shrinking effectively drive the singularity in the x^i-directions, just as the time-varying dilaton drives the singularity in the AdS/CFT cosmological context [2–4]. Based on our discussion here, it would seem that the near singularity region in the bulk null AdS cosmologies [2], having a sufficiently high local energy density, is filled with (relatively light) string oscillator states. Assuming that the bulk string theory has no qualitatively new features stemming from the world sheet coupling to the D3-brane 5-form flux, this system might be qualitatively similar to the present case. It would then seem that interaction effects between the various string modes could become non-negligible near the null singularity in the bulk. In these cases however, the string (gauge) coupling $g_s(x^+) = e^{\Phi(x^+)} = g_{\rm YM}^2(x^+)$ vanishes near the singularity so that bulk string interactions might be suppressed, modulo bulk reflections of possible gauge theory renormalization effects as discussed (for AdS cosmologies with spacelike singularities) in [4]. Analogs of such solutions here with null Kasner-like dilatonic cosmological singularities satisfy $R_{++} = \frac{1}{2}(\partial_+\Phi)^2$. With $e^\Phi = g_s(x^+)^\alpha$, the Kasner exponents a, b, now satisfy $a^2 + 2a + \frac{D-4}{2}(-b^2 + 2b + 2ab) = \frac{\alpha^2}{2}$, and similarly for multiple Kasner exponent solutions.

In general, one expects[6] that null singularities have no α' corrections: the lightlike nature forbids any nonzero covariant contraction contributing to a higher derivative correction to the effective action that might correct the singular region of the spacetime. As long as string interaction (g_s) effects also give only local covariant corrections to the low energy effective action, these will also vanish. This might lead us to imagine that null singularities are perhaps not resolved at all since possible stringy corrections vanish, i.e. these are not allowed singularities in string theory. In this sense, these are quite different from spacelike singularities where higher derivative corrections (i.e. stringy effects) yield increasingly important corrections near the singularity. This argument relies on the presence of the null isometry: however the low energy null isometry could be invalid in the near singularity region, e.g. broken by stringy effects.

From the world sheet analysis above (and in [1]), we see detailed distinctions between the behavior of string modes depending on the Kasner exponents leading us to suspect that such a general no-go argument need not be strictly true. Most notably, since string oscillator modes are being excited, in particular, with a set of highly stringy modes being light, it is conceivable that nonlocal stringy effects become important near the singularity. The rough intuition is that a string mode of high oscillation number corresponds to a highly extended (or long and wiggly) string: such highly extended strings would in general be expected to intersect and thus interact nontrivially, fitting naturally within a second quantized framework incorporating interactions. This would be consistent with the idea that the low energy notion of spacetime does not exist in the vicinity of the singularity. Thus the possibility of a low energy mechanism for null singularity resolution via higher derivative corrections to a low energy effective description need not exist either. Similar features can of course be recalled from investigations of e.g. flop transitions in Calabi-Yau spaces.

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