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Dark energy simulacrum in nonlinear electrodynamics

Lance Labun and Johann Rafelski

Department of Physics, University of Arizona, Tucson, Arizona, 85721 USA,
and Ludwig-Maximillians-Universität München, 85748 Garching, Germany

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Quasiconstant external fields in nonlinear electromagnetism generate a global contribution proportional to $g^{\mu\nu}$ in the energy-momentum tensor, thus a simulacrum of dark energy. To provide a thorough understanding of the origin and strength of its effects, we undertake a complete theoretical and numerical study of the energy-momentum tensor $T^{\mu\nu}$ for nonlinear electromagnetism. The Euler-Heisenberg nonlinearity due to quantum fluctuations of spinor and scalar matter fields is considered and contrasted with the properties of classical nonlinear Born-Infeld electromagnetism. We address modifications of charged particle kinematics by strong background fields.

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I. INTRODUCTION

Recent analysis has constrained the dark energy to the characteristics of a cosmological constant: equation of state $w \equiv p/\rho \approxeq -1$ and spatially homogeneous distribution [1,2]. This means that the dark energy is present in the energy-momentum tensor proportional to $g^{\mu\nu}$ as has often been discussed in the context of vacuum energy [3]. Proportionality to $g^{\mu\nu}$ does not exclude that dark energy originates in properties of ponderable fields and matter; in the energy-momentum tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}V_{\text{eff}}, \tag{1}$$

a nonvanishing trace $T^\mu_\mu = g^{\mu\nu}T_{\mu\nu}$ is a simulacrum of the dark energy, that is, a tangible but not necessarily complete analogy: the properties and consequences are the same, but ultimately the generating mechanisms may differ between the observed dark energy and the model T^μ_μ. The possibility that the cosmological dark energy is consequence of a (more complete) theory of ponderable matter and fields motivates a deeper probe of well-controlled situations in which the phenomenologically similar energy-momentum trace arises.

As has often been noted, the small magnitude of the dark energy suggests a weakly broken symmetry as its source. Hence the natural starting point for the present investigation is a theory with a traceless energy-momentum tensor. Extensions of this theory are analyzed for the generation of an energy-momentum trace and by comparison with the starting theory the physics of the energy-momentum trace better understood. An effort in this direction has been made in the context of the vacuum structure of quantum chromodynamics (QCD) [4], and we show here that quantum electrodynamics (QED) has similar features which are more easily accessible.

Our point of departure is therefore Maxwell electromagnetism, whose energy-momentum tensor,

$$T^{\text{Max}}_{\mu\nu} = g_{\mu\nu}\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - F_{\mu\lambda}F_\nu{}^\lambda \tag{2}$$

is manifestly traceless. Even in absence of external sources the Maxwell field equations are incomplete due to interaction with the electron-positron vacuum fluctuations which are present at the length scale $\lambda\!\!\!^-_e \equiv \hbar/m_ec$. At distances of comparable magnitude ($\lambda \simeq \lambda\!\!\!^-_e$) these are vacuum polarization effects which impact precision atomic physics experiments. For long distance ($\lambda \gg \lambda\!\!\!^-_e$) one obtains the nonlinear effective theory of the photon studied in depth by Euler, Kockel, and Heisenberg, and Schwinger [5,6].

This nonlinear Euler-Heisenberg (EH) theory of electromagnetism is just one of many possible effective actions. Beyond the EH-QED framework, we can imagine writing down a more complete theory containing all effective interactions, reducing in the long wavelength limit at the classical level to Maxwell's equations. Born-Infeld (BI) theory, designed to regulate point-particle-induced divergences [7], can be thought of as an effort to provide a more complete theory of electromagnetism, though it too is now considered as an effective theory [8] arising from string theory.

In nonlinear electromagnetism, the Maxwell energy-momentum tensor Eq. (2) is modified by two quantities: a dielectric function ε, which scales the contribution of the Maxwell energy-momentum in the total, and the trace of energy-momentum tensor T^μ_μ, which is a new contribution. We have elsewhere observed that T^μ_μ arising in QED can in this regard be viewed as the modification of the vacuum energy by the presence of electromagnetic fields [9], and, analogously, the connection of dark energy with a form of vacuum energy has been discussed before [3,4]. In the external field framework of the EH theory, the trace T^μ_μ is a global constant energy density proportional to $g^{\mu\nu}$ and thus serves as a simulacrum of a cosmological constant in the "universe" spanned by the external field. In this work we investigate the magnitude and some consequences of a ponderable T^μ_μ-as-cosmological-constant, evaluate the dielectric function and compare the behavior of EH and BI nonlinear theories.

Now, as is well known, a theory with a traceless energy-momentum tensor is invariant under scale changes. Maxwell's electromagnetism is the prime example: the energy-momentum tensor is traceless and indeed the classical theory of radiation is scale invariant. In a gauge theory, the related conformal symmetry can be spontaneously broken, the study of which has a long and distinguished history, originating with the rise of QCD as the theory of strong interactions [10,11]. We will rederive here some results of the extensions of these studies to QED [12,13].

Scale invariance can be broken by the explicit appearance of a dimensionful quantity in the theory, such as the mass of the electron in QED. In fact, any nonlinear electromagnetism requires a scale with the dimension of an electrical field eE_0, in order to render the Lagrangian dimensionally consistent. We express this scale in terms of a mass M:

$$eE_0 \equiv \frac{(Mc^2)^2}{\hbar c}.$$

For the Euler-Heisenberg (EH) effective action the introduction of M is a natural step to take as the nonlinearity is of quantum origin and $M \simeq m/\sqrt{\alpha}$, where $\alpha = 1/137$ is the usual fine structure constant. For the BI theory it is a matter of convenience to use mass rather than length as the scale, converting one into another using $\hbar$.

If indeed BI theory is a weak-field limit of string theory in which the nonlinearity is a consequence of high mass quantum fluctuations [8] the appearance of $\hbar$ would be appropriate, and the associated scale could be as large as the Planck mass, $M_{\rm Pl} = 1.2 \times 10^{19}$ GeV. It should be noted that current experimental limits as well as EH nonlinearity probes a scale below 100 MeV, thus a string related BI theory maybe quite removed from the present experimental reality. We compare EH and BI theories mainly because their behavior is very different.

In QED and BI, the presence of scale which breaks the conformal symmetry is explicit, and the energy-momentum trace is not "anomalous," unlike the case of QCD. The scale of the nonlinearity is, as we shall show, the determining factor of T^μ_μ. This fact is the simple yet important and original theoretical observation presented in this paper. Having thus suggested the interconnection of dark energy, conformal symmetry and the presence of scale in the theory, we leave issues specific to conformal symmetry to future work and here focus on the physics of T^μ_μ and its origins in nonlinearity of the theory.

We derive in Sec. II the field energy-momentum tensor and explicitly connect its trace to nonlinearity of the electromagnetic theory. To compare relative magnitudes and suggest new constraints on a Born-Infeld-type completion of electromagnetism, in Sec. III we evaluate the BI modifications to the Maxwell energy-momentum tensor. In clarification of conflicting statements present in the literature, we begin our discussion of quantum electrodynamics in Sec. IV with a new derivation of the relationship between the electron-positron condensate and the energy-momentum trace based on the explicit origin of the trace in nonlinearity of the theory. Extending a technique of resummation of the action introduced by Müller, *et al.* [14], we then provide complete numerical evaluations of the condensate, energy-momentum trace and dielectric function for QED and spin-0 quantum electrodynamics. These evaluations display striking analytical features not before apparent in the Euler-Heisenberg functions. We compare BI and QED contributions to T^μ_μ and show that QED vacuum fluctuations remain dominant given the experimental constraints.

In Sec. V of this report, we discuss the kinematics of charged particles moving in external fields. The vacuum of a nonlinear theory is studied as a ponderous medium with nonlinear response. The Lorentz force is preserved, but the breaking of the superposition principle results in effective potentials for charged particles moving in external fields that are not automatically obtained from the Lorentz force.

II. ENERGY-MOMENTUM TENSOR OF NONLINEAR ELECTROMAGNETISM

Setting from now on $\hbar = c = 1$, we can consider the effects of $E_0 := M^2/e$ mass M or length $l = 1/M$ scale, where M can be as large as a string theory scale or as small as the mass of the electron. The consequences are best seen writing the effective action in the form

$$\bar{V}_{\rm eff} \equiv -\mathcal{S} + M^4 \bar{f}_{\rm eff}\left(\frac{\mathcal{S}}{M^4}, \frac{\mathcal{P}}{M^4}\right) \underset{M\to\infty}{\longrightarrow} -\mathcal{S} \tag{3}$$

presented here as a function of the Lorentz scalar and pseudoscalar

$$\mathcal{S} := \tfrac{1}{4}F_{\kappa\lambda}F^{\kappa\lambda} = \tfrac{1}{2}(B^2 - E^2); \tag{4a}$$

$$\mathcal{P} := \tfrac{1}{4}F^*_{\kappa\lambda}F^{\kappa\lambda} = E \cdot B. \tag{4b}$$

As noted, classical, linear electromagnetism must constitute the limit of Eq. (3) for fields small as measured in units of E_0. It should be noted that $\bar{f}_{\rm eff}$ consists only of nonlinear terms, specifically excluding linear terms such as $\mathcal{S}\ln m/\mu$ which introduce another scale μ. When such terms are admitted, the bar is left off $f_{\rm eff}$ and $V_{\rm eff}$.

A. Dielectric function and Trace

To understand the implications of the dimensioned scale we consider the explicit form of the energy-momentum tensor (1), separating the traceless Maxwell part. For a general function $V_{\rm eff}(\mathcal{S}, \mathcal{P})$, we obtain

$$T_{\mu\nu} = \left(-\frac{\partial V_{\rm eff}}{\partial \mathcal{S}}\right)(g_{\mu\nu}\mathcal{S} - F_{\mu\lambda}F_\nu{}^\lambda) - g_{\mu\nu}\left(V_{\rm eff} - \mathcal{S}\frac{\partial V_{\rm eff}}{\partial \mathcal{S}} - \mathcal{P}\frac{\partial V_{\rm eff}}{\partial \mathcal{P}}\right). \tag{5}$$

Comparison with Eq. (2) shows the energy-momentum tensor of Maxwell's electromagnetism is modified by a dielectric function, $-\partial V_{\rm eff}/\partial\mathcal{S}$, to be discussed below. Using Eq. (3) we simplify the second term to

$$\left(V_{\rm eff} - \mathcal{S}\frac{\partial V_{\rm eff}}{\partial \mathcal{S}} - \mathcal{P}\frac{\partial V_{\rm eff}}{\partial \mathcal{P}}\right) = \frac{1}{4}M\frac{\partial \bar{f}_{\rm eff}}{\partial M}, \tag{6}$$

This form is very useful because it provides a simple means of calculating the trace directly from the effective action. The importance of Eq. (6) lies in its distillation of the physical source of conformal symmetry breaking in any nonlinear theory of electromagnetism:

Terms linear in the invariant $\mathcal{S}$ cannot contribute to the right side of Eq. (6) since they cancel explicitly on the left side of Eq. (6). Such contributions must therefore be omitted from $V_{\rm eff}$ in the study of energy-momentum trace, and hence we have introduced the barred $\bar{f}_{\rm eff}$ to denote the nonlinear components of the effective potential. Letting $V^{(1)}_{\rm eff}$ denote the remaining linear terms in the Lagrangian, we have the decomposition

$$V_{\rm eff} = V^{(1)}_{\rm eff} + M^4\bar{f}_{\rm eff}. \tag{7}$$

The lowest power in $\mathcal{S}$, $\mathcal{P}$ is 2 due to preservation of both parity and charge conjugation symmetry, which, respectively, require an action even under parity transformations and even in the coupling e, hence even in the field strengths (already true of any Lorentz scalar). A nonzero imaginary part of the action entails breaking of time reversal symmetry, though the trace $\mathcal{T}$ will be small under laboratory conditions. These symmetry arguments imply that for field strengths below the critical scale $E \ll m^2$, the energy-momentum trace must be at least 4th order in the fields.

Solving the partial differential equation

$$V_{\rm eff} - \mathcal{S}\frac{\partial V_{\rm eff}}{\partial \mathcal{S}} - \mathcal{P}\frac{\partial V_{\rm eff}}{\partial \mathcal{P}} = 0 \tag{8}$$

displays one obvious class of nonlinear Lagrangians that have traceless energy-momentum tensors, namely,

$$V_{\rm eff} = \mathcal{S}\sum_{n=-\infty}^{+\infty} a_n\left(\frac{\mathcal{S}}{\mathcal{P}}\right)^n. \tag{9}$$

Comparison with Eq. (3) reveals the reason: such Lagrangians are conformal. Since this class is nonperturbative in at least one of the field invariants and we are interested in having an energy-momentum trace, we exclude theories of the form in Eq. (9) as well any other nonperturbative actions that may satisfy Eq. (8), despite their inherent interest.

In view of Eqs. (5) and (6), we summarize

$$T_{\mu\nu} = \varepsilon T^{\rm Max}_{\mu\nu} + g_{\mu\nu}\tfrac{1}{4}\mathcal{T} \tag{10a}$$

$$\varepsilon \equiv -\frac{\partial V_{\rm eff}}{\partial \mathcal{S}}, \qquad T^\mu_\mu \equiv \mathcal{T} = -M\frac{d\bar{f}_{\rm eff}}{dM}. \tag{10b}$$

Here ε is the dielectric function, $T^{\rm Max}_{\mu\nu}$ the Maxwell energy-momentum tensor, and $\mathcal{T}$ the energy-momentum trace. Interestingly, $\mathcal{T}/4 \leftrightarrow \lambda/2$ provides a dark energy or Einstein-like cosmological constant, while the traceless part is the same as in Maxwell theory, up to the multiplicative dielectric function.

Equation (10b) though derived here for the case of electromagnetism has a much wider domain of validity. The implications of the separation in Eq. (10) have been previously noted in context of the photon propagation effects [15,16], but $\mathcal{T}$ was not given in the form Eq. (10b), nor has in any form the trace $\mathcal{T}$ been computed. We shall below demonstrate how the concomitant identities Eq. (10b) provide new physical insight in understanding the quantum effects and their connection to nonlinearity of the action.

The alternative and equivalent representation, often used in the electromagnetism of nonlinear media (for example see Sec. 8 of [17]),

$$T^{\mu\nu} = H^{\mu\lambda}F^\nu{}_\lambda - g^{\mu\nu}\mathcal{L} \tag{11}$$

is using the displacement tensor

$$H^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} = \frac{\partial\mathcal{L}}{\partial\mathcal{S}}F^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial\mathcal{P}}\tilde{F}^{\mu\nu}. \tag{12}$$

The trace is now distributed into several components

$$\mathcal{T} = H^{\mu\nu}F_{\mu\nu} - 4\mathcal{L} = \vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H} - 4\mathcal{L}. \tag{13}$$

The origins and properties of $\mathcal{T}$ are obscured by the constitutive relation $H^{\mu\nu}(F^{\alpha\beta})$. The expression Eq. (10) evidences the departure from the classical theory more clearly in the context we consider.

B. Stress-energy density

Some further notable properties of the stress-energy density of the nonlinear electromagnetic (EM) field will be collected here. The energy density in the frame of the metric, i.e. the quantity entering Einstein's equations, is

$$T^{00} = \tfrac{\varepsilon}{2}(E^2 + B^2) + \tfrac{1}{4}\mathcal{T}.$$

The EM-stresses T^{ij} have the same structure as in the Maxwell theory, as is already evident from the format of Eq. (16):

$$T^{ij} = \varepsilon T^{ij}_{\rm Max} - \delta_{ij}\tfrac{1}{4}\mathcal{T} \tag{14}$$

The trace $\mathcal{T}$ acts to compensate the forces $T^{ij}_{\rm Max}$ tearing the field sources apart in Maxwell electromagnetism. For this reason, for example, in BI theory the T^{ij} vanishes allowing a stable charged particle without material stresses. A sufficient condition for this to be true is that the point particle solution satisfies $\lim_{r\to 0} r^3 T^{00} = 0$ [18].

Because the energy-momentum tensor is conserved we have $T_{\mu\nu,\nu} \equiv \partial T_{\mu\nu}/\partial x^\nu = 0$, which is a covariant relation true in any frame. A differential conservation law leads to an integral conservation law by integration over the observer's hypersurface:

$$\int_1^2 d^4x \frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0, \quad \text{or} \quad \int_1 d^3\sigma_\nu T^{\mu\nu} = \int_2 d^3\sigma_\nu T^{\mu\nu}, \tag{15}$$

i.e. the energy-momentum flow through surface 1 is the same as later through surface 2. It is common to choose an observer at rest in laboratory so that $d^3\sigma_\nu = u_\nu d^3x$, with $u^\nu = (1, 0, 0, 0)$.

For nearly homogeneous fields we can omit the 3-volume and consider a conserved four-momentum density of the EM field

$$p^\mu_{\rm Max} = u_\nu T^{\mu\nu}_{\rm Max} \to ((E^2 + B^2)/2, \vec{E} \times \vec{B})$$

finding the well-known result for the rest energy and Poynting vector of the classical field. This result is easily generalized to the nonlinear electromagnetism:

$$p^\mu = \left(\varepsilon \frac{E^2 + B^2}{2} + \frac{\mathcal{T}}{4}, \varepsilon \vec{E} \times \vec{B}\right), \tag{16}$$

showing the appropriateness of calling ε the dielectric function, since it plays the role of the dielectric constant $\vec{E} = \varepsilon \vec{D}$ when considering electric charge in vacuum. For Maxwell electromagnetism $\varepsilon = 1$ and $\mathcal{T} = 0$.

It is an elementary exercise to show that in the Maxwell limit the proper energy density, or its "mass density," is

$$p^{\rm Max}_\mu p^\mu_{\rm Max} = \frac{(E^2 + B^2)^2}{4} - (\vec{E} \times \vec{B})^2 \tag{17a}$$

$$= \frac{(E^2 - B^2)^2}{4} + (\vec{E} \cdot \vec{B})^2 = \mathcal{S}^2 + \mathcal{P}^2. \tag{17b}$$

Generalizing to nonlinear EM theory, we find the local mass density of the field to be

$$u_f \equiv \sqrt{p_\mu p^\mu} = \sqrt{(\mathcal{S}^2 + \mathcal{P}^2)\varepsilon^2 + (\mathcal{T}/4)^2}. \tag{18}$$

$\mathcal{T}/4 \leftrightarrow \lambda/2$ provides thus both a "dark energy" and a "mass density" of the electromagnetic field.

III. BORN-INFELD ELECTROMAGNETISM

As demonstrated in the preceding discussion, an intrinsically nonlinear theory of electromagnetism (in most cases) entails an energy-momentum trace, and we begin by studying a nonquantum example of a nonlinear alternative to Maxwellian theory. Historically, Born-Infeld electromagnetism was introduced in order to solve the infinite self-energy (and self-stress) problem of a pointlike electron arising in consideration of the radial electric field E_r of a point charge q:

$$U = \int d^3x \frac{1}{2} E_r^2 \to \infty, \quad \text{for } E_r = \frac{q}{r^2}$$

To remedy this, Born and Infeld took inspiration from special relativity, considering the action [note that we follow the modern convention, opposite in sign to the original paper [7], and also, recall remarks about the scale M above Eq. (3)]:

$$V^{\rm (BI)}_{\rm eff} = M^4(1 - \sqrt{1 + 2\mathcal{S}/M^4 - (\mathcal{P}/M^4)^2}) \tag{19}$$

$$= M^4(\sqrt{-g} - \sqrt{-h}), \tag{20}$$

$$h = \det h_{\mu\nu}, \qquad h_{\mu\nu} = g_{\mu\nu} + \frac{F_{\mu\nu}}{M^2}, \tag{21}$$

where the particular combination of S, P terms derives from the extension of the space-time metric with the antisymmetric field tensor, $F_{\mu\nu}$. In the weak field (infinite mass) limit Maxwell's theory indeed arises, $V^{\rm (BI)} \to -\mathcal{S}$.

For the Born-Infeld case the dielectric function is

$$\varepsilon_{\rm BI} = -\frac{\partial V^{\rm (BI)}_{\rm eff}}{\partial S} = [1 + 2\mathcal{S}/M^4 - (\mathcal{P}/M^4)^2]^{-1/2}, \tag{22}$$

which exhibits a formal analogy to the γ-factor familiar from special relativity, though with two different limits as $\mathcal{S}$ or $\mathcal{P}$ respectively approach the limiting value M^4 (see Fig. 1, left). The dielectric function goes over from suppression ($\varepsilon < 1$) to augmentation ($\varepsilon > 1$), when the magnetic component of the field becomes subdominant, which corresponds to crossing the line $2\mathcal{S} = \mathcal{P}^2$ from the lower right.

As presented, Eq. (19) is in the form required by Eq. (3). However, the BI action contains no terms linear in $\mathcal{S}$; $\mathcal{T}$ is identically $-M(dV_{\rm eff}/dM)$. We obtain for the BI energy-momentum trace using the relation Eq. (6)

$$\mathcal{T}^{\rm (BI)} = 4M^4(\varepsilon_{\rm BI}(1 + \mathcal{S}/M^4 1) - 1),$$

$$= 4M^4\left(\sqrt{\frac{1 + 2\mathcal{S}/M^4 + (\mathcal{S}/M^4)^2}{1 + 2\mathcal{S}/M^4 - (\mathcal{P}/M^4)^2}} - 1\right) \tag{23}$$

which in the latter form is manifestly positive-definite, just like the cosmological constant. For small fields we expand the second form in Eq. (23) to obtain

$$\mathcal{T}^{\rm (BI)} \to \frac{2}{M^4} \frac{\mathcal{S}^2 + \mathcal{P}^2}{\sqrt{1 + 2\mathcal{S}/M^4}}. \tag{24}$$

In Fig. 1 we show the dielectric function and energy-momentum trace for strengths up to the maximum field strength. The functional behavior of both is smooth, though a $\mathcal{S}$, $\mathcal{P}$ functional asymmetry develops at large values of the fields.

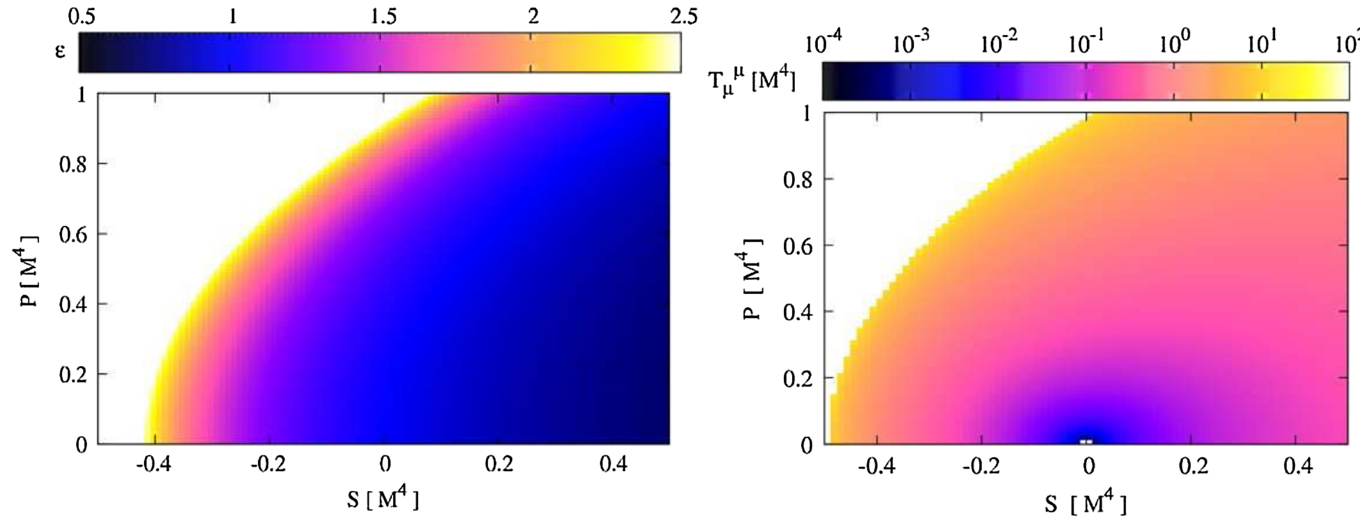

FIG. 1 (color online). The departure of the Born-Infeld energy-momentum tensor from that of the Maxwell $T^{\mu\nu}_{\rm Max}$: At left the dielectric function, Eq. (22) and at right the trace, Eq. (23). Field strength invariants and energy densities are in units of M^4.

In contrast to the classical theory analyzed here, we note a recent report suggesting that the quantized BI theory as studied on the lattice may be conformally symmetric [19].

IV. EULER-HEISENBERG ELECTROMAGNETISM

The Euler-Heisenberg effective action is well known:

$$V^{\rm f}_{\rm eff} = \int_{0+\delta}^{\infty} \frac{ds e^{-m^2 s}}{8\pi^2 s^3}(1 - eas\cot(eas)ebs\coth(ebs)) \tag{25}$$

for Dirac fermions, and

$$V^{\rm s}_{\rm eff} = \int_{0+\delta}^{\infty} \frac{ds e^{-m^2 s}}{16\pi^2 s^3}(eas\csc(eas)ebs\,{\rm csch}(ebs) - 1) \tag{26}$$

for charged scalars, in which

$$a := \sqrt{\sqrt{\mathcal{S}^2 + \mathcal{P}^2} - \mathcal{S}}, \quad \text{and} \quad b := \sqrt{\sqrt{\mathcal{S}^2 + \mathcal{P}^2} + \mathcal{S}}. \tag{27}$$

The characteristic strength of fluctuations in the matter field is made explicit in the appearance of m which is the mass of the matter particle which has been integrated out; in QED it is the mass of the electron.

For vanishing $\mathcal{P}$ we have $a \to \sqrt{|\mathcal{S}| - \mathcal{S}}$ and $b \to \sqrt{|\mathcal{S}| + \mathcal{S}}$. Thus when $E = 0$ we find $a \to 0$, $b \to |B|$ and when $B = 0$ we find $a \to |E|$, $b \to 0$. In this sense, b plays the role of the generalized (Lorentz invariant) magnetic field and a that of the generalized electric field. The constant subtraction ± 1 removes the (divergent) zero point energy of free electrons and positrons. The difference in normalization reflects the doubling of the number of degrees of freedom for spin-$1/2$ particles, and the overall sign corresponds to the difference in sign of vacuum fluctuations between bosons and fermions.

A. The condensate $\langle\bar\psi\psi\rangle$ and trace $\mathcal{T}$

Equation (10b) provides a direct means of calculating the energy-momentum trace, but its connection to vacuum structure in QED and its deformation by the applied fields which induce the EH nonlinearity is encoded in $m(dV_{\rm eff}/dm)$. Consider the Feynman boundary condition Green's function of the fluctuating matter field in presence of the electromagnetic field, which determines the vacuum fluctuations,

$$m\frac{\partial V_{\rm eff}(x, m)}{\partial m} = im \lim_{\epsilon\to 0} {\rm tr}[S_F(x+\epsilon, x-\epsilon, m) - S_F^{(0)}] \tag{28}$$

Here ϵ is a timelike vector, and $S_F^{(0)}$ is the free field Feynman Green's function. In general

$$S_F(x, x') = -i\langle T(\psi(x')\bar\psi(x))\rangle. \tag{29}$$

We rewrite the right side of Eq. (28) using the elementary form of Wick's decomposition theorem

$$T(\psi(x')\bar\psi(x)) = :\psi(x')\bar\psi(x): + \langle 0|T(\psi(x')\bar\psi(x))|0\rangle,$$

where the normal ordering is with respect to the "no-field" vacuum. Taking the expectation value of this relation at a single space-time point in the "with-field" vacuum $|\rangle$ we find

$$\begin{aligned}\langle:\psi\bar\psi:\rangle &= \langle|T(\psi(x)\bar\psi(x))|\rangle - \langle 0|T(\psi(x)\bar\psi(x))|0\rangle \\ &= iS_F(x,x) - iS_F^0(x,x)\end{aligned} \tag{30}$$

with the same ϵ-limit as in Eq. (28) implied for equal propagator arguments. Usually the trace is implied by commuting the fields and the normal ordering symbols omitted,

$$m\langle :\psi\bar{\psi}:\rangle \to -m\langle\bar{\psi}\psi\rangle,$$

hiding the important operational definition Eq. (30) of what we now recognize as the fermionic condensate. Thus, we see that the condensate derives from the difference of normal ordering in the no-field (also called perturbative) vacuum and the with-field vacuum. Furthermore, one must keep in mind that the subtraction of the unperturbed vacuum term follows directly from application of the rules of QED and is not a consequence of arbitrary removal of zero-point energy. For this reason our discussion has no bearing on the zero-point energy of quantum field theory or its gravitational coupling.

Equations (29), (28), and (30) then combine with the result

$$m\frac{dV_{\rm eff}}{dm} = -m\langle\bar{\psi}\psi\rangle, \tag{31}$$

a known and widely used relation in nonperturbative QCD. We will evaluate it for the case of electrons in the presence of external electromagnetic fields.

We undertook the derivation of the Fermi condensate in the preceding section in order to emphasize that the condensate is not in general the energy-momentum trace, though there is a relationship. Using Eqs. (7), (10b), and (31) we find

$$\mathcal{T} = m\frac{dV^{(1)}_{\rm eff}}{dm} + m\langle\bar{\psi}\psi\rangle. \tag{32}$$

If and only if $V^{(1)}_{\rm eff}$ is just the action of classical electromagnetism, the first term on the right-hand side, linear in $\mathcal{S}$, vanishes.

However, the coupling to a fluctuating quantum field complicates the issue significantly in that it produces a contribution linear in $\mathcal{S}$ with a coefficient which is a function of m and renormalization scale. It is the logarithmic divergence of Eqs. (25) and (26) which in the limit $\delta \to 0$ is also proportional to $\mathcal{S}$ and thus appears in $V^{(1)}_{\rm eff}$. The standard procedure is to absorb the divergence into the definition of the charge in the process of charge renormalization. Equations (25) and (26) assume a cutoff regulator which of course also introduces a scale. Alternatively one can use dimensional regularization, the spinor effective potential can be written

$$V^{\rm f}_{\rm eff} = \int_0^\infty \frac{ds e^{-m^2 s}}{8\pi^2 s^{3-\epsilon}}(1 - eas\cot(eas)ebs\coth(ebs)) \tag{33}$$

and similar for the charged scalar case.

For any finite ϵ, Eq. (33) is finite and can be differentiated with respect to m, and hence the quantity $m(dV^{\rm f,s}_{\rm eff}/dm)$ is finite, allowing the consideration of a vanishing ϵ. In particular, the differentiation $m(dV_{\rm eff}/dm)$ renders the lowest order contribution $m(dV^{(1)}_{\rm eff}/dm)$ finite. The condensate and energy-momentum trace are thereby independent of renormalization procedure (as they should be), and obey the nontrivial relationship expressed by Eq. (32). The finiteness of $\mathcal{T}$ for fermions is also discussed at length by Adler *et al* [12].

Separation of the term linear in $\mathcal{S}$ in the Fermi case shows

$$m\frac{dV^{(1)}_{\rm eff}}{dm} = -\frac{e^2}{8\pi^2}\frac{b^2-a^2}{3}\int_0^\infty \frac{ds}{s}e^{-m^2 s} = \frac{2\alpha}{3\pi}\mathcal{S}. \tag{34}$$

Thus for the EH effective action, the decomposition in Eq. (7) becomes

$$m\frac{dV^{\rm f}_{\rm eff}}{dm} = m\frac{d\bar{V}^{\rm f}_{\rm eff}}{dm} + \frac{2\alpha}{3\pi}\mathcal{S} \tag{35}$$

and in turn with emphasis on properties of the vacuum:

$$\mathcal{T}^{\rm f} = \frac{2\alpha}{3\pi}\langle\mathcal{S}\rangle + m\langle\bar{\psi}\psi\rangle. \tag{36}$$

Equation (36) corresponds to Eq. (2.17) in [12]. It is of importance to note that there is considerable cancellation between the two terms on the right-hand side [4].

The essential relation Eq. (36) must be preserved at any number of loops in the effective action in its suitable generalization. In particular, if the condensate were evaluated to two loops, the coefficient of the first term must become the two-loop β function, i.e.

$$\frac{2\alpha}{3\pi} \to \beta(\alpha) = \frac{2\alpha}{3\pi} + \frac{\alpha^2}{2\pi^2} + \ldots. \tag{37}$$

Our result Eq. (36) is not obvious if one evaluates the energy-momentum trace after renormalization has been carried out. The logarithmically divergent term in $V_{\rm eff}$ is

$$V^{(1)}_{\rm eff} = -\frac{e^2}{8\pi^2}\frac{b^2-a^2}{3}\int_{0+\delta}^\infty \frac{ds}{s}e^{-m^2 s} = \frac{\alpha}{3\pi}\mathcal{S}\ln(m^2\delta) \tag{38}$$

in which $\delta = 1/M^2$, some large mass or momentum extraneous to QED. Before presenting the EH action, this term is absorbed in the process of charge renormalization. To restore its contribution one must realize the mass dependence of charge renormalization. The relation of Eq. (38) to the QED β function [12] demonstrates why use of the renormalized $V_{\rm eff}$ with the incorrect identification $\mathcal{T} = mdV_{\rm eff}/dm$ leads to the correct result as shown in Eq. (36), just as it was developed for QCD [4,10,11].

B. Properties of $\mathcal{T}$ in QED

We obtain the explicit form of the condensate in external fields combining Eq. (31) and (25)

$$-m\langle\bar\psi\psi\rangle = m^2\int_0^\infty \frac{dse^{-m^2s}}{4\pi^2s^2} \times (eas\cot(eas)ebs\coth(ebs)-1). \quad (39)$$

The condensate vanishes in the absence of field, as it should, since $x\cot x\to 1$ and $x\coth x\to 1$ for $x\to 0$. The term quadratic in the fields has been discussed above, it must be subtracted to arrive at the integral representation of the energy-momentum trace:

$$\mathcal{T}^{\rm f} = -m^2\int_0^\infty \frac{dse^{-m^2s}}{4\pi^2s^2} \times\left(eas\cot(eas)ebs\coth(ebs)-1-\frac{e^2}{3}(b^2-a^2)s^2\right). \quad (40)$$

To study the integrals Eq. (39) and (40), we introduce a transformation that will be helpful in dealing with the nonanalyticities generated by the electric field. The detailed calculations are carried out in the Appendix, and we simply state here their results. The integral representations obtained along the way have better convergence properties than Eq. (39) or Eq. (40), particularly at strong fields B, $E\sim E_0$.

In a magnetic background field, the condensate can be written

$$-m\langle\bar\psi\psi\rangle = -\frac{m^4}{2\pi^2\beta'}\int_0^\infty \frac{\ln(1-e^{-\beta' s})}{s^2+1}ds, \quad (41)$$

where $\beta' := \pi m^2/eB = \pi/(B/E_0)$, with $E_0 = m^2/e$. The energy-momentum trace is obtained by removal of the leading term quadratic in the field [recall Eq. (36)], with the result that

$$\mathcal{T}^{\rm f} = -\frac{m^4}{2\pi^2\beta'}\int_0^\infty \frac{s^2\ln(1-e^{-\beta' s})}{1+s^2}ds \quad (42)$$

is manifestly positive definite.

For the electric field, the poles are resummed into a logarithmic winding point, with the result

$$-m\langle\bar\psi\psi\rangle = \frac{m^4}{2\pi^2\beta}\int_0^\infty \frac{\ln(1-e^{-\beta s})}{1-s^2-i\epsilon}ds \quad (43)$$

where $\beta := \pi/(E/E_0)$. The trace in the pure electric background is

$$\mathcal{T}^{\rm f} = \frac{m^4}{2\pi^2\beta}\int_0^\infty \frac{s^2\ln(1-e^{-\beta s})}{1-s^2-i\epsilon}ds. \quad (44)$$

Eqs. (43) and (44) could equivalently be obtained by taking $B\to iE$ in the respective magnetic expressions. The condensate behaves as $(eE)^2$ for small fields, but the poles displayed in Eq. (A7) give the condensate and the energy-momentum trace a nonzero imaginary part, which can be evaluated from Eq. (43) or (44) recalling the identity

$$\frac{1}{x-i\epsilon} = {\rm PV}\frac{1}{x} + i\pi\delta(x).$$

As befits its role contributing to the proper mass of the nonlinear electromagnetic field [recall Eq. (18)],

$$\mathrm{Im}\mathcal{T}^{\rm f} = -m^2\frac{eE}{4\pi^2}\sum_{n=1}^\infty \frac{1}{n}e^{-((n\pi E_0)/E)} \quad (45a)$$

$$= \frac{m^4}{4\pi\beta}\ln(1-e^{-\beta}), \quad (45b)$$

is manifestly negative and strongly suppressed for field strengths less than $0.1E_0$. This is consistent with direct differentiation of the positive imaginary part of the action $V^{\rm f}_{\rm eff}$ evaluated by Schwinger

$$\mathrm{Im}\,V^{\rm f}_{\rm eff} = \frac{(eE)^2}{8\pi^3}\sum_{n=1}^\infty \frac{1}{n^2}e^{-((n\pi E_0)/E)}.$$

Observe that $m\langle\bar\psi\psi\rangle$ in external magnetic fields is negative, as is manifest in Eq. (41), while the energy-momentum trace, Eq. (42) is positive. With the perspective that the energy-momentum trace represents the energy arising from deformation of the vacuum, a negative value would imply that the vacuum state is unstable, for example, to the spontaneous generation of strong magnetic fields if $\mathcal{T}$ were identically $m\langle\bar\psi\psi\rangle$.

The energy-momentum trace in general combinations of electric and magnetic can also be cast Eq. (A9) reminiscent of our prior study of the special cases of electric and magnetic fields. Using b in the definition of β' and a in β, the numerically useful representation is

$$\mathcal{T}^{\rm (QED)} = \frac{m^4}{2\pi^2}(I_a+I_b) \quad (46a)$$

$$I_a = \int_0^\infty \frac{dss^2}{1-s^2-i\epsilon}\sum_{k=1}^\infty \frac{e^{-sk\beta}}{k\beta}\frac{\pi k\beta}{\beta'}\coth\frac{\pi k\beta}{\beta'}, \quad (46b)$$

$$I_b = \int_0^\infty \frac{dss^2}{1+s^2}\sum_{k=1}^\infty \frac{e^{-sk\beta'}}{k\beta'}\frac{\pi k\beta'}{\beta}\coth\frac{\pi k\beta'}{\beta}. \quad (46c)$$

A further resummation (see the Appendix) displays a more statistical form of the integrands:

$$I_a = -b\sum_{n,\sigma}\int_0^\infty \frac{s+\frac{1}{2}\ln(\frac{1-x+i\epsilon}{1+x-i\epsilon})}{e^{\beta\mathcal{H}_\sigma}-1}ds \quad (47a)$$

$$I_b = a\sum_{n,\sigma}\int_0^\infty \frac{s-\arctan s}{e^{\beta'\mathcal{H}'_\sigma}-1}ds \quad (47b)$$

where $\sigma = \pm 1$ and $n = 0, 1, 2\ldots$ in accordance with the Landau levels apparent in the quasi-Hamiltonians

$$m^2\mathcal{H}_\sigma = m^2s + (2n+1-\sigma)eb \quad (48a)$$

$$m^2\mathcal{H}'_\sigma = m^2s + (2n+1-\sigma)ea. \quad (48b)$$

The slower convergence of I_a (I_b) for $b/a \ll 1$ ($a/b \ll 1$) means Eq. (47) is not as advantageous as Eq. (46) for

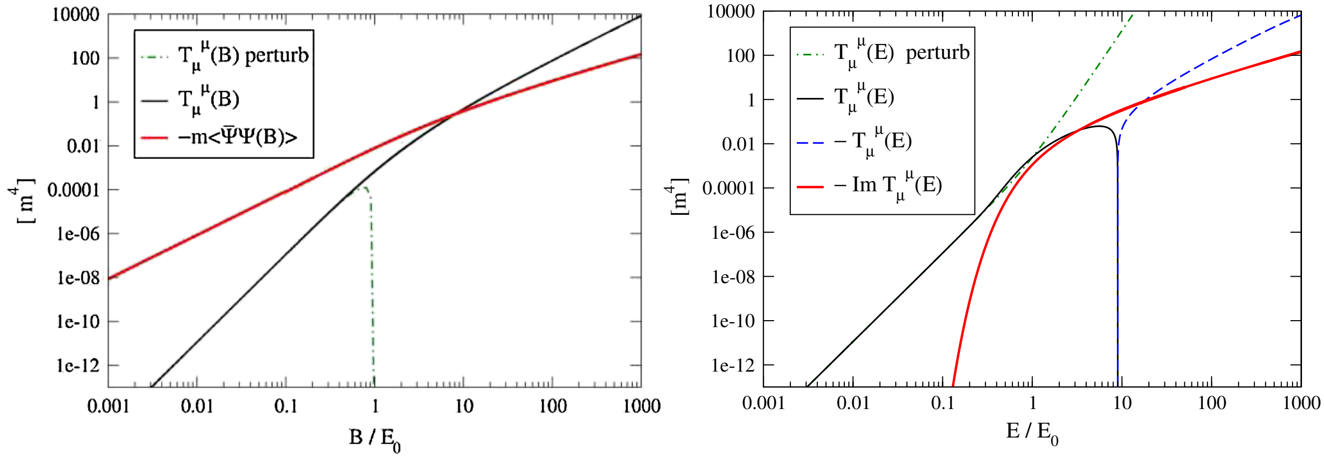

FIG. 2 (color online). The trace T^μ_μ normalized by the electron mass m_e^4 for spin-1/2 matter fields in magnetic (left panel) and electric (right panel) fields, evaluated perturbatively (dashed-dotted line) from the first two terms in Eq. (50) and exactly (solid line) from Eqs. (42) and (44). At left, the condensate in magnetic-only backgrounds is included for comparison of magnitudes, and at right, the imaginary part present for electric fields is included. The trace changes sign from positive to negative at $E = 9E_0$ and for higher fields the negative $(-T^\mu_\mu)$ is plotted.

numerical work. These expressions clarify the electric and magnetic field limits and thereby the correspondence between Figs. 2 and 3 below when one also recalls $z\coth z \to 1$ for $z \to 0$.

C. Numerical evaluation of QED $\mathcal{T}$

We evaluate numerically the condensate and the energy-momentum trace for arbitrary constant, homogeneous background electromagnetic field. We consider first a magnetic-only background field in Eq. (39), a form which is easily integrated numerically. The behavior is displayed in Fig. 2 as the solid upper (red) line. Noting that $f(x) = (x\coth x - 1)/x^2$ is essentially constant for $x < 1$, we recover the quadratic dominance $-m\langle\bar\psi\psi\rangle \propto (eB)^2$ for small fields. Figure 2 confirms the small field (mass-dominant) quadratic behavior and large field (field-dominant) linear behavior [20]. The results of numerically integrating (43) also appear in Fig. 2, but since the result is opposite in sign to the magnetic field case, we show the negative of the result as the lower (black) curve.

Equations (44) and (42) are plotted on the right in Fig. 2, and interestingly, we see that at $E = 9E_0$ $\mathcal{T}$ changes sign from positive at low fields to negative at high fields. This result is not apparent in the perturbative expansion, but can be understood from the Cauchy-Riemann equations in view of the rapidly changing imaginary part. Having the appearance of a π phase flip, the feature may be related to the rapid dissolution (attosecond timescale) of fields at magnitudes surpassing E_0 [21]. The methods developed here are not suited to the dynamics implied by such fields and the processes used to obtain such field strengths. For comparison, we also plot the weak field expansions, derived from the original EH expression with the power-law semiconvergent expansions of cot and coth. The first two terms for the condensate and energy-momentum trace are

$$-m\langle\bar\psi\psi\rangle \approx \frac{m^4}{12\pi^2}\frac{\mathcal{E}^2}{E_0^2} - \frac{m^4}{90\pi^2}\frac{\mathcal{E}^4}{E_0^4} + \dots \tag{49}$$

$$\mathcal{T}^{\,\mathrm{f}} \approx \frac{m^4}{90\pi^2}\frac{\mathcal{E}^4}{E_0^4} - \frac{4m^4}{315\pi^2}\frac{\mathcal{E}^6}{E_0^6} + \dots \tag{50}$$

where $\mathcal{E}^2 = B^2$ or $\mathcal{E}^2 = -E^2$ for magnetic or electric fields, respectively.

The condensate and energy-momentum trace for more general field configurations are evaluated using the rapidly convergent sums in Eq. (A9), and the results displayed in

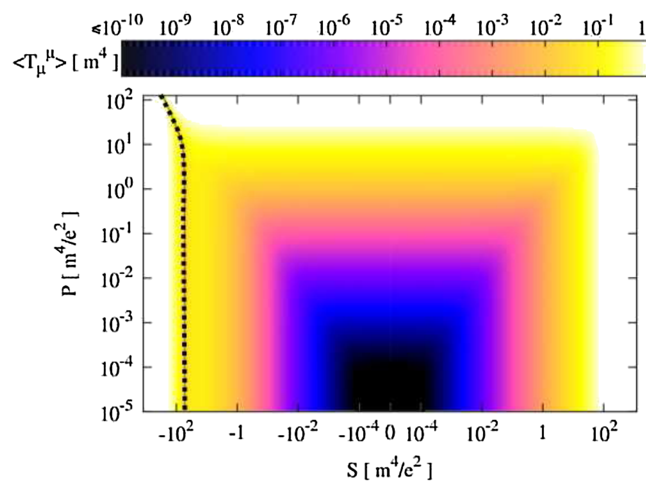

FIG. 3 (color online). The spinor energy-momentum trace for general E, B fields, parametrized by the Lorentz invariants, $\mathcal{S}$, $\mathcal{P}$defined below Eq. (3). As only $\mathcal{P}^2$ appears in V_{eff}, we plot only positive $\mathcal{P}$. Although the figure has a logarithmic scale, the trace crosses zero at the dotted line, going from positive to negative for very large electric fields $\mathcal{S} \ll -1$, as seen also in Fig. 2.

Fig. 3. An examination of the condensate for arbitrary field configurations, which does include the term linear in $\mathcal{S}$ (but is not displayed here), shows nontrivial features near the line $\mathcal{S} = \mathcal{P}$. This is reflecting on the expectation [22], of the influence of zero modes, known to be present when the field is self-dual, i.e. $\mathcal{S} = \mathcal{P}$. Comparison of the appearance of Fig. 3 with the BI result, Fig. 1 reveals a profound difference in these results, with the vacuum fluctuation effective action being more "edgy," and suggesting the possibility of a more singular behavior in the full multiloop strong field case.

The weak field expansions in general electromagnetic backgrounds give

$$-m\langle\bar\psi\psi\rangle \approx \frac{m^4}{6\pi^2}\mathcal{S} - \frac{m^4}{90\pi^2}(4\mathcal{S}^2 + 7\mathcal{P}^2) + \ldots \tag{51}$$

$$\mathcal{T}^{\,\mathrm{f}} \approx \frac{m^4}{90\pi^2}(4\mathcal{S}^2 + 7\mathcal{P}^2) - \frac{4m^4}{315\pi^2}\mathcal{S}(8\mathcal{S}^2 + 13\mathcal{P}^2) + \ldots \tag{52}$$

using $\mathcal{S}$, $\mathcal{P}$ normalized to the natural field strength, E_0. We can compare the perturbative EH result (52) with the Born-Infeld weak field $\mathcal{T}$ Eq. (23), but because the coefficients of $\mathcal{S}^2$ and $\mathcal{P}^2$ do not match in QED, we obtain two values for the BI-type limiting mass, in obvious notation:

$$M_{\mathcal{S}} = \left(\frac{45}{16\alpha^2}\right)^{1/4} m = 15.16m, \tag{53a}$$

$$\text{or}\quad M_{\mathcal{P}} = \left(\frac{45}{28\alpha^2}\right)^{1/4} m = 13.18m. \tag{53b}$$

D. $\mathcal{T}$ of scalar QED

The Euler-Heisenberg-Schwinger calculation of the effective action is easily extended to the case of a charged scalar field. The spin-0 particle has no Pauli spin coupling $\sigma_{\mu\nu}F^{\mu\nu}$ in the Hamiltonian, and the proper time integral is evaluated for the Klein-Gordon equation with covariant derivative,

$$(D^2 + m^2)\phi = 0, \qquad D_\mu \equiv \partial_\mu - ieA_\mu, \tag{54}$$

leading to Eq. (26) displayed above. The condensate takes the slightly different form, $m^2\langle\phi\phi^*\rangle$. Analogous manipulation of the proper time representations as above (see Sec. IVA) leads to an identity similar to (31):

$$m^2\langle\phi\phi^*\rangle = m\frac{\partial V^{\mathrm{s}}_{\mathrm{eff}}}{\partial m}. \tag{55}$$

In turn, we have

$$\mathcal{T}^{\,\mathrm{s}} = \frac{\alpha}{6\pi}\mathcal{S} - m^2\langle\phi\phi^*\rangle, \tag{56}$$

which displays for the scalar field the large cancellation between the photon and matter condensates. Then, since the explicit form of the energy-momentum tensor, Eq. (5), remains valid, we need only manipulate the action of scalar electrodynamics.

The alternating sign in the meromorphic expansions of the csc and csch functions (see the Appendix) leads upon partial integration to the opposite sign, or opposite "statistics" in the logarithm $\ln(1 \pm e^{-\beta s})$, as was already remarked upon by [14]. Thus, we have for magnetic-only background fields

$$m^2\langle\phi\phi^*\rangle = \frac{m^4}{4\pi^2\beta'}\int_0^\infty \frac{\ln(1+e^{-\beta' s})}{1+s^2}ds \tag{57}$$

and for electric-only fields

$$m^2\langle\phi\phi^*\rangle = \frac{m^4}{4\pi^2\beta}\int_0^\infty \frac{\ln(1+e^{-\beta s})}{1-s^2-i\epsilon}ds. \tag{58}$$

Results of numerically integrating Eqs. (57) and (58) appear in Fig. 4. As before, the condensate appears with opposite signs when comparing the electric and magnetic backgrounds, and the pole structure remains consistent for electric and magnetic fields, irrespective of particle type. The imaginary part induced by the electric background similarly becomes

$$\mathrm{Im}m\frac{dV^{\mathrm{s}}_{\mathrm{eff}}}{dm} = m^2\frac{eE}{8\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-((n\pi E_0)/E)} \tag{59a}$$

$$= \frac{-m^4}{8\pi\beta}\ln(1+e^{-\beta}), \tag{59b}$$

which is again negative in continued agreement with the role of $\mathcal{T}$ in the proper mass-energy of the nonlinear electromagnetic field.

Finally, the energy-momentum trace generated by the scalar field quantum fluctuations is

$$\mathcal{T}^{\,\mathrm{s}} = \frac{m^4}{4\pi^2\beta'}\int_0^\infty \frac{s^2\ln(1+e^{-\beta' s})}{1+s^2}ds, \tag{60}$$

in the magnetic background, and

$$\mathcal{T}^{\,\mathrm{s}} = -\frac{m^4}{4\pi^2\beta}\int_0^\infty \frac{s^2\ln(1+e^{-\beta s})}{1-s^2-i\epsilon}ds \tag{61}$$

in the electric background. Similarly positive for all but the highest electric fields, the scalar energy-momentum trace is exhibited in Fig. 4 on right. The effect of the fermionic statistics is apparent in the shift of the zero crossing to $E \simeq 20.2E_0$. In the figure, we also compare the weak field expansions,

$$m^2\langle\phi\phi^*\rangle \approx \frac{m^4}{48\pi^2}\frac{\mathcal{E}^2}{E_0^2} - \frac{7m^4}{1440\pi^2}\frac{\mathcal{E}^4}{E_0^4} + \ldots \tag{62}$$

$$\mathcal{T}^{\,\mathrm{s}} \approx \frac{7m^4}{1440\pi^2}\frac{\mathcal{E}^4}{E_0^4} - \frac{31m^4}{5040\pi^2}\frac{\mathcal{E}^6}{E_0^6} + \ldots \tag{63}$$

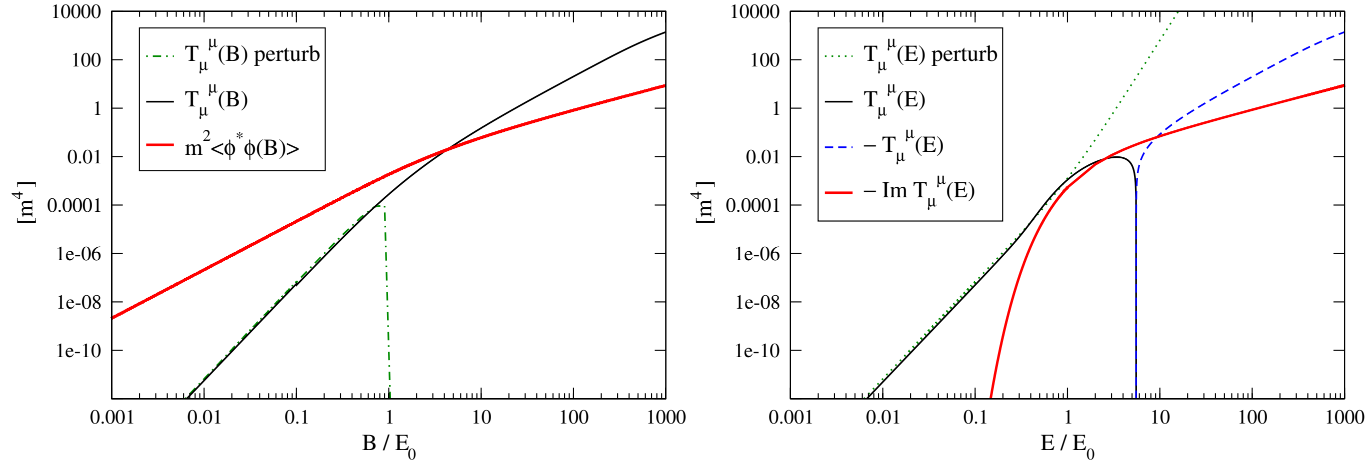

FIG. 4 (color online). For spinless particle fluctuations, we display the energy-momentum trace in units of m_e^4 for both magnetic-only (left) and electric-only (right) backgrounds. Again, the condensate is displayed for comparison in the magnetic case, and the imaginary part of the trace present in the electric background. Note the change in sign of the trace is now at $E = 5.5E_0$.

where $\mathcal{E}^2 = B^2$ or $\mathcal{E}^2 = -E^2$ for magnetic or electric fields, respectively.

The results for general electric and magnetic backgrounds are

$$\mathcal{T}^s = -\frac{m^4}{4\pi^2}(I_a + I_b); \tag{64a}$$

$$I_a = \int_0^\infty \frac{ds s^2}{1 - s^2 - i\epsilon} \sum_{k=1}^{\infty} \frac{(-1)^k e^{-sk\beta}}{k\beta} \frac{\pi k\beta}{\beta'} \operatorname{csch} \frac{\pi k\beta}{\beta'}, \tag{64b}$$

$$I_b = \int_0^\infty \frac{ds s^2}{1 + s^2} \sum_{k=1}^{\infty} \frac{(-1)^k e^{-sk\beta'}}{k\beta'} \frac{\pi k\beta'}{\beta} \operatorname{csch} \frac{\pi k\beta'}{\beta}. \tag{64c}$$

which is plotted in Fig. 5. The statistical form analogous to

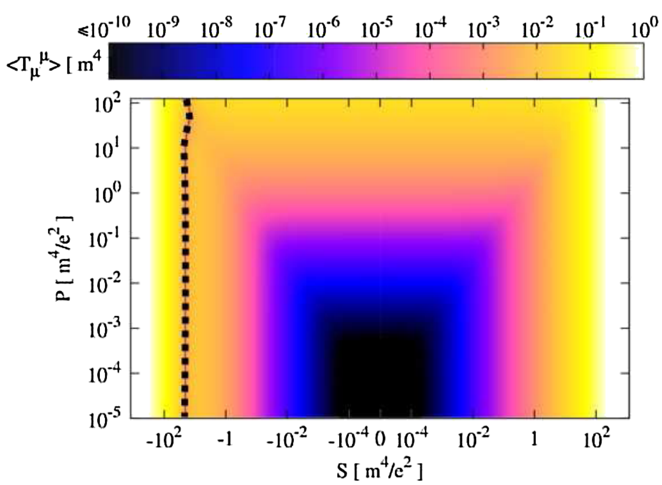

FIG. 5 (color online). The scalar energy-momentum trace for general E, B fields, Eq. (64), parametrized by the Lorentz invariants. The dotted vertical line indicates the change in sign of the trace to negative values for very large electric fields. The transition is present up to arbitrary values of $\mathcal{P}$ as in Fig. 3.

Eq. (47) can be obtained from that expression by changing to the fermionic + sign in the denominator and omitting the spin sum and σ term in the quasi-Hamiltonians Eq. (48); the spectral functions

$$s + \frac{1}{2}\ln\left(\frac{1 - x + i\epsilon}{1 + x - i\epsilon}\right) \quad \text{and} \quad s - \arctan s$$

remain unchanged. The weak field expansions for general electromagnetic backgrounds are

$$m^2\langle\phi\phi^*\rangle \approx \frac{m^4}{24\pi^2}\mathcal{S} - \frac{m^4}{360\pi^2}(7\mathcal{S}^2 + \mathcal{P}^2) + \ldots \tag{65}$$

$$\mathcal{T}^{\,s} \approx \frac{m^4}{360\pi^2}(7\mathcal{S}^2 + \mathcal{P}^2) - \frac{m^4}{630\pi^2}\mathcal{S}(31\mathcal{S}^2 + 11\mathcal{P}^2)\ldots \tag{66}$$

using $\mathcal{S}$, $\mathcal{P}$ normalized to the natural field strength, E_0. Again, we compare with the Born-Infeld weak-field energy-momentum trace [see Eq. (23)], and as above, the coefficients of $\mathcal{S}^2$ and $\mathcal{P}^2$ are not the same. For the scalar quantum theory then, the two corresponding values for the BI-type limiting mass are

$$M_{\mathcal{S}} = \left(\frac{45}{7\alpha^2}\right)^{1/4} m = 18.64m, \tag{67a}$$

$$\text{and} \quad M_{\mathcal{P}} = \left(\frac{45}{\alpha^2}\right)^{1/4} m = 30.32m. \tag{67b}$$

E. Euler-Heisenberg dielectric function

As we will discuss below, the kinematical and gravitational effects of a trace contribution to the electromagnetic energy-momentum differ strikingly from the classical Maxwell energy-momentum Eq. (2). Although this fact should make experimental verification of the presence of the trace term easy in principle, the relative strength of the classical contribution cannot be ignored in the study of real physical systems. Thus, we complete the analysis of the energy-momentum tensor of Euler-Heisenberg electromagnetism with evaluation of the dielectric function $\varepsilon = -\partial V_{\rm eff}/\partial\mathcal{S}$.

The EH actions are corrections to the classical $-\mathcal{S}$, so the total dielectric functions have the form

$$\varepsilon^{\rm f} - 1 = -\frac{\partial V^{\rm f}_{\rm eff}}{\partial\mathcal{S}} := \Delta\varepsilon^{\rm f}, \tag{68a}$$

$$\varepsilon^{\rm s} - 1 = -\frac{\partial V^{\rm s}_{\rm eff}}{\partial\mathcal{S}} := \Delta\varepsilon^{\rm s}, \tag{68b}$$

requiring differentiation of the expressions Eq. (25) and (26), via the partial differentials $(\partial a/\partial\mathcal{S})\partial/\partial a$ and $(\partial b/\partial\mathcal{S})\partial/\partial b$, where

$$\frac{\partial a}{\partial\mathcal{S}} = \frac{-a}{a^2+b^2}, \quad \text{and} \quad \frac{\partial b}{\partial\mathcal{S}} = \frac{b}{a^2+b^2}.$$

Considering first the fermionic case Eq. (25), we find

$$\Delta\varepsilon^{\rm f} = \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\frac{eas\cot(eas)ebs\coth(ebs)}{a^2+b^2} \times (\coth ebs - ebs\,\mathrm{csch}^2 ebs - \cot eas + eas\csc^2 eas - \frac{2}{3}(b+a)s^2)e^{-m^2 s}, \tag{69}$$

for which renormalization only requires subtraction of the logarithmic divergence, since the zero-field constant is differentiated away. Obtaining the meromorphic expansion of the integrand by differentiating the Sitaramachandrarao identity Eq. (A8), used above in Eq. (46), we have the numerically more convenient representation

$$\Delta\varepsilon^{\rm f} = \frac{e^4}{2\pi^2}\frac{ab}{a^2+b^2}\int_0^\infty se^{-m^2 s}(K_a+K_b)ds, \tag{70a}$$

$$K_a := a^2\sum_{k=1}^\infty\left(\frac{k\pi\coth(k\pi b/a)}{((eas)^2-k^2\pi^2)^2} - \frac{(b/a)\mathrm{csch}^2(k\pi b/a)}{(eas)^2-k^2\pi^2}\right) \tag{70b}$$

$$K_b := -b^2\sum_{k=1}^\infty\left(\frac{k\pi\coth(k\pi a/b)}{((ebs)^2+k^2\pi^2)^2} + \frac{(a/b)\mathrm{csch}^2(k\pi a/b)}{(ebs)^2+k^2\pi^2}\right). \tag{70c}$$

This form also provides the imaginary part

$$\mathrm{Im}\varepsilon^{\rm f} = \frac{\alpha}{2\pi}\frac{\beta\beta'}{\beta^2+\beta'^2}\sum_{k=1}^\infty\frac{e^{-k\beta}}{k\pi}\left(\coth\frac{k\pi\beta}{\beta'} - \frac{2\beta}{\beta'}\mathrm{csch}^2\frac{k\pi\beta}{\beta'}\right) \tag{71}$$

which is again a reflection the instability of strong electric fields, as confirmed in Fig. 7 by its suppression in dominantly magnetic fields. The polarization function in general field configurations can also be exhibited in the quasistatistical form of Eq. (47), but the stronger singularity s^{-3} in the proper time variable persisting in the absence of the m-differentiation makes for simpler "spectral" functions

$$\ln(1-s^2+i\epsilon) \quad \text{and} \quad \ln(1+s^2)$$

for the electric and magneticlike integrals K_a and K_b.

The weak field expansion of the dielectric function can be obtained by straightforward differentiation of the expansion of the effective action, giving

$$\Delta\varepsilon^{\rm f} \approxeq -\frac{\alpha}{90\pi}\frac{e^2}{m^4}8\mathcal{S} + \frac{2\alpha}{315\pi}\frac{e^4}{m^8}(24\mathcal{S}^2+13\mathcal{P}^2)+\ldots \tag{72}$$

For completeness, we exhibit the dielectric functions for magnetic-only

$$\Delta\varepsilon^{\rm f}(B) = -\frac{2\alpha}{\pi}\int_0^\infty ds\frac{s(s^2+2)}{(s^2+1)^2}\sum_{k=1}^\infty\frac{e^{-k\beta' s}}{k^2\pi^2} \tag{73}$$

and electric-only

$$\Delta\varepsilon^{\rm f}(E) = -\frac{2\alpha}{\pi}\int_0^\infty ds\frac{s(s^2-2)}{(s^2-1)^2}\sum_{k=1}^\infty\frac{e^{-k\beta s}}{k^2\pi^2}. \tag{74}$$

backgrounds, recalling $\beta' \to \pi m^2/eB$ and $\beta \to \pi m^2/eE$ in the respective limits.

For the scalar case Eq. (25),

$$\Delta\varepsilon^{\rm s} = -\int_0^\infty\frac{e^{-m^2 s}ds}{16\pi^2 s^3}\frac{eas\csc(eas)ebs\,\mathrm{csch}(ebs)}{a^2+b^2} \times\left(ebs\coth ebs - eas\cot eas - \frac{b+a}{3}s^2\right), \tag{75}$$

again renormalized by subtraction of the logarithmic divergence. The identity used above in Eq. (64), provides the numerically more convenient representation

$$\Delta\varepsilon^{\rm s} = -\frac{e^4}{4\pi^2}\frac{ab}{a^2+b^2}\int_0^\infty ds\,se^{-m^2 s}(K_a+K_b) \tag{76a}$$

$$K_a := a^2\sum_{k=1}^\infty(-1)^k\mathrm{csch}\left(\frac{k\pi b}{a}\right)\left(\frac{k\pi}{((eas)^2-k^2\pi^2)^2} - \frac{(b/a)\coth(k\pi b/a)}{(eas)^2-k^2\pi^2}\right) \tag{76b}$$

$$K_b := -b^2\sum_{k=1}^\infty(-1)^k\mathrm{csch}\left(\frac{k\pi a}{b}\right)\left(\frac{k\pi}{((ebs)^2+k^2\pi^2)^2} + \frac{(a/b)\coth(k\pi a/b)}{(ebs)^2+k^2\pi^2}\right) \tag{76c}$$

displaying the imaginary part

$$\mathrm{Im}\varepsilon^{\rm s} = -\frac{\alpha}{4\pi}\frac{\beta\beta'}{\beta^2+\beta'^2}\sum_{k=1}^\infty\frac{(-1)^k}{k\pi}\mathrm{csch}\left(\frac{k\pi\beta}{\beta'}\right) \times\left(1-\frac{2\beta}{\beta'}\coth\frac{k\pi\beta}{\beta'}\right)e^{-k\beta} \tag{77}$$

which is positive only for dominantly electric ($0 < \mathcal{P} < -\mathcal{S}$) fields. The limits of magnetic- and electric-only back-

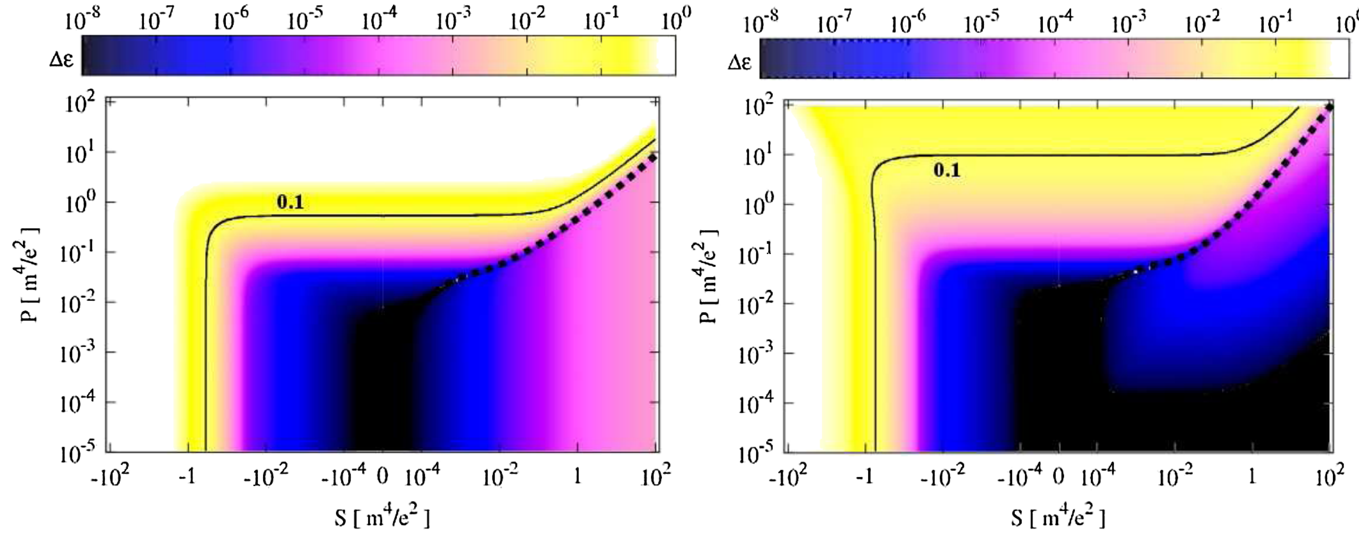

FIG. 6 (color online). The dielectric function for spin-1/2 (left panel) and spin-0 (right panel) quantum fluctuations. The value is the correction to the Maxwell 1, and as in BI theory, a dominantly magnetic field (below the dotted line) gives a negative correction suppressing the Maxwell energy-momentum.

ground are obtained immediately from Eqs. (73) and (74) by multiplication with $-1/2$ and insertion of an alternating $(-1)^k$ in the sum. The weak field expansion is

$$\Delta\varepsilon^{s} \approxeq -\frac{\alpha}{1440\pi}\frac{e^2}{m^4}14\mathcal{S} + \frac{\alpha}{20\,160\pi}\frac{e^4}{m^8}(279\mathcal{S}^2 + 77\mathcal{P}^2) + \ldots \quad (78)$$

Numerical evaluations of the EH corrections to the spinor and scalar dielectric functions for general field strengths are displayed in Fig. 6. $\Delta\varepsilon$ again reflects the unusually square character of the EH integrals, though in agreement with Born-Infeld theory, dominantly magnetic fields suppress ($\Delta\varepsilon < 0$) the Maxwell tensor. However, the boundary for which this magnetic suppression is present differs between fermionic and scalar electrodynamics, being approximately $\mathcal{S} \propto \mathcal{P}^2$ in the former case and $\mathcal{S} \propto \mathcal{P}$ in the latter. Indeed, the transition to augmentation

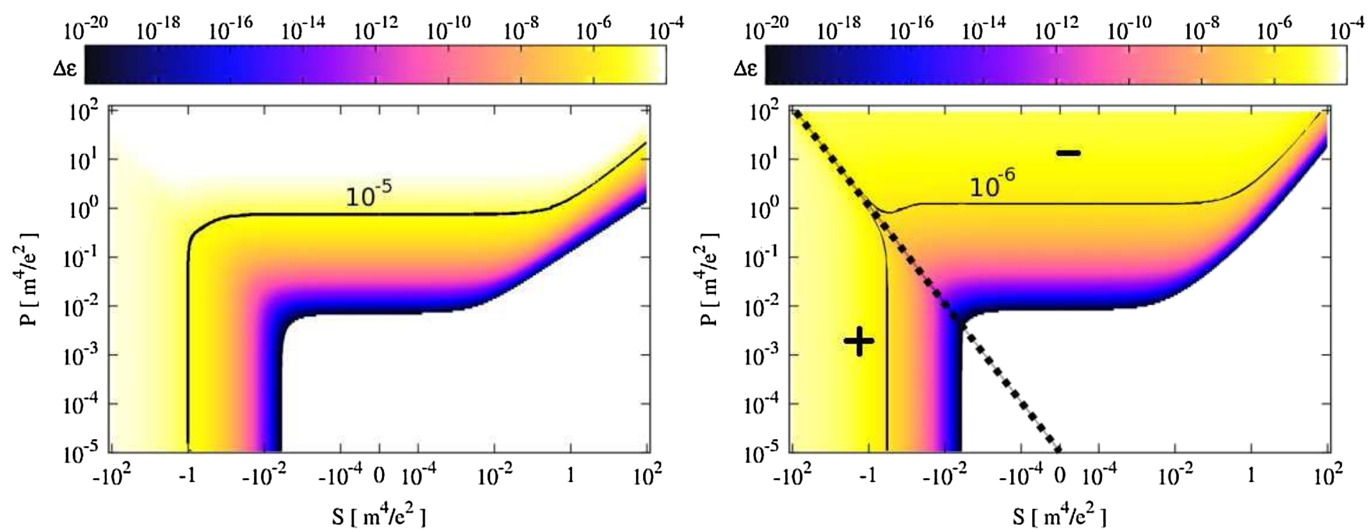

FIG. 7 (color online). The imaginary part of the dielectric function for spin-1/2 (left panel) and spin-0 (right panel) quantum fluctuations. For spin-1/2, the dielectric function is consistently positive; however, as indicated on the plot itself in the scalar dielectric function, the imaginary part is positive to the left and negative to the right of the dotted line at $\mathcal{S} = -\mathcal{P}$. The vanishing of $\Delta\varepsilon^{s}$ for the anti-self-dual $\mathcal{S} = -\mathcal{P}$ field configuration is a striking difference from the spinor case, which is suppressed when $\mathcal{S} > \mathcal{P}^2$. Details are suppressed when $\Delta\varepsilon < 10^{-20}$.

($\Delta\varepsilon > 0$) appears to arise in conjunction with the growing imaginary part, as seen in Fig. 7.

V. KINEMATICAL EFFECTS OF $\mathcal{T}$

With numerics providing the magnitude of the induced vacuum fluctuation, we can accurately evaluate physical situations in which the modification of the Maxwell energy-momentum tensor has observable consequences. A fundamental (i.e. vacuum) nonlinearity of electromagnetism is phenomenologically identical to a ponderable medium with nonlinear dielectric response. As may be verified by direct calculation (see also discussion in Sec. 8 of [17]), the formal structure of the Lorentz force is therefore unaltered

$$f^{\mu} = j_{\nu}F^{\mu\nu}, \tag{79}$$

as it is dictated by the necessity of gauge invariance in the coupling of EM potentials to charged matter. This point is to be contrasted with the modification of particle properties.

Violation of the superposition principle [23,24] entails an interaction between the background field and the field generated by the charged matter. Such an interaction can be extracted by careful study of the Lorentz force [9], but it is more easily evaluated using the weak field expansion of the EH effective potential.

As an example, take a large ($r > \lambda_e$) charged sphere in a strong background magnetic field, which could provide a rough model for an α particle in the atmosphere of a highly magnetized neutron star. In the rest frame of a nonrelativistic charged probe particle, we take the background magnetic field as constant $\vec{B} = B\hat{z}$ and the electric field as the particle's Coulomb field $\vec{E} = Ze\hat{r}/r^2$. Integrating the energy of the combined field configuration, T^{00}, over the volume with a short distance cutoff at the Compton wavelength λ, the leading contribution is the trace $\mathcal{T} \to u_{\rm eff}$:

$$u_{\rm eff} = \int d^4x \frac{2\alpha^2}{45m_e^4}(7\mathcal{P}^2 + 4\mathcal{S}^2) = \frac{2\alpha^2}{45m_e^4}\frac{4\pi}{3\lambda}(ZeB)^2, \tag{80}$$

keeping only the nonlinear-sourced cross terms. The coefficient of the Maxwell energy $-\frac{\partial V_{\rm eff}}{\partial \mathcal{S}}$ also induces cross terms subleading at $\mathcal{O}(\alpha^3)$. The cutoff arises since at distances shorter than λ we must use quantum dynamics to describe the probe particle, consideration of which would be inconsistent with the classical particle dynamics.

This interaction energy is positive, independent of the sign of the charge, and comparable to the gravitational potential of a neutron star with dipolar magnetic field. As $u_{\rm grav}$ is negative and $\propto r^{-1}$ and the effective (scalar) potential goes with $B^2 \propto r^{-6}$,

$$\frac{u_{\rm eff}}{u_{\rm grav}} = \frac{8\pi(\alpha Z)^2}{135'26 - 9mu\lambda}\left(\frac{eB_{\rm surf}}{m_e^2}\right)^2\frac{R_{\rm surf}^6}{r^6}\left(\frac{1.48M_{\odot}m}{r}\right)^{-1} \tag{81}$$

converting Newton's constant into the convenient units $G = 1.48$ km/solarmass. The λ cutoff in Eq. (80) cancels against particle mass m, making Eq. (81) independent of both the mass of the particle m and the cutoff λ. Remarkably, at the surface of a $1.5M_{\odot}$, 14 km radius star with critical surface field $B_{\rm surf} = B_c$, the nonlinear-electromagnetic effective potential is 34 times the gravitational potential, resulting in a large repulsive, quasi-Lorentz-scalar potential for charged particles entering the strong field region.

For a relativistic particle, a consistent treatment and thorough discussion of the force due to vacuum fluctuations in a strong magnetic field are given in [9].

VI. DISCUSSION AND CONCLUSIONS

The central motivation of this study is the observation that externally applied fields in nonlinear electromagnetism have a dark energy-like contribution to the energy-momentum tensor. We therefore examined the physics giving rise to the trace of the energy-momentum tensor as an avenue of insight into the origin of the observed dark energy in the universe. As T^{μ}_{μ} in our study is generated by quantum-induced nonlinearity of the electromagnetic field the physics of dark energy is accessible to laboratory experiment probing electromagnetism at high fields.

We derived the energy-momentum tensor for general nonlinear electromagnetic theories and emphasized the form Eq. (10). We considered the relationship of the trace with the matter condensate and obtained a result, Eq. (36) which amounts to removal of the leading term in $\mathcal{S}$ in $\langle\bar{\psi}\psi\rangle$. This reduces the numerical results by a factor of about 100 and along with this, the physical dark energy effect of the energy-momentum trace is greatly reduced.

Employing the resummation technique introduced in [14], we numerically evaluated the deviations from the Maxwell tensor, the dielectric function and the trace for Born-Infeld electromagnetism and the Euler-Heisenberg effective action with both Fermi and Bose matter fields. We believe that these are the first presentations in literature of the matter condensate and energy-momentum trace arising from the Euler-Heisenberg action at field strengths well beyond critical. The dielectric function is found in both BI and EH to suppress the Maxwell tensor in the presence of dominantly magnetic fields. The dielectric response of the vacuum thereby enhances the observable consequences of the presence of the energy-momentum trace.

The Born-Infeld theory is at first sight an interesting source of energy-momentum trace. However, a lower limit on the limiting BI electric field strength obtained 30 years ago from the study of precision atomic and muonic spectra

[23] requires $M^2/e \geq 1.710^{22}$ V/m, implying $M \geq$ 60 MeV. Contemporary $g-2$ experimental results, if analyzed with the objective to set a limit on the BI scale, would very probably push this limit further up. As $g-2$ of the electron in strong fields is in itself a project requiring study of the two-loop Lagrangian [25], such analysis takes us far beyond the scope of this paper, and we leave the investigation to the future. The scale of the energy-momentum trace of BI-type must be rather large and the effect at best comparable to the effect of vacuum fluctuations expressed by the Euler-Heisenberg effective action. Even though the energy-momentum trace is suppressed by the QED coupling, $(\alpha/\pi)^2 = 6 \times 10^{-6}$ (see Sec. IV), the effective scale $40m_e \simeq 20$ MeV implies that quantum fluctuations remain dominant compared with any other current theoretical framework, given the experimental constraints.

This is consistent with BI being a high-cutoff theory arising from more fundamental matter properties. Hence, we devoted the largest part of this report to expanding our prior study of electron fluctuations in the vacuum, the energy-momentum trace has previously been discussed as a signal of vacuum deformation [9]. The energy density in the trace is then interpreted as the shift of the vacuum energy induced by the applied field, and in concordance with this interpretation, the trace is positive definite when extracted correctly from the effective action. The smooth BI $\mathcal{T}$ highlights the extraordinary form of the quantum-induced trace. Prior considerations of the analytic structure of the Euler-Heisenberg effective action had not made apparent the near singular boundaries present in the condensate and the trace. The present evaluations suggest further investigations into the strong field vacuum phase structure.

We provided further the first complete numerical calculations of the electron-positron condensates of spinor and scalar QED for arbitrary fields and have addressed the contradictory statements in literature relating to claims that $\mathcal{T}$ and $-m\langle\bar{\psi}\psi\rangle$ are equal, and we found in our nonperturbative study a clear difference originating in the scale dependence of charge renormalization. We emphasize that our evaluation of the condensate and of the energy-momentum trace are completely independent of renormalization procedure.

The energy-momentum trace of QED and its effects exist anywhere and everywhere an electromagnetic field is present. For instance, the observed dark energy density

$$\frac{\Lambda}{4\pi G} \simeq (2.325\ \text{meV})^4 \simeq 6.09 \times 10^{-10}\ \text{J/m}^3 \qquad (82)$$

requires a magnetic field of 108 T. To claim such a magnetic field spanning most of the universe has gone as yet undetected would be extremely far-fetched, and the measured intergalactic magnetic field on the order of μG is clearly insufficient to explain the observed dark energy.

However, the lesson for cosmology is that the external field framework reveals a physical interpretation of the energy-momentum trace as the observable—and hence gravitating—energy of a false vacuum. As the external field is global in extent, this vacuum energy is indistinguishable from a cosmological constant. Thus, the QED $\mathcal{T}$ provides an experimentally tangible simulacrum of the "cosmological constant"

We left open in this work the question of how the structure of the QCD vacuum, which is very strongly deformed by glue and quark fluctuations, relates to the trace $\mathcal{T}$ and responds to an applied electromagnetic field. Some discussion of this question, including its relation to dark energy, has been already offered [4]. Combining the quantum vacuum with general relativity is a very delicate question [3]. Our report establishes an important connection, tying already-recognized quantum vacuum effects to dark energy.

To summarize, we have studied the energy-momentum tensor of nonlinear electrodynamics emphasizing an explicit relationship of the dark energylike trace of the energy-momentum tensor Eq. (10b) to the nonlinearity of the theory. In the consideration of electrodynamics as a quantum gauge theory, the connection provided a new derivation of a nonperturbative identity between the energy-momentum trace and the gauge and matter condensates, Eq. (36). The Euler-Heisenberg effective action provided a natural example for the numerical evaluation of the condensate and energy-momentum trace, the results of which are displayed for both fermionic and scalar fields in Figs. 3 and 5. Finally, we briefly explored the implications of an energy-momentum trace for charged-particle kinematics.

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APPENDIX: IMPROVING CONVERGENCE OF EULER-HEISENBERG INTEGRALS

In this appendix, we display the steps in the transformation of the proper time integrals Eqs. (25) and (26) into the more rapidly convergent representations used for numerics in the text.

The effective action for electrodynamics displays nonanalyticities that, generating an imaginary part of the action, are associated with the instability of the vacuum. However, our method of resumming the poles is very useful for improving the overall convergence of integrals of the Euler-Heisenberg form, and we start with the case of only a magnetic field being present, for which

$$-m\langle\bar{\psi}\psi\rangle = \frac{m^2}{4\pi^2}\int_0^\infty \frac{ds}{s^2}(eBs\coth eBs - 1) \qquad \text{(A1)}$$

is analytic on the real axis. We use the (subtracted) meromorphic expansions,

$$x\coth x - 1 = 2x^2\sum_{k=1}^{\infty}\frac{1}{x^2 + k^2\pi^2} \qquad \text{(A2)}$$

$$= \frac{x^2}{3} - 2x^4\sum_{k=1}^{\infty}\frac{1}{(k\pi)^2}\frac{1}{x^2 + k^2\pi^2}. \qquad \text{(A3)}$$

Inserting Eq. (A2) in Eq. (A1) we obtain

$$-m\langle\bar{\psi}\psi\rangle = \frac{m^2(eB)^2}{2\pi^2}\int_0^\infty ds\sum_{k=1}^{\infty}\frac{e^{-m^2s}}{(eBs)^2 + (k\pi)^2}. \qquad \text{(A4)}$$

All terms are individually absolutely convergent, so we reorder the sum and integral following the procedure in [14]. After rescaling $s \to sk\pi/eB$, the k sum is evaluated in closed form and we obtain

$$-m\langle\bar{\psi}\psi\rangle = -\frac{m^4}{2\pi^2\beta'}\int_0^\infty \frac{\ln(1 - e^{-\beta' s})}{1 + s^2}ds, \qquad \text{(A5)}$$

which is Eq. (41).

Equation (A3) allows us to remove the quadratic term in Eq. (40), and, rescaling and resumming, we find Eq. (42). A further integration by parts results in

$$\mathcal{T}^{\,\mathrm{f}} = \frac{m^4}{2\pi^2}\int_0^\infty \frac{s - \arctan s}{e^{\beta' s} - 1}ds \qquad \text{(A6)}$$

which is the $a \to 0$ limit of Eq. (47), but we retain the form Eq. (42) for clarity in the associated discussion of signs.

Turning now to $B = 0$, i.e. electric field only with $a \to |E|$, we see in the meromorphic expansion

$$x\cot x - 1 = 2x^2\sum_{k=1}^{\infty}\frac{1}{x^2 - k^2\pi^2}$$

the singularities that indicate the instability of the system to produce real pairs. We assign to the mass a small imaginary component $m^2 \to m^2 + i\epsilon$ so that

$$-m\langle\bar{\psi}\psi\rangle = \frac{m^2(eE)^2}{2\pi^2}\int_0^\infty ds\sum_{k=1}^{\infty}\frac{e^{-m^2s}}{(eEs)^2 - (k\pi)^2 + i\epsilon}, \qquad \text{(A7)}$$

whence resummation produces Eq. (43). Removing the leading term in meromorphic expansion for the case of the electric field by use of

$$x\cot x - 1 = -\frac{x^2}{3} + 2x^4\sum_{k=1}^{\infty}\frac{1}{k^2\pi^2}\frac{1}{x^2 - k^2\pi^2},$$

we obtain Eq. (44).

For general fields, the proper time integrals are rewritten using the Sitaramachandrarao identity [Eq. (6) in [26]]

$$xy\coth x\cot y = 1 + \frac{x^2 - y^2}{3} - 2x^3y\sum_{k=1}^{\infty}\frac{1}{k\pi}\frac{\coth(k\pi y/x)}{x^2 + k^2\pi^2} + 2y^3x\sum_{k=1}^{\infty}\frac{1}{k\pi}\frac{\coth(k\pi x/y)}{y^2 - k^2\pi^2} \qquad \text{(A8)}$$

with the result

$$\mathcal{T}^{(\mathrm{QED})} = \frac{m^4}{2\pi^2}(I_a + I_b); \qquad \text{(A9a)}$$

$$I_a = e^4a^3b\int_0^\infty ds s^2e^{-s}\sum_{k=1}^{\infty}\frac{\coth(k\pi b/a)}{k\pi(k^2\pi^2 - (eas)^2)}, \qquad \text{(A9b)}$$

$$I_b = e^4b^3a\int_0^\infty ds s^2e^{-s}\sum_{k=1}^{\infty}\frac{\coth(k\pi a/b)}{k\pi(k^2\pi^2 + (ebs)^2)}. \qquad \text{(A9c)}$$

Rescaling converts these expressions to those found in Eq. (46).

The quasistatistical representation of I_a in Eq. (47) is derived by exchanging the sum and the integral in order to integrate by parts,

$$I_a = -b\int_0^\infty ds\left(s + \frac{1}{2}\ln\left(\frac{1 - x + i\epsilon}{1 + x - i\epsilon}\right)\right) \times \sum_{k=1}^{\infty}\coth\left(\frac{k\pi a}{b}\right)e^{-((k\pi)/(ea))m^2s}. \qquad \text{(A10)}$$

We break up the coth function:

$$\coth x = \sum_{\sigma=\pm1}\frac{e^{\sigma x}}{1 - e^{-2x}}e^{-x}. \qquad \text{(A11)}$$

Expanding the denominator as a power series, the sum in Eq. (A10) becomes

$$\sum_{k,n,\sigma}\exp\left(-\frac{k\pi}{ea}(m^2s + (2n + 1 - \sigma)eb)\right). \qquad \text{(A12)}$$

Since $\exp(-k\pi b/a) < 1$, the n sum is absolutely convergent, though slowly when $b/a \ll 1$. We can exchange the order of summation and do the k sum:

$$I_a = -b\int_0^\infty ds\left(s + \frac{1}{2}\ln\left(\frac{1 - x + i\epsilon}{1 + x - i\epsilon}\right)\right)\sum_{n,\sigma}\frac{e^{-\beta\mathcal{H}_\sigma}}{1 - e^{-\beta\mathcal{H}_\sigma}} \qquad \text{(A13)}$$

with $\beta \equiv \pi m^2/ea$ and $m^2\mathcal{H}_\sigma$ the inner expression in Eq. (A12), thus obtaining Eq. (47).

For the scalar case, Eq. (26), we require identities paralleling those used in the spinor integrations:

$$x\,\mathrm{csch}x - 1 = -\frac{1}{6}x^2 - 2x^4 \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} \frac{(-1)^k}{x^2 + k^2\pi^2} \qquad \text{(A14a)}$$

$$= 2x^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{x^2 + k^2\pi^2} \qquad \text{(A14b)}$$

$$x\csc x - 1 = \frac{1}{6}x^2 + 2x^4 \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} \frac{(-1)^k}{x^2 - k^2\pi^2} \qquad \text{(A15a)}$$

$$= 2x^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{x^2 - k^2\pi^2}. \qquad \text{(A15b)}$$

The representation for general fields uses an identity closely related to Eq. (A8) [see Eq. (20) of [26]]:

$$xy\,\mathrm{csch}x\csc y = 1 + \frac{x^2 - y^2}{6} - 2x^3 y \sum_{k=1}^{\infty} \frac{(-1)^k}{k\pi} \times \frac{\mathrm{csch}(k\pi y/x)}{x^2 + k^2\pi^2} + 2xy^3 \sum_{k=1}^{\infty} \frac{(-1)^k}{k\pi} \frac{\mathrm{csch}(k\pi x/y)}{y^2 - k^2\pi^2}. \qquad \text{(A16)}$$

For the further resummation resulting in the statistical representation, the csch function is expanded analogously

$$\mathrm{csch}\,x = \frac{2}{1 - e^{-2x}} e^{-x} = e^{-x} \sum_{n=0}^{\infty} e^{-2nx} \qquad \text{(A17)}$$

but the absence of cosh in the numerator means no sum over σ is introduced. The remaining procedure is the same.

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