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Decoherence in an interacting quantum field theory: The vacuum case

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We apply the decoherence formalism to an interacting scalar field theory. In the spirit of the decoherence literature, we consider a "system field" and an "environment field" that interact via a cubic coupling. We solve for the propagator of the system field, where we include the self-energy corrections due to the interaction with the environment field. In this paper, we consider an environment in the vacuum state ($T = 0$). We show that neglecting inaccessible non-Gaussian correlators increases the entropy of the system as perceived by the observer. Moreover, we consider the effect of a changing mass of the system field in the adiabatic regime, and we find that at late times no additional entropy has been generated.

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I. INTRODUCTION

A. Outline

We aim to apply the decoherence formalism to an interacting quantum field theoretical model. The main idea in the framework of decoherence ([1–4], for reviews see [5–9]) is that a macroscopic system cannot be separated from its environment. The conventional strategy is to assume the existence of a distinct system, environment, and observer. If the observer and the environment are weakly coupled, we are allowed to integrate the environment out to study its average effect on the system perceived by the observer. Alternatively one could say that the environmental degrees of freedom are inaccessible to the observer. This averaging process is an intrinsically nonunitary operation, which consequently gives rise to an increase in entropy of the system. A quantum system with a large entropy, in turn, corresponds to an effectively classical system.

It is however difficult to realistically apply the decoherence machinery to quantum field theory: this requires in general involved out-of-equilibrium, finite temperature, interacting quantum field theoretical computations. It is of course widely appreciated that entropy can be generated as a result of an incomplete knowledge of a system. We thus need to keep in mind what quantities are actually measured in quantum field theory: all information in a system that can in principle be observed by a "perfect observer" is contained in the n-point correlators of the system. However, realistic observers are not capable of measuring irreducible n-point functions of arbitrary order as they are limited by the sensitivity of their apparatus. Therefore, it is important to realize that inaccessible higher order correlators, from the observer's perspective, yield an increase in entropy of the system. We thus propose the following viewpoint when applying the decoherence program to quantum field theory[1]: Neglecting observationally inaccessible correlators will give rise to an increase in entropy of the system as perceived by the observer.

As an example, consider some interacting quantum field theory where information is stored in either two-point or Gaussian correlators or in higher order, non-Gaussian correlators. The latter are generated generically in any interacting field theory. If we assume that the information stored in these non-Gaussian correlators is barely accessible in experiments, then neglecting this information will give rise to an increase in the entropy. From the Gaussian correlators, we can fix the entropy uniquely [14,16]. As before, a quantum system with a considerable amount of entropy corresponds to a classical system.[2] We emphasize that this definition can be improved if e.g. three- or four-point correlators are accessible through experiments such that knowledge of these correlators is included in the definition of the entropy [14]. It is important to stress that this procedure does not require a nonunitary process of tracing out environmental degrees of freedom.

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[1]Older work can already be interpreted in a similar spirit [10,11]. Here, we propose it as a strict procedure [12,13] of how to study entropy generation in the 2PI formalism, also see [14]. Recently, during the last stages of writing this paper, we found an interesting article by Giraud and Serreau [15] addressing the question of entropy production in an interacting field theory from a similar perspective.

[2]Our definition of classicality differs from the approach used in some of the literature, where, for example, coherent states with large occupation numbers are also considered to have many classical properties, even though these states have vanishing entropy.

In order to apply these rather abstract ideas to a scalar field toy model, let us outline our paper. We will consider the following interacting scalar field theory:

$$S[\phi,\chi] = \int d^D x \mathcal{L}[\phi,\chi] = \int d^D x \mathcal{L}_0[\phi] + \mathcal{L}_0[\chi] + \mathcal{L}_{\rm int}[\phi,\chi], \tag{1}$$

where

$$\mathcal{L}_0[\phi] = -\tfrac{1}{2}\partial_\mu\phi(x)\partial_\nu\phi(x)\eta^{\mu\nu} - \tfrac{1}{2}m_\phi^2(t)\phi^2(x), \tag{2a}$$

$$\mathcal{L}_0[\chi] = -\tfrac{1}{2}\partial_\mu\chi(x)\partial_\nu\chi(x)\eta^{\mu\nu} - \tfrac{1}{2}m_\chi^2\chi^2(x), \tag{2b}$$

$$\mathcal{L}_{\rm int}[\phi,\chi] = -\frac{\lambda}{3!}\chi^3(x) - \frac{1}{2}h\chi^2(x)\phi(x), \tag{2c}$$

where $\eta_{\mu\nu} = {\rm diag}(-1, 1, 1, \ldots)$ is the D-dimensional Minkowski metric. Here, $\phi(x)$ will play the role of the system, interacting with an environment $\chi(x)$, where we assume that $\lambda \gg h$ such that the environment is in thermal equilibrium at temperature T. In this paper, we study an environment of temperature $T = 0$, i.e., an environment in its vacuum state and we postpone the finite temperature corrections to a future publication. We assume that $\langle\hat{\phi}\rangle = 0 = \langle\hat{\chi}\rangle$, which can be realized by suitably renormalizing the tadpoles.

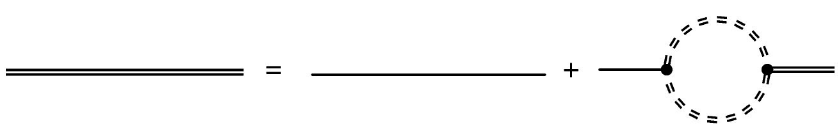
Another application of our calculation is baryogenesis in an early Universe setting where the system is driven out of equilibrium by a changing mass term $m_\phi^2(t)$ generated by a time dependent Higgs-like scalar field during a symmetry breaking. For electroweak baryogenesis, we can neglect the Universe's expansion during the phase transition and our assumption to work in Minkowski spacetime is well justified.

As we study out-of-equilibrium quantum field theory, we work in the Schwinger-Keldysh formalism. The two particle irreducible (2PI) effective action then captures the effect of perturbative loop corrections to the various propagators $\imath\Delta_\phi$ and $\imath\Delta_\chi$. Of course we will discuss these equations of motion in greater detail in the main text, but when we omit all indices and arguments, they have the following structure:

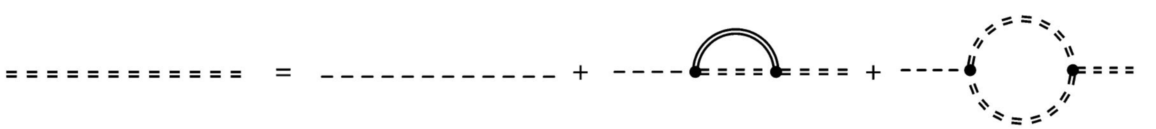
$$(\partial^2 - m_\phi^2)\imath\Delta_\phi - \int M_\phi \imath\Delta_\phi = \imath\delta^D, \tag{3a}$$

$$(\partial^2 - m_\chi^2)\imath\Delta_\chi - \int M_\chi \imath\Delta_\chi = \imath\delta^D, \tag{3b}$$

where M_ϕ and M_χ are the corresponding self-masses. These two equations are non-Gaussian due to the coupling of the two fields with coupling constant h. Multiplying Eq. (3a) by $\Delta_{0,\phi} = (\partial^2 - m_\phi^2)^{-1}\delta^D$ and Eq. (3b) by $\Delta_{0,\chi} = (\partial^2 - m_\chi^2)^{-1}\delta^D$ and integrating, one gets the following Schwinger-Dyson equations:



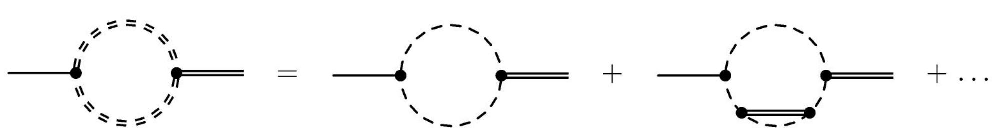
(4a)



(4b)

The dressed ϕ propagators (double solid lines) can be written as the sum of the tree level propagators (solid lines) and the self-mass corrections due to interaction with the dressed χ field (double dashed lines), and vice versa for the environment field.

Let us at this point explicitly state the two main assumptions of our work. First, we assume that our observer is only sensitive to Gaussian correlation functions. This implies that we use only these correlators to calculate the entropy [14]. Secondly, we neglect the backreaction from the system field on the environment field, i.e., we assume that we can neglect the self-mass corrections due to the ϕ field in Eq. (4)b. This assumption thus implies that the environment remains at temperature $T = 0$. In particular, the self-mass corrections to the system propagators in Eq. (4)a are now given by



(5)

The first diagram contains the leading order self-mass correction, at order $\mathcal{O}(h^2/\omega_\phi^2)$, where $\omega_\phi^2 = m_\phi^2 + k^2$. It effectively is a Gaussian correction to the ϕ propagators as it acts just like a known source to ϕ. Note that in Eq. (5) we dropped a diagram at order $\mathcal{O}(h^2\lambda^2/\omega_\phi^4)$ since it is irrelevant for the argument presented below.

Looking at Eq. (5) we see that these assumptions are well justified by perturbative arguments provided there are no secular effects: the backreaction of the system field on the environment field and the first intrinsically non-Gaussian correction occurs only at order $\mathcal{O}(h^4/\omega_\phi^4)$, which can be appreciated by examining the second Feynman diagram on the right-hand side of Eq. (5). The 3-point function of the system, an intrinsically non-Gaussian correlator, is at one-loop level also perturbatively suppressed at order $\mathcal{O}(h^3/\omega_\phi^3)$.

Finally, let us just mention that the concept of the pointer basis is frequently discussed in the decoherence literature. The pointer basis of our theory in the highly squeezed limit is the field amplitude basis [14], occurring, for example, in cosmological perturbation theory. Intuitively, this can be appreciated as follows: the Hamiltonian of a squeezed state is dominated by the potential term. A system in interaction with an environment at temperature T minimizes its free energy $F = H - TS$. The system will realize this by increasing its entropy S, mainly due to increasing the spread in momentum $\langle\hat{\pi}^2\rangle$ since that hardly affects the Hamiltonian, whereas it does significantly affect the entropy. In other words, the field amplitude basis is robust under the process of decoherence, qualifying it as a proper pointer basis. Note that ϕ is a pointer basis only in the statistical sense, such that there is a well-defined probability distribution function from which a measurement is drawn.

Having discussed the setup of our theory, the assumptions used and their justification, let us direct our attention to discussing potential applications. Our results are relevant for several research areas: the study of decoherence of cosmological perturbations, of out-of-equilibrium quantum field theory, and of baryogenesis.

B. Decoherence of cosmological perturbations

Although applications of the framework of decoherence are mainly directed toward experimental efforts (for example related to the increasingly relevant field of quantum computing), it also concerns quantum field theory (see e.g. [17]). Research efforts are primarily focused on addressing the fundamental question of the decoherence of cosmological perturbations; see e.g. [16,18–36]. One of the most important consequences of the inflationary paradigm is that it provides us with a causal explanation of how initial density inhomogeneities can be laid out on super-Hubble scales that seed the large scale structure we observe in the Universe today in, for example, clusters of galaxies. The decoherence formalism applied to cosmological perturbations aims at describing the transition between the quantum nature of the initial density inhomogeneities as a consequence of inflation and the classical stochastic behavior as assumed by large scale structure theory.

In the literature, specific models, for example, assume that during inflation the UV (or sub-Hubble) modes of a field, once integrated out, decohere the IR (or super-Hubble) modes because the former modes are inaccessible observationally ([37–39], however also see [40]). A similar split of UV and IR modes has been made in the context of stochastic inflation, see, e.g., [41–46]. In [47] vacuum fluctuations decohere the mean field, turning it into a classical stochastic field. In [16] it is argued that self-mass corrections to the equation of motion for the statistical propagator can be rewritten in terms of a stochastic noise term that in turn decoheres the system. In [31] it was shown that isocurvature modes decohere the adiabatic mode.

C. Nonequilibrium quantum field theory

In recent years, the study of nonequilibrium quantum field theory has become more and more tractable (for review articles, see [48,49]). A central ingredient in performing these studies is the two particle irreducible action, from which quantum corrections to propagators can be investigated. Out-of-equilibrium $\lambda\phi^4(x)$ theory has extensively been studied in, for example, [50–58]. The dynamics of nonequilibrium fermions has been addressed in, e.g., [59].

An interesting study has been performed in [60], where one also studies, under certain assumptions, the dynamics of a system field that interacts via a cubic coupling with a thermal bath, which we also consider. Their thermal bath consists of two scalar fields with different masses. Very recently, another interesting calculation for $\lambda\phi^4(x)$ self-interaction has been performed in [15] where one calculates a decoherence parameter and thermalization of an initial pure state.

Calzetta and Hu consider in [61,62] also an out-of-equilibrium $\lambda\phi^4(x)$ theory. What we would refer to as "Gaussian–von Neumann entropy" is referred to as "correlation entropy" in [62]. They prove an H theorem for a quantum mechanical $O(N)$ model.

Renormalizing the Kadanoff-Baym equations is a subtle business. In $\lambda\phi^4(x)$ theory it has been examined in different contexts in [63–70]. We will also come to address the question of renormalizing our cubically interacting field theory. Our main finding is that the structure of the renormalized equations of motion differs from the unrenormalized equations, which has in general to our knowledge not previously been considered in the literature.

Furthermore, imposing initial conditions at some finite time t_0 results in additional infinities that have to be renormalized separately according to the authors of [71–74]. Another interesting study has been performed by

Garny and Muller in [75] in which the renormalized Kadanoff-Baym equations, again in $\lambda\phi^4(x)$, are numerically integrated by imposing non-Gaussian initial conditions. We differ in our approach as we consider the memory effects from the interacting theory at times before t_0. We can then impose appropriate Gaussian initial conditions at t_0 without encountering initial time divergences.

D. Baryogenesis

This work is in part inspired by fundamental questions concerning the problem of entropy in field theory, and in part by electroweak scale baryogenesis. The problem is to calculate axial vector currents generated by a *CP* violating advancing phase interface of a true vacuum bubble at the electroweak phase transition. These currents then feed in hot sphalerons, thus biasing baryon production. The axial currents are difficult to calculate reliably, since a controlled calculation would include nonequilibrium dynamics in a finite temperature plasma in the presence of a nonadiabatically changing mass parameter. In this paper we neither include a plasma at finite temperature (this will be done in a future companion paper), nor do we consider scattering of fermions on a nonadiabatically changing phase interface. Yet there are important similarities between the problem we address here and baryogenesis: our interacting scalar field model (2) mimics the Yukawa part of the Lagrangian of the standard model, whereby one scalar field plays the role of the Higgs field, while the other is a heavy fermion (top quark or a chargino of a supersymmetric theory). The role of the axial current is taken by the entropy which are both sensitive to quantum coherence and the phase interface is a time dependent mass parameter $m_\phi^2(t)$. The importance of quantum coherence in baryogenesis is also treated in [76–78], where a coherent mixture of fermions has been used to generate baryons in grand unified theories during preheating after inflation. However, the authors of [76–78] treat the interactions phenomenologically in the relaxation time approximation.

Quantum mechanical scattering on bubble walls in a thermal bath may become the dominant mechanism for baryon production when the walls are thin and has been addressed in several papers in the mid 1990s [79–84], mostly in the context of baryogenesis within the standard model. Currently, the consensus is that so far no satisfactory solution to the problem has been found. Recently Herranen, Kainulainen, and Rahkila [85–87] have reinvigorated interest in the problem, which has gained on timeliness by the upcoming LHC experiments. Their approach is based on the observation that the constraint equations for fermions and scalars admit a third particle shell at a vanishing energy, $k_0 = 0$. The authors show that this third shell can be used to correctly reproduce the Klein paradox both for fermions and bosons in a step potential, and hope that their intrinsically off-shell formulation can be used to include interactions in a field theoretical settings, for which off-shell physics is essential. The authors have studied both fermionic [85,86] and bosonic [87] quantum mechanical reflection in the presence of scatterings. However, not all scatterings implied by the Kadanoff-Baym equations are taken into account. One important motivation for the present paper is to work within a set of approximations where all relevant terms in the Kadanoff-Baym equations are kept.

II. ENTROPY AND PROPAGATORS

A. The statistical propagator and entropy

There is a connection between the statistical propagator and the Gaussian entropy of a system ([14], also see [10,16]). In quantum field theory, one can calculate many propagators, with different properties associated with each, but not all of them are independent. In this work, we will be primarily interested in solving for the statistical propagator of the system. Let us mention that the information contained in the statistical propagator is also encoded in the two Wightman functions. Generically, in the presence of quantum fluctuations, one needs complete knowledge of the causal propagator in order to solve for the statistical propagator. In the simple free theory example we consider in Appendix A, we can directly solve for the statistical propagator however and no prior knowledge of the causal propagator is required.

The statistical propagator describes how states are populated and is in the Heisenberg picture defined by

$$F_\phi(x;x') = \frac{1}{2}\mathrm{Tr}[\hat{\rho}(t_0)\{\hat{\phi}(x'),\hat{\phi}(x)\}] = \frac{1}{2}\mathrm{Tr}[\hat{\rho}(t_0)(\hat{\phi}(x')\hat{\phi}(x)+\hat{\phi}(x)\hat{\phi}(x'))], \tag{6}$$

given some density matrix operator $\hat{\rho}(t_0)$. The causal propagator roughly describes the number of accessible states and is given by the commutator of the two fields:

$$\imath\Delta^c_\phi(x;x') = \mathrm{Tr}(\hat{\rho}(t_0)[\hat{\phi}(x),\hat{\phi}(x')]) = \mathrm{Tr}[\hat{\rho}(t_0)(\hat{\phi}(x)\hat{\phi}(x')-\hat{\phi}(x')\hat{\phi}(x))]. \tag{7}$$

In spatially homogeneous backgrounds, we can Fourier transform, e.g., the statistical propagator as follows:

$$F_\phi(k,t,t') = \int d(\vec{x}-\vec{x}')F_\phi(x;x')e^{-\imath\vec{k}\cdot(\vec{x}-\vec{x}')}, \tag{8}$$

which in the case we will consider in this paper only depends on $k = \|\vec{k}\|$. It is only the statistical propagator and its various time derivatives that determine the entropy. In short, the entropy is fixed by the area in phase space Δ the state of the system occupies and is given by

$$\Delta_k^2(t) = 4[F(k,t,t')\partial_t\partial_{t'}F(k,t,t') - \{\partial_t F(k,t,t')\}^2]|_{t=t'}. \tag{9}$$

Throughout the paper, and, in particular, in this equation we set $\hbar = 1$. We also set $c = 1$. The entropy per mode then follows as

$$S_k(t) = \frac{\Delta_k(t)+1}{2}\log\left(\frac{\Delta_k(t)+1}{2}\right) - \frac{\Delta_k(t)-1}{2}\log\left(\frac{\Delta_k(t)-1}{2}\right). \tag{10}$$

Finally, it is interesting to note that the phase space area can be related to an effective phase space particle number density per mode or the statistical particle number density per mode as

$$n_k(t) = \frac{\Delta_k(t)-1}{2}. \tag{11}$$

In Appendix A we illustrate our ideas by studying a nontrivial exact case: quantum scattering due to a changing mass in the free case, i.e., the interaction coefficients h and λ in Eq. (2c) are switched off. For a free scalar field with a smoothly changing mass term, we show that $\Delta_k(t) = 1$ and hence no entropy has been generated by the mass change. Secondly, we point out that the reader should not confuse the statistical particle number density in Eq. (11) with the parameter $|\beta_k|^2$ characterizing nonadiabaticity of the mass change in Eq. (A25b), which in the literature is often referred to as a particle number as well [88]. This parameter is nonzero, and possibly large, simply because the asymptotic in and out vacua differ.

B. Propagators in the Schwinger-Keldysh formalism

The material included in this section may well be familiar to the experienced reader, but we include it nevertheless for pedagogical reasons and in order to clearly establish our notation. Let us consider the expectation value of an operator $\hat{Q}(t)$ in the Heisenberg picture, given a density matrix operator $\hat{\rho}(t_0)$:

$$\langle\hat{Q}(t)\rangle = \mathrm{Tr}[\hat{\rho}(t_0)\hat{Q}(t)] = \mathrm{Tr}\left[\hat{\rho}(t_0)\left\{\bar{T}\exp\left(\imath\int_{t_0}^{t} dt'\hat{H}(t')\right)\right\}\hat{Q}(t_0) \times\left\{T\exp\left(-\imath\int_{t_0}^{t} dt'\hat{H}(t')\right)\right\}\right], \tag{12}$$

where $t_0 < t$ denotes an initial time, $\bar{T}$ and T denote the antitime ordering and time ordering operations, respectively, and $\hat{H}(t)$ denotes the Hamiltonian. If $\hat{Q}$ in the Scrödinger picture depends explicitly on time, we should replace $\hat{Q}(t_0)$ by $\hat{Q}_S(t)$.

The Schwinger-Keldysh formalism, or closed time path (CTP) formalism, or in-in formalism, is based on the original papers by Schwinger [89] and Keldysh [90] and is particularly useful for nonequilibrium quantum field theory (also see [91–95]). According to the CTP formalism, the expectation value above can be calculated from the in-in generating functional in the path integral formulation:

$$\mathcal{Z}[J_+^\phi, J_-^\phi, J_+^\chi, J_-^\chi, \rho(t_0)] = \int \mathcal{D}\phi_0^+\mathcal{D}\phi_0^-\mathcal{D}\chi_0^+\mathcal{D}\chi_0^-\langle\phi_0^+,\chi_0^+|\hat{\rho}(t_0)|\phi_0^-,\chi_0^-\rangle\int_{\phi_0^+}^{\phi_0^-}\mathcal{D}\phi^+\mathcal{D}\phi^-\delta[\phi^+(t_f)-\phi^-(t_f)] \times\int_{\chi_0^+}^{\chi_0^-}\mathcal{D}\chi^+\mathcal{D}\chi^-\delta[\chi^+(t_f)-\chi^-(t_f)]\exp\left[\imath\int d^{D-1}x\int_{t_0}^{t_f}dt'(\mathcal{L}[\phi^+,\chi^+,t'] - \mathcal{L}[\phi^-,\chi^-,t'] + J_+^\phi\phi^+ + J_-^\phi\phi^- + J_+^\chi\chi^+ + J_-^\chi\chi^-)\right], \tag{13}$$

where the Lagrangian is given in Eq. (1). We can use the well-known Schwinger-Keldysh contour depicted in Figs. 1 and 2. It runs from t_0 up to t_f, where both times can in principle be extended to negative and positive infinity, respectively (as depicted in Fig. 2). As we will come to discuss, the two paths are not equivalent in an interacting quantum field theory, where memory effects play an important role. In this paper, we will extend t_0 to negative infinity at some point, but let us for the moment keep it finite. Clearly, these contours are closely related to

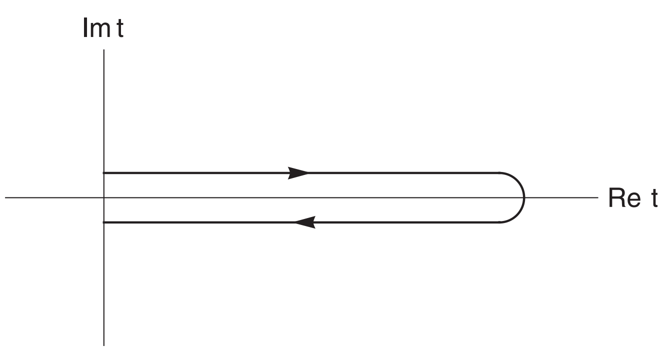


FIG. 1. Schwinger-Keldysh contour with finite initial time t_0 and final time t_f.

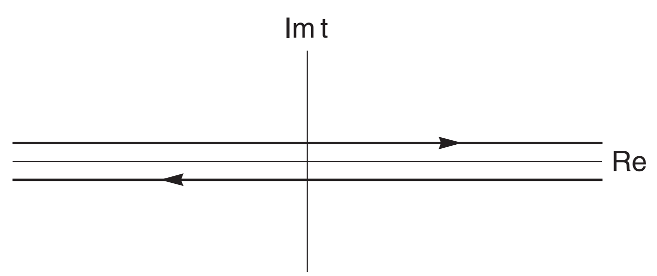


FIG. 2. Schwinger-Keldysh contour where the initial and final times in Fig. 1 have been extended to negative and positive infinity, respectively.

the two evolution operators in Eq. (12). The fields ϕ and χ and their corresponding sources J^ϕ and J^χ split up on the upper (+) and lower (−) parts of the contour, where necessarily the conditions $\phi^+(t_f) = \phi^-(t_f)$ and $\chi^+(t_f) = \chi^-(t_f)$ apply. These conditions are indeed enforced by the two functional δ distributions. The first functional integrals in (13) are over the initial configuration space at t_0, where the system is specified by the density operator (density matrix) $\hat{\rho}(t_0)$. The path integrals in (13) run over the Schwinger-Keldysh contour in Figs. 1 and 2.

Expectation values of n-point functions are obtained by varying the generating functional (13) as follows:

$$\mathrm{Tr}[\hat{\rho}(t_0)\bar{T}[\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)]T[\hat{\phi}(y_1)\cdots\hat{\phi}(y_k)]] = \frac{\delta^{n+k}Z[J,\rho(t_0)]}{\imath\delta J^\phi_-(x_1)\cdots\imath\delta J^\phi_-(x_n)\imath\delta J^\phi_+(y_1)\cdots\imath\delta J^\phi_+(y_k)}\bigg|_{J=0},$$

provided that $x_j^0 \le t_f$ and $y_j^0 \le t_f$ for all j, and where $J = (J^\phi_\pm, J^\chi_\pm)$. We can now define the following propagators:

$$\imath\Delta^{++}_\phi(x;x') = \mathrm{Tr}[\hat{\rho}(t_0)T[\hat{\phi}(x')\hat{\phi}(x)]] = \mathrm{Tr}[\hat{\rho}(t_0)\hat{\phi}^+(x')\hat{\phi}^+(x)] = \frac{\delta^2 Z[J,\rho(t_0)]}{\imath\delta J^\phi_+(x)\imath\delta J^\phi_+(x')}\bigg|_{J=0}, \tag{14a}$$

$$\imath\Delta^{--}_\phi(x;x') = \mathrm{Tr}[\hat{\rho}(t_0)\bar{T}[\hat{\phi}(x')\hat{\phi}(x)]] = \mathrm{Tr}[\hat{\rho}(t_0)\hat{\phi}^-(x')\hat{\phi}^-(x)] = \frac{\delta^2 Z[J,\rho(t_0)]}{\imath\delta J^\phi_-(x)\imath\delta J^\phi_-(x')}\bigg|_{J=0}, \tag{14b}$$

$$\imath\Delta^{-+}_\phi(x;x') = \mathrm{Tr}[\hat{\rho}(t_0)\hat{\phi}(x)\hat{\phi}(x')] = \mathrm{Tr}[\hat{\rho}(t_0)\hat{\phi}^-(x)\hat{\phi}^+(x')] = \frac{\delta^2 Z[J,\rho(t_0)]}{\imath\delta J^\phi_-(x)\imath\delta J^\phi_+(x')}\bigg|_{J=0}, \tag{14c}$$

$$\imath\Delta^{+-}_\phi(x;x') = \mathrm{Tr}[\hat{\rho}(t_0)\hat{\phi}(x')\hat{\phi}(x)] = \mathrm{Tr}[\hat{\rho}(t_0)\hat{\phi}^-(x')\hat{\phi}^+(x)] = \frac{\delta^2 Z[J,\rho(t_0)]}{\imath\delta J^\phi_+(x)\imath\delta J^\phi_-(x')}\bigg|_{J=0}. \tag{14d}$$

We define the various propagators for the χ field analogously. In the absence of a condensate for χ all mixed two-point functions, such as

$$\langle\Omega|\hat{\phi}(x')\hat{\chi}(x)|\Omega\rangle, \tag{15}$$

vanish by virtue of the interaction term (2c). In Eq. (14), $\imath\Delta^{++}_\phi(x;x') \equiv \imath\Delta^F_\phi(x;x')$ denotes the Feynman or time ordered propagator and $\imath\Delta^{--}_\phi(x;x')$ represents the antitime ordered propagator. The two Wightman functions are given by $\imath\Delta^{-+}_\phi(x;x')$ and $\imath\Delta^{+-}_\phi(x;x')$. From Eq. (1) we infer that the free Feynman propagator obeys

$$\mathcal{D}_x\imath\Delta^{++}_{\phi,0}(x;x') \equiv (\partial_x^2 - m^2)\imath\Delta^{++}_{\phi,0}(x;x') = \imath\delta^D(x-x'), \tag{16}$$

where $\partial_x^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and where the same identity holds for $\imath\Delta^{++}_{\chi,0}(x;x')$. In the presence of interactions the equation of motion for the Feynman propagator becomes much more involved and we will discus it shortly. One can easily show that at tree level the Wightman functions obey the homogeneous equation:

$$\mathcal{D}_x\imath\Delta^{+-}_{\phi,0}(x;x') = 0 = \mathcal{D}_x\imath\Delta^{-+}_{\phi,0}(x;x'). \tag{17}$$

Identical relations hold for the free χ propagators. The four propagators defined above are not independent. The Wightman functions, for example, constitute the time ordered and antitime ordered propagators:

$$\imath\Delta^{++}_\phi(x;x') = \theta(t-t')\imath\Delta^{-+}_\phi(x;x') + \theta(t'-t)\imath\Delta^{+-}_\phi(x;x'), \tag{18a}$$

$$\imath\Delta^{--}_\phi(x;x') = \theta(t'-t)\imath\Delta^{-+}_\phi(x;x') + \theta(t-t')\imath\Delta^{+-}_\phi(x;x'), \tag{18b}$$

where $t = x^0$, $t' = x^{0\prime}$, and where this identity holds for the χ propagators as well. Appreciate that

$$\imath\Delta^{++}_\phi(x;x') + \imath\Delta^{--}_\phi(x;x') = \imath\Delta^{-+}_\phi(x;x') + \imath\Delta^{+-}_\phi(x;x'), \tag{18c}$$

$$\imath\Delta^{-+}_\phi(x;x') = \imath\Delta^{+-}_\phi(x';x), \tag{18d}$$

are exact identities and they are also satisfied by the χ propagators. We write the four Green's functions in the 2×2 Keldysh propagator matrix form:

$$\imath\mathcal{G}_\phi(x;x') = \begin{pmatrix} \imath\Delta^{++}_\phi & \imath\Delta^{+-}_\phi \\ \imath\Delta^{-+}_\phi & \imath\Delta^{--}_\phi \end{pmatrix}, \tag{19}$$

which at tree level obeys

$$\mathcal{D}_x\imath\mathcal{G}_{\phi,0}(x;x') = \imath\sigma^3\delta^D(x-x'), \tag{20}$$

where $\sigma^3 = \mathrm{diag}(1,-1)$ is the third Pauli matrix, which we can also write as

$$(\sigma^3)^{ab} = a\delta^{ab}, \tag{21}$$

where $a, b = \pm$.

Let us define some more Green's functions. In Sec. II A we already defined the causal and statistical propagator, but let us for completeness list them again. The causal

Green's function, also known as the Pauli-Jordan or Schwinger or spectral two-point function, $\imath\Delta^c_\phi \equiv \imath\Delta^{PJ}_\phi \equiv \mathcal{A}_\phi \equiv \rho_\phi$, is given by

$$\begin{aligned}\imath\Delta^c_\phi(x;x') &= \mathrm{Tr}(\hat\rho(t_0)[\hat\phi(x),\hat\phi(x')]) \\ &= \imath\Delta^{-+}_\phi(x;x') - \imath\Delta^{+-}_\phi(x;x'),\end{aligned} \tag{22}$$

and the statistical or Hadamard two-point function, $F_\phi \equiv \Delta^H_\phi$, is given by

$$\begin{aligned}F_\phi(x;x') &= \tfrac{1}{2}\mathrm{Tr}[\hat\rho(t_0)\{\hat\phi(x'),\hat\phi(x)\}] \\ &= \tfrac{1}{2}(\imath\Delta^{-+}_\phi(x;x') + \imath\Delta^{+-}_\phi(x;x')).\end{aligned} \tag{23}$$

The retarded ($\imath\Delta^r$) and advanced ($\imath\Delta^a$) propagators are defined as

$$\begin{aligned}\imath\Delta^r_\phi(x;x') &= \imath\Delta^{++}_\phi(x;x') - \imath\Delta^{+-}_\phi(x;x') \\ &= -[\imath\Delta^{--}_\phi(x;x') - \imath\Delta^{-+}_\phi(x;x')] \\ &= \theta(t-t')\imath\Delta^c_\phi(x;x'),\end{aligned} \tag{24a}$$

$$\begin{aligned}\imath\Delta^a_\phi(x;x') &= \imath\Delta^{++}_\phi(x;x') - \imath\Delta^{-+}_\phi(x;x') \\ &= -[\imath\Delta^{--}_\phi(x;x') - \imath\Delta^{+-}_\phi(x;x')] \\ &= -\theta(t'-t)\imath\Delta^c_\phi(x;x').\end{aligned} \tag{24b}$$

Moreover, we can express all propagators $\imath\Delta^{ab}_\phi$ solely in terms of the causal and statistical propagators:

$$\imath\Delta^{+-}_\phi(x;x') = F_\phi(x;x') - \tfrac{1}{2}\imath\Delta^c_\phi(x;x'), \tag{25a}$$

$$\imath\Delta^{-+}_\phi(x;x') = F_\phi(x;x') + \tfrac{1}{2}\imath\Delta^c_\phi(x;x'), \tag{25b}$$

$$\imath\Delta^{++}_\phi(x;x') = F_\phi(x;x') + \tfrac{1}{2}\mathrm{sgn}(t-t')\imath\Delta^c_\phi(x;x'), \tag{25c}$$

$$\imath\Delta^{--}_\phi(x;x') = F_\phi(x;x') - \tfrac{1}{2}\mathrm{sgn}(t-t')\imath\Delta^c_\phi(x;x'). \tag{25d}$$

Since $F^\dagger_\phi = F_\phi$ and $(\imath\Delta^c_\phi)^\dagger = -\imath\Delta^c_\phi$, the relations above correspond to splitting the various Green's functions into their Hermitian and anti-Hermitian parts (for that reason we do not put an $\imath$ in front of F_ϕ). The definitions of the retarded, advanced, causal, and the statistical propagators and the relations between them easily extend to the χ field.

C. The Kadanoff-Baym equations

In order to study the effect of perturbative loop corrections on classical expectation values, one often considers the effective action. In this section we will calculate the 2PI effective action, using the Schwinger-Keldysh formalism outlined above. The 2PI effective action is the relevant functional to consider because it captures the interaction of the ϕ and χ fields in the right way. Varying the 2PI effective action with respect to the propagators yields the so-called Kadanoff-Baym equations that govern their dynamics. These equations of motion contain the nonlocal scalar self-energy corrections or self-mass corrections to the propagator.

In the present section, we shall mainly follow [48,94,96,97]. We can extract the Feynman rules from the interaction part of the tree level action (2c):

$$\mathcal{L}_{\mathrm{int}}[\phi,\chi] = -\sum_{a=\pm} a\bigg(\frac{\lambda}{3!}(\chi^a(x))^3 + \frac{1}{2}h(\chi^a(x))^2\phi^a(x)\bigg). \tag{26}$$

The Feynman propagator is promoted to $\imath\mathcal{G}_\phi$ and each vertex has two polarities: plus ($+$) and minus ($-$), such that the minus vertex gains an extra minus sign $+\imath\lambda$ as compared to the standard perturbation theory plus vertex $-\imath\lambda$.

The 2PI effective action can be obtained as a double Legendre transform from the generating functional W for connected Green's functions with respect to the linear source J and another quadratic source (see, e.g., [48]). In the absence of field condensates the background fields vanish:

$$\bar\phi^a \equiv \langle\Omega|\hat\phi^a|\Omega\rangle = 0, \tag{27a}$$

$$\bar\chi^a \equiv \langle\Omega|\hat\chi^a|\Omega\rangle = 0, \tag{27b}$$

in which case the variation with respect to the linear or quadratic sources can easily be related. In particular, the definitions of the four propagators in Eq. (14) remain valid. The effective action formally reads [48,93,98,99]

$$\begin{aligned}&\Gamma[\bar\phi^a,\bar\chi^a,\imath\Delta^{ab}_\phi,\imath\Delta^{ab}_\chi] \\ &= S[\bar\phi^a,\bar\chi^a] + \frac{\imath}{2}\mathrm{Tr}\ln[(\imath\Delta^{ab}_\phi)^{-1}] + \frac{\imath}{2}\mathrm{Tr}\ln[(\imath\Delta^{ab}_\chi)^{-1}] \\ &\quad + \frac{\imath}{2}\mathrm{Tr}\frac{\delta^2 S[\bar\phi^a,\bar\chi^a]}{\delta\bar\phi^a\delta\bar\phi^b}\imath\Delta^{ab}_\phi + \frac{\imath}{2}\mathrm{Tr}\frac{\delta^2 S[\bar\phi^a,\bar\chi^a]}{\delta\bar\chi^a\delta\bar\chi^b}\imath\Delta^{ab}_\chi \\ &\quad + \Gamma^{(2)}[\bar\phi^a,\bar\chi^a,\imath\Delta^{ab}_\phi,\imath\Delta^{ab}_\chi].\end{aligned} \tag{28}$$

Here, $\Gamma^{(2)}$ denotes the 2PI contribution to the effective action. Moreover, we omitted the dependence on all variables for notational convenience. Several Feynman diagrams contribute to the effective action of which the one- and two-loop order contributions are given in Fig. 3. We can now write the effective action up to two loops as

$$\begin{aligned}\Gamma[\imath\Delta^{ab}_\phi,\imath\Delta^{ab}_\chi] &= \Gamma_0[\imath\Delta^{ab}_\phi,\imath\Delta^{ab}_\chi] + \Gamma_1[\imath\Delta^{ab}_\phi,\imath\Delta^{ab}_\chi] \\ &\quad + \Gamma_2[\imath\Delta^{ab}_\phi,\imath\Delta^{ab}_\chi],\end{aligned} \tag{29}$$

where the subscript denotes the number of loops and where

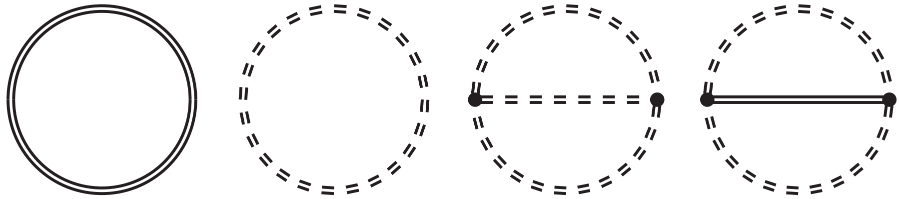


FIG. 3. Contributions to the 2PI effective action up to two-loop order. The double solid lines denote ϕ propagators, whereas the double dashed lines correspond to χ propagators.

$$\Gamma_0[\imath\Delta^{ab}_\phi, \imath\Delta^{ab}_\chi] = \int d^Dx d^Dx' \sum_{a,b=\pm} \frac{a}{2}(\partial_x^2 - m_\phi^2)\delta^D(x-x')\delta^{ab}\imath\Delta^{ba}_\phi(x';x) + \int d^Dx d^Dx' \sum_{a,b=\pm} \frac{a}{2}(\partial_x^2 - m_\chi^2)\delta^D(x-x')\delta^{ab}\imath\Delta^{ba}_\chi(x';x), \tag{30a}$$

$$\Gamma_1[\imath\Delta^{ab}_\phi, \imath\Delta^{ab}_\chi] = -\frac{\imath}{2}\,\mathrm{Tr}\ln[\imath\Delta^{aa}_\phi(x;x)] - \frac{\imath}{2}\,\mathrm{Tr}\ln[\imath\Delta^{aa}_\chi(x;x)], \tag{30b}$$

$$\Gamma_2[\imath\Delta^{ab}_\phi, \imath\Delta^{ab}_\chi] = \frac{\imath\lambda^2}{12}\int d^Dx d^Dx' \sum_{a,b=\pm} ab(\imath\Delta^{ab}_\chi(x';x))^3 + \frac{\imath h^2}{4}\int d^Dx d^Dx' \sum_{a,b=\pm} ab(\imath\Delta^{ab}_\chi(x;x'))^2\imath\Delta^{ab}_\phi(x';x), \tag{30c}$$

where Tr denotes a trace over both spacetime variables and the Keldysh indices $\pm$. The equations of motion for the propagators result as usual from the variational principle:

$$\frac{\delta\Gamma[\imath\Delta^{ab}_\phi, \imath\Delta^{ab}_\chi]}{\delta\imath\Delta^{ab}_\phi} = 0, \tag{31a}$$

$$\frac{\delta\Gamma[\imath\Delta^{ab}_\phi, \imath\Delta^{ab}_\chi]}{\delta\imath\Delta^{ab}_\chi} = 0. \tag{31b}$$

Explicitly, they yield

$$\frac{a}{2}(\partial_x^2 - m_\phi^2)\delta^D(x-x')\delta^{ab} - \frac{\imath}{2}[\imath\Delta^{ab}_\phi(x;x')]^{-1} + \frac{\imath h^2}{4}ab(\imath\Delta^{ab}_\chi(x;x'))^2 = 0, \tag{32a}$$

$$\frac{a}{2}(\partial_x^2 - m_\chi^2)\delta^D(x-x')\delta^{ab} - \frac{\imath}{2}[\imath\Delta^{ab}_\chi(x;x')]^{-1} + \frac{\imath\lambda^2}{4}ab(\imath\Delta^{ab}_\chi(x;x'))^2 + \frac{\imath h^2}{2}ab\imath\Delta^{ab}_\chi(x;x')\imath\Delta^{ab}_\phi(x;x') = 0. \tag{32b}$$

We will bring these equations into a more familiar form by multiplying by $2a\imath\Delta^{bc}_\phi(x';x'')$ and $2a\imath\Delta^{bc}_\chi(x';x'')$, respectively, and then integrating over x' and summing over $b=\pm$. This results in the following one-loop Kadanoff-Baym [100] equations for the elements of the Keldysh propagator $\imath\mathcal{G}(x;x')$:

$$(\partial_x^2 - m_\phi^2)\imath\Delta^{ab}_\phi(x;x') - \sum_{c=\pm} c\int d^Dx_1 M^{ac}_\phi(x;x_1)\imath\Delta^{cb}_\phi(x_1;x') = a\delta^{ab}\imath\delta^D(x-x'), \tag{33a}$$

$$(\partial_x^2 - m_\chi^2)\imath\Delta^{ab}_\chi(x;x') - \sum_{c=\pm} c\int d^Dx_1 M^{ac}_\chi(x;x_1)\imath\Delta^{cb}_\chi(x_1;x') = a\delta^{ab}\imath\delta^D(x-x'), \tag{33b}$$

where the self-masses at one loop have the form:

$$\imath M^{ac}_\phi(x;x_1) = -2ac\frac{\delta\Gamma_2[\imath\Delta^{ab}_\phi, \imath\Delta^{ab}_\chi]}{\delta\imath\Delta^{ca}_\phi(x_1;x)} = -\frac{\imath h^2}{2}(\imath\Delta^{ac}_\chi(x;x_1))^2, \tag{34a}$$

$$\imath M^{ac}_\chi(x;x_1) = -2ac\frac{\delta\Gamma_2[\imath\Delta^{ab}_\phi, \imath\Delta^{ab}_\chi]}{\delta\imath\Delta^{ca}_\chi(x_1;x)} = -\frac{\imath\lambda^2}{2}(\imath\Delta^{ac}_\chi(x;x_1))^2 - \imath h^2\imath\Delta^{ac}_\chi(x;x_1)\imath\Delta^{ac}_\phi(x;x_1), \tag{34b}$$

where in the last step we used the Hermiticity symmetry of the operator $\imath\mathcal{G}$, according to which $\imath\Delta^{ac}(x;x') = \imath\Delta^{ca}(x';x)$. The Feynman diagrams contributing to the one-loop self-mass are given in Fig. 4. We have chosen the definition of (34) such that the structure of the self-mass resembles that of the propagators. The factor $1/2$ in (34) originates from the symmetry factor of the one-loop self-mass diagram.

Equation (33a) consists of the following four equations:

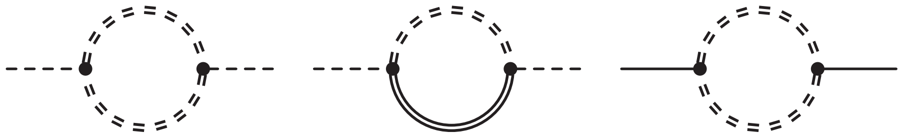


FIG. 4. Contributions to the self-masses up to one-loop order. Again, the double solid lines denote ϕ propagators, whereas the double dashed lines correspond to χ propagators. Hence, the first two Feynman diagrams contribute to the self-mass of $\chi(x)$, and only the third diagram contributes to the self-mass of $\phi(x)$.

$$(\partial_x^2 - m_\phi^2)\imath\Delta_\phi^{++}(x;x') - \int d^D y[\imath M_\phi^{++}(x;y)\imath\Delta_\phi^{++}(y;x') - \imath M_\phi^{+-}(x;y)\imath\Delta_\phi^{-+}(y;x')] = \imath\delta^D(x-x'), \tag{35a}$$

$$(\partial_x^2 - m_\phi^2)\imath\Delta_\phi^{+-}(x;x') - \int d^D y[\imath M_\phi^{++}(x;y)\imath\Delta_\phi^{+-}(y;x') - \imath M_\phi^{+-}(x;y)\imath\Delta_\phi^{--}(y;x')] = 0, \tag{35b}$$

$$(\partial_x^2 - m_\phi^2)\imath\Delta_\phi^{-+}(x;x') - \int d^D y[\imath M_\phi^{-+}(x;y)\imath\Delta_\phi^{++}(y;x') - \imath M_\phi^{--}(x;y)\imath\Delta_\phi^{-+}(y;x')] = 0, \tag{35c}$$

$$(\partial_x^2 - m_\phi^2)\imath\Delta_\phi^{--}(x;x') - \int d^D y[\imath M_\phi^{-+}(x;y)\imath\Delta_\phi^{+-}(y;x') - \imath M_\phi^{--}(x;y)\imath\Delta_\phi^{--}(y;x')] = -\imath\delta^D(x-x'), \tag{35d}$$

but in the light of Eq. (18), only two of them are independent. Note that we have another set of four equations of motion for the χ field. In the end, we will be interested in solving this equation of motion in Fourier space, e.g.,

$$\imath\Delta_\phi^{ab}(x;x') = \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}}\imath\Delta_\phi^{ab}(\vec{k},t,t')e^{\imath\vec{k}(\vec{x}-\vec{x}')}, \tag{36a}$$

$$\imath\Delta_\phi^{ab}(\vec{k},t,t') = \int d^{D-1}(\vec{x}-\vec{x}')\imath\Delta_\phi^{ab}(x;x')e^{-\imath\vec{k}(\vec{x}-\vec{x}')}. \tag{36b}$$

Such that Eq. (35) transforms into

$$(\partial_t^2 + k^2 + m_\phi^2)\imath\Delta_\phi^{++}(k,t,t') + \int_{-\infty}^{\infty} dt_1[\imath M_\phi^{++}(k,t,t_1)\imath\Delta_\phi^{++}(k,t_1,t') - \imath M_\phi^{+-}(k,t,t_1)\imath\Delta_\phi^{-+}(k,t_1,t')] = \imath\delta(t-t'), \tag{37a}$$

$$(\partial_t^2 + k^2 + m_\phi^2)\imath\Delta_\phi^{+-}(k,t,t') + \int_{-\infty}^{\infty} dt_1[\imath M_\phi^{++}(k,t,t_1)\imath\Delta_\phi^{+-}(k,t_1,t') - \imath M_\phi^{+-}(k,t,t_1)\imath\Delta_\phi^{--}(k,t_1,t')] = 0, \tag{37b}$$

$$(\partial_t^2 + k^2 + m_\phi^2)\imath\Delta_\phi^{-+}(k,t,t') + \int_{-\infty}^{\infty} dt_1[\imath M_\phi^{-+}(k,t,t_1)\imath\Delta_\phi^{++}(k,t_1,t') - \imath M_\phi^{--}(k,t,t')\imath\Delta_\phi^{-+}(k,t_1,t')] = 0, \tag{37c}$$

$$(\partial_t^2 + k^2 + m_\phi^2)\imath\Delta_\phi^{--}(k,t,t') + \int_{-\infty}^{\infty} dt_1[\imath M_\phi^{-+}(k,t,t_1)\imath\Delta_\phi^{+-}(k,t_1,t') - \imath M_\phi^{--}(k,t,t_1)\imath\Delta_\phi^{--}(k,t_1,t')] = -\imath\delta(t-t'). \tag{37d}$$

Note that we have extended $t_0 \to -\infty$ in the equation above. Again, we have an analogous set of equations of motion for the χ field. In principle we can solve these coupled equations of motion only numerically in full generality. Our strategy is to push the analytical calculation forward as far as possible, before relying on numerical methods. Before we make an important simplifying assumption, let us first consider the renormalization of our theory.

III. RENORMALIZING THE KADANOFF-BAYM EQUATIONS

In order to renormalize equation of motion (35) or (37) above, we need to Fourier transform also with respect to the difference of the time variables:

$$\imath\Delta_\phi^{ab}(x;x') = \int \frac{d^D k}{(2\pi)^D}\imath\Delta_\phi^{ab}(k^\mu)e^{\imath k\cdot(x-x')}, \tag{38a}$$

$$\imath\Delta_\phi^{ab}(k^\mu) = \int d^D(x-x')\imath\Delta_\phi^{ab}(x;x')e^{-\imath k\cdot(x-x')}. \tag{38b}$$

There is a subtlety: for the moment we neglect the time dependence in the mass term. We only use this assumption to renormalize. In the end it turns out that we need a mass independent counterterm to cancel all divergences in our theory, which allows us to consider a time varying mass term again. In fact, as we assume there is no residual dependence on the average time coordinate $(t+t')/2$ in $\imath\Delta_\phi^{ab}(k^\mu)$, Eq. (38) coincides with a Wigner transform. Fourier transforming equation of motion (35) yields

$$(-k_\mu k^\mu - m_\phi^2 - \imath M_\phi^{++}(k^\mu))\imath\Delta_\phi^{++}(k^\mu) + \imath M_\phi^{+-}(k^\mu)\imath\Delta_\phi^{-+}(k^\mu) = \imath, \tag{39a}$$

$$(-k_\mu k^\mu - m_\phi^2 - \imath M_\phi^{++}(k^\mu))\imath\Delta_\phi^{+-}(k^\mu) + \imath M_\phi^{+-}(k^\mu)\imath\Delta_\phi^{--}(k^\mu) = 0, \tag{39b}$$

$$(-k_\mu k^\mu - m_\phi^2 + \imath M_\phi^{--}(k^\mu))\imath\Delta_\phi^{-+}(k^\mu) - \imath M_\phi^{-+}(k^\mu)\imath\Delta_\phi^{++}(k^\mu) = 0, \tag{39c}$$

$$(-k_\mu k^\mu - m_\phi^2 + \imath M_\phi^{--}(k^\mu))\imath\Delta_\phi^{--}(k^\mu) - \imath M_\phi^{-+}(k^\mu)\imath\Delta_\phi^{+-}(k^\mu) = -\imath. \tag{39d}$$

Here and henceforth, we use the notation $k_\mu k^\mu = -k_0^2 + k^2$ to distinguish the four-vector length from the spatial three-vector length $k = \| \vec{k} \|$. Because of the convolution, the equations of motion above are local in Fourier space. Let us remind the reader again that analogous equations hold for the χ propagators. As already announced in the Introduction, we shall not solve the dynamical equations for both ϕ and χ propagators. Instead, we shall assume the following hierarchy of couplings:

$$h \ll \lambda \tag{40}$$

and expand the solution in powers of $h/\lambda \ll 1$. In fact, we shall solve the system only at order $(h/\lambda)^0$. This does not imply that the $h\chi^2\phi$ interaction is unimportant: we will only assume that λ is large such that the χ field is thermalized by its strong self-interaction. This allows us to approximate the solutions of the dynamical equations for χ as thermal propagators which we derived in the appendix in Eq. (B10), see [101]:

$$\imath\Delta_\chi^{++}(k^\mu) = \frac{-\imath}{k_\mu k^\mu + m_\chi^2 - \imath\epsilon} + 2\pi\delta(k_\mu k^\mu + m_\chi^2) n_\chi^{\rm eq}(|k_0|), \tag{41a}$$

$$\imath\Delta_\chi^{--}(k^\mu) = \frac{\imath}{k_\mu k^\mu + m_\chi^2 + \imath\epsilon} + 2\pi\delta(k_\mu k^\mu + m_\chi^2) n_\chi^{\rm eq}(|k^0|), \tag{41b}$$

$$\imath\Delta_\chi^{+-}(k^\mu) = 2\pi\delta(k_\mu k^\mu + m_\chi^2)[\theta(-k^0) + n_\chi^{\rm eq}(|k^0|)], \tag{41c}$$

$$\imath\Delta_\chi^{-+}(k^\mu) = 2\pi\delta(k_\mu k^\mu + m_\chi^2)[\theta(k^0) + n_\chi^{\rm eq}(|k^0|)], \tag{41d}$$

where the Bose-Einstein distribution is given by

$$n_\chi^{\rm eq}(k^0) = \frac{1}{e^{\beta k^0} - 1}, \qquad \beta = \frac{1}{k_B T}, \tag{42}$$

with k_B denoting the Stefan-Boltzmann constant and T the temperature. Let us remark that assumption (40) allows us to compute the quantum corrections to the ϕ propagators as it depends solely on χ propagators running in the loop. We neglect the backreaction of the system field on the environment field, such that the latter remains in thermal equilibrium at temperature T. This assumption is perturbatively well justified as already discussed in the Introduction. Furthermore, we neglected for simplicity the $\mathcal{O}(\lambda^2)$ correction to the propagators above that will slightly change the equilibrium of the environment field. Note finally that, in our approximation scheme, the dynamics of the system propagators is effectively influenced only by the usual 1PI self-mass correction.

In this paper, we consider only an environment field χ in its vacuum state at $T = 0$ and we postpone the finite temperature corrections to a future publication. Any divergences, if present, originate from the vacuum contributions to the self-masses, i.e., the vacuum propagators at $T = 0$ are a useful case to consider anyway:

$$\imath\Delta_\chi^{++}(k^\mu) = \frac{-\imath}{k_\mu k^\mu + m_\chi^2 - \imath\epsilon}, \tag{43a}$$

$$\imath\Delta_\chi^{--}(k^\mu) = \frac{\imath}{k_\mu k^\mu + m_\chi^2 + \imath\epsilon}, \tag{43b}$$

$$\imath\Delta_\chi^{+-}(k^\mu) = 2\pi\delta(k_\mu k^\mu + m_\chi^2)\theta(-k^0), \tag{43c}$$

$$\imath\Delta_\chi^{-+}(k^\mu) = 2\pi\delta(k_\mu k^\mu + m_\chi^2)\theta(k^0). \tag{43d}$$

We evaluate the Feynman self-mass $\imath M_\phi^{++}(x;x')$ following from Eq. (34a) where we make the simplifying assumption $m_\chi \to 0$. Let us briefly expatiate justifying this assumption as at first sight it seems that $m_\chi \to 0$ makes our approximation scheme more susceptible to undesired backreaction effects.[3] It is *a priori* not at all clear that the backreaction is negligible: if we examine the second Feynman diagram on the right-hand side of Eq. (4)b we see that the leading order backreaction occurs at order $\mathcal{O}(h^2/\omega_\chi^2)$. Since in our setup $\omega_\chi^2 = k^2 + m_\chi^2 = k^2$, it is clear that the backreaction on deep IR Fourier modes of the environment field is perturbatively unsuppressed. Despite that, it does not spoil the perturbative arguments employed in the Introduction: the influence of the environment field on the system field is still perturbatively under control. In order to see this, let us consider the first non-Gaussian contribution on the right-hand side of Eq. (5). Indeed, one can show that the IR part of the inner loop is phase space suppressed: the IR part of this integral is given by $\int_0^{\bar\mu} d^4q[(k^\nu k_\nu - m_\phi^2)(q_\sigma + k_\sigma) \times (q^\sigma + k^\sigma)]^{-1} \sim \bar\mu^2/m_\phi^2$ when $m_\phi > h \simeq \bar\mu$. Nevertheless, we admit it would be worthwhile to examine these integrals for $m_\chi \neq 0$ and see whether the results presented in this paper are robust under this change.

We thus need to evaluate

$$\imath\Delta_\chi^{++}(x;x') = \int \frac{d^D k}{(2\pi)^D} \imath\Delta_\chi^{++}(k^\mu) e^{\imath k(x-x')}. \tag{44}$$

This integral can be performed in arbitrary dimensions by making use of two straightforward contour integrations and [102,103]:

[3] We thank Julien Serreau for this useful comment.

$$\int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} e^{i\vec{k}\cdot\vec{x}} f(k) = \frac{2}{(4\pi)^{D-1/2}} \times \int_0^\infty dk k^{D-2} \frac{J_{D-3/2}(kx)}{(\frac{1}{2}kx)^{D-3/2}} f(k), \tag{45}$$

which is valid for any function $f(k)$ that depends solely on $k = \| \vec{k} \|$. $J_\mu(kx)$ is a Bessel function of the first kind. This yields

$$\imath\Delta_\chi^{++}(x;x') = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}} \frac{1}{\Delta x_{++}^{D-2}(x;x')}. \tag{46}$$

Here, $\Delta x_{++}^2(x,x')$ is one of the distance functions between two spacetime points x and x' frequently used in the Schwinger-Keldysh formalism and given by

$$\Delta x_{++}^2(x,x') = -(|t-t'| - i\varepsilon)^2 + \|\vec{x}-\vec{x}'\|^2, \tag{47a}$$

$$\Delta x_{+-}^2(x,x') = -(t-t' + i\varepsilon)^2 + \|\vec{x}-\vec{x}'\|^2, \tag{47b}$$

$$\Delta x_{-+}^2(x,x') = -(t-t' - i\varepsilon)^2 + \|\vec{x}-\vec{x}'\|^2, \tag{47c}$$

$$\Delta x_{--}^2(x,x') = -(|t-t'| + i\varepsilon)^2 + \|\vec{x}-\vec{x}'\|^2. \tag{47d}$$

We thus immediately find from Eqs. (34a) and (46)

$$\imath M_\phi^{++}(x;x') = -\frac{\imath h^2}{2} \frac{\Gamma^2(\frac{D}{2}-1)}{16\pi^D} \frac{1}{\Delta x_{++}^{2D-4}(x;x')}. \tag{48}$$

The other self-masses can be obtained from this expression using the appropriate ε pole prescription as indicated in Eq. (47). We will now rewrite this expression slightly in order to extract the divergence. For an arbitrary exponent $\beta \neq D$, $\beta \neq 2$, we can easily derive

$$\frac{1}{\Delta x_{++}^{\beta}(x;x')} = \frac{1}{(\beta-2)(\beta-D)} \partial^2 \frac{1}{\Delta x_{++}^{\beta-2}(x;x')}. \tag{49}$$

Furthermore, recall [104,105]

$$\partial^2 \frac{1}{\Delta x_{++}^{D-2}(x;x')} = \frac{4\pi^{D/2}}{\Gamma(\frac{D-2}{2})} \imath\delta^D(x-x'). \tag{50a}$$

Let us also recall the similar identities for the other distance functions:

$$\partial^2 \frac{1}{\Delta x_{--}^{D-2}(x;x')} = -\frac{4\pi^{D/2}}{\Gamma(\frac{D-2}{2})} \imath\delta^D(x-x'), \tag{50b}$$

$$\partial^2 \frac{1}{\Delta x_{+-}^{D-2}(x;x')} = 0, \tag{50c}$$

$$\partial^2 \frac{1}{\Delta x_{-+}^{D-2}(x;x')} = 0. \tag{50d}$$

We now arrange Eq. (48), using (49) and (50a)

$$\imath M_\phi^{++}(x;x') = -\frac{\imath h^2\Gamma^2(\frac{D}{2}-1)}{64\pi^D} \frac{1}{(D-3)(D-4)} \times \Big[\partial^2 \Big\{ \frac{1}{\Delta x_{++}^{2D-6}(x;x')} - \frac{\mu^{D-4}}{\Delta x_{++}^{D-2}(x;x')} \Big\} + \frac{4\pi^{D/2}\mu^{D-4}}{\Gamma(\frac{D-2}{2})} \imath\delta^D(x-x') \Big]. \tag{51}$$

Here, the scale μ has been introduced on dimensional grounds. If we Taylor expand the term in curly brackets[4] around $D=4$, we find

$$\imath M_\phi^{++}(x;x') = -\frac{\imath h^2\Gamma(\frac{D}{2}-1)\mu^{D-4}}{16\pi^{D/2}(D-3)(D-4)} \imath\delta^D(x-x') + \frac{\imath h^2}{128\pi^4} \partial^2 \Big[\frac{\log(\mu^2\Delta x_{++}^2(x;x'))}{\Delta x_{++}^2(x;x')} \Big] + \mathcal{O}(D-4). \tag{52}$$

We have been able to separate a local $(D-4)^{-1}$ divergence and a nonlocal finite term to the self-mass. In order to precisely cancel the divergence, we can thus add a local counterterm, i.e., an ordinary mass term of the form:

$$\imath M_{\phi,\mathrm{ct}}^{\pm\pm}(x;x') = \mp \frac{\imath h^2\Gamma(\frac{D}{2}-1)\mu^{D-4}}{16\pi^{D/2}(D-3)(D-4)} \imath\delta^D(x-x'). \tag{53}$$

The relative sign difference of $\imath M_{\phi,\mathrm{ct}}^{--}(x;x')$ is due to Eq. (50b). We are left with the following renormalized self-mass:

$$\imath M_{\phi,\mathrm{ren}}^{++}(x;x') = -\frac{\imath h^2\Gamma^2(\frac{D}{2}-1)}{64\pi^D} \frac{1}{(D-3)(D-4)} \times \partial^2 \Big\{ \frac{1}{\Delta x_{++}^{2D-6}(x;x')} - \frac{\mu^{D-4}}{\Delta x_{++}^{D-2}(x;x')} \Big\}. \tag{54}$$

We will now perform a spatial Fourier transform in order to solve for the dynamics this term generates:

$$\imath M_{\phi,\mathrm{ren}}^{++}(\vec{k},t,t') = \int d^{D-1}(\vec{x}-\vec{x}') \imath M_{\phi,\mathrm{ren}}^{++}(x;x') e^{-\imath\vec{k}(\vec{x}-\vec{x}')}. \tag{55}$$

By introducing a regulator in order to dispose of the overall surface terms arising from two partial integrations, we can easily convert the partial derivatives. Using several analytic extensions, we obtain

[4]Note that in the minimal subtraction scheme, one would also expand the term multiplying the Dirac delta function around $D=4$, which gives rise, once integrated at the level of the equation of motion, to a finite local contribution to the mass of ϕ.

$$\imath M^{++}_{\phi,\mathrm{ren}}(k,t,t') = \frac{\imath h^2\Gamma^2(\frac{D}{2}-1)2^{(D-13)/2}\pi^{-(D+1)/2}}{k^{(D-3)/2}(D-3)(D-4)}(\partial_t^2+k^2) \times\left[-\frac{\mu^{D-4}2^{(3-D)/2}\pi^{1/2}k^{(D-5)/2}}{\Gamma(\frac{D-2}{2})}e^{-\imath k(|\Delta t|-\imath\epsilon)} + \frac{k^{D-4}(\imath|\Delta t|+\epsilon)^{(5-D)/2}}{2^{D-4}\Gamma(D-3)}K_{(D-5)/2}(k(i|\Delta t|+\epsilon))\right], \tag{56}$$

where $k = \|\vec{k}\|$, $\Delta t = t - t'$ and where $K_\nu(z)$ is the modified Bessel function of the second kind. We expand this result around $D = 4$:

$$\imath M^{++}_{\phi,\mathrm{ren}}(k,t,t') = \frac{\imath h^2}{32\pi^2\sqrt{2k\pi}}(\partial_t^2+k^2) \times\left[\sqrt{\frac{\pi}{2k}}e^{-\imath k(|\Delta t|-\imath\epsilon)}\left(\gamma_e + \log\left[\frac{k}{2\imath\mu^2(|\Delta t|-\imath\epsilon)}\right]\right) - \sqrt{\imath|\Delta t|+\epsilon}\,\partial_\nu K_\nu(\imath k(|\Delta t|-\imath\epsilon))|_{\nu=1/2}\right] + \mathcal{O}(D-4). \tag{57}$$

Here, γ_E is the Euler-Mascheroni constant. Moreover, the scale μ introduced earlier combines nicely with the other terms to make the argument of the logarithm dimensionless as it should. Indeed, we need to find an expression for the derivative with respect to the order ν of $K_\nu(z)$. Starting from the general expansion:

$$K_\nu(z) = \frac{\pi\csc(\pi\nu)}{2}\sum_{k=0}^{\infty}\left\{\frac{1}{\Gamma(k-\nu+1)k!}\left(\frac{z}{2}\right)^{2k-\nu} - \frac{1}{\Gamma(k+\nu+1)k!}\left(\frac{z}{2}\right)^{2k+\nu}\right\}, \tag{58}$$

we immediately derive

$$\partial_\nu K_\nu(z)|_{\nu=1/2} = -\sqrt{\frac{\pi}{2z}}e^z[\mathrm{Chi}(2z) - \mathrm{Shi}(2z)] = -\sqrt{\frac{\pi}{2z}}e^z\left[\gamma_E + \log(2z) + \int_0^{2z}dt\frac{\cosh t - 1}{t} - \int_0^{2z}dt\frac{\sinh t}{t}\right], \tag{59}$$

where Chi$(2z)$ and Shi$(2z)$ are the hyperbolic cosine and hyperbolic sine integral functions, respectively, defined by the expressions on the second line. In our case, the variable z is imaginary, so it proves useful to extract an $\imath$ and convert this expression to the somewhat more familiar sine and cosine integral functions, defined by

$$\mathrm{si}(z) = -\int_z^\infty dt\frac{\sin t}{t}, \tag{60a}$$

$$\mathrm{ci}(z) = -\int_z^\infty dt\frac{\cos t}{t}. \tag{60b}$$

We finally arrive at

$$\imath M^{++}_{\phi,\mathrm{ren}}(k,t,t') = \frac{\imath h^2}{64k\pi^2}(\partial_t^2+k^2) \times\left[e^{-\imath k|\Delta t|}\left(\gamma_E + \log\left[\frac{k}{2\imath\mu^2(|\Delta t|-\imath\epsilon)}\right]\right) + e^{\imath k|\Delta t|}(\mathrm{ci}(2k(|\Delta t|-\imath\epsilon)) - \imath\mathrm{si}(2k(|\Delta t|-\imath\epsilon)))\right] + \mathcal{O}(D-4), \tag{61}$$

where we have set the ϵ regulators in the exponents to zero as the expression is well defined. Rather than going several times through the calculation above to determine the other self-masses, we make use of a few analytic extensions. Observe, for example, that if $\Delta t > 0$, $\Delta x_{++}(x,x')$ and $\Delta x_{-+}(x,x')$ coincide; hence the expressions for self-masses $\imath M^{++}_{\phi,\mathrm{ren}}(k,t,t')$ and $\imath M^{-+}_{\phi}(k,t,t')$ should also coincide in that region. All we need to do is to sensibly analytically extend to $\Delta t < 0$. We will thus need

$$\mathrm{si}(-z) = -\mathrm{si}(z) - \pi. \tag{62}$$

If $\Delta t < 0$, we have to carefully make use of the ϵ pole prescription in the cosine integral function:

$$\begin{aligned}\mathrm{ci}(-2k(-\Delta t+\imath\epsilon)) &= -\int_{-2k(-\Delta t+\imath\epsilon)}^{\infty}dt\frac{\cos t}{t} \\ &= -\left[\int_{-2k(-\Delta t)}^{-\imath\epsilon}dt + \int_{-\imath\epsilon}^{2k(-\Delta t)}dt + \int_{2k(-\Delta t)}^{\infty}dt\right]\frac{\cos t}{t} \\ &= -\log(\imath\epsilon) + \log(-\imath\epsilon) + \mathrm{ci}(2k(-\Delta t)) \\ &= -\imath\pi + \mathrm{ci}(2k(-\Delta t)).\end{aligned} \tag{63}$$

We thus find the following expressions for the renormalized self-masses:

$$\imath M^{ab}_{\phi,\mathrm{ren}}(k,t,t') = (\partial_t^2+k^2)\imath Z^{ab}_{\phi}(k,t,t'), \tag{64a}$$

where

$$Z_\phi^{\pm\pm}(k,t,t') = \frac{h^2}{64k\pi^2}\bigg[e^{\mp\imath k|\Delta t|}\bigg(\gamma_{\rm E} + \log\bigg[\frac{k}{2\mu^2|\Delta t|}\bigg] \mp \imath\frac{\pi}{2}\bigg) + e^{\pm\imath k|\Delta t|}({\rm ci}(2k|\Delta t|) \mp \imath{\rm si}(2k|\Delta t|))\bigg], \quad (64b)$$

$$Z_\phi^{\mp\pm}(k,t,t') = \frac{h^2}{64k\pi^2}\bigg[e^{\mp\imath k\Delta t}\bigg(\gamma_{\rm E} + \log\bigg[\frac{k}{2\mu^2|\Delta t|}\bigg] \mp \imath\frac{\pi}{2}{\rm sgn}(\Delta t)\bigg) + e^{\pm\imath k\Delta t}({\rm ci}(2k|\Delta t|) \mp \imath{\rm sgn}(\Delta t){\rm si}(2k|\Delta t|))\bigg]. \quad (64c)$$

First, appreciate that $\imath M_\phi^{-+}(k,t,t')$ and $\imath M_\phi^{+-}(k,t,t')$ need not be renormalized. The reason is that these expressions do not contain a divergence in $D=4$, which can be seen from Eqs. (50c) and (50d). Moreover, the local counterterm which we add to renormalize $\imath M_\phi^{--}(k,t,t')$ contains the opposite sign as compared to $\imath M_\phi^{++}(k,t,t')$ because of Eq. (50b), which we already stated in Eq. (53). Finally, we have sent all ϵ regulators to zero as the expression above is well defined in the limit $\Delta t \to 0$.

We performed two independent checks of the calculation above. First, one can renormalize via a calculation in Fourier space (rather than position space). We show that the two results agree in Appendix C. Second, one can calculate the retarded self-mass directly from the position space result using (52) and compare with the result obtained from (64). We show that the two results agree in Appendix D.

If one were to evaluate the two time derivatives in the expressions above, one would find a divergent answer in the limit when $\Delta t \to 0$. We also show this in Appendix C. This does not reflect an incorrect renormalization procedure. It is crucial to extract the two time derivatives as presented above in order to properly take the effect of the self-masses into account as only now $Z_\phi^{ab}(k,t,t')$ is finite at coincidence $\Delta t \to 0$. Indeed, this is most easily seen in position space.[5]

Let us compare these expressions with existing literature. In, e.g., [48,66] it is derived that the renormalized equations for $\lambda\phi^4(x)$ theory have an identical structure as the unrenormalized equations. In our theory, clearly, the structure of the two equations changes as we need to extract an operator of the form $(\partial_t^2 + k^2)$, as derived in Eq. (64).

IV. DECOUPLING THE KADANOFF-BAYM EQUATIONS

Having renormalized our theory, we are ready to massage the Kadanoff-Baym equations (37) in two different ways. First, we will write Kadanoff-Baym equations in terms of the causal and statistical propagator such that they decouple. This is of course a vital step required to solve the Kadanoff-Baym equations in the next section. Second, we show that when we write the equations in terms of the advanced and retarded propagators, the one-loop contributions preserve causality as they should.

Note that the structure of the self-mass (34) is such that we can construct relations analogous to Eq. (25), which of course hold identically for χ:

$$Z_\phi^{+-}(k,t,t') = Z_\phi^F(k,t,t') - \tfrac{1}{2}\imath Z_\phi^c(k,t,t'), \quad (65a)$$

$$Z_\phi^{-+}(k,t,t') = Z_\phi^F(k,t,t') + \tfrac{1}{2}\imath Z_\phi^c(k,t,t'), \quad (65b)$$

$$Z_\phi^{++}(k,t,t') = Z_\phi^F(k,t,t') + \tfrac{1}{2}{\rm sgn}(t-t')\imath Z_\phi^c(k,t,t'), \quad (65c)$$

$$Z_\phi^{--}(k,t,t') = Z_\phi^F(k,t,t') - \tfrac{1}{2}{\rm sgn}(t-t')\imath Z_\phi^c(k,t,t'), \quad (65d)$$

such that we find from (64):

$$Z_\phi^F(k,t,t') = \frac{1}{2}[Z_\phi^{-+}(k,t,t') + Z_\phi^{+-}(k,t,t')] = \frac{h^2}{64k\pi^2}\bigg[\cos(k\Delta t)\bigg(\gamma_{\rm E} + \log\bigg[\frac{k}{2\mu^2|\Delta t|}\bigg] + {\rm ci}(2k|\Delta t|)\bigg) + \sin(k|\Delta t|)\bigg({\rm si}(2k|\Delta t|) - \frac{\pi}{2}\bigg)\bigg], \quad (66a)$$

$$Z_\phi^c(k,t,t') = \imath[Z_\phi^{+-}(k,t,t') - Z_\phi^{-+}(k,t,t')] = \frac{h^2}{64k\pi^2}\bigg[-2\cos(k\Delta t){\rm sgn}(\Delta t)\bigg({\rm si}(2k|\Delta t|) + \frac{\pi}{2}\bigg) + 2\sin(k\Delta t)\bigg({\rm ci}(2k|\Delta t|) - \gamma_{\rm E} - \log\bigg[\frac{k}{2\mu^2|\Delta t|}\bigg]\bigg)\bigg]. \quad (66b)$$

The expressions for χ differ due to (34). We can derive a system of two closed equations for the causal and statistical propagator by adding and subtracting Eqs. (37c) and (37b). In order to obtain the equation of motion for the causal propagator (7), we subtract (37b) from (37c) to find

[5] One can easily recognize that the structure of the renormalized self-masses in Eq. (64) is identical to the d'Alembertian in Fourier space. The presence of $Z_\phi^{ab}(k,t,t')$ induces time dependence in the propagator. A similar phenomenon has been observed in [106], where this phenomenon is referred to as a "finite wave function renormalization," in which the effect of gravitons on fermions in an expanding Universe is investigated.

$$(\partial_t^2 + k^2 + m_\phi^2)\imath\Delta_\phi^c(k,t,t') + \frac{1}{2}\int_{-\infty}^{\infty} dt_1[\{\imath M_\phi^{-+}(k,t,t_1) - \imath M_\phi^{+-}(k,t,t_1)\}\mathrm{sgn}(t_1 - t') + \imath M_{\phi,\mathrm{ren}}^{++}(k,t,t_1) - \imath M_{\phi,\mathrm{ren}}^{--}(k,t,t_1)]\imath\Delta_\phi^c(k,t_1,t') = 0. \quad (67)$$

Using Eqs. (64) and (65) we find

$$(\partial_t^2 + k^2 + m_\phi^2)\Delta_\phi^c(k,t,t') - (\partial_t^2 + k^2)\int_{t'}^{t} dt_1 Z_\phi^c(k,t,t_1)\Delta_\phi^c(k,t_1,t') = 0. \quad (68)$$

Note that Eq. (68) is causal, in the sense that no knowledge in the future of the maximum of t, t' is needed to specify $\imath\Delta_\phi^c(k,t,t')$. Moreover, at one loop the evolution of $\imath\Delta_\phi^c$ requires only knowledge of the Green's functions in the time interval between t' and t, and is thus independent of the initial conditions at $t_0 = -\infty$. Finally note that we deleted the $\imath$ in front of $\imath\Delta_\phi^c$ in the equation of motion above to stress that Δ_ϕ^c is real to prepare this equation for numerical integration.

In order to get an equation for the statistical Hadamard function (6), we add Eq. (35b) to (35c) to get

$$(\partial_t^2 + k^2 + m_\phi^2)F_\phi(k,t,t') + \frac{1}{2}\int_{-\infty}^{\infty} dt_1[\imath M_{\phi,\mathrm{ren}}^{++}(k,t,t_1) - \imath M_\phi^{+-}(k,t,t_1) + \imath M_\phi^{-+}(k,t,t_1) - \imath M_{\phi,\mathrm{ren}}^{--}(k,t,t_1)]F_\phi(k,t_1,t') + \frac{1}{4}\int_{-\infty}^{\infty} dt_1[\{\imath M_\phi^{+-}(k,t,t_1) + \imath M_\phi^{-+}(k,t,t_1)\}\mathrm{sgn}(t_1 - t') - \imath M_{\phi,\mathrm{ren}}^{++}(k,t,t_1) - \imath M_{\phi,\mathrm{ren}}^{--}(k,t,t_1)]\imath\Delta_\phi^c(k,t_1,t') = 0. \quad (69)$$

Again using (64) and (65) we find the relevant differential equation for the statistical propagator:

$$(\partial_t^2 + k^2 + m_\phi^2)F_\phi(k,t,t') - (\partial_t^2 + k^2) \times \left[\int_{-\infty}^{t} dt_1 Z_\phi^c(k,t,t_1)F_\phi(k,t_1,t') - \int_{-\infty}^{t'} dt_1 Z_\phi^F(k,t,t_1)\Delta_\phi^c(k,t_1,t')\right] = 0. \quad (70)$$

We can thus say that the equations of motion for the causal and statistical propagator have decoupled in the following sense: the differential equations haven been brought in triangular form. Note that Eqs. (68) and (70) together with the causal and statistical self-masses in Eq. (66) represent a closed causal system of equations suitable for integration in terms of an initial value problem. Given the knowledge of F and $\imath\Delta^c$ for both χ and ϕ, all other Green's functions can be reconstructed from Eq. (25). This strategy was used (see [48] and references therein) to study the dynamics of out-of-equilibrium quantum statistical (scalar and fermionic) field theories. Indeed, we will solve Eqs. (68) and (70) numerically in the next section. We emphasize however that the form of Eqs. (68) and (70) differs from the ones found in [48]. The renormalized equations of motion have a different structure than the unrenormalized ones, which is not taken into account in, e.g., [48,66].

Before doing so, let us show that the one-loop self-masses do not spoil causality in another way: the retarded and advanced Green's functions only receive information from the past and future light cone, respectively. Now subtracting Eq. (37b) from (37a) one obtains

$$(\partial_t^2 + k^2 + m_\phi^2)\imath\Delta_\phi^r(k,t,t') + \int_{-\infty}^{\infty} dt_1 \imath M_{\phi,\mathrm{ren}}^r(k,t,t_1)\imath\Delta_\phi^r(k,t_1,t') = \imath\delta(t - t'). \quad (71)$$

Making use of Eq. (D4) we find

$$\imath M_{\phi,\mathrm{ren}}^r(k,t,t_1) = -(\partial_t^2 + k^2)\theta(t - t_1)Z_\phi^c(k,t,t_1). \quad (72)$$

Equation of motion (71) transforms into

$$(\partial_t^2 + k^2 + m_\phi^2)\imath\Delta_\phi^r(k,t,t') - (\partial_t^2 + k^2)\int_{-\infty}^{t} dt_1 Z_\phi^c(k,t,t_1)\imath\Delta_\phi^r(k,t_1,t') = \imath\delta(t - t'). \quad (73)$$

The retarded self-mass gets contributions only from within the past light cone, i.e., when $t_1 < t$.

Similar to Eq. (73), we can subtract Eq. (37c) from (37a) to obtain the equation of motion for the advanced propagator:

$$(\partial_t^2 + k^2 + m_\phi^2)\imath\Delta_\phi^a(k,t,t') + \int_{-\infty}^{\infty} dt_1[(\imath M_{\phi,\mathrm{ren}}^{++}(k,t,t_1) - \imath M_\phi^{-+}(k,t,t_1))\imath\Delta_\phi^{++}(k,t_1,t') - (\imath M_\phi^{+-}(k,t,t_1) - \imath M_{\phi,\mathrm{ren}}^{--}(k,t,t_1))\imath\Delta_\phi^{-+}(k,t_1,t')] = \imath\delta(t - t'). \quad (74)$$

This yields

$$(\partial_t^2 + k^2 + m_\phi^2)\imath\Delta_\phi^a(k,t,t') - (\partial_t^2 + k^2)\int_{t}^{\infty} dt_1 Z_\phi^c(k,t,t_1)\imath\Delta_\phi^a(k,t_1,t') = \imath\delta(t - t'), \quad (75)$$

where we find an analogous relation for the advanced self-mass:

$$\imath M^{a}_{\phi,\mathrm{ren}}(k,t,t_1) = -(\partial_t^2 + k^2)\theta(t_1 - t)Z^{c}_{\phi}(k,t,t_1). \quad (76)$$

As expected, $\imath M^{a}_{\phi,\mathrm{ren}}(k,t,t_1)$ acquires contributions from the future only, i.e., when $t_1 > t$. Rather than solving for the causal propagator, we could alternatively solve for the retarded propagator or the advanced propagator. We will however not pursue this in the present work.

V. NUMERICALLY SOLVING THE KADANOFF-BAYM EQUATIONS

Let us once more explicitly write down the equations of motion of the causal and statistical propagators (68) and (70) we will numerically tackle in this section:

$$(\partial_t^2 + k^2 + m_\phi^2)\Delta^c_\phi(k,t,t') - (\partial_t^2 + k^2)\int_{t'}^{t} dt_1 Z^c_\phi(k,t,t_1)\Delta^c_\phi(k,t_1,t') = 0, \quad (77a)$$

$$(\partial_t^2 + k^2 + m_\phi^2)F_\phi(k,t,t') - (\partial_t^2 + k^2)\left[\int_{-\infty}^{t} dt_1 Z^c_\phi(k,t,t_1)F_\phi(k,t_1,t') - \int_{-\infty}^{t'} dt_1 Z^F_\phi(k,t,t_1)\Delta^c_\phi(k,t_1,t')\right] = 0. \quad (77b)$$

The causal and statistical self-masses are given in Eq. (66). In particular, we will be interested in two cases:

$$m_\phi(t) = m_0 = \mathrm{const}, \quad (78a)$$

$$m_\phi^2(t) = A + B\tanh(\rho t - 50), \quad (78b)$$

where we let A and B take different values. Let us take a closer look at the two equations of motion above. Clearly, we first need to determine the causal propagator. Note that equations of motion (77) depend on two variables, i.e., for each t', we have to solve this equation of motion.[6] The self-mass corrections contribute only through a "memory kernel" (memory integral over time) between t' and t. The boundary conditions for determining the causal propagator are as follows:

$$\Delta^c_\phi(t,t) = 0, \quad (79a)$$

$$\partial_t \Delta^c_\phi(t,t')|_{t=t'} = -1. \quad (79b)$$

Condition (79a) has to be satisfied by definition and condition (79b) follows from the Wronskian normalization condition due to the commutation relations.

Once we have solved for the causal propagator, we turn our attention to the second equation (77b). Suppose we would not have sent $t_0 \to -\infty$. The equation for the statistical propagator then would have been of the following form:

$$(\partial_t^2 + k^2 + m_\phi^2)F_\phi(k,t,t') - (\partial_t^2 + k^2)\left[\int_{t_0}^{t} dt_1 Z^c_\phi(k,t,t_1)F_\phi(k,t_1,t') - \int_{t_0}^{t'} dt_1 Z^F_\phi(k,t,t_1)\Delta^c_\phi(k,t_1,t')\right] = 0. \quad (80)$$

Clearly, Eqs. (77b) and (80) are not equivalent. Equation (77b) contains a memory kernel from the infinite past up to t and t', whereas Eq. (80) only contains memory kernels from t_0 onward. This corresponds to an interaction that is switched on nonadiabatically at time t_0. To understand this, consider replacing the coupling constant h hidden in the self-masses (66) with[7]

$$h \to h\theta(t_1 - t_0). \quad (81)$$

The step function would then have transformed Eq. (77b) to (80) which mimics switching on the interaction between the two fields at some finite time t_0. The two standard Schwinger-Keldysh contours presented in Figs. 1 and 2 are thus not equivalent in interacting quantum field theories where memory effects play an important role. Alternatively, we could say that nonlocality, generic for any interacting quantum field theory, enforces the memory kernel to start at the infinite past. This effect has, in the context of electromagnetic radiation, been recognized and investigated by Serreau [107]. In the work of Borsanyi and Reinosa [67,68], the memory integral, extended to negative infinity, plays an important role too. They suggest to use that in connection with a generalized dissipation-fluctuation theorem.

Needless to say, we have to start at some finite time in our numerical analysis. We therefore make the assumption to approximate the propagators in the memory kernels from the negative past to t_0 with the free propagators:

[6] Alternatively, we could have written down the equations of motion of the causal and statistical propagator where the operator acts on the other leg of the propagator, on t'. Then we would have to solve these four equations of motion simultaneously. Needless to say the two methods are completely equivalent.

[7] Note that $t > t_1$ by construction.

$$(\partial_t^2 + k^2 + m_\phi^2)F_\phi(k,t,t') - (\partial_t^2 + k^2)\bigg[\int_{-\infty}^{t_0} dt_1 Z_\phi^c(k,t,t_1)F_\phi^{\rm free}(k,t_1,t') + \int_{t_0}^{t} dt_1 Z_\phi^c(k,t,t_1)F_\phi(k,t_1,t') - \int_{-\infty}^{t_0} dt_1 Z_\phi^F(k,t,t_1)\Delta_\phi^{c,\rm free}(k,t_1,t') - \int_{t_0}^{t'} dt_1 Z_\phi^F(k,t,t_1)\Delta_\phi^c(k,t_1,t')\bigg] = 0, \quad (82)$$

where $F_\phi^{\rm free}(k, t_1, t')$ and $\Delta_\phi^{c,\rm free}(k, t_1, t')$ are the free propagators obtained in Eq. (A9). This approximation induces an error of the order $\mathcal{O}(h^4/\omega_\phi^4)$. An alternative approach has been outlined in [75] where, for $\lambda\phi^4(x)$ theory, non-Gaussian initial conditions at t_0 are imposed. We can explicitly evaluate the infinite past memory kernel:

$$M_F^{\rm free}(k,t,t',t_0) = (\partial_t^2 + k^2)\int_{-\infty}^{t_0} dt_1 [Z_\phi^c(k,t,t_1)F_\phi^{\rm free}(k,t_1,t') - Z_\phi^F(k,t,t_1)\Delta_\phi^{c,\rm free}(k,t_1,t')] = \frac{h^2}{32\pi^2\omega_{\rm in}}\int_{-\infty}^{t_0} dt_1 \frac{\cos(kt + \omega_{\rm in}t' - (k+\omega_{\rm in})t_1)}{t - t_1}. \quad (83)$$

We change variables to $\tau = t - t_1$ to find

$$M_F^{\rm free}(k,t,t',t_0) = \frac{h^2}{32\pi^2\omega_{\rm in}}\int_{t-t_0}^{\infty} d\tau \frac{\cos((k+\omega_{\rm in})\tau - \omega_{\rm in}(t-t'))}{\tau} = -\frac{h^2}{32\pi^2\omega_{\rm in}}[\cos(\omega_{\rm in}(t-t'))\mathrm{ci}((k+\omega_{\rm in})(t-t_0)) + \sin(\omega_{\rm in}(t-t'))\mathrm{si}((k+\omega_{\rm in})(t-t_0))]. \quad (84)$$

We postpone the discussion of imposing boundary conditions for F_ϕ at t_0 to Sec. VA. Equation (77) transforms into

$$(\partial_t^2 + k^2 + m_\phi^2)\Delta_\phi^c(k,t,t') - (\partial_t^2 + k^2)\int_{t'}^{t} dt_1 Z_\phi^c(k,t,t_1)\Delta_\phi^c(k,t_1,t') = 0, \quad (85a)$$

$$(\partial_t^2 + k^2 + m_\phi^2)F_\phi(k,t,t') - M_F^{\rm free}(k,t,t',t_0) - (\partial_t^2 + k^2)\bigg[\int_{t_0}^{t} dt_1 Z_\phi^c(k,t,t_1)F_\phi(k,t_1,t') - \int_{t_0}^{t'} dt_1 Z_\phi^F(k,t,t_1)\Delta_\phi^c(k,t_1,t')\bigg] = 0. \quad (85b)$$

We have now prepared the problem for numerical integration. In the numerical code, we take $t_0 = 0$ and we let ρt and $\rho t'$ run between 0 and 100. In order to solve differential equation (85b), we thus need to evaluate two more memory kernels. The one involving the causal propagator can be computed immediately. Once we have solved for the statistical propagator, our life becomes much easier as we can immediately find the phase space area via relation (9). The phase space area fixes the entropy.

Differential equation (85b) merits another remark. If we let $t \to t_0$, we encounter a logarithmic divergence in $M_F^{\rm free}(k, t, t', t_0)$ as $\mathrm{ci}(x) \propto \log(x)$ as $x \to 0$. This divergence is only apparent. Intuitively, this should of course be the case as we introduced the boundary time t_0 by hand and no divergences should arise consequently. If $h = $ const, the time t_0 is introduced as a fictitious time, and hence observables cannot depend on t_0. Of course, neglecting the memory integral from negative past infinity to t_0 introduces a dependence on t_0. Thus, removing the distant memory integrals completely is equivalent to setting $h \to h\theta(t_1 - t_0)$ as in Eq. (81). We will prove that this logarithmic divergence is only apparent rigorously by rewriting Eq. (85a) for the causal propagator and (85b) for the statistical propagator in a different form, and by using the symmetry properties of the propagators. Focusing first on the equation of motion for the causal propagator (85a), note that we can transfer the t derivative to a t_1 derivative by using the fact that the causal self-mass (66b) is a function of $\Delta t = t - t_1$ only:

$$\partial_t^2 \int_{t'}^{t} dt_1 Z_\phi^c(k,t,t_1)\Delta_\phi^c(k,t_1,t') = -\partial_t\bigg[\int_{t'}^{t} dt_1 \partial_{t_1} Z_\phi^c(k,t,t_1)\Delta_\phi^c(k,t_1,t')\bigg] = \partial_t\bigg[\int_{t'}^{t} dt_1 Z_\phi^c(k,t,t_1)\partial_{t_1}\Delta_\phi^c(k,t_1,t')\bigg] = -Z_\phi^c(k,t,t') + \int_{t'}^{t} dt_1 Z_\phi^c(k,t,t_1)\partial_{t_1}^2\Delta_\phi^c(k,t_1,t'), \quad (86)$$

where we partially integrated in the third line [the boundary terms vanish by virtue of Eq. (79a) and $Z_\phi^c(k, t, t) = 0$], and we used the commutation relations in the fourth. We transform the equation of motion of the statistical propagator (85b) analogously to find

$$(\partial_t^2 + k^2 + m_\phi^2)\Delta_\phi^c(k,t,t') + Z_\phi^c(k,t,t') - \int_{t'}^{t} dt_1 Z_\phi^c(k,t,t_1)(\partial_{t_1}^2 + k^2)\Delta_\phi^c(k,t_1,t') = 0, \tag{87a}$$

$$\begin{aligned}(\partial_t^2 + k^2 + m_\phi^2)F_\phi(k,t,t') &- M_F^{\rm free}(k,t,t',t_0) - [\partial_t Z_\phi^c(k,t,t_0)F_\phi(k,t_0,t') - \partial_t Z_\phi^F(k,t,t_0)\Delta_\phi^c(k,t_0,t') - Z_\phi^F(k,t,t') \\ &+ Z_\phi^c(k,t,t_0)\partial_{t_0}F_\phi(k,t_0,t') - Z_\phi^F(k,t,t_0)\partial_{t_0}\Delta_\phi^c(k,t_0,t')] - \int_{t_0}^{t} dt_1 Z_\phi^c(k,t,t_1)(\partial_{t_1}^2 + k^2)F_\phi(k,t_1,t') \\ &+ \int_{t_0}^{t'} dt_1 Z_\phi^F(k,t,t_1)(\partial_{t_1}^2 + k^2)\Delta_\phi^c(k,t_1,t') = 0.\end{aligned} \tag{87b}$$

These two differential equations should be completely equivalent to Eq. (85). In fact, we have found a nontrivial test of our numerical code: the results of Eq. (85) and of the two equations above should agree. We will show this in due course.

Now, we can see another logarithmic divergence appearing in $\partial_t Z_\phi^c(k,t,t_0)$ in Eq. (87b) when we send $t \to t_0$. The reader can easily verify that the logarithmic divergences in $M_F^{\rm free}(k,t,t',t_0)$ and $\partial_t Z_\phi^c(k,t,t_0)F_\phi(k,t_0,t')$ in Eq. (87b) above cancel to leave a finite result when $t \to t_0$ if we set $2F_\phi(k,t_0,t') = \cos(\omega_{\rm in}(t_0 - t'))/\omega_{\rm in}$. We thus find that at order $\mathcal{O}(h^2/\omega_\phi^2)$ no divergences at t_0 remain and we expect that a similar treatment would cure these types of apparent divergences at higher order: clearly, t_0 has been introduced by hand so this should not lead to any irregularities.

Let us finally make some remarks about the literature. The authors of [71–74] study out-of-equilibrium $\lambda\phi^4(x)$. They encounter, after renormalization, a residual divergence in their theory at the surface of initial boundary conditions at t_0 which they choose to renormalize separately. We differ in their approach as we do not find these residual divergences. The infinite past memory kernel precisely takes care of these as can be appreciated from the previous discussion. This is also the case in the approach of [67,68] mentioned before.

A. Constant mass solutions

The constant mass case is interesting as we can make nontrivial statements based on some analytical calculations. The bottom line is that the generated entropy is constant. The argument is rather simple. When $m_\phi =$ const, we have

$$F_\phi(k,t,t') = F_\phi(k,t-t'). \tag{88}$$

Using a few Fourier transforms (38), we have

$$F_\phi(k,0) = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} F_\phi(k^\mu), \tag{89a}$$

$$\partial_t F_\phi(k,\Delta t)|_{\Delta t=0} = -\imath \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} k^0 F_\phi(k^\mu), \tag{89b}$$

$$\partial_{t'}\partial_t F_\phi(k,\Delta t)|_{\Delta t=0} = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} k_0^2 F_\phi(k^\mu). \tag{89c}$$

The right-hand sides no longer contain any time dependence. Hence, the left-hand sides are also time independent. This implies that the phase space area Δ_k is constant, and so is the generated entropy. As we insert some finite t_0, we expect to observe some transient dependence of the entropy on time because we approximated the propagators in the infinite past memory kernel with free propagators. When this behavior has died out, the entropy should settle to its constant value derived from Eqs. (9), (10), and (89). Indeed, this constant entropy does not necessarily equal 0. We interpret this nonzero entropy as the entropy generated by the coupling to the second field, which in the effective action acts as a source for F_ϕ. Effectively, this opens up phase space for the system field that previously was inaccessible for it. More accessible phase space for the system field in turn implies that less information about the system field is accessible to us, and hence we observe an increase in entropy.

In order to evaluate the integrals above, we have derived the statistical propagator in Fourier space in Appendix E. The result is

$$\begin{aligned}F_\phi(k^\mu) = -\frac{\imath}{2}{\rm sgn}(k^0)\theta(k_0^2 - k^2)\Bigg[&\frac{1}{k_\mu k^\mu + m_\phi^2 + \frac{h^2}{32\pi^2}(\log(\frac{|k_\mu k^\mu|}{4\mu^2}) + 2\gamma_{\rm E}) - \frac{\imath h^2}{32\pi}{\rm sgn}(k^0)\theta(k_0^2 - k^2)} \\ &- \frac{1}{k_\mu k^\mu + m_\phi^2 + \frac{h^2}{32\pi^2}(\log(\frac{|k_\mu k^\mu|}{4\mu^2}) + 2\gamma_{\rm E}) + \frac{\imath h^2}{32\pi}{\rm sgn}(k^0)\theta(k_0^2 - k^2)}\Bigg].\end{aligned} \tag{90}$$

To gain some intuitive understanding, we depicted the statistical propagator for $h/\rho = 3/2$ and for $h/\rho = 4$ in Fig. 5. When $h/\rho = 0$, we have a δ-function dispersion relation as usual but in the presence of a nonzero coupling, the δ function broadens to a so-called "quasiparticle peak" of a Breit-Wigner form. For $h/\rho = 4$, this peak is still well pronounced, but when we enter the strongly coupled regime, this simple picture breaks down when the resonance

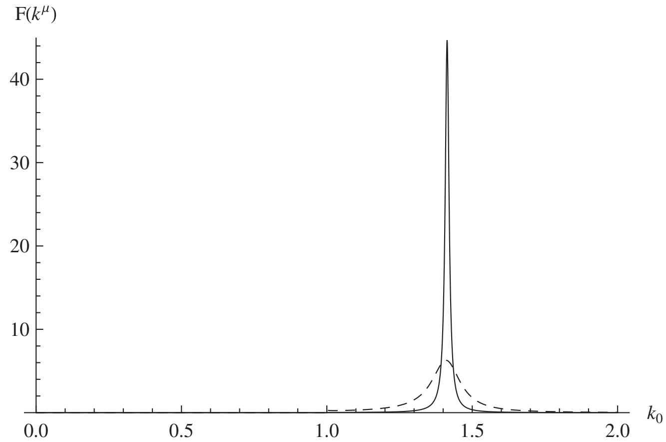


FIG. 5. Statistical propagator in Fourier space for a small coupling $h/\rho = 3/2$ (solid line) and a larger one $h/\rho = 4$ (dashed line). Because of a nonzero coupling, we observe that the δ function, present in the original dispersion relation, has broadened to a "quasiparticle peak," roughly of a Breit-Wigner form. If the coupling increases, the quasiparticle peak broadens further. Clearly, when $h \gg \omega_\phi$ in the strongly coupled regime, we have a "collection of quasiparticles." We used $k/\rho = 1$, $m_\phi/\rho = 1$, and $\mu/\rho = 1$.

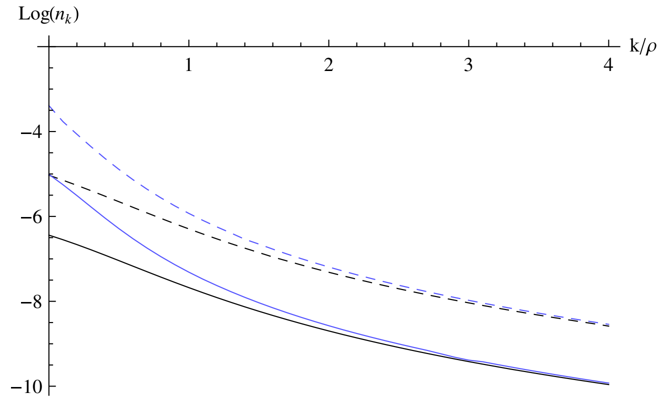


FIG. 6 (color online). Statistical particle number density n_k as a function of k/ρ. We used $h/\rho = 1$ and $m_\phi/\rho = 1$ (black solid line), $h/\rho = 2$ and $m_\phi/\rho = 1$ (black dashed line), $h/\rho = 1$ and $m_\phi/\rho = 0.5$ [gray (blue) line] and $h/\rho = 2$ and $m_\phi/\rho = 0.5$ [gray (blue) dashed line]. In the UV n_k vanishes irrespective of the value of h/ρ or m_ϕ/ρ. Particles are only produced by the interaction in the statistical sense in the IR. The mass only influences the IR behavior. Moreover, one can show by appropriate rescalings of the functions above that n_k is given by Eq. (94) in the UV.

becomes broad and we can no longer sensibly talk about a "quasiparticle," but rather we should think of a "collection of quasiparticles."

The integrals in Eq. (89) can now be evaluated numerically to yield the appropriate initial conditions. For example when $k/\rho = 1$, $m_\phi/\rho = 1$, and $h/\rho = 4$, we find

$$F_\phi(k/\rho = 1, \Delta t)|_{\Delta t=0} = 0.351\,96, \tag{91a}$$

$$\partial_t F_\phi(k/\rho = 1, \Delta t)|_{\Delta t=0} = 0, \tag{91b}$$

$$\partial_{t'}\partial_t F_\phi(k/\rho = 1, \Delta t)|_{\Delta t=0} = 0.731\,20. \tag{91c}$$

Clearly, Eq. (91b) holds for all values of k, m_ϕ, and h as the integrand is an odd function of k^0, which can be appreciated from Eqs. (89b) and (90). The numerical value of the phase space area in this case follows from (9) and (91) as

$$\Delta_{\rm ms} = 1.014\,61 > 1. \tag{92}$$

Hence also

$$S_{\rm ms} = 0.043\,27 > 0, \tag{93}$$

where the subscript ms is an abbreviation for "mixed state."

From the phase space area $\Delta_{\rm ms}$ we can easily obtain the statistical particle number density (11). It is interesting to study its behavior as a function of k. Figure 6 clearly shows that in the deep UV the particle number density vanishes: the interaction between the two fields only produces particles (in the statistical sense) in the IR. Moreover, using Fig. 6, we can show

$$n_k(h, m_\phi) \to n_{\rm UV}\left(\frac{h}{k}\right) = \zeta\frac{h^2}{k^2}, \tag{94}$$

in the deep UV. In fact, we can estimate the constant of proportionality ζ appearing in Eq. (94) as $\zeta \simeq 0.0008$ which turns out to be insensitive to the value of the mass of the system field m_ϕ and the coupling h. The mass only influences the IR behavior, as expected, which can also be appreciated from Fig. 6. Note finally that the formal divergence of derived quantities, such as the total particle number per volume $N/V = \int d^3k/(2\pi)^3 n_k$ or the total entropy per volume $S/V = \int d^3k/(2\pi)^3 S_k$, does not pose any problems for the dynamics we are about to solve since these quantities do not enter the equations of motion.

So far, we postponed the discussion of imposing boundary conditions for numerically determining the statistical propagator. We just proved that, independently of how one imposes initial conditions, the phase space area should settle to a constant value and for a specific choice of parameters, we have been able to calculate this constant in Eq. (92). One could think of at least two separate ways of imposing boundary conditions: "pure state initial conditions" and what we will henceforth refer to as "mixed state initial conditions." If we constrain the statistical propagator to occupy the minimal allowed phase space area initially, we set

$$F_\phi(t_0, t_0) = \frac{1}{2\omega_{\rm in}}, \tag{95a}$$

$$\partial_t F_\phi(t, t_0)|_{t=t_0} = 0, \tag{95b}$$

$$\partial_{t'}\partial_t F_\phi(t, t')|_{t=t'=t_0} = \frac{\omega_{\rm in}}{2}, \tag{95c}$$

and $\omega_{\rm in}$ is determined from Eq. (A11). This yields

$$\Delta_k(t_0) = 1, \tag{96}$$

such that $S_k(t_0) = 0$. Physically, this means that despite the fact that interactions enlarge the accessible phase space of the system field, we force it to occupy a minimal area initially and let it evolve.[8] Alternatively, we can impose mixed state initial conditions, i.e., the values calculated from Eqs. (89) and (90):

$$F_\phi(k, t_0, t_0) = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} F_\phi(k^\mu), \tag{97a}$$

$$\partial_t F_\phi(k, t, t')|_{t=t'=t_0} = 0, \tag{97b}$$

$$\partial_{t'}\partial_t F_\phi(k, t, t')|_{t=t'=t_0} = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} k_0^2 F_\phi(k^\mu), \tag{97c}$$

such that initially

$$\Delta_k(t_0) = \Delta_{\rm ms} = \text{const}, \tag{98}$$

and also $S_{\rm ms} > 0$. Clearly, boundary conditions (97) can only be evaluated numerically for each choice of parameters. Needless to say we are completely free to impose any other type of initial conditions as well, but we consider the two cases above to be physically well motivated if the system is close to its minimum energy state.

In Fig. 7, we show the phase space evolution for both pure state initial conditions (black line) and mixed state initial conditions (red line). For pure state initial conditions, the evolution is precisely as anticipated. The phase space area increases from its minimal area $\Delta_k(t_0) = 1$, to the asymptotic value $\Delta_{\rm ms}$ calculated in Eq. (92). For mixed state initial conditions, we first observe some transient behavior which eventually decays. We then smoothly evolve to $\Delta_{\rm ms}$. The initial transient is due to our assumption to approximate the propagators in the memory kernel from past infinity to t_0 with free propagators. As is apparent from (84), its effect becomes less important as time elapses.

From the evolution of the phase space area (for pure state initial conditions), we can immediately find the time evolution of the entropy $S_k(t)$ in Fig. 8. This shows that entropy has been generated by interaction with an environment that is in the vacuum state assuming that some observer is only sensitive to Gaussian correlators. The entropy eventually settles to its asymptotic value $S_{\rm ms}$ calculated from $\Delta_{\rm ms}$ and Eq. (10). As can be anticipated, the generated entropy per mode is small: both system and environment are in a state close to the minimum energy state ($T = 0$).

We conclude that when the mass m_ϕ is a constant, no further entropy is generated if we start with mixed state

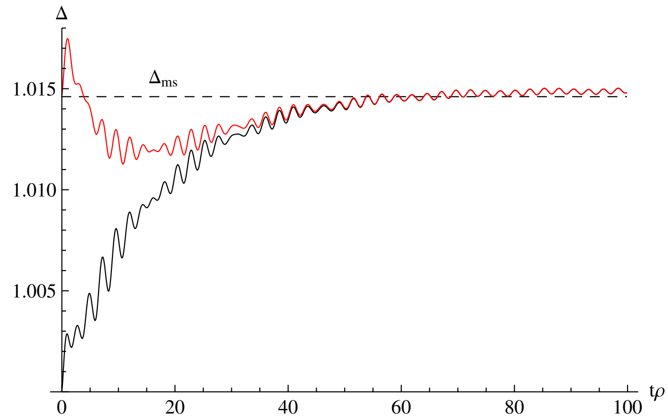


FIG. 7 (color online). Phase space evolution for constant m_ϕ for pure state initial conditions $\Delta_k(t_0) = 1$ (black line) and mixed state initial conditions [gray (red) line] $\Delta_k(t_0) = \Delta_{\rm ms}$. In both cases the phase space area settles to the constant value $\Delta_{\rm ms}$, indicated by the dashed line and calculated in Eq. (92). We use $k/\rho = 1$, $m_\phi/\rho = 1$, $h/\rho = 4$, and $N = 2000$.

initial conditions. However, we do observe a generation of entropy if we start with pure state initial conditions. This increase in entropy can be understood by the system's tendency to evolve toward the vacuum state of the interacting theory.

Let us return once more to Fig. 7. The fact that our numerical asymptote is located slightly above the one calculated from (92) can be attributed to the accuracy of the implementation of the infinite past memory kernel. This can be appreciated from Fig. 9 where we test the accuracy of our code. Clearly, the numerically found asymptote decreases toward $\Delta_{\rm ms}$ as accuracy increases. Moreover, observe that the initial violent oscillations in

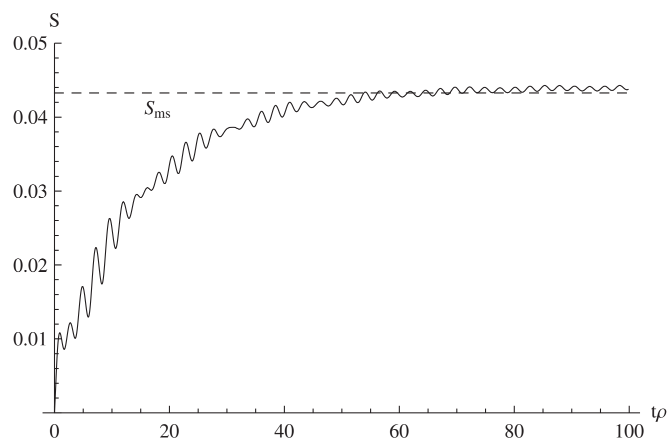


FIG. 8. Entropy generation for the system field ϕ through interaction with the environment χ in the vacuum state. As both fields are in a vacuum state, the entropy generation is relatively small. We used pure state initial conditions $S_k(t_0) = 0$ and $k/\rho = 1$, $m_\phi/\rho = 1$, $h/\rho = 4$ and $N = 2000$. The entropy settles to a constant value $S_{\rm ms}$ calculated from the value of $\Delta_{\rm ms}$ of Fig. 7 and Eq. (10).

[8]If we would not include the infinite past memory kernel $M_F^{\rm free}(k, t, t', t_0)$ and indeed consider a coupling between two fields that is switched on nonadiabatically at some finite time t_0 as previously discussed, the pure state initial condition would be the natural choice for this problem.

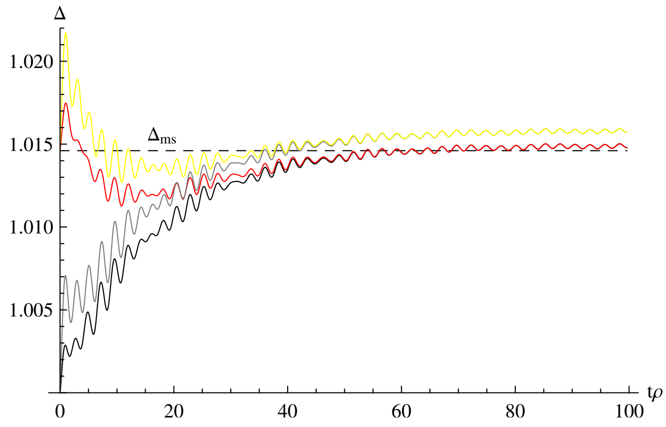


FIG. 9 (color online). Convergence for the phase space evolution presented in Fig. 7. The black and red (the darker ones) lines are identical to the ones in Fig. 7 and most accurate ($N = 2000$). The gray and yellow lines (the lighter ones) are calculated with $N = 1000$. The other parameters are kept fixed. Clearly, the difference between $\Delta_{\rm ms}$ and the numerical asymptotes decreases as the accuracy increases.

$\Delta_k(t)$ decrease as accuracy improves. Also, we have chosen $\omega\Delta t$, where Δt is the step size of the numerical integration, such that we resolve all the oscillations. For a $N = 2000$ run at $t\rho = 100$, we have $\omega\Delta t \simeq 0.071$ for the parameters used in Fig. 7. Also, we can observe a "beating" phenomenon that persists even if the accuracy increases (and that can hence not be attributed to numerical artifacts). It is caused by a frequency mismatch by approximating the propagators in the infinite past memory kernel by free propagators. Finally, let us discuss Fig. 10. Here, we show the difference between the phase space evolution calculated from Eq. (85) and from (87). The dashed line is more accurate than the dotted one, which in turn is more accurate than the solid line. Clearly, the differences disappear when the accuracy improves. This confirms our numerical analysis in a nontrivial way.

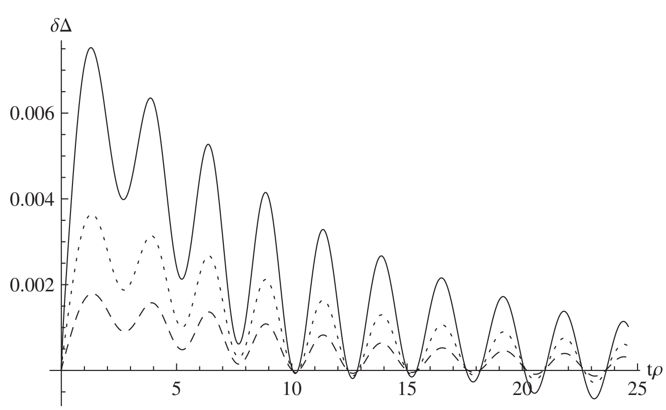


FIG. 10. Test of our numerical code. This plot shows the difference of the phase space area calculated from Eqs. (85) and (87) for different values of N but the same values for the other parameters $k/\rho = 1$, $m_\phi/\rho = 1$, and $h/\rho = 4$ for pure state initial conditions. We used $N = 1000$ at $t = 100$ (solid line), $N = 1000$ at $t = 50$ (dotted line), and $N = 1000$ at $t = 25$ (dashed line). The difference between the two methods disappears as the accuracy of the numerical evolution increases.

B. The decoherence time scale

We define the decoherence time scale to be the characteristic time it takes for the phase space area $\Delta_k(t)$ to settle to its constant mixed state value $\Delta_{\rm ms}$. One can suppose that such a process is described by a differential equation of the following form:

$$\delta\dot{\Delta}_k(t) + \Gamma_k(h, \omega_\phi)\delta\Delta_k(t) = 0, \tag{99}$$

where $\delta\Delta_k(t) = \Delta_k(t) - \Delta_{\rm ms}$ and where $\Gamma_k(h, \omega_\phi)$ is the decoherence rate. This equation is equivalent to $\dot{n}_k = -\Gamma_k(n_k - n_{\rm ms})$, where n_k is defined in Eq. (11) and $n_{\rm ms}$ is the stationary n corresponding to $\Delta_{\rm ms}$. We anticipate that the decoherence rate depends both on the coupling constant and on the energy of our system field.[9] The following intuitive picture is helpful: the solution of Eq. (99) results in an exponential decay to the mixed state value $\delta\Delta_k(t) \propto \exp[-\Gamma_k(h, \omega_\phi)t]$. Furthermore, a stronger coupling h should result in a larger value of $\Gamma_k(h, \omega_\phi)$. However, a larger energy $\omega_\phi^2 = m_\phi^2 + k^2$ should be reflected in a smaller value of $\Gamma_k(h, \omega_\phi)$. On dimensional grounds, we thus anticipate

$$\Gamma_k(h, \omega_\phi) = \frac{h^2}{\omega_\phi}\gamma, \tag{100}$$

where $\gamma = \text{const}$. Let us now test this expected scaling relation.

Looking back at Fig. 7, we see in the first few time steps that $\Delta_k(t)$ oscillates. Clearly, the time scale of these fluctuations has nothing to do with the decoherence time scale, but rather can be fully attributed to numerical accuracy. To capture the decoherence time scale correctly, we thus consider the difference $\Delta_k^{\rm ms}(t) - \Delta_k^{\rm ps}(t)$ of the evolution of the phase space area $\Delta_k^{\rm ms}(t)$ using mixed state initial conditions and $\Delta_k^{\rm ps}(t)$ using pure state initial conditions. On a logarithmic scale, we observe in Fig. 11 an exponential decay toward $\Delta_{\rm ms}$ (solid line). Moreover, the slope should not depend on the particular choice of initial conditions. To this end, we also calculate the difference of $\Delta_k^{\rm ms}(t) - \Delta_k^{\rm nps}(t)$, where $\Delta_k^{\rm nps}(t)$ follows from setting $1 < \Delta_k^{\rm nps}(t_0) < \Delta_k^{\rm ms}(t_0)$ initially. In order to do this, we kept the value of $F_\phi(k, t_0, t_0)$ identical to the value it had for the mixed state boundary conditions but reduced the value of $\partial_{t'}\partial_t F_\phi(k, t, t')|_{t=t'=t_0}$ such that the inequality $1 < \Delta_k^{\rm nps}(t_0) < \Delta_k^{\rm ms}(t_0)$ is satisfied. The resulting decoherence rates precisely coincide as seen in Fig. 11, where we plot

[9] Ideally, we should of course take the $\mathcal{O}(h^2/\omega_\phi^2)$ to ω_ϕ through the dispersion relation into account.

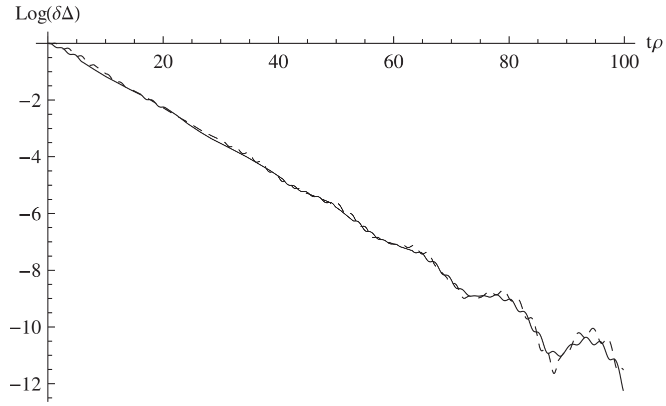


FIG. 11. Exponential approach to $\Delta_{\rm ms}$. We study, for different initial conditions, differences of $\Delta_k(t)$ on a logarithmic scale defined by Eq. (101). Clearly, the decoherence rate does not depend on the initial conditions. We used $k/\rho = 1$, $m_\phi/\rho = 1$, $h/\rho = 4$, and $N = 2000$.

$$\log(\delta\Delta) \equiv \log\left(\frac{\Delta_k^{\rm ms}(t) - \Delta_k^{\rm ps}(t)}{\Delta_k^{\rm ms}(t_0) - \Delta_k^{\rm ps}(t_0)}\right), \qquad (101)$$

and likewise for $\Delta_k^{\rm nps}(t)$ (dashed line).

We can repeat the steps outlined above for a different choice of parameters m and h. By rescaling the obtained decoherence rates by a factor of ω_ϕ/h^2, we can test the scaling relation (99). All decoherence rates now precisely overlap as we depicted in Fig. 12. We can thus estimate the value of the constant of proportionality γ appearing in Eq. (100):

$$\Gamma_k(h, \omega_\phi) = (0.0101 \pm 0.0003)\frac{h^2}{\omega_\phi}. \qquad (102)$$

This relation gives the decoherence rate for our particular model. This result is nothing but the single particle decay rate:

$$\Gamma_{\phi\to\chi\chi} = -\frac{{\rm Im}(\imath M^{++})}{\omega_\phi} = \frac{1}{32\pi}\frac{h^2}{\omega_\phi}, \qquad (103)$$

where we have used Eq. (C2) and e.g. [60,108]. Let us compare the result (102) to the literature. Let us remark that most of the calculations have been performed in an expanding Universe setting, or with a different model, so it is hard to compare this result quantitatively. In [39] it was found that, for a different model during inflation, the decoherence rate is proportional to the spatial volume, which we do not find.

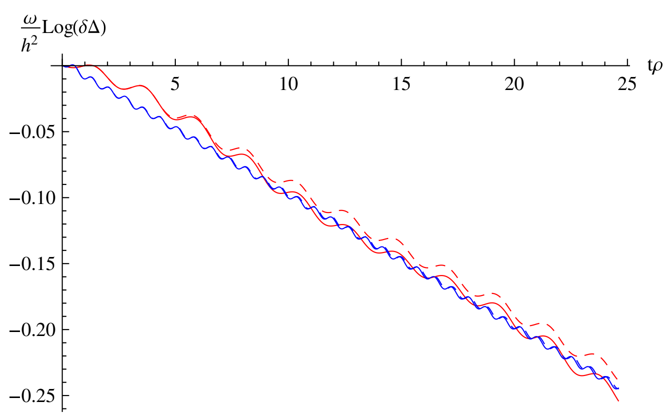


FIG. 12 (color online). Decoherence time scale. We confirm the scaling relation for the decoherence time scale anticipated in Eq. (100). For all plots we took $k/\rho = 1$ and $N = 1000$, and furthermore we used $m_\phi/\rho = 1$, $h/\rho = 4$ (solid red line), $m_\phi/\rho = 1$, $h/\rho = 1$ (dashed red line), $m_\phi/\rho = 4$, $h/\rho = 4$ (solid blue line, higher frequency), and $m_\phi/\rho = 4$, $h/\rho = 1$ (dashed blue line, higher frequency).

C. Changing mass solutions

Finally, let us discuss the evolution of the phase space area when $m_\phi^2(t)$ is changing according to Eq. (78b). The analytic expression for the statistical propagator in Fourier space we previously derived in Eq. (90) is no longer valid. Introducing a time dependent mass $m_\phi^2(t)$, generated by a time dependent Higgs-like scalar field, breaks the time translation invariance of the problem. Consequently, the statistical propagator $F(k, t, t')$ no longer depends only on the time difference of its time variables $\Delta t = t - t'$, because considering $m_\phi^2(t)$ introduces a proper time dependence on the average time coordinate $\tau = (t + t')/2$ in the problem. When the mass is changing rapidly, we can only rely on numerical methods. However, asymptotically, where the mass settles again to a constant value, the analysis performed in the previous section should still apply.

We impose mixed state boundary conditions as in Eq. (97) such that $\Delta_k(t_0) = \Delta_{\rm ms}$ initially. Of course, the value of the mass inserted to calculate these initial conditions is the value of the initial mass, valid before the mass jump.

If the mass changes nonadiabatically, this results in a significant particle creation according to the discussion in Appendix A. We can thus identify the following regimes:

$$|\beta_k|^2 \ll 1 \qquad \text{adiabatic regime}, \qquad (104a)$$

$$|\beta_k|^2 \gg 1 \qquad \text{nonadiabatic regime}, \qquad (104b)$$

where β_k is given is Eq. (A25b). For the parameters we used in Figs. 13 and 15 we are in the adiabatic regime ($|\beta_k|^2 = 3.5 \times 10^{-4}$).

In Fig. 13 we study a mass increase from $m_\phi/\rho = 0.75$ to $m_\phi/\rho = 2$ (black line). This decreases the phase space area and consequently the entropy decreases, which we depicted in Fig. 14. Intuitively, a larger mass of the ϕ field reduces the effect of the quantum corrections of the χ field. Hence $\Delta_{\rm ms}^{(2)} < \Delta_{\rm ms}^{(1)}$, where $\Delta_{\rm ms}^{(2)}$ is the constant phase space

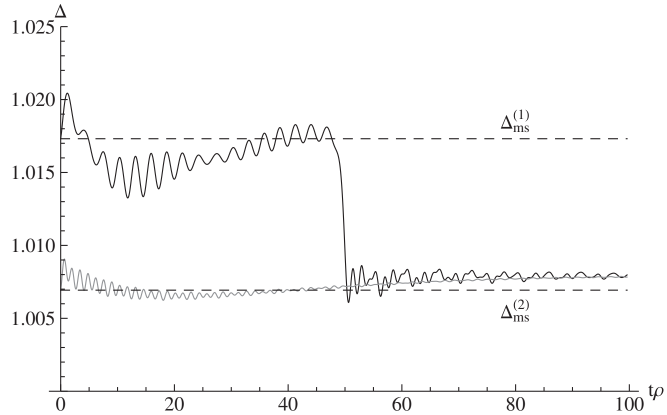


FIG. 13. Phase space area decrease due to a mass increase from $m_\phi/\rho = 0.75$ to $m_\phi/\rho = 2$ (solid black line). We used $h/\rho = 4$, $k/\rho = 1$, and $N = 2000$. We thus observe a slight entropy decrease. The solid gray line denotes the constant mass phase space evolution for $m_\phi/\rho = 2$. As the two asymptotes coincide, we conclude no entropy has been generated at late times by the mass change. $\Delta_{\rm ms}^{(1)}$ and $\Delta_{\rm ms}^{(2)}$ are the constant mixed phase space areas calculated for $m_\phi/\rho = 0.75$ and $m_\phi/\rho = 2$, respectively.

area calculated for $m_\phi/\rho = 2.0$ from three Fourier integrals as in Eq. (91). Likewise, $\Delta_{\rm ms}^{(1)}$ corresponds to the phase space area calculated for $m_\phi/\rho = 0.75$. The relevant behavior to compare with is the constant mass phase space evolution for $m_\phi/\rho = 2$ which we also depicted in Fig. 13 in gray. Clearly, the late time asymptotes of the two functions coincide and we conclude that, also at late times, no entropy has been generated. As we are in the deep adiabatic regime, this is to be expected.

Now let us study the opposite: a mass decrease from $m_\phi/\rho = 2$ to $m_\phi/\rho = 0.75$. This is depicted (black line) in Fig. 15. Clearly, the phase space area increases from $\Delta_{\rm ms}^{(2)}$ to $\Delta_{\rm ms}^{(1)}$ and we plotted the resulting entropy increase in

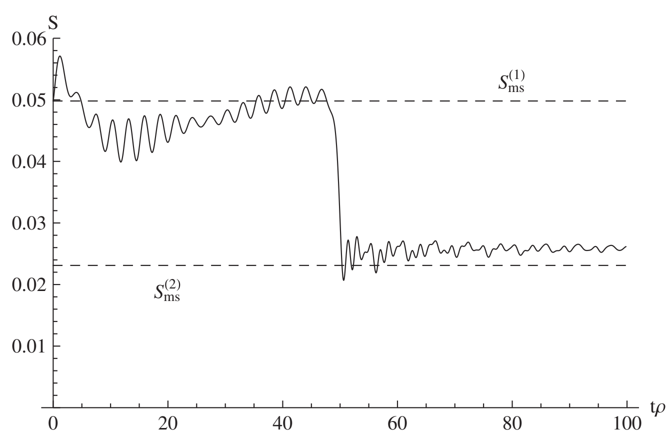


FIG. 14. Entropy decrease for the case presented in Fig. 13. Because of the mass increase, the phase space area decreases which results consequently in a drop in the entropy.

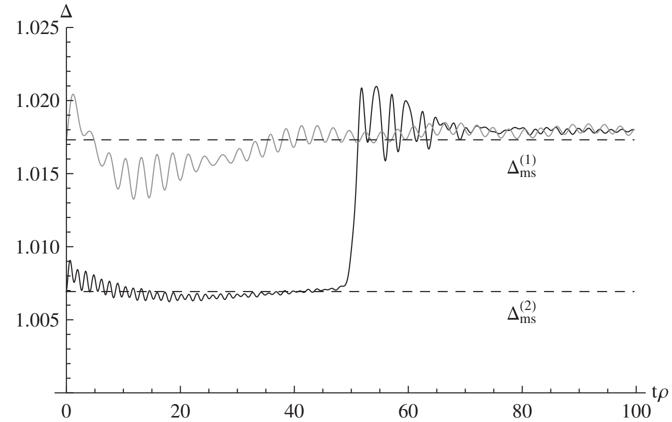


FIG. 15. Phase space area increase due to a mass decrease from $m_\phi/\rho = 2$ to $m_\phi/\rho = 0.75$ (solid black line). The other parameters are the same as in Fig. 13. The solid gray line denotes the constant mass phase space evolution for $m_\phi/\rho = 0.75$. Again the two asymptotes coincide and no entropy has been generated at late times by the mass change.

Fig. 16. Again, we compared this evolution with the phase space area calculated for a constant mass $m_\phi/\rho = 0.75$ (gray line). As the two asymptotes also coincide in this case, we conclude that no entropy has been generated at late times by the mass change. Of course, it would be very interesting to see what happens when we study the same process in the nonadiabatic regime and we hope to address this question in a future publication.

If we compare the evolution of the entropy in time in these two cases with the free case $S_k(t) = 0$, the interacting case reveals much more interesting behavior. First, due to the presence of an environment field, the constant value to which the entropy settles asymptotically is different from zero, unlike the free case. Second, a changing mass induces dynamics: the entropy depends on time and evolves from one value $S_{\rm ms}^{(1)}$ to another $S_{\rm ms}^{(2)}$ or vice versa.

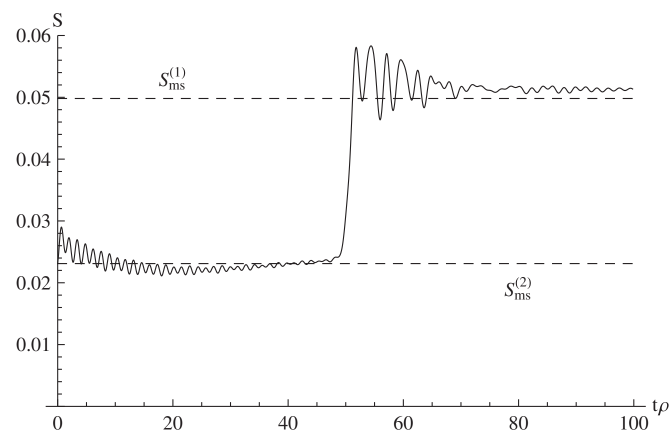


FIG. 16. Entropy increase for the case presented in Fig. 15. Clearly, a mass decrease gives rise to a slight entropy increase.

VI. CONCLUSION

We apply the decoherence framework to quantum field theory. We consider two scalar fields, one corresponding to a "system field," the second to an "environment field," in interaction via a cubic coupling. Here, we consider an environment in its vacuum state ($T = 0$) and postpone finite temperature contributions to a future publication. We neglect the backreaction of the environment field on the system field, assuming that the former remains at $T = 0$.

We advocate the following point of view regarding a sensible application of decoherence to quantum field theory: for some observer inaccessible higher order correlators give rise, once neglected, to an increase in entropy S_k of the system. This is inspired by realizing that correlators are measured in quantum field theories and that higher order irreducible n-point functions are usually perturbatively suppressed. In this work, we assume that our observer is only sensitive to Gaussian correlators and will hence neglect higher order, non-Gaussian correlators. Neglecting the information stored in these higher order correlators gives rise to an increase of the entropy of the system. If the system initially occupies the minimal area in phase space [characterized by a pure state with $S_k(t_0) = 0$], we numerically calculate the evolution of the entropy $S_k(t)$ in Fig. 8. Also, we calculate the asymptotic value of the phase space area $\Delta_{\rm ms}$ and the entropy $S_{\rm ms}$ as a function of the coupling h, the mass m_ϕ, and k to which these functions evolve. This increase in entropy can be understood from the system's tendency to evolve toward the vacuum state of the interacting theory. Even though we do not solve the full 2PI equations at one loop, we have strong numerical evidence that within our approximation scheme, the system evolves toward its correct stationary interacting vacuum state. If the system, however, starts out initially occupying the state characterized by $S_{\rm ms}$, we observe no further increase in the entropy. Furthermore, we calculate the decoherence rate in Eq. (102) which charaterizes the exponential rate at which the system approaches its stationary state.

Secondly, we study the effect of a time dependent mass $m_\phi^2(t)$ of the system field. Now, calculating $S_k(t)$ can only be addressed numerically. Starting out at mixed state initial conditions $S_k(t_0) = S_{\rm ms} > 0$, we observe an entropy increase (decrease) due to a mass decrease (increase) as depicted in Figs. 14 and 16, respectively. By comparing with the constant mass evolution for the entropy, we conclude that no additional entropy has been generated asymptotically by the mass jump. As we study mass changes in the deep adiabatic regime, it remains to be investigated whether this statement also holds in the nonadiabatic regime.

We also would like to draw a few somewhat more technical conclusions. It is important to stress that in interacting field theories where memory effects play a crucial role, one cannot just insert initial conditions at some arbitrary finite time t_0, because one then neglects the memory effects existing from the infinite asymptotic past to t_0. In numerical computations however, one has to start at some finite time. We therefore approximate the propagators in the memory integral from the past infinity to t_0 by free propagators, inducing a perturbatively suppressed error.

Also, it has not been previously appreciated in the literature that renormalizing the vacuum contribution in the Kadanoff-Baym equations can actually significantly change the structure of these equations. In order to properly take account of the renormalized self-mass contribution we had to extract two time derivatives which can readily be seen from Eq. (64).

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APPENDIX A: QUANTUM EFFECTS OF A CHANGING MASS: FREE CASE

It is interesting to compare our results to a nontrivial exact case: scattering of a changing mass field in the spirit of Birrell and Davies [88]. The solutions presented here stem from the cosmological particle creation literature (based on [109,110]) and are originally due to Bernard and Duncan [111]. Let us consider the action of a free scalar field:

$$S[\phi] = \int d^4x \Big\{ -\frac{1}{2}\partial_\mu\phi(x)\partial_\nu\phi(x)\eta^{\mu\nu} - \frac{1}{2}m_\phi^2(t)\phi^2(x)\Big\}, \tag{A1}$$

where, as usual, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric, and where we consider the following behavior of the mass $m_\phi(t)$ of the scalar field:

$$m_\phi^2(t) = (A + B\tanh(\rho t)). \tag{A2}$$

From (A1), it follows that

$$(\partial_t^2 - \partial_i^2 + m_\phi^2(t))\phi(x) = 0. \tag{A3}$$

The vacuum causal and statistical propagators follow from (6) and (7) where $\hat{\rho}(t_0) = |0\rangle\langle 0|$ as

$$\imath\Delta^c_\phi(x;x') = \langle 0|[\hat\phi(x), \hat\phi(x')]|0\rangle, \tag{A4a}$$

$$F_\phi(x;x') = \tfrac{1}{2}\langle 0|\{\hat\phi(x'), \hat\phi(x)\}|0\rangle. \tag{A4b}$$

Let us quantize our fields in D dimensions by making use of creation and annihilation operators:

$$\hat\phi(x) = \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} (\hat a_{\vec k}\phi_k(t)e^{i\vec k\cdot\vec x} + \hat a^\dagger_{\vec k}\phi^*_k(t)e^{-i\vec k\cdot\vec x}). \tag{A5}$$

The annihilation operator acts as usual on the vacuum:

$$\hat a_{\vec k}|0\rangle = 0, \tag{A6a}$$

and we impose the following commutation relations:

$$[\hat a_{\vec k}, \hat a^\dagger_{\vec k'}] = (2\pi)^{D-1}\delta^{D-1}(\vec k - \vec k'). \tag{A6b}$$

Hence the equation of motion for the mode functions $\phi_k(t)$ of $\phi(x)$, defined by relation (A5), follows straightforwardly as

$$(\partial_t^2 + k^2 + m^2_\phi(t))\phi_k(t) = 0, \tag{A7}$$

where $k = \| \vec k \|$. The mode functions determine the causal and statistical propagators from (A4) completely:

$$\imath\Delta^c_\phi(k,t,t') = \phi_k(t)\phi^*_k(t') - \phi_k(t')\phi^*_k(t), \tag{A8a}$$

$$F_\phi(k,t,t') = \tfrac{1}{2}\{\phi_k(t')\phi^*_k(t) + \phi_k(t)\phi^*_k(t')\}. \tag{A8b}$$

Let us at this point for completeness calculate the constant mass causal and statistical propagators in Fourier space. We just insert a constant mass m_ϕ, rather than a changing one as in Eq. (A2):

$$\imath\Delta^c_\phi(k,t,t') = -\frac{\imath}{\omega}\sin(\omega(t-t')), \tag{A9a}$$

$$F_\phi(k,t,t') = \frac{1}{2\omega}\cos(\omega(t-t')), \tag{A9b}$$

where $\omega^2 = m^2_\phi + k^2$. Let us now return to the changing mass case. The physical picture is clear: we would like to study reflection and transmission, i.e., scattering, of an incoming wave due to the changing mass. Before solving this equation of motion exactly, let us first solve for the asymptotic mode functions to gain intuitive understanding. In the asymptotic past ($t \to -\infty$), Eq. (A7) is solved by

$$\phi^{\rm in}_k(t) = \frac{1}{\sqrt{2\omega_{\rm in}}}\exp[-\imath\omega_{\rm in}t], \tag{A10}$$

i.e., one right-moving or incoming wave with frequency:

$$\omega_{\rm in} = (k^2 + A - B)^{1/2}. \tag{A11}$$

In the infinite asymptotic future, the solution necessarily is an appropriately normalized linear superposition of a left- and right-moving wave:

$$\phi^{\rm out}_k(t) = \alpha_k\frac{1}{\sqrt{2\omega_{\rm out}}}\exp[-\imath\omega_{\rm out}t] + \beta_k\frac{1}{\sqrt{2\omega_{\rm out}}} \times \exp[\imath\omega_{\rm out}t], \tag{A12}$$

where

$$\omega_{\rm out} = (k^2 + A + B)^{1/2}, \tag{A13}$$

and where

$$\|\alpha_k\|^2 - \|\beta_k\|^2 = 1, \tag{A14}$$

for a consistent canonical quantization. In both asymptotic regions we can now immediately calculate the statistical propagator from Eq. (A8b):

$$F_{\rm in}(k,t,t') = \frac{1}{2\omega_{\rm in}}\cos(\omega_{\rm in}(t-t')), \tag{A15a}$$

$$F_{\rm out}(k,t,t') = \frac{1}{2\omega_{\rm out}}[(|\alpha_k|^2 + |\beta_k|^2) \times \cos(\omega_{\rm out}(t-t')) + \alpha_k\beta^*_k e^{-\imath\omega_{\rm out}(t+t')} + \alpha^*_k\beta_k e^{\imath\omega_{\rm out}(t+t')}]. \tag{A15b}$$

Using Eq. (9), we can calculate the area in phase space the in and out states occupy:

$$\Delta^{\rm in}_k(t) = 1, \tag{A16a}$$

$$\Delta^{\rm out}_k(t) = 1. \tag{A16b}$$

Hence for the entropy we find

$$S^{\rm in}_k(t) = 0 = S^{\rm out}_k(t). \tag{A17}$$

We conclude that in both asymptotic regions the entropy is zero and no entropy has been generated by changing the mass.

However, we can do better than study the asymptotic behavior only. Birrell and Davies study cosmological particle creation in Sec. 3.4 of their book [88] in a simple, conveniently chosen cosmological setting. They consider a scale factor as a function of conformal time $a(\eta)$ which behaves as

$$a^2(\eta) = A + B\tanh(\rho\eta). \tag{A18}$$

This represents an asymptotically static universe with a smooth expansion connecting these two asymptotic regions. Indeed, the equation of motion (in conformal time) for the mode functions Birrell and Davies consider coincides precisely with (A7). The solution to (A7) which behaves as a positive frequency mode in the asymptotic past ($t \to -\infty$) can be expressed in terms of Gauss' hypergeometric function ${}_2F_1$:

$$\phi_k^{\rm in}(t) = \frac{1}{\sqrt{2\omega_{\rm in}}} \exp\left[-\imath\omega_+ t - \imath\frac{\omega_-}{\rho}\log\{2\cosh(\rho t)\}\right] {}_2F_1\left(1+\imath\frac{\omega_-}{\rho}, \imath\frac{\omega_-}{\rho}; 1-\imath\frac{\omega_{\rm in}}{\rho}; \frac{1}{2}\{1+\tanh(\rho t)\}\right), \tag{A19}$$

such that

$$\lim_{t\to-\infty}\phi_k^{\rm in}(t) = \frac{1}{\sqrt{2\omega_{\rm in}}}\exp[-\imath\omega_{\rm in}t], \tag{A20}$$

where we defined $\omega_{\rm in}$ and $\omega_{\rm out}$ in Eqs. (A11) and (A13), respectively, and

$$\omega_\pm = \tfrac{1}{2}(\omega_{\rm out} \pm \omega_{\rm in}). \tag{A21}$$

Alternatively, the modes which reduce to positive frequency modes in the out region are given by

$$\phi_k^{\rm out}(t) = \frac{1}{\sqrt{2\omega_{\rm out}}} \exp\left[-\imath\omega_+ t - \imath\frac{\omega_-}{\rho}\log\{2\cosh(\rho t)\}\right] {}_2F_1\left(1+\imath\frac{\omega_-}{\rho}, \imath\frac{\omega_-}{\rho}; 1+\imath\frac{\omega_{\rm out}}{\rho}; \frac{1}{2}\{1+\tanh(\rho t)\}\right), \tag{A22}$$

such that

$$\lim_{t\to\infty}\phi_k^{\rm out}(t) = \frac{1}{\sqrt{2\omega_{\rm out}}}\exp[-\imath\omega_{\rm out}t]. \tag{A23}$$

We can rewrite the hypergeometric functions using Eqs. (15.3.3) and (15.3.6) of [112] and identify

$$\phi_k^{\rm in}(t) = \alpha_k\phi_k^{\rm out}(t) + \beta_k\phi_k^{\rm out*}(t), \tag{A24}$$

where

$$\alpha_k = \left(\frac{\omega_{\rm out}}{\omega_{\rm in}}\right)^{1/2}\frac{\Gamma(1-\imath\omega_{\rm in}/\rho)\Gamma(-\imath\omega_{\rm out}/\rho)}{\Gamma(-\imath\omega_+/\rho)\Gamma(1-\imath\omega_+/\rho)}, \tag{A25a}$$

$$\beta_k = \left(\frac{\omega_{\rm out}}{\omega_{\rm in}}\right)^{1/2}\frac{\Gamma(1-\imath\omega_{\rm in}/\rho)\Gamma(\imath\omega_{\rm out}/\rho)}{\Gamma(\imath\omega_-/\rho)\Gamma(1+\imath\omega_-/\rho)}. \tag{A25b}$$

Having the mode functions at our disposal, we can find (the rather cumbersome expressions for) the exact causal and statistical propagators. The statistical and causal propagators can however neatly be visualized. Figure 17 shows the causal propagator with a constant mass from Eq. (A9a) for comparison to the changing mass case. In Figs. 18 and 19 we show the exact causal propagator for a relatively small increase of the mass (from $m_\phi/\rho = 0$ to $m_\phi/\rho = 1$) and a larger one (from $m_\phi/\rho = 0$ to $m_\phi/\rho = 4$) for one particular Fourier mode only ($k/\rho = 1$). Figures 20–22 show the analogous statistical propagators.

We can easily relate the statistical propagator to the phase space area by making use of Eq. (9). It will not come as a surprise to the reader that we find

$$\Delta_k(t) = 1, \tag{A26}$$

and hence

$$S_k(t) = 0, \tag{A27}$$

also for all intermediate times. A final remark is in order. The reader should not confuse $|\beta_k|^2$ calculated from Eq. (A25b) with the phase space particle number density or statistical number density (11). Although the mass is changing, the phase space particle density remains zero but $|\beta_k|^2$, which in the literature is often referred to as a particle number, can change significantly as can be appre-

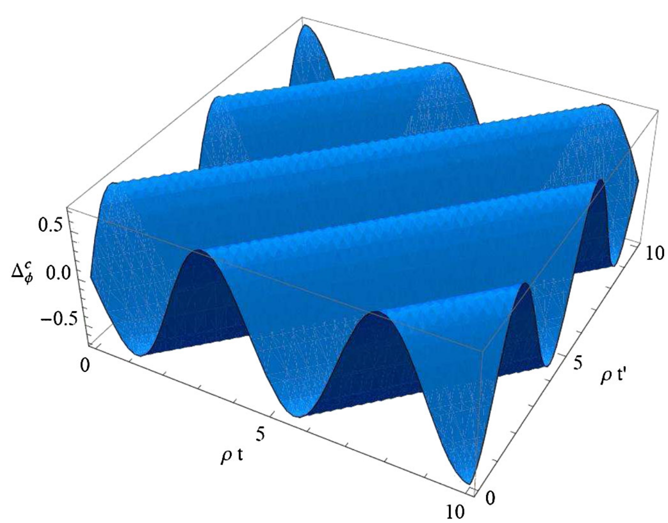

FIG. 17 (color online). Causal propagator with a constant mass. Parameters: $k/\rho = 1$, $m_\phi/\rho = 1$.

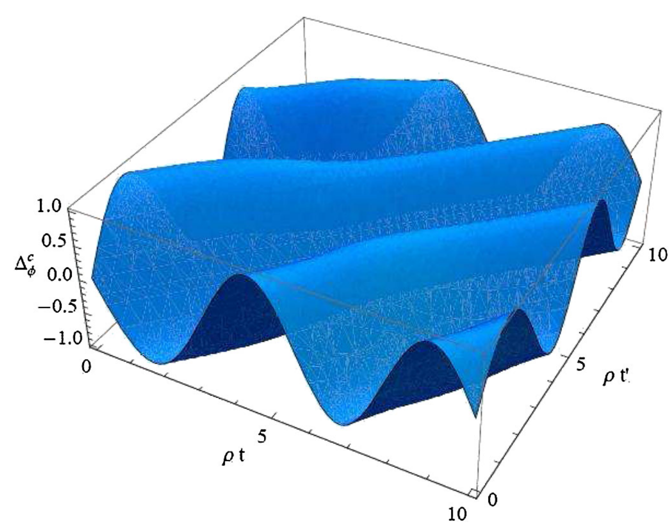

FIG. 18 (color online). Causal propagator for a small change in the mass. Parameters: $k/\rho = 1$, $A/\rho^2 = B/\rho^2 = 1/2$.

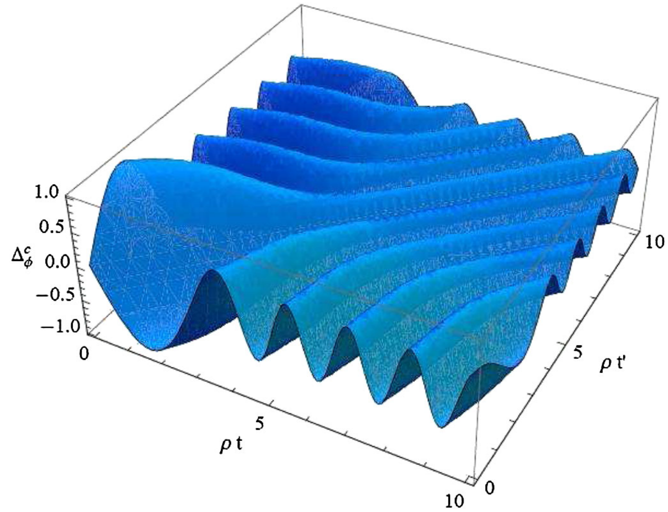

FIG. 19 (color online). Causal propagator for a large change in the mass. Parameters: $k/\rho = 1$, $A/\rho^2 = B/\rho^2 = 2$.

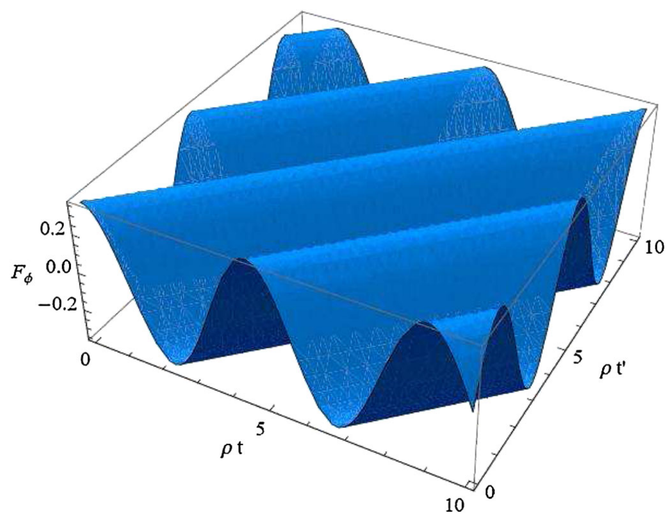

FIG. 20 (color online). Statistical propagator with a constant mass. Parameters: $k/\rho = 1$, $m_\phi/\rho = 1$.

ciated from Fig. 23. This is just caused by the fact that the in and out vacua differ. We plot the behavior of $|\beta_k|^2$ as a function of $m_{\rm out}/\rho$ in both the adiabatic regime ($|\beta_k|^2 \ll 1$) and nonadiabatic regime ($|\beta_k|^2 \gg 1$).

This simple example suggests the following: (i) the area in phase space a state occupies is a good quantitative measure of the entropy, (ii) the statistical propagator contains all the information required to calculate this phase space area, and (iii) a changing mass does not change the entropy for a free scalar field. If we contrast this result with the calculations performed in the main body of the paper, it is important to realize that just a changing mass, in the absence of interactions, produces no entropy, whereas we have shown that the entropy can change in the interacting case.

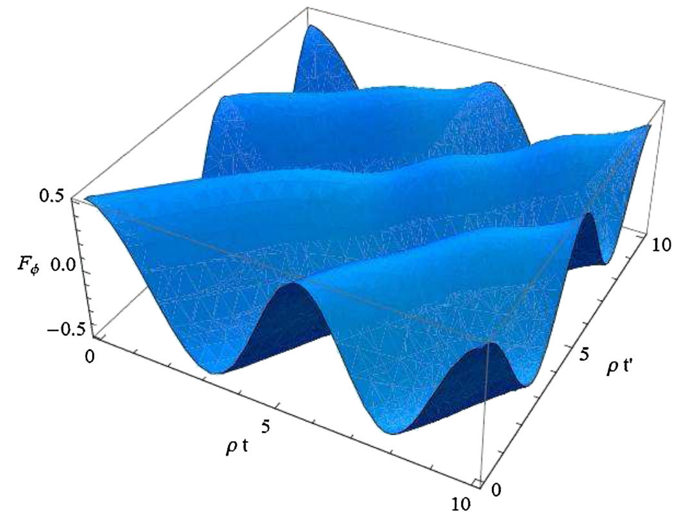

FIG. 21 (color online). Statistical propagator for a small change in the mass. Parameters: $k/\rho = 1$, $A/\rho^2 = B/\rho^2 = 1/2$.

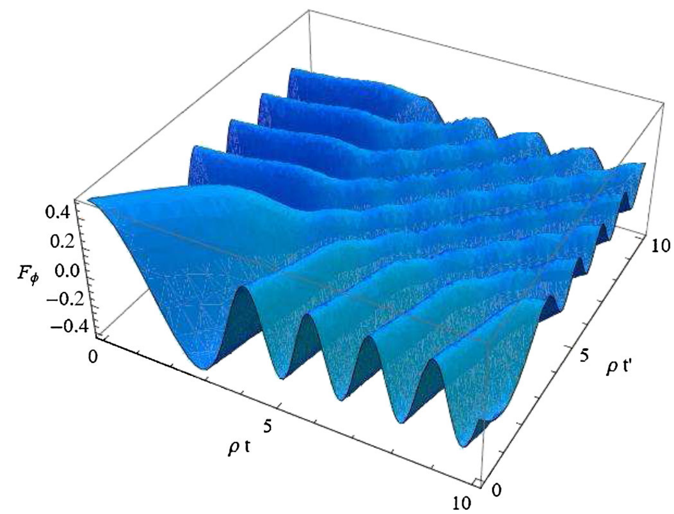

FIG. 22 (color online). Statistical propagator for a large change in the mass. Parameters: $k/\rho = 1$, $A/\rho^2 = B/\rho^2 = 2$.

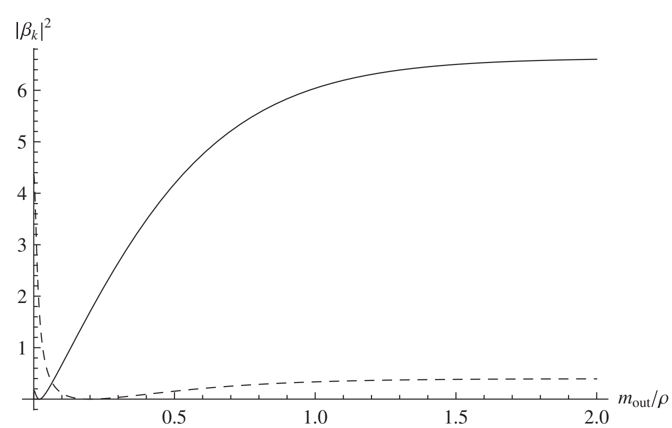

FIG. 23. This plot shows $|\beta_k|^2$ as a function of the final mass $m_{\rm out}/\rho$, for fixed $k/\rho = 0.01$. The dashed line shows the adiabatic regime ($m_{\rm in}/\rho = 0.2$), whereas the solid line shows the nonadiabatic regime ($m_{\rm in}/\rho = 0.02$).

APPENDIX B: EVALUATION OF THE THERMAL PROPAGATORS

In this Appendix, we will for pedagogical reasons derive the thermal propagators from first principles. The four thermal propagators should solve the standard differential equation (19). We start by summarizing the conditions the thermal propagators have to satisfy in position space:

$$\imath\Delta_\chi^{++}(x;x') + \imath\Delta_\chi^{--}(x;x') = \imath\Delta_\chi^{-+}(x;x') + \imath\Delta_\chi^{+-}(x;x'), \tag{B1a}$$

$$\imath\Delta_\chi^{++}(x;x') - \imath\Delta_\chi^{--}(x;x') = \mathrm{sgn}(t-t')(\imath\Delta_\chi^{-+}(x;x') - \imath\Delta_\chi^{+-}(x;x')), \tag{B1b}$$

$$\imath\Delta_\chi^{-+}(x;x') = \imath\Delta_\chi^{+-}(x';x), \tag{B1c}$$

$$\imath\Delta_\chi^{-+}((t-\imath\beta,\bar{\mathbf{x}});x') = \imath\Delta_\chi^{+-}(x';x), \tag{B1d}$$

$$[\chi(t,\bar{\mathbf{x}}), \dot\chi(t,\bar{\mathbf{x}}')] = \partial_{t'}\{\imath\Delta_\chi^{-+}(x;x') - \imath\Delta_\chi^{+-}(x;x')\}|_{t=t'} = \imath\delta^{(3)}(\bar{\mathbf{x}}-\bar{\mathbf{x}}'). \tag{B1e}$$

Here, the first condition [identical to (18c)] and the second relate the sum and the difference of the time ordered and antitime ordered propagators to the two Wightman functions, respectively. The third condition is just identical to (18d). Condition (B1d) is the well-known KMS condition or Kubo-Martin-Schwinger condition, see [113,114]. The KMS condition corresponds to periodic boundary conditions in the imaginary time direction due to assuming a thermal density matrix operator $\hat\rho_{\mathrm{th}} \propto \exp[-\beta\hat H]$. The final equation arises from requiring standard commutation relations. Fourier transforming the equations above according to (38) yields

$$\imath\Delta_\chi^{++}(k^\mu) + \imath\Delta_\chi^{--}(k^\mu) = \imath\Delta_\chi^{-+}(k^\mu) + \imath\Delta_\chi^{+-}(k^\mu), \tag{B2a}$$

$$\imath\Delta_\chi^{++}(k^\mu) - \imath\Delta_\chi^{--}(k^\mu) = \mathrm{P}\left[\frac{-2\imath}{k_\mu k^\mu + m_\chi^2}\right], \tag{B2b}$$

$$\imath\Delta_\chi^{-+}(k^\mu) = \imath\Delta_\chi^{+-}(-k^\mu), \tag{B2c}$$

$$\imath\Delta_\chi^{-+}(k^\mu) = e^{\beta k^0}\imath\Delta_\chi^{+-}(k^\mu), \tag{B2d}$$

$$\imath\Delta_\chi^{-+}(k^\mu) - \imath\Delta_\chi^{+-}(k^\mu) = 2\pi\mathrm{sgn}(k^0)\delta(k_\mu k^\mu + m_\chi^2). \tag{B2e}$$

To obtain the second relation (B2b), we recall

$$\mathrm{sgn}\,(x) = \mathrm{P}\int_{-\infty}^{\infty} dk \frac{1}{\imath\pi k} e^{\imath kx}, \tag{B3}$$

where P denotes the Cauchy principal value. Relations (B2d) and (B2e) trivially yield the two thermal Wightman functions:

$$\imath\Delta_\chi^{+-}(k^\mu) = 2\pi\mathrm{sgn}(k^0)\delta(k_\mu k^\mu + m_\chi^2) n_\chi^{\mathrm{eq}}(k^0), \tag{B4a}$$

$$\imath\Delta_\chi^{-+}(k^\mu) = 2\pi\mathrm{sgn}(k^0)\delta(k_\mu k^\mu + m_\chi^2)(1 + n_\chi^{\mathrm{eq}}(k^0)), \tag{B4b}$$

where $n_\chi^{\mathrm{eq}}(k^0)$ is the Bose-Einstein distribution given by (42). In order to solve for the time ordered and antitime ordered propagators, let us make the following general *Ansätze*:

$$\imath\Delta_\chi^{++}(k^\mu) = \frac{-\imath}{k_\mu k^\mu + m_\chi^2 - \imath\epsilon} + \delta(k_\mu k^\mu + m_\chi^2) f(k^0), \tag{B5a}$$

$$\imath\Delta_\chi^{--}(k^\mu) = \frac{\imath}{k_\mu k^\mu + m_\chi^2 + \imath\epsilon} + \delta(k_\mu k^\mu + m_\chi^2) g(k^0). \tag{B5b}$$

The functions $f(k^0)$ and $g(k^0)$ do not depend on k^i due to the delta function. We have already chosen the time ordered and antitime ordered pole prescription. This is particularly convenient because, as we will appreciate in a moment, this allows us to easily recover the familiar vacuum solutions when $T \to 0$. We will return to this subtlety shortly. Condition (B2b) immediately implies

$$f(k^0) = g(k^0), \tag{B6}$$

where we have made use of the Dirac identity:

$$\frac{1}{x+\imath\epsilon} = \mathrm{P}\frac{1}{x} - \imath\pi\delta(x). \tag{B7}$$

Because of the time ordering, $\imath\Delta^{++}(x;x') = \imath\Delta^{++}(x';x)$ such that $\imath\Delta^{++}(k^\mu) = \imath\Delta^{++}(-k^\mu)$. This consideration likewise applies for the antitime ordered propagator and suggests that the most economic way of writing $f(k^0)$ is in terms of $|k^0|$. We observe

$$\tfrac{1}{2} + n_\chi^{\mathrm{eq}}(k^0) = \mathrm{sgn}(k^0)(\tfrac{1}{2} + n_\chi^{\mathrm{eq}}(|k^0|)). \tag{B8}$$

Using the relation above and condition (B2a):

$$f(k^0) = 2\pi n_\chi^{\mathrm{eq}}(|k^0|). \tag{B9}$$

The thermal propagators are thus given by

$$\imath\Delta_\chi^{++}(k^\mu) = \frac{-\imath}{k_\mu k^\mu + m_\chi^2 - \imath\epsilon} + 2\pi\delta(k_\mu k^\mu + m_\chi^2) n_\chi^{\mathrm{eq}}(|k_0|), \tag{B10a}$$

$$\imath\Delta_\chi^{--}(k^\mu) = \frac{\imath}{k_\mu k^\mu + m_\chi^2 + \imath\epsilon} + 2\pi\delta(k_\mu k^\mu + m_\chi^2) n_\chi^{\mathrm{eq}}(|k^0|), \tag{B10b}$$

$$\imath\Delta_\chi^{+-}(k^\mu) = 2\pi\delta(k_\mu k^\mu + m_\chi^2)[\theta(-k^0) + n_\chi^{\mathrm{eq}}(|k^0|)], \tag{B10c}$$

$$\imath\Delta_\chi^{-+}(k^\mu) = 2\pi\delta(k_\mu k^\mu + m_\chi^2)[\theta(k^0) + n_\chi^{\mathrm{eq}}(|k^0|)]. \tag{B10d}$$

If we now let $T \to 0$ to obtain the familiar vacuum solutions, we find

$$\imath\Delta_\chi^{++}(k^\mu) = \frac{-\imath}{k_\mu k^\mu + m_\chi^2 - \imath\epsilon}, \tag{B11a}$$

$$\imath\Delta_\chi^{--}(k^\mu) = \frac{\imath}{k_\mu k^\mu + m_\chi^2 + \imath\epsilon}, \tag{B11b}$$

$$\imath\Delta_\chi^{+-}(k^\mu) = 2\pi\delta(k_\mu k^\mu + m_\chi^2)\theta(-k^0), \tag{B11c}$$

$$\imath\Delta_\chi^{-+}(k^\mu) = 2\pi\delta(k_\mu k^\mu + m_\chi^2)\theta(k^0). \tag{B11d}$$

Clearly, writing the thermal propagators in the form (B10) above facilitates obtaining the vacuum solutions easily. The reason is that $n_\chi^{\rm eq}(|k^0|) \to 0$ when $T \to 0$, whereas this statement does not hold for $n_\chi^{\rm eq}(k^0)$.

The freedom to choose a different pole prescription such as the advanced or retarded pole prescription is just an equivalent way of writing the thermal propagators, which can easily be verified by making use of the Dirac identity (B7). The vacuum and thermal contributions to $\imath\Delta_\chi^{++}(k^\mu)$ and $\imath\Delta_\chi^{--}(k^\mu)$ separate only so neatly when we use the time ordered and antitime ordered contours to evaluate these propagators, respectively.

APPENDIX C: ALTERNATIVE METHOD OF RENORMALIZING THE SELF-MASSES

In this Appendix, we find the correctly renormalized self-masses by means of an alternative Fourier space calculation. From Eqs. (34a) and (43a) we immediately deduce

$$\begin{aligned}
\imath M_\phi^{++}(k^\mu) &= -\frac{\imath h^2}{2}\int d^D(x-x')(\imath\Delta_\chi^{++}(x;x'))^2 e^{-\imath k(x-x')} \\
&= \frac{\imath h^2}{2}\int \frac{d^D k'}{(2\pi)^D}\frac{1}{k'_\mu k'^\mu + m_\chi^2 - \imath\epsilon} \\
&\quad\times \frac{1}{(k_\mu - k'_\mu)(k^\mu - k'^\mu) + m_\chi^2 - \imath\epsilon} \\
&= -\frac{h^2}{2}\int_0^1 dx \int \frac{d^D l}{(2\pi)^D} \\
&\quad\times \frac{1}{(l_\mu l^\mu + k_\mu k^\mu x(1-x) + m_\chi^2 - \imath\epsilon)^2},
\end{aligned} \tag{C1}$$

where we used Feynman's trick (see e.g. [115]), performed a Wick rotation, and defined $l_\mu = k'_\mu - x k_\mu$ as the new Euclideanized integration variable. The integral can now straightforwardly be performed, which yields when $m_\chi \to 0$:

$$\begin{aligned}
\imath M_\phi^{++}(k^\mu) &= \frac{h^2(D-2)(k_\mu k^\mu - \imath\epsilon)^{(D-4)/2}2^{1-2D}\pi^{(3-D)/2}\Gamma(\frac{D-2}{2})}{\Gamma(\frac{D}{2})\Gamma(\frac{D-1}{2})\sin(\frac{\pi D}{2})} \\
&= \frac{h^2\mu^{4-D}}{16\pi^2(D-4)} - \frac{h^2}{32\pi^2}\left(2 - \gamma_{\rm E} - \log\left(\frac{k_\mu k^\mu - \imath\epsilon}{4\pi\mu^2}\right)\right) + \mathcal{O}(D-4),
\end{aligned} \tag{C2}$$

where in the last line we have expanded around $D = 4$ as usual and we have again introduced a scale μ to make the argument of the logarithm dimensionless. Observe that the (numerical value of the) divergent term coincides with Eq. (51) as it should. Note that for $\imath M_\phi^{--}(k^\mu)$, we would have to use the other Wick rotation (in order not to cross the poles) giving us the desired minus sign difference just as in the position space calculation. Finally, we need to perform the k^0 integral in order to derive the self-mass in Fourier space. The relevant integral is

$$\begin{aligned}
\imath M_{\phi,\rm ren}^{++}(k,t,t') &= \frac{h^2}{32\pi^2}\int_{-\infty}^{\infty}\frac{dk^0}{2\pi}e^{-\imath k^0\Delta t}\left(\gamma_{\rm E} - 2 + \log\left(\frac{k_\mu k^\mu - \imath\epsilon}{4\pi\mu^2}\right)\right) \\
&= -\frac{h^2}{64\pi^3}\left[2\pi\delta(\Delta t)\{\log(4\pi\mu^2) + 2 - \gamma_{\rm E}\} - \int_{-\infty}^{\infty} dk^0 e^{-\imath k^0\Delta t}\{\log|k^2 - (k^0)^2| - \imath\pi\theta((k^0)^2 - k^2)\}\right].
\end{aligned} \tag{C3}$$

In order to make the inverse Fourier integral convergent, we introduce ϵ regulators where appropriate. As an intermediate result, we present

$$\begin{aligned}
\imath M_{\phi,\rm ren}^{++}(k,t,t') = -\frac{h^2}{32\pi^2}\bigg[&\delta(\Delta t)\{\log(4\pi\mu^2) + 2 - \gamma_{\rm E}\} + [\gamma_{\rm E} + \log(-\imath\{|\Delta t| + \imath\epsilon\})]\frac{\cos(k\{|\Delta t| + \imath\epsilon\})}{-\imath\pi(|\Delta t| + \imath\epsilon)} \\
&+ [\gamma_{\rm E} + \log(\imath\{|\Delta t| - \imath\epsilon\})]\frac{\cos(k\{|\Delta t| - \imath\epsilon\})}{\imath\pi(|\Delta t| - \imath\epsilon)} + \frac{1}{2}\left(\frac{e^{-\imath k(|\Delta t| - \imath\epsilon)}}{|\Delta t| - \imath\epsilon} - \frac{e^{\imath k(|\Delta t| + \imath\epsilon)}}{|\Delta t| + \imath\epsilon}\right)\bigg].
\end{aligned} \tag{C4}$$

Clearly, the ϵ regulators in the logarithms and exponents are redundant and can be sent to zero. Using the Dirac rule in Eq. (B7) once more, we finally arrive at

$$\imath M^{++}_{\phi,\mathrm{ren}}(k,t,t') = -\frac{h^2}{32\pi^2}\left[\delta(\Delta t)\{\gamma_E + 2 + \log(4\pi\mu^2\Delta t^2)\} + \frac{e^{-\imath k|\Delta t|}}{|\Delta t| - \imath\epsilon}\right]. \tag{C5}$$

This result is divergent when we let $\Delta t \to 0$. As already discussed in the paper in Sec. III, the correct way to deal with this is to extract two time derivatives acting on e.g. $Z^{++}_\phi(k,t,t')$. We then remain with a perfectly finite result, which can be appreciated from Eq. (64).

To prove this, let us indeed evaluate the two time derivatives in Eq. (64). The result is

$$\begin{aligned}\imath M^{++}_{\phi,\mathrm{ren}}(k,t,t') &= -\frac{h^2}{32\pi^2}\left[\delta(\Delta t)\{\log(4\mu^2\Delta t^2)\} + \frac{e^{-\imath k|\Delta t|}}{|\Delta t| - \imath\epsilon}\right] - \frac{h^2}{32\pi^2}\delta(\Delta t)[\gamma_E + 2 + \log(\pi)] \\ &= -\frac{h^2}{32\pi^2}\left[\delta(\Delta t)\{\gamma_E + 2 + \log(4\pi\mu^2\Delta t^2)\} + \frac{e^{-\imath k|\Delta t|}}{|\Delta t| - \imath\epsilon}\right].\end{aligned} \tag{C6}$$

The first line contains two elements. The first is obtained directly from evaluating the double time derivative in Eq. (64). The second contribution originates from expanding the term multiplying the delta function in Eq. (51) around $D = 4$, corresponding to the minimal subtraction renormalization scheme. Indeed, the second line of Eq. (C6) is identical to (C5) as it should be. This shows that the position space and Fourier space calculations yield identical results; however the former calculation proves to be superior to the latter as the two extracted time derivatives appear naturally in that case.

APPENDIX D: RETARDED SELF-MASS

The retarded self-mass $\imath M^r_{\phi,\mathrm{ren}}(x;x')$ can be obtained by means of an independent calculation by making use of Eq. (52):

$$\begin{aligned}\imath M^r_{\phi,\mathrm{ren}}(x;x') &= \imath M^{++}_{\phi,\mathrm{ren}}(x;x') - \imath M^{+-}_{\phi}(x;x') = \frac{\imath h^2}{128\pi^4}\partial^2\left[\frac{\log(\mu^2\Delta x^2_{++}(x;x'))}{\Delta x^2_{++}(x;x')} - \frac{\log(\mu^2\Delta x^2_{+-}(x;x'))}{\Delta x^2_{+-}(x;x')}\right] \\ &= \frac{\imath h^2}{1024\pi^4}\partial^4[\log^2(\mu^2\Delta x^2_{++}(x;x')) - 2\log(\mu^2\Delta x^2_{++}(x;x')) - \log^2(\mu^2\Delta x^2_{+-}(x;x')) \\ &\quad + 2\log(\mu^2\Delta x^2_{+-}(x;x'))] \\ &= \frac{h^2}{256\pi^3}\partial^4[\theta(\Delta t^2 - r^2)\theta(\Delta t)\{1 - \log(\mu^2(\Delta t^2 - r^2))\}],\end{aligned} \tag{D1}$$

where as before $r = \| \vec{x} - \vec{x}' \|$. In Fourier space we find after some partial integrations:

$$\begin{aligned}\imath M^r_{\phi,\mathrm{ren}}(k,t,t') &= \frac{h^2}{256\pi^3}(\partial_t^2 + k^2)^2\int d^3(\vec{x} - \vec{x}')\theta(\Delta t^2 - r^2)\theta(\Delta t)[1 - \log(\mu^2(\Delta t^2 - r^2))]e^{-\imath\vec{k}\cdot(\vec{x} - \vec{x}')} \\ &= \frac{h^2}{64k\pi^2}(\partial_t^2 + k^2)^2\theta(\Delta t)\Delta t^2\left[\frac{\sin(k\Delta t) - k\Delta t\cos(k\Delta t)}{(k\Delta t)^2}(1 - \log(\mu^2\Delta t^2)) - \int_0^1 dx x\sin(k\Delta t x)\log(1 - x^2)\right].\end{aligned} \tag{D2}$$

The last integral is performed in e.g. [116]:

$$\begin{aligned}\imath M^r_{\phi,\mathrm{ren}}(k,t,t') = \frac{h^2}{64k^3\pi^2}(\partial_t^2 + k^2)^2\theta(\Delta t)\Bigg[&(k\Delta t\cos(k\Delta t) - \sin(k\Delta t))\left(\mathrm{ci}(2k\Delta t) - \gamma_E - \log\left(\frac{k}{2\mu^2\Delta t}\right) - 1\right) \\ &+ (\cos(k\Delta t) + k\Delta t\sin(k\Delta t))\left(\frac{\pi}{2} + \mathrm{si}(2k\Delta t)\right) - 2\sin(k\Delta t)\Bigg].\end{aligned} \tag{D3}$$

Since the term in square brackets is proportional to $(\Delta t)^2$ as $\Delta t \to 0$, the $\theta(\Delta t)$ commutes through one of the $(\partial_t^2 + k^2)$ operators. Evaluating it further yields

$$\imath M^r_{\phi,\mathrm{ren}}(k,t,t') = \frac{h^2}{32k\pi^2}(\partial_t^2 + k^2)\theta(\Delta t)\left[\cos(k\Delta t)\left(\frac{\pi}{2} + \mathrm{si}(2k\Delta t)\right) - \sin(k\Delta t)\left(\mathrm{ci}(2k\Delta t) - \gamma_E - \log\left(\frac{k}{2\mu^2\Delta t}\right)\right)\right]. \tag{D4}$$

If we determine the retarded self-mass directly from Eq. (64), we find perfect agreement with the result above, representing yet another consistency check of (64).

APPENDIX E: THE STATISTICAL PROPAGATOR IN FOURIER SPACE

Let us now calculate the statistical propagator in Fourier space. The starting point is the equation of motion (39) in Fourier space. We can straightforwardly write the two Wightman functions as

$$\imath\Delta_\phi^{-+}(k^\mu) = \frac{-\imath M_\phi^{-+}(k^\mu)\imath\Delta_\phi^{a}(k^\mu)}{k_\mu k^\mu + m_\phi^2 + \imath M_{\phi,\mathrm{ren}}^{r}(k^\mu)}, \quad \text{(E1a)}$$

$$\imath\Delta_\phi^{+-}(k^\mu) = \frac{-\imath M_\phi^{+-}(k^\mu)\imath\Delta_\phi^{a}(k^\mu)}{k_\mu k^\mu + m_\phi^2 + \imath M_{\phi,\mathrm{ren}}^{r}(k^\mu)}, \quad \text{(E1b)}$$

where we have made use of the definition of the advanced propagator (24b) in Fourier space:

$$\imath\Delta_\phi^{a}(k^\mu) = \frac{-\imath}{k_\mu k^\mu + m_\phi^2 + \imath M_{\phi,\mathrm{ren}}^{a}(k^\mu)}. \quad \text{(E2)}$$

Moreover, we made use of

$$\begin{aligned}\imath M_{\phi,\mathrm{ren}}^{r}(k^\mu) &= \imath M_{\phi,\mathrm{ren}}^{++}(k^\mu) - \imath M_\phi^{+-}(k^\mu)\\ &= \imath M_\phi^{-+}(k^\mu) - \imath M_{\phi,\mathrm{ren}}^{--}(k^\mu),\end{aligned} \quad \text{(E3a)}$$

$$\begin{aligned}\imath M_{\phi,\mathrm{ren}}^{a}(k^\mu) &= \imath M_{\phi,\mathrm{ren}}^{++}(k^\mu) - \imath M_\phi^{-+}(k^\mu)\\ &= \imath M_\phi^{+-}(k^\mu) - \imath M_{\phi,\mathrm{ren}}^{--}(k^\mu).\end{aligned} \quad \text{(E3b)}$$

Clearly, we need to evaluate some self-masses in Fourier space. The simplest method of determining e.g. $\imath M_\phi^{+-}(k^\mu)$ or $\imath M_\phi^{-+}(k^\mu)$ is to use the retarded self-mass in Fourier space and $\imath M_{\phi,\mathrm{ren}}^{++}(k^\mu)$ which we already derived before in Appendix C. Using expressions (72) and (D4) we can derive

$$\begin{aligned}\imath M_{\phi,\mathrm{ren}}^{r}(k^\mu) &= \int_{-\infty}^{\infty} d\Delta t e^{\imath k^0\Delta t}\imath M_{\phi,\mathrm{ren}}^{r}(k,\Delta t)\\ &= -\frac{h^2}{64k\pi^2}(-k_0^2+k^2)\int_0^\infty d\Delta t\Big[e^{\imath(k+k^0)\Delta t}\Big\{-\imath(\mathrm{ci}(2k\Delta t)-\log(2k\Delta t)-\gamma_{\mathrm{E}})-\frac{\pi}{2}-\mathrm{si}(2k\Delta t)\Big\}\\ &\quad + e^{\imath(k^0-k)\Delta t}\Big\{\imath(\mathrm{ci}(2k\Delta t)-\log(2k\Delta t)-\gamma_{\mathrm{E}})-\frac{\pi}{2}-\mathrm{si}(2k\Delta t)\Big\} - 2\imath\log(2\mu\Delta t)(e^{\imath(k+k^0+\imath\epsilon)\Delta t}-e^{\imath(k^0-k+\imath\epsilon)\Delta t})\Big],\end{aligned} \quad \text{(E4)}$$

where $k_0^2=(k^0)^2$ and where we have used two partial integrations and disposed ourselves of the boundary terms by introducing an ϵ regulator where necessary. We can now use

$$\int_0^\infty dx\log(\beta x)e^{\imath\alpha x} = -\frac{\imath}{\alpha}\Big[\log\Big(\frac{-\imath\alpha+\epsilon}{\beta}\Big)+\gamma_{\mathrm{E}}\Big], \quad \text{(E5)}$$

and moreover we write

$$e^{\imath(k^0\pm k)\Delta t} = \frac{-\imath}{k^0\pm k}\partial_t e^{\imath(k^0\pm k)\Delta t}, \quad \text{(E6)}$$

to prepare for another partial integration. Equation (E4) evaluates to

$$\begin{aligned}\imath M_{\phi,\mathrm{ren}}^{r}(k^\mu) = -\frac{h^2}{64k\pi^2}\Big[&2(k^0-k)\Big(\log\Big(\frac{-\imath(k+k^0)+\epsilon}{2\mu}\Big)\\ &+\gamma_{\mathrm{E}}\Big) - 2(k^0+k)\Big(\log\Big(\frac{-\imath(k^0-k)+\epsilon}{2\mu}\Big)\\ &+\gamma_{\mathrm{E}}\Big) + \int_0^\infty d\Delta t\frac{2k^0}{\Delta t}\\ &\times(e^{\imath(k^0+k)\Delta t}-e^{\imath(k^0-k)\Delta t})\Big].\end{aligned} \quad \text{(E7)}$$

For $\alpha,\beta\in\mathbb{R}$, we can use

$$\lim_{z\downarrow 0}\int_z^\infty d\Delta t\Big[\frac{\cos(\alpha\Delta t)-1}{\Delta t}-\frac{\cos(\beta\Delta t)-1}{\Delta t}\Big] = \log\Big(\frac{|\beta|}{|\alpha|}\Big), \quad \text{(E8)}$$

to evaluate the remaining integrals. The result is

$$\imath M_{\phi,\mathrm{ren}}^{r}(k^\mu) = \frac{h^2}{32\pi^2}\Big[\log\Big(\frac{-k_0^2+k^2-\imath\,\mathrm{sgn}(k^0)\epsilon}{4\mu^2}\Big)+2\gamma_{\mathrm{E}}\Big]. \quad \text{(E9)}$$

From $\imath M_{\phi,\mathrm{ren}}^{r}(k^\mu)=\imath M_{\phi,\mathrm{ren}}^{++}(k^\mu)-\imath M_\phi^{+-}(k^\mu)$ we can immediately find the Wightman self-masses. We take $\imath M_{\phi,\mathrm{ren}}^{++}(k^\mu)$ from Eq. (C2), but we have to make sure we use the same subtraction scheme as in our position space calculation in Eq. (52). We therefore modify Eq. (C2) slightly to

$$\imath M_{\phi,\mathrm{ren}}^{++}(k^\mu) = \frac{h^2}{32\pi^2}\Big[\log\Big(\frac{-k_0^2+k^2-\imath\epsilon}{4\mu^2}\Big)+2\gamma_{\mathrm{E}}\Big]. \quad \text{(E10)}$$

We thus find

$$\imath M^{+-}_{\phi}(k^\mu) = -\frac{\imath h^2}{16\pi}\theta(-k^0 - k), \tag{E11a}$$

$$\imath M^{-+}_{\phi}(k^\mu) = -\frac{\imath h^2}{16\pi}\theta(k^0 - k). \tag{E11b}$$

As a check, we consider the following relation that has to be satisfied:

$$\imath M^{++}_{\phi,\mathrm{ren}}(k^\mu) + \imath M^{--}_{\phi,\mathrm{ren}}(k^\mu) = \imath M^{+-}_{\phi}(k^\mu) + \imath M^{-+}_{\phi}(k^\mu). \tag{E12}$$

Of course, $\imath M^{++}_{\phi,\mathrm{ren}}(k^\mu)$ is given in Eq. (E10) which also allows us to derive the antitime ordered self-mass:

$$\imath M^{--}_{\phi,\mathrm{ren}}(k^\mu) = -\frac{h^2}{32\pi^2}\left[\log\left(\frac{-k_0^2 + k^2 + \imath\epsilon}{4\mu^2}\right) + 2\gamma_{\mathrm{E}}\right], \tag{E13}$$

where $\imath M^{--}_{\phi,\mathrm{ren}}(k^\mu)$ contains an additional minus sign because of the Wick rotation (see Appendix C for details). We thus find

$$\imath M^{+-}_{\phi}(k^\mu) + \imath M^{-+}_{\phi}(k^\mu) = -\frac{\imath h^2}{16\pi}\theta(k_0^2 - k^2). \tag{E14}$$

This is in perfect agreement with Eq. (E11).

Using Eqs. (E1), (E10), and (E11) we can derive our solutions for the two Wightman functions in Fourier space:

$$\imath\Delta^{+-}_{\phi}(k^\mu) = \imath\theta(-k^0 - k)\Bigg[\frac{1}{k_\mu k^\mu + m_\phi^2 + \frac{h^2}{32\pi^2}(\log(\frac{|k_\mu k^\mu|}{4\mu^2}) + 2\gamma_{\mathrm{E}}) - \frac{\imath h^2}{32\pi}\mathrm{sgn}(k^0)\theta(k_0^2 - k^2)} - \frac{1}{k_\mu k^\mu + m_\phi^2 + \frac{h^2}{32\pi^2}(\log(\frac{|k_\mu k^\mu|}{4\mu^2}) + 2\gamma_{\mathrm{E}}) + \frac{\imath h^2}{32\pi}\mathrm{sgn}(k^0)\theta(k_0^2 - k^2)}\Bigg], \tag{E15a}$$

$$\imath\Delta^{-+}_{\phi}(k^\mu) = -\imath\theta(k^0 - k)\Bigg[\frac{1}{k_\mu k^\mu + m_\phi^2 + \frac{h^2}{32\pi^2}(\log(\frac{|k_\mu k^\mu|}{4\mu^2}) + 2\gamma_{\mathrm{E}}) - \frac{\imath h^2}{32\pi}\mathrm{sgn}(k^0)\theta(k_0^2 - k^2)} - \frac{1}{k_\mu k^\mu + m_\phi^2 + \frac{h^2}{32\pi^2}(\log(\frac{|k_\mu k^\mu|}{4\mu^2}) + 2\gamma_{\mathrm{E}}) + \frac{\imath h^2}{32\pi}\mathrm{sgn}(k^0)\theta(k_0^2 - k^2)}\Bigg]. \tag{E15b}$$

The limit $h \to 0$ in the equations above nicely agrees with the vacuum Wightman propagators in Eqs. (43c) and (43d). Hence the statistical propagator in Fourier space reads

$$F_{\phi}(k^\mu) = -\frac{\imath}{2}\mathrm{sgn}(k^0)\theta(k_0^2 - k^2)\Bigg[\frac{1}{k_\mu k^\mu + m_\phi^2 + \frac{h^2}{32\pi^2}(\log(\frac{|k_\mu k^\mu|}{4\mu^2}) + 2\gamma_{\mathrm{E}}) - \frac{\imath h^2}{32\pi}\mathrm{sgn}(k^0)\theta(k_0^2 - k^2)} - \frac{1}{k_\mu k^\mu + m_\phi^2 + \frac{h^2}{32\pi^2}(\log(\frac{|k_\mu k^\mu|}{4\mu^2}) + 2\gamma_{\mathrm{E}}) + \frac{\imath h^2}{32\pi}\mathrm{sgn}(k^0)\theta(k_0^2 - k^2)}\Bigg]. \tag{E16}$$

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