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Curing the UV/IR mixing for field theories with translation-invariant star products

Adrian Tanasa[1,2,*] and Patrizia Vitale[3,4,†]

[1]*Centre de Physique Théorique, CNRS UMR 7644, École Polytechnique, 91128 Palaiseau, France*
[2]*Institutul de Fizică şi Inginerie Nucleară Horia Hulubei, P.O. Box MG-6, 077125 Măgurele, România*
[3]*Laboratoire de Physique Théorique, Université Paris XI, 91405 Orsay Cedex, France*
[4]*Dipartimento di Scienze Fisiche, Università di Napoli Federico II and INFN, Sezione di Napoli, Via Cintia 80126 Napoli, Italy*
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The ultraviolet/infrared (UV/IR) mixing of noncommutative field theories has been recently shown to be a generic feature of translation-invariant associative products. In this paper we propose to take into account the quantum corrections of the model to modify in this way the noncommutative action. This idea was already used to cure the UV/IR mixing for theories on Moyal space. We show that in the present framework also, this proposal proves successful for curing the mixing. We achieve this task by explicit calculations of one and higher loops Feynman amplitudes. For the sake of completeness, we compute the form of the new action in the matrix base for the Wick-Voros product.

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I. INTRODUCTION AND MOTIVATION

Noncommutative geometry [1] is an appealing framework for the quantization of gravity. At the Planck scale, the quantum nature of the underlying space-time replaces a local interaction by a specific nonlocal effective interaction in the ordinary Minkowski space [2].

Noncommutative quantum field theories (for general reviews, see [3,4]) can be interpreted as limits of matrix models or of string theory models. The first use of noncommutative geometry in string theory was in the formulation of open string theory [5]. Noncommutativity is here natural just because an open string has two ends and an interaction which involves two strings joining at their end points shares all the formal similarities to noncommutative matrix multiplication. In this context, one also has the Seiberg-Witten map [6], which maps the noncommutative vector potential to a conventional Yang-Mills vector potential, explicitly exhibiting the equivalence between these two classes of theories.

Probably the simplest context in which noncommutativity arises is in a limit in which a large background antisymmetric tensor potential dominates the background metric. In this limit, the world-volume theories of Dirichlet branes become noncommutative [7,8]. Noncommutativity was also recently proved to arise as some limit of loop quantum gravity models. There, the effective dynamics of matter fields coupled to three-dimensional quantum gravity is described after integration over the gravitational degrees of freedom by some noncommutative quantum field theory [9]. In a different context, some three-dimensional noncommutative space emerging in the context of three-dimensional Euclidean quantum gravity was also studied in [10].

In condensed matter physics, noncommutative theories can be of particular interest when describing effective nonlocal interactions, as is the case, for example, of the fractional quantum Hall effect. where different authors proposed that a good description of this phenomenon can be obtained using noncommutative rank 1 Chern-Simons theory [11].

Nevertheless, when going from commutative to noncommutative theories, locality is lost and one can wonder, in this situation, if renormalizability can be restored. Indeed, when describing theories on the noncommutative Moyal space (the most studied noncommutative space), a new type of nonlocal divergence occurs, the UV/IR mixing [12]. This new divergence is nonlocal and cannot be absorbed by counterterms at the level of the two-point function.

Despite this important difficulty, solutions exist for renormalizability to be restored. This is achieved for the ϕ^4 theory by modifying the propagation part of the initial action, such that this new type of divergence is cured. A first type of modification adds a harmonic oscillator term in the propagator [13]. A different type of modification was proposed in [14], where the quantum correction $1/p^2$ was included in the bare action in momentum space. The physical interpretation of this model in the long distance regime was studied in [15]. Such a modified scalar propagator appears also in recent work on non-Abelian gauge theory in the context of the Gribov-Zwanziger result [16]. Both these noncommutative models were proved renormalizable at any order in perturbation theory, but the latter is also manifestly translation invariant. Several field theoretical properties have been further investigated, for both these noncommutative models (see [17–29] and references therein). Furthermore, some algebraic geometrical properties of the parametric representation of the Grosse-Wulkenhaar models have been investigated in [30].

A part from the Moyal product, other noncommutative products have been investigated to construct noncommutative field theories on flat space-time. One of them is the Wick-Voros (WV) product [31] which corresponds to nor-

*atanasa@cpht.polytechnique.fr
†vitale@na.infn.it

mal ordering in deformation quantization [32], as opposed to symmetric ordering which is instead related to the Moyal product. Recently, a scalar ϕ^4 theory was studied, with this kind of noncommutativity [33]. Computing the nonplanar tadpole Feynman amplitude, it was shown that the UV/IR mixing appears in this framework as well, although the propagator and the vertex have different forms than in the Moyal case. This result was extended to generic translation-invariant products on flat space-time [34], showing in a one-loop calculation that the UV/IR mixing stays unmodified for the whole class of products.

In this paper we propose to modify the Euclidean action of scalar field theories with quartic interaction on noncommutative $\mathbb{R}^d$, with generalized translation-invariant star products, along the lines of [14], in order to cure the UV/IR mixing. Namely, we compute the one-loop quantum correction for the propagator and we modify the scalar action accordingly. We then compute, for the modified action, the one-loop quantum corrections for the propagator and for the vertex and we show that the mixing has disappeared when inserting the modified nonplanar tadpoles into "bigger" nonplanar graphs.

At tree level we show that, for the new parameter lying in a certain range determined by the noncommutativity scale, the propagator may be decomposed as a sum of Klein-Gordon (KG) propagators, some of which with negative sign. In a commutative setting this is a signal of illness of the theory since, when performing a Wick rotation to the Minkowski space, these new fields lead to negative norm states, which in turn can be rephrased into loss of unitarity of the S matrix. Whether or not the same conclusions can be drawn for our model is an interesting open problem, mainly because the Minkowskian analogue of an Euclidean theory is not uniquely defined in the NC setting, but also because there is no general agreement on the definition of particle states, commutation relations, and the S matrix formalism itself (cf. [33,35] and references therein). We will come back to this argument in the paper.

Quantum field theories (QFT) with the WV product (which, as already stated, are a particular class of the field theories we treat here) have already been investigated in the literature [36]. This product has been studied within the coherent states framework [37] and in relation to matrix models and Chern-Simons theory [38,39]. Black holes have also been defined using such a noncommutative product [40]. Finally, let us also state that a different approach for studying QFT with WV product has been undertaken in [41].

From a mathematical point of view, translation-invariant $\star$ products on the (hyper)plane are all equivalent to the Moyal product in the sense of formal series [42] as they all share the same underlying Poisson bracket. Nevertheless, they are not *a priori* physically equivalent, as they yield QFTs with different quantum actions. Furthermore, one cannot relate these QFTs by simple field redefinitions, as we will show in the sequel.

The paper is structured as follows. In the next section we recall the definition and some basic properties of translation-invariant generalizations of the Moyal product on Euclidean $\mathbb{R}^d$. The results of [34] are recalled, showing the appearance of the UV/IR mixing. In Sec. III we write the action of the modified model we propose. We also derive the associated Feynman rules, namely, the modified propagator and modified vertex. We also discuss the issue of ghost states in a Minkowskian formulation.

We then compute the one-loop quantum corrections for the propagator and the vertex of the proposed model. This allows one to show that inserting nonplanar tadpoles into some higher loop graphs leads to IR convergent Feynman amplitudes, thus curing the problem of the UV/IR mixing. In Sec. IV we recall the matrix basis for the WV product and we compute the expression of the modified action in this basis. Finally, the last section presents some conclusions and perspectives.

II. TRANSLATION-INVARIANT PRODUCTS

In this section we review translation-invariant star products on the space $\mathbb{R}^d$, as derived in [34]. As we will see, examples of such products are the Moyal product and the less known WV one (or normal ordered product). For the present purposes we present the products in terms of their integral kernel in Fourier space, although other forms are available. A generic star product on $\mathbb{R}^d$ may be represented as

$$(\phi \star \psi)(x) = \frac{1}{(2\pi)^{d/2}} \int d^d p d^d q d^d k e^{ip\cdot x} \times \tilde{\phi}(q)\tilde{\psi}(k)K(p, q, k), \tag{2.1}$$

where K can be a distribution and $\tilde{\phi}(q)$ is the Fourier transform of $f = \phi$. The product of d vectors is understood with the Minkowskian or Euclidean metric: $p \cdot x = p_i x^i$. The usual pointwise product is also of this kind for $K(p, q, k) = \delta^d(k - p + q)$. Translation invariance requires that the product obey

$$\mathcal{T}_b(\phi) \star \mathcal{T}_b(\psi) = \mathcal{T}_b(\phi \star \psi), \tag{2.2}$$

where $\mathcal{T}_b(f)(x) = f(x + b)$ represents the translation by the vector b. At the level of Fourier transform we have

$$\widetilde{\mathcal{T}_b \phi}(q) = e^{ibp}\tilde{\phi}(q). \tag{2.3}$$

It may be seen that, for the product (2.1) to be invariant, the kernel must be of the form

$$K(p, q, k) = e^{\alpha(p,q)}\delta(k - p + q), \tag{2.4}$$

where α is a generic function of p and q, further constrained by associativity and cyclicity. We therefore consider products that can be expressed as

$$(\phi \star \psi)(x) = \frac{1}{(2\pi)^{d/2}} \int d^d p d^d q e^{ip\cdot x} \tilde{\phi}(q) \tilde{\psi}(p-q) e^{\alpha(p,q)}. \tag{2.5}$$

Except for the commutative case, d has to be even because of translation invariance (besides degenerate cases, where one of the dimensions commutes with the other ones).

Let us also emphasize here that translation invariance requires as well the commutator of coordinates to be constant, as in the Moyal case. We explicitly show this at the end of this section, in Eq. (2.31).

When $\alpha = 0$, one has the usual pointwise product. One then has two important examples of noncommutative associative products which are of the form above. They are both borrowed by ordinary phase space quantization and correspond to different ordering choices. One is the Moyal product, quite well studied in the literature. It corresponds to symmetric ordering in deformation quantization of phase space (Weyl quantization) and in Fourier transform it acquires the form

$$(\phi \star_M \psi)(x) = \frac{1}{(2\pi)^{d/2}} \int d^d p d^d q \tilde{\phi}(q) \tilde{\psi}(p-q) \times e^{ip\cdot x} e^{i/2 p_i \theta \Sigma^{ij} q_j}, \tag{2.6}$$

thus giving

$$\alpha_M(p,q) = -\frac{i}{2}\theta \Sigma^{ij} q_i p_j. \tag{2.7}$$

We have denoted by Σ the $d \times d$ block-diagonal antisymmetric matrix

$$\Sigma = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & . & \\ & & & . \end{pmatrix}$$

and by θ some constant noncommutativity parameter.

The other example is the WV product. It corresponds to normal ordering in deformation quantization of phase space and in Fourier transform it reads

$$(\phi \star_{\rm WV} \psi)(x) = \frac{1}{(2\pi)^{d/2}} \int d^d p d^d q \tilde{\phi}(q) \tilde{\psi}(p-q) \times e^{ip\cdot x} e^{-\theta q_- \cdot (p_+ - q_+)} \tag{2.8}$$

with

$$p^i_\pm = \frac{p^i_1 \pm i p^i_2}{\sqrt{2}}, \qquad i = 1, \ldots, d/2. \tag{2.9}$$

The function α is therefore given by

$$\alpha_{\rm WV}(p,q) = -\theta q_- \cdot (p_+ - q_+). \tag{2.10}$$

Further restrictions on K come from the associativity requirement which reads

$$\int d^d k K(p,k,q) K(k,r,s) = \int d^d k K(p,r,k) K(k,s,q). \tag{2.11}$$

This is nothing but the usual cocycle condition in the Hochschild cohomology. For more details on cohomological aspects we refer the interested reader to [34]. In terms of α, Eq. (2.11) reads

$$\alpha(p,q) + \alpha(q,r) = \alpha(p,r) + \alpha(p-r, q-r). \tag{2.12}$$

From this cocycle relation then follow

$$\alpha(p,p) = \alpha(0,0) = \alpha(p,0) \qquad \alpha(0,p) = \alpha(0,-p)$$
$$\alpha(p,q) = -\alpha(q,p) + \alpha(0, q-p) \tag{2.13}$$

and

$$\alpha(p+q, p) + \alpha(-p-q, -q) + \alpha(0, p+q) - \alpha(0,p) - \alpha(0,q) = 0. \tag{2.14}$$

The second relation of (2.13) ensures also the trace property. We have indeed

$$\int d^d x \phi \star \psi = \int d^d x d^d p d^d q e^{\alpha(p,q)} e^{ip\cdot x} \tilde{\phi}(q) \tilde{\psi}(p-q) = \int d^d q e^{\alpha(0,q)} \tilde{\phi}(q) \tilde{\psi}(-q) = \int d^d x \psi \star \phi. \tag{2.15}$$

For the product to be commutative, α has to satisfy the condition

$$\alpha(p,q) = \alpha(p, p-q), \tag{2.16}$$

which may be regarded as a coboundary condition (see again [34]). Finally, let us notice that the Moyal and WV products are related by the following relation:

$$\alpha_M(p,q) = \alpha_{\rm WV}(p,q) - \frac{\theta}{2} q \cdot (p-q). \tag{2.17}$$

As already stated above, translation-invariant products have been introduced in [34] in the context of noncommutative scalar field theories with quartic interaction, on $\mathbb{R}^d$ with Minkowskian metric. Here we choose to work with the Euclidean metric. We keep the notation d for the number of space-time dimensions; nevertheless, when explicit divergence analysis is performed, we refer to the case

$$d = 4.$$

The action reads

$$S = \int d^d x \left[\frac{1}{2}(-\partial_\mu \phi \star \partial_\mu \phi + m^2 \phi \star \phi) + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right]. \tag{2.18}$$

In momentum space the propagator is thus

$$G_0^{(2)}(p) = \frac{e^{-\alpha(0,p)}}{p^2 + m^2} \tag{2.19}$$

whereas for the vertex we have

$$V_\star = V_0 e^{\alpha(k_1+k_2,k_1)+\alpha(k_3+k_4,k_3)+\alpha(0,k_1+k_2)}, \tag{2.20}$$

and we have denoted by V_0 the ordinary commutative vertex

$$V_0 = \frac{\lambda}{4!}(2\pi)^d \delta^d\bigg(\sum_{a=1}^{4} k_a\bigg). \tag{2.21}$$

Interestingly, a propagator of the form (2.19) with the function α specified in (2.10) was already found in [43] in a different approach.

To obtain the four-point correlation function at tree level we just attach to the vertex four propagators. We thus have (up to a constant)

$$G_0^{(4)} = \frac{e^{\alpha(k_1+k_2,k_1)+\alpha(k_3+k_4,k_3)+\alpha(0,k_1+k_2)-\sum_{a=1}^{4}\alpha(0,k_a)}}{\prod_{a=1}^{4}(k_a^2+m^2)}\delta\bigg(\sum_{a=1}^{4} k_a\bigg). \tag{2.22}$$

Let us now make the following important remark. In order to obtain the usual Feynman rules of a Moyal QFT, one can try by reabsorbing the phase $e^{-\alpha(0,p)}$ of (2.19) by a proper field redefinition. Nevertheless, this would not reproduce the vertex form of Moyal QFT. We thus conclude that, by a simple field redefinition one does not have equivalence between the general class of QFTs we deal with in this paper and the Moyal one. In [33] it was shown that an equivalence between Moyal and WV Minkowskian QFTs can be established at the level of the S matrix only by implementing the appropriate twisted Poincaré symmetry for each of them.

Before going further, we also give some explanations on the planarity of the Feynman graphs used in this work. As in the Moyal case, the vertex has a symmetry under cyclic permutation of the incoming/outgoing fields at some vertex. Furthermore, one can also use some matrix base to reexpress these products (for the WV case, see Sec. IV). For all these reasons, an appropriate way to represent Feynman graphs is through ribbon graphs. If the genus of the manifold on which the respective graph is denoted is vanishing, the respective graph is planar. Furthermore, another important notion is the one of faces broken by external legs. In the rest of the paper, by a slight abuse of language we will call nonplanar graphs also the planar graphs with more than one face broken by external legs.

Consider now the two graphs of Figs. 1 and 2.

For the planar case of Fig. 1, the correction is obtained using three propagators of the form (2.19), one with momentum p, one with momentum $-p$, one with momentum q, and the vertex (2.20) with assignments $k_1 = -k_4 = p$ and $k_2 = -k_3 = q$ and, of course, the integration in q. We have (up to a constant)

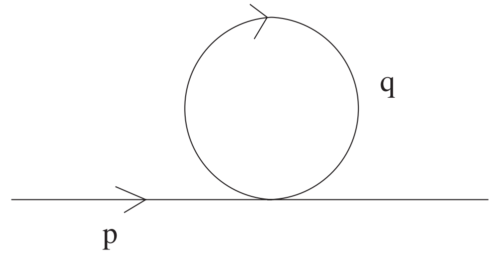

FIG. 1. The planar tadpole graph.

$$G_{1,P}^{(2)} = \frac{e^{-\alpha(0,p)}}{(p^2+m^2)^2}\int d^d q \frac{e^{\sigma(p,q)}}{(q^2+m^2)}, \tag{2.23}$$

where we have used the fact that the three exponential factors of the vertex combine with two out of three exponential factors of the propagators to yield

$$\sigma(p,q) = \alpha(p+q,p) + \alpha(-p-q,-q) + \alpha(0,p+q) - \alpha(0,q) - \alpha(0,p). \tag{2.24}$$

Using the cocycle condition (2.14), we have then

$$\sigma(p,q) = 0. \tag{2.25}$$

Notice that, with respect to the commutative case, the only correction is in the factor $e^{-\alpha(0,p)}$ which is the correction of the free propagator. The ultraviolet divergences here are thus identical to the commutative and to the Moyal case.

Consider now the nonplanar case in Fig. 2. The structure is the same as in the planar case, but this time the assignments are

$$k_1 = -k_3 = p \quad \text{and} \quad k_2 = -k_4 = q. \tag{2.26}$$

We have (up to a constant)

$$G_{1,\mathrm{NP}}^{(2)} = \int d^d q \frac{e^{-\alpha(0,p)+\alpha(p+q,p)-\alpha(p+q,q)}e^{\sigma(p,q)}}{(p^2+m^2)^2(q^2+m^2)}, \tag{2.27}$$

with the same notation as above. Therefore $\sigma(p,q) = 0$ and the one-loop corrections to the propagator in the nonplanar case can be rewritten as

$$G_{1,\mathrm{NP}}^{(2)} = \frac{e^{-\alpha(0,p)}}{(p^2+m^2)^2}, \qquad \int d^d q \frac{e^{\omega(p,q)}}{(q^2+m^2)}, \tag{2.28}$$

where we have introduced the antisymmetric function

$$\omega(p,q) = \alpha(p+q,p) - \alpha(p+q,q). \tag{2.29}$$

For the Moyal product this term is the oscillating phase. For general translation-invariant products this function has

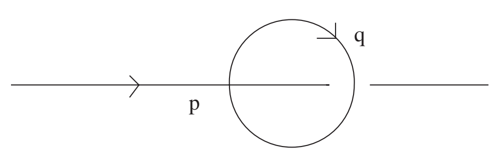

FIG. 2. The nonplanar tadpole graph.

been computed explicitly in [34], using the properties of the function α. It does not depend on the specific translation-invariant product but only on the cohomology class of α. Except for numerical factors, one has, for all translation-invariant products,

$$\omega(p, q) = i p_i \theta \Sigma^{ij} q_j = 2\alpha_M(p, q). \tag{2.30}$$

Details on the derivation may be found in [34]. The function $\alpha(0, p)$ which appears in (2.28) is not integrated in the loop, therefore it does not influence the convergence properties of the graphs.

In fact the function ω is nothing but the Poisson structure of the underlying classical space, the germ of deformation of the commutative product towards the star product. It determines the noncommutativity of space-time coordinates. A straightforward calculation gives

$$\begin{aligned} x^i \star x^j - x^j \star x^i &= -\frac{\partial^2 \alpha}{\partial p_i \partial q_j}(0,0) + \frac{\partial^2 \alpha}{\partial p_j \partial q_i}(0,0) \\ &= \omega^{ij} = i\theta\Sigma^{ij}. \end{aligned} \tag{2.31}$$

Using (2.30), one can prove that the Feynman integral (2.28) is UV finite but has a

$$\frac{C_1}{(\theta p)^2} + m^2 C_2 \log(\theta p)^2 + F(p) \tag{2.32}$$

behavior in the IR regime of the external momentum p. We have denoted by C_1 and C_2 some constants and by $F(p)$ some analytic function at $p = 0$ (see [14] for a detailed analysis).

III. THE PROPOSED MODEL AND QUANTUM CORRECTIONS—CURING THE UV/IR MIXING

In this section we write the action for the proposed model in the noncommutative setting described previously. We then compute quantum corrections of the modified model (one and higher number of loops) which show the way in which the UV/IR is manifestly cured.

As already stated in the Introduction, the modification we propose for the model of the previous section is dictated by the quantum corrections. Thus, we add to the action the supplementary term:

$$\delta S[\phi] = \frac{a}{2\theta^2}\int d^d x (\partial_\mu)^{-1}\phi \star (\partial_\mu)^{-1}\phi. \tag{3.1}$$

The complete action, in coordinate space, reads then

$$\begin{aligned} S = \int d^d x\bigg[\frac{1}{2}\bigg(\partial_\mu\phi \star \partial_\mu\phi + \frac{a}{\theta^2}\partial_\mu^{-1}\phi \star \partial_\mu^{-1}\phi \\ + m^2 \phi\star\phi\bigg) + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\bigg], \end{aligned} \tag{3.2}$$

with

$$\partial_\mu^{-1}\phi(x) = \int d^\mu x' \phi(x'). \tag{3.3}$$

The supplementary term is better understood in momentum space. Observing that for the star product (2.5)

$$\int d^d x f(x) \star g(x) = \int d^d p \tilde{f}(p)\tilde{g}(-p) e^{\alpha(0,p)}, \tag{3.4}$$

we have

$$\delta S[\phi] = \frac{a}{2\theta^2}\int d^d p \frac{e^{\alpha(0,p)}}{p^2}\tilde{\phi}(p)\tilde{\phi}(-p). \tag{3.5}$$

This leads to a modification of the propagator (2.19) in the form

$$G_0^{(2)}(p) = \frac{e^{-\alpha(0,p)}}{p^2 + m^2 + \frac{a}{(\theta p)^2}}. \tag{3.6}$$

Note that, as in [14] the new parameter a is taken to be positive [such that the propagator (3.6) is positively defined]. The vertex contribution remains the same as in (2.20) [since the term (3.5) is only quadratic in the field].

A few remarks are needed here. The first one is related to the possibility of decomposing the propagator (3.6) as a sum of conventional Klein-Gordon propagators. We use the formula

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}B\frac{1}{A+B}, \tag{3.7}$$

for

$$A = p^2 + m^2, \qquad B = \frac{a}{\theta^2 p^2}. \tag{3.8}$$

Thus, the propagator (3.6) writes

$$\begin{aligned} &\frac{e^{-\alpha(0,p)}}{p^2+m^2} - \frac{e^{-\alpha(0,p)}}{p^2+m^2}\frac{a}{\theta^2 p^2(p^2+m^2)+a} \\ &= \frac{e^{-\alpha(0,p)}}{p^2+m^2} - \frac{e^{-\alpha(0,p)}}{p^2+m^2}\frac{a}{\theta^2(p^2+m_1^2)(p^2+m_2^2)}, \end{aligned} \tag{3.9}$$

where $-m_1^2$ and $-m_2^2$ are the roots of the denominator of the second term in the left-hand side considered as a second order equation in p^2, namely,

$$\begin{aligned} m_1^2 &= \frac{\theta^2 m^2 - \sqrt{\theta^4 m^4 - 4\theta^2 a}}{2\theta^2} \\ m_2^2 &= \frac{\theta^2 m^2 + \sqrt{\theta^4 m^4 - 4\theta^2 a}}{2\theta^2} \end{aligned} \tag{3.10}$$

with

$$0 < a < \theta^2 m^4/4. \tag{3.11}$$

Also

$$m^2 > m_2^2 > \frac{m^2}{2} > m_1^2 > 0. \tag{3.12}$$

Note that this decomposition was already made (for the Moyal case) in [24]. We now go further and decompose the second term on the right-hand side of (3.9); after some algebra this finally leads to rewrite the propagator (3.6) as an alternate sum,

$$G_0^{(2)}(p) = \frac{a/\theta^2}{m_2^2 - m_1^2}\left(\frac{1}{m_1^2(p^2 + m_2^2)} - \frac{1}{m_2^2(p^2 + m_1^2)}\right). \tag{3.13}$$

In similar situations, in the commutative framework (for example QFT with higher derivatives) an equivalent description is introduced in terms of KG fields. The ones responsible for the negative propagators are the ghost fields. They lead, when analytically continuing the model to the Minkowskian setting, to states of negative norm. They are not independent fields, therefore one could resort to the original formulation in terms of one field (for scalar theories as the one considered here) but this usually causes the loss of unitarity for the S matrix.

Some care is needed in order to extend this analysis to our model. First of all, we notice that it is only applicable when the parameter a is smaller than $\theta^2 m^4/4$ [see Eq. (3.11)]: for a greater than this value the two masses become complex valued.

On a more general footing, we have two main differences between commutative and noncommutative field theory: one is that the Minkowskian version of a NC field theory is not uniquely defined. Important features, such as the UV/IR mixing of the Euclidean formulation may be completely absent in some Minkowskian formulations [44]. The second important point is that the very concept of particle state is in general not well defined in a nonlocal theory. This remark is particularly relevant for our model, where the asymptotic regime is not attained for small noncommutativity [15].

However, let us stick to the usual Wick rotation. Then, it was shown in [45] that perturbative nonunitarity manifests as soon as time-space noncommutativity is present, independently from the details of the model, although a more careful analysis indicates that unitarity can be restored [46]. Later on, it has been argued with a nonperturbative study [47] that unitarity loss is a direct consequence of the UV/IR mixing. What is the interplay between that kind of nonunitarity, which is somehow inherent to NC theories with UV/IR mixing, and the one we are facing here where the modification to the kinetic term was introduced precisely to cancel the mixing, is a delicate issue. In order to see the consequences of the ghost fields appearing in (3.13), one should carefully define the S matrix in the appropriate Minkowski formulation, and then study its properties. This is an interesting problem which deserves further investigation.

Let us also make a second remark with respect to the Euclidean theory treated here. Even though, for massive theories, the quantum correction of the propagator (2.32) has subleading logarithmic divergence, one does not need to take it into consideration when proposing a model which may be *perturbatively* renormalizable. We will come back to this point at the end of the next section.

Since we have modified the propagator (2.19) to (3.6) but not the vertex (2.20), the one-loop corrections to the propagator represented by the graphs of Figs. 1 and 2 may be derived as in (2.23) and (2.27) and we find, respectively,

$$G_{1,P}^{(2)} = \frac{e^{-\alpha(0,p)}}{(p^2 + m^2 + \frac{a}{\theta^2 p^2})^2}\int d^d q \frac{e^{\sigma(p,q)}}{(q^2 + m^2 + \frac{a}{\theta^2 q^2})}, \tag{3.14}$$

$$G_{1,\mathrm{NP}}^{(2)} = \frac{e^{-\alpha(0,p)}}{(p^2 + m^2 + \frac{a}{\theta^2 p^2})^2}\int d^d q \frac{e^{\omega(p,q)+\sigma(p,q)}}{(q^2 + m^2) + \frac{a}{\theta^2 q^2}} \tag{3.15}$$

with $\sigma(p,q) = 0$.

Let us consider now the one-loop correction to four-point functions. For the planar case in Fig. 3 we have

$$G_{1,P}^{(4)} = \frac{\lambda^2}{8} G_0^{(4)} \int d^d q d^d k \frac{\delta(K - q - k)e^{\sigma(q,k)}}{(q^2 + m^2 + \frac{a}{\theta^2 q^2})(k^2 + m^2 + \frac{a}{\theta^2 k^2})}, \tag{3.16}$$

with G_0^4 the four-points Schwinger function at tree level, given by (2.22), while $\sigma(q,k) = 0$ [with σ defined in (2.24)]. Note that we have also denoted by K the total incoming momentum.

For the nonplanar case, one possible graph is shown in Fig. 4. We have then

$$G_{1,\mathrm{NP}}^{(4)} = \frac{\lambda^2}{8} \tilde{G}_0^4 \int d^d q d^d k \frac{\delta(K - q - k)e^{\omega(q,k)+\sigma(k,q)}}{(q^2 + m^2 + \frac{a}{\theta^2 q^2})(k^2 + m^2 + \frac{a}{\theta^2 k^2})}, \tag{3.17}$$

with $\omega(q,k)$ defined in (2.29) and (2.30) while the remaining exponential factors rearrange in such a way to yield $\sigma(k,q)$ vanishing. The other nonplanar graphs are obtained similarly, with a relabeling of the external paths.

For $d = 4$, the last integral behaves like a logarithm in the external momenta. Indeed, this can be seen from the fact that, in the UV regime of the loop momentum q, the corrections in $a/(\theta^2 q^2)$ are neglectable. Solving the δ function, the integral (3.17) behaves like

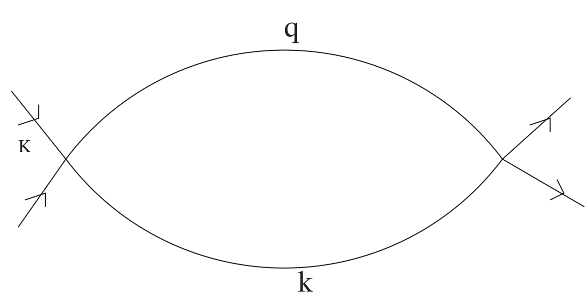

FIG. 3. Planar one-loop four-point graph of incoming momentum K.

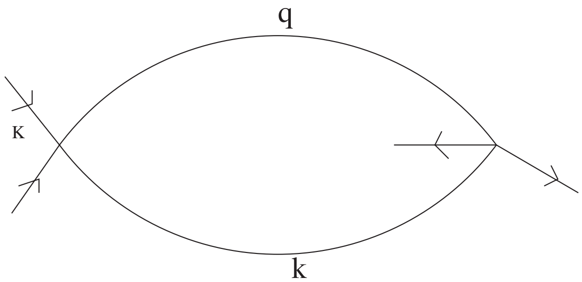

FIG. 4. A nonplanar one-loop four-points graph of total incoming momentum K.

$$\int d^4q \frac{e^{\omega(q,k)}}{q^4}. \tag{3.18}$$

Because of the form (2.30) of the factor $\omega(q, k)$, one can easily obtain the logarithm behavior in the external momenta. Nevertheless, these logarithms are harmless for *perturbative* renormalization. We will come back to this point at the end of this section.

Let us now compute some two-loop contributions to the four-point Schwinger function. The first of them is the one due to the graph in Fig. 5. This graph is obtained by inserting the nonplanar tadpole (3.15) of momentum p into the planar graph in Fig. 3. One has

$$G^{(4)}_{2,P} = G^{(4)}_0 \int d^4p \frac{e^{\omega(p,q)+\sigma(p,q)}}{p^2 + m^2 + \frac{a}{\theta^2 p^2}} \times \int d^dq d^4k \frac{\delta(K-q-k)e^{\sigma(q,k)}}{(q^2+m^2+\frac{a}{\theta^2q^2})^2(k^2+m^2+\frac{a}{\theta^2k^2})}, \tag{3.19}$$

and $\sigma = 0$. Let us emphasize that again, the cancellation of the exponentials in σ is obtained thanks to the cocycle condition (2.14). It is this cancellation that makes the Feynman integrals have the same behavior as in the Moyal case. Let us now investigate the behavior of the Feynman amplitude of the more general graph obtained from inserting a chain of N nonplanar tadpoles in the planar graph of Fig. 3 (see Fig. 6).

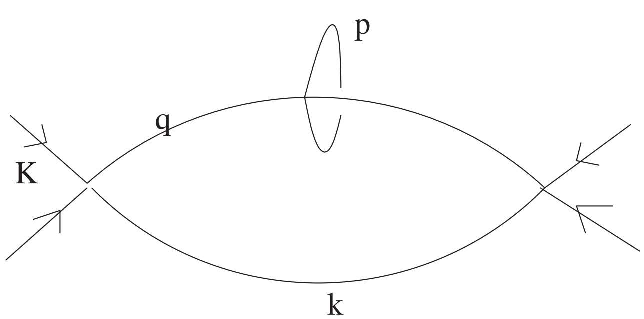

FIG. 5. A two-loop graph obtained by inserting into the bubble graph of Fig. 3 a nonplanar tadpole of momentum p.

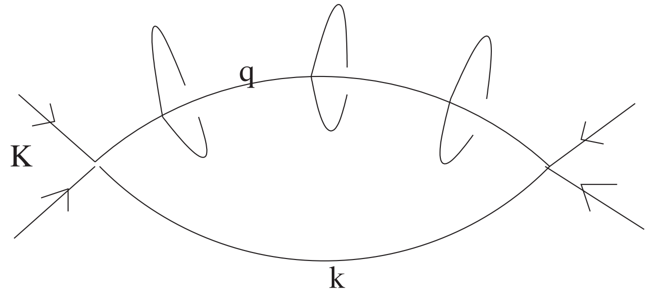

FIG. 6. A nonplanar graph obtained from the insertion into the bubble graph of Fig. 3 of a chain of nonplanar tadpoles of momenta p_i ($i = 1, \ldots, N$).

The integral to investigate writes indeed

$$\int \prod_{i=1}^N d^dp_i \frac{e^{\omega(p_i,q)+\sigma(p_i,q)}}{p_i^2+m^2+\frac{a}{\theta^2p_i^2}} \times \int d^dqd^dk\left(\frac{1}{q^2+m^2+\frac{a}{\theta^2q^2}}\right)^{N+1} \frac{\delta(K-q-k)e^{\sigma(q,k)}}{k^2+m^2+\frac{a}{\theta^2q^2}} \tag{3.20}$$

with $\sigma(p_i, q) = \sigma(q, k) = 0$. This is the generalization of the Feynman amplitude (3.19). Let us now have a closer look at the structure of the divergences of this general integral. For $d = 4$, when performing the integrations in the momenta p_i ($i = 1, \ldots, N$) and placing ourselves in the IR regime of the momentum q, each of these integrals leads to a $1/\theta^2q^2$ behavior (as proved above). The integral (3.20) thus becomes

$$\int d^4q\left(\frac{1}{\theta^2q^2}\right)^N\left(\frac{1}{q^2+m^2+\frac{a}{\theta^2q^2}}\right)^{N+1} \times \frac{1}{(q-K)^2+m^2+\frac{a}{\theta^2(q-K)^2}}. \tag{3.21}$$

Note that if $a = 0$ this integral is IR divergent for $N > 1$ (for $N = 1$ the mass m prevents the divergence to appear). Nevertheless, if $a \neq 0$, in the IR regime of q the dominant term is the a/θ^2q^2 in the propagators and the integral leads to an IR finite behavior.

Let us now discuss the appearance of some logarithmic divergences at the level of the two- and four-point functions, logarithms on the external momenta of the respective graph. These divergences are harmless when dealing with *perturbative* renormalizability. To illustrate this, let us consider the example of Fig. 7, where one has to deal with a chain of N nonplanar bubble graphs inserted into some "bigger" nonplanar graph. Since, as already explained above, the correction proposed here in the propagator is irrelevant in the UV regime (where this analysis is now performed), the Feynman integral gives the same behavior as in a commutative theory, namely,

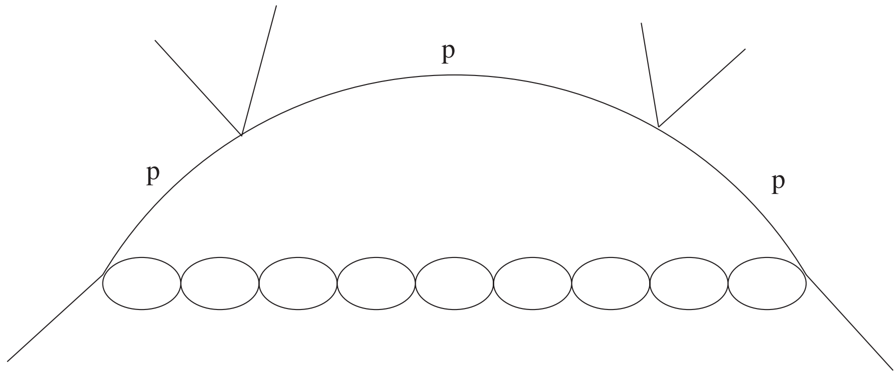

FIG. 7. Insertion of a chain of the bubble graph into some bigger graph.

$$\int d^4p \frac{1}{(p^2+m^2)^3} \log^N \frac{p^2}{m^2} \approx N!. \tag{3.22}$$

This is a (large) finite number which appears as a difficulty in summing perturbation theory—the renormalon problem (see for example [48]). The situation is analogous for the nonplanar tadpoles insertions. These logarithms should however be taken into consideration when defining a model which is requested to be nonperturbatively renormalizable. Let us also remark that, in [14], these logarithm divergences have not appeared because of the use of some appropriate scale decomposition (the multiscale analysis being used there for the proof of perturbative renormalization).

IV. THE MODIFIED ACTION FOR THE WV PRODUCT IN THE MATRIX BASIS

For the WV product it is possible to rewrite the model in a suitably defined matrix basis. The basis is a variation of the one described in [49] for the Moyal product and it was introduced, up to our knowledge, in [38] where the WV product was used on $\mathbb{R}^2$ to build a fuzzy version of the disk. In the following we review the derivation of the matrix basis as in [38] and we adapt it to the present notation; we then derive the model under consideration in the matrix basis. It is convenient to consider the plane as a complex space with $z = (x + iy)/\sqrt{2}$. The quantized versions of z and $\bar{z}$ are the usual annihilation and creation operators, $a = (\hat{x} + i\hat{y})/\sqrt{2}$ and $a^\dagger = (\hat{x} - i\hat{y})/\sqrt{2}$ with a slightly unusual normalization, so that their commutation rule is

$$[a, a^\dagger] = \theta. \tag{4.1}$$

Given the function $\phi(\bar{z}, z)$ consider its Taylor expansion:

$$\phi(\bar{z}, z) = \sum_{m,n=0}^{\infty} \phi_{mn}^{\mathrm{Tay}} \bar{z}^m z^n. \tag{4.2}$$

To this function we associate the operator

$$\Omega_\theta(\phi) := \hat{\phi} = \sum_{m,n=0}^{\infty} \phi_{mn}^{\mathrm{Tay}} a^{\dagger m} a^n. \tag{4.3}$$

We have thus "quantized" the plane using a normal ordering prescription. The map Ω_θ is invertible. It can be efficiently expressed defining the *coherent* states:

$$a|z\rangle = z|z\rangle. \tag{4.4}$$

One then has

$$\Omega_\theta^{-1}(\hat{\phi}) = \phi(\bar{z}, z) = \langle z|\hat{\phi}|z\rangle. \tag{4.5}$$

The maps Ω and Ω^{-1} yield a procedure of going back and forth from functions to operators. Moreover, the product of operators being noncommutative, a noncommutative $\star$ product between functions is implicitly defined as

$$(\phi \star \phi')(\bar{z}, z) = \Omega^{-1}(\Omega(\phi)\Omega(\phi')). \tag{4.6}$$

It is possible to see that the WV product (2.8) is exactly of this form, namely,

$$(\phi \star_{\mathrm{WV}} \psi)(\bar{z}, z) = \langle z|\hat{\phi}\ \hat{\psi}\ |z\rangle. \tag{4.7}$$

There is another useful basis on which it is possible to represent the operators and hence the functions—the matrix basis. As we already stated above, it is similar to the one introduced for the Moyal product in [49] and subsequently used in [13]. Consider the number operator

$$N = a^\dagger a, \tag{4.8}$$

and its eigenvectors which we indicate by $|n\rangle$:

$$N|n\rangle = n\theta|n\rangle. \tag{4.9}$$

We can then express the operators within a density matrix notation:

$$\hat{\phi} = \sum_{m,n=0}^{\infty} \phi_{mn}|m\rangle\langle n|. \tag{4.10}$$

Applying the dequantization map (4.5) to $\hat{\phi}$, we associate to it

$$\phi(z,\bar{z}) = \langle z|\phi|z\rangle = \sum_{m,n=0}^{\infty} \phi_{mn}\langle z|m\rangle\langle n|z\rangle$$
$$:= \sum_{m,n=0}^{\infty} \phi_{mn} v_{mn}(z,\bar{z}) \tag{4.11}$$

to be compared with (4.2), the same function in a different basis. Observing that

$$\langle z|n\rangle = e^{-(\bar{z}z/2\theta)} \frac{\bar{z}}{\sqrt{n!\theta^n}}, \tag{4.12}$$

we have

$$v_{nm}(\bar{z},z) = e^{-(\bar{z}z/2\theta)} \frac{\bar{z}^n z^m}{\sqrt{n!m!\theta^{n+m}}}. \tag{4.13}$$

The elements of the density matrix basis have a very simple multiplication rule:

$$|m\rangle\langle n|p\rangle\langle q| = \delta_{np}|m\rangle\langle q|; \tag{4.14}$$

this leads to

$$v_{mn}(\bar{z},z) \star_{\mathrm{WV}} v_{pq}(\bar{z},z) = \delta_{np} v_{mq}(\bar{z},z). \tag{4.15}$$

Moreover, one has

$$\int d^2 z v_{nm}(\bar{z},z) = \delta_{nm}\theta\pi. \tag{4.16}$$

The functions $v_{nm}(\bar{z},z)$ thus form an orthogonal basis in the noncommutative algebra of functions on the plane, with the WV product, the matrix basis [in analogy with its operator counterpart, (4.14)]. The connection between the expansions (4.3) and (4.10) is given by

$$a = \sum_{n=0}^{\infty} \sqrt{(n+1)\theta}\, |n\rangle\langle n+1| \tag{4.17}$$

$$a^\dagger = \sum_{n=0}^{\infty} \sqrt{(n+1)\theta}\, |n+1\rangle\langle n|. \tag{4.18}$$

Thus, looking at their symbols

$$\begin{aligned} z &= \langle z|a|z\rangle = \sum_n \sqrt{(n+1)\theta}\, v_{nn+1}(\bar{z},z) \\ \bar{z} &= \langle z|a^\dagger|z\rangle = \sum_n \sqrt{(n+1)\theta}\, v_{n+1n}(\bar{z},z), \end{aligned} \tag{4.19}$$

we then have

$$\phi_{mn}^{\mathrm{Tay}} = \sum_{l=0}^{\min\{m,n\}} (-1)^l \frac{\phi_{m-l,n-l}}{l!\sqrt{(m-l)!(n-l)!\theta^{m+n}}}. \tag{4.20}$$

In the density matrix basis, using (4.13), the product (2.8) [or (4.7)] simplifies to an infinite row by column matrix multiplication:

$$(\phi \star_{\mathrm{WV}} \psi)_{mn} = \sum_{k=1}^{\infty} \phi_{mk}\psi_{kn}. \tag{4.21}$$

Using the expansion (4.11) and (4.16) it is easy to see that

$$\int d^2 z \phi(\bar{z},z) = \pi\theta\, \mathrm{Tr}\Phi = \pi\theta \sum_{n=0}^{\infty} \phi_{nn}, \tag{4.22}$$

where we have introduced the infinite matrix Φ with entries $\{\phi_{ij}\}$.

Let us generalize this basis to higher (even) dimensions. In d dimensions we need $d/2$ copies of the WV plane, with $d/2$ pairs of complex coordinates z^i, $\bar{z}^i$. As for the Moyal hyperplane, coordinates describing different two-planes, commute among themselves. We define

$$\begin{aligned} &v_{\vec{m}\,\vec{n}}(\bar{z}_1,\ldots,\bar{z}_{d/2},z_1,\ldots,z_{d/2}) \\ &\quad = \langle z_1,\ldots z_{d/2}|m_1,\ldots,m_{d/2}\rangle \\ &\qquad \times \langle n_1,\ldots,n_{d/2}|z_1,\ldots z_{d/2}\rangle = \Pi_{i=1}^{d/2} v^i_{m_i n_i} \end{aligned} \tag{4.23}$$

with

$$v^i_{m_i n_i} = \langle z_i|m_i\rangle\langle n_i|z_i\rangle = e^{-(\bar{z}_i z_i/2\theta)} \frac{\bar{z}_i^{m_i} z_i^{n_i}}{\sqrt{n_i!m_i!\theta^{n_i+m_i}}}. \tag{4.24}$$

The functions $v_{\vec{m}\,\vec{n}}$ form an orthogonal basis in the algebra of noncommutative functions on the hyperplane, as it may be easily checked that

$$v_{\vec{m}\,\vec{n}} \star_{\mathrm{WV}} v_{\vec{p}\,\vec{q}} = v_{\vec{m}\,\vec{q}}\delta_{\vec{n}\,\vec{p}}. \tag{4.25}$$

Moreover, one has

$$\int d^d z v_{\vec{m}\,\vec{n}} = (\pi\theta)^{d/2}\delta_{\vec{m}\,\vec{n}}. \tag{4.26}$$

The field ϕ is therefore expanded as

$$\phi(\bar{z},z) = \sum_{\substack{m_i,n_i=0\\ i=1,\ldots,d}}^{\infty} \phi_{\vec{m}\,\vec{n}} v_{\vec{m}\,\vec{n}}(z,\bar{z}). \tag{4.27}$$

Moreover, one has

$$\phi(\bar{z},z) \star_{\mathrm{WV}} \psi(\bar{z},z) = \sum \phi_{\vec{m}\,\vec{n}}\psi_{\vec{n}\,\vec{q}} v_{\vec{m}\,\vec{q}} \tag{4.28}$$

and also

$$\int d^d z \phi(\bar{z},z) = (\pi\theta)^d \sum \phi_{\vec{m}\,\vec{n}}\delta_{\vec{m}\,\vec{n}}. \tag{4.29}$$

Having established this basis it is now possible to express the action of our model (3.2) in matrix notation. The mass and the interaction terms are just row by column multiplication. We have

$$\begin{aligned} &\frac{m^2}{2}\int d^d z \phi \star_{\mathrm{WV}} \phi + \frac{\lambda}{4!}\int d^d z \phi \star_{\mathrm{WV}} \phi \star_{\mathrm{WV}} \phi \star_{\mathrm{WV}} \phi \\ &\quad = (\pi\theta)^{d/2}\Big(\frac{\mu^2}{2}\sum \phi_{\vec{m}\,\vec{n}}\phi_{\vec{n}\,\vec{q}}\delta_{\vec{m}\,\vec{q}} \\ &\qquad + \frac{\lambda}{4!}\sum \phi_{\vec{m}\,\vec{n}}\phi_{\vec{n}\,\vec{q}}\phi_{\vec{q}\,\vec{p}}\phi_{\vec{p}\,\vec{r}}\delta_{\vec{m}\,\vec{r}}\Big). \end{aligned} \tag{4.30}$$

Let us consider the kinetic term. We first observe that

$$\frac{1}{2}\int d^dz\partial_\mu\phi \star_{\rm WV} \partial_\mu\phi = \int d^dz\sum_i \partial_{z_i}\phi \star_{\rm WV} \partial_{\bar z_i}\phi \tag{4.31}$$

and

$$\partial_{z_i}\phi = \frac{1}{\theta}[\bar z_i, \phi]_{\star_{\rm WV}} \tag{4.32}$$

$$\partial_{\bar z_i}\phi = \frac{1}{\theta}[z_i, \phi]_{\star_{\rm WV}}. \tag{4.33}$$

Equations (4.17) and (4.18), suitably generalized to the d case, imply in turn that z_i and $\bar z_i$ are expanded in the matrix basis as

$$z_i = \sum_{\vec n}\sqrt{(n_i+1)\theta}\, v^i_{n_i n_i+1}\Pi_{j\neq i}v^j_{n_j n_j} \tag{4.34}$$

$$\bar z_i = \sum_{\vec n}\sqrt{(n_i+1)\theta}\, v^i_{n_i+1 n_i}\Pi_{j\neq i}v^j_{n_j n_j}. \tag{4.35}$$

We then have

$$\begin{aligned}\partial_{z_i}\phi &= \frac{1}{\theta}\sum_{\vec m\,\vec n}\phi_{\vec m\,\vec n}(\sqrt{m_i+1}v^i_{m_i+1 n_i} \\ &\quad - \sqrt{n_i}v^i_{m_i n_i-1})\Pi_{j\neq i}v^i_{m_i n_i} \\ \partial_{\bar z_i}\phi &= \frac{1}{\theta}\sum_{\vec m\,\vec n}\phi_{\vec m\,\vec n}(\sqrt{m_i}v^i_{m_i-1 n_i} \\ &\quad - \sqrt{n_i+1}v^i_{m_i n_i+1})\Pi_{j\neq i}v^i_{m_i n_i}.\end{aligned} \tag{4.36}$$

The kinetic term (4.31) becomes

$$\begin{aligned}&\frac{(\pi\theta)^d}{\sqrt{\theta}}\sum_{\substack{\vec m\,\vec n\\ \vec p\,\vec q}}\phi_{\vec m\,\vec n}\phi_{\vec p\,\vec q}\sum_i(\delta_{m_i q_i}\delta_{n_i p_i}(-m_i-n_i-1) \\ &\quad + \delta_{m_i+1 q_i}\delta_{n_i p_i-1}\sqrt{q_i p_i} \\ &\quad + \delta_{n_i-1 p_i}\delta_{m_i q_i+1}\sqrt{(q_i+1)(p_i+1)})\Pi_{j\neq i}\delta_{m_j q_j}\delta_{n_j p_j}.\end{aligned} \tag{4.37}$$

Let us consider now the supplementary term we have added in (3.2):

$$\frac{1}{\theta^d}\int d^dz\Big(\int d^dz'\phi(\bar z', z')\Big)\star_{\rm WV}\Big(\int d^dz'\phi(\bar z', z')\Big). \tag{4.38}$$

To compute the two indefinite integrals above, we use the Taylor expansion (4.3). We thus arrive at the following expression:

$$\begin{aligned}&\frac{1}{\theta^d}\sum_{\substack{\vec m\,\vec n\\ \vec r\,\vec s}}\phi^{\rm Tay}_{\vec m\,\vec n}\phi^{\rm Tay}_{\vec r\,\vec s}\Pi_i\frac{1}{(m_i+1)(n_i+1)(r_i+1)(s_i+1)} \\ &\quad\times\int d^dz\bar z_i^{m+1}z_i^{n+1}\star_{\rm WV}\bar z_i^{r+1}z_i^{s+1}.\end{aligned} \tag{4.39}$$

We then use the matrix basis to evaluate the star product and we obtain

$$\begin{aligned}&\frac{1}{\theta^d}\sum_{\substack{\vec m\,\vec n\\ \vec r\,\vec s}}\phi^{\rm Tay}_{\vec m\,\vec n}\phi^{\rm Tay}_{\vec r\,\vec s}\sum_{\vec p}\int d^dz\Pi_i\frac{\bar z_i^{p_i+m_i}z_i^{p_i+n_i-r_i+s_i}e^{-((\bar z_i z_i)/\theta)}}{(m_i+1)(n_i+1)(r_i+1)(s_i+1)} \\ &\quad\times\frac{(p_i+n_i+1)!\sqrt{(p_i+m_i+1)!(p_i+n_i-r_i+s_i+1)!\theta^{m_i+n_i+r_i+s_i+4}}}{p_i!(p_i+n_i-r_i)!\sqrt{(p_i+m_i)!(p_i+n_i-r_i+s_i)!\theta^{2p_i+m_i+n_i-r_i+s_i}}}.\end{aligned} \tag{4.40}$$

The integral may be easily performed and, after some algebra, we obtain for the supplementary term the expression

$$\sum_{\substack{\vec m\,\vec n\\ \vec r\,\vec s}}\phi^{\rm Tay}_{\vec m\,\vec n}\phi^{\rm Tay}_{\vec r\,\vec s}\sum_{\vec p}\Pi_i\frac{(p_i+m_i+1)!(p_i+n_i+1)!\theta^{r_i-m_i-p_i}}{(m_i+1)(n_i+1)(r_i+1)(s_i+1)p_i!(p_i+n_i-r_i)!}\delta_{m_i+r_i,n_i+s_i}. \tag{4.41}$$

Finally, using (4.20) to rewrite the coefficients $\phi^{\rm Tay}$, we get

$$\begin{aligned}&\sum_{\substack{\vec m\,\vec n\\ \vec r\,\vec s\,\vec p}}\Pi_i\frac{(p_i+m_i+1)!(p_i+n_i+1)!\theta^{r_i-m_i-p_i}}{(m_i+1)(n_i+1)(r_i+1)(s_i+1)p_i!(p_i+n_i-r_i)!}\delta_{m_i+r_i,n_i+s_i} \\ &\quad\times\sum_{\vec l=0}^{\min\{\vec m,\vec n\}}\sum_{\vec k=0}^{\min\{\vec r,\vec s\}}\Pi_j\frac{(-1)^{l_j+k_j}\theta^{-m_j-r_j}\phi_{\vec m-\vec l\,\vec n-\vec l}\phi_{\vec r-\vec k\,\vec s-\vec k}}{l_j!k_j!\sqrt{(m_j-l_j)!(n_j-l_j)!(r_j-k_j)!(s_j-k_j)!}},\end{aligned} \tag{4.42}$$

where, with an abuse of notation, $\sum_{\vec l=0}^{\min\{\vec m,\vec n\}}$ stands for $\sum_{\substack{l_i=0\\ i=1,\dots,d}}^{\min\{m_i,n_i\}}$. Finally, summing all the contributions that we obtain from (4.30), (4.31), and (4.42), the complete action (3.2) of our model with supplementary term is rewritten in the matrix basis as

$$S[\phi] = (\pi\theta)^{d/2}\Big(\frac{m^2}{2}\sum \phi_{\vec{m}\,\vec{n}}\phi_{\vec{n}\,\vec{q}}\delta_{\vec{m}\,\vec{q}} + \frac{\lambda}{4!}\sum \phi_{\vec{m}\,\vec{n}}\phi_{\vec{n}\,\vec{q}}\phi_{\vec{q}\,\vec{p}}\phi_{\vec{p}\,\vec{r}}\delta_{\vec{m}\,\vec{r}}\Big) + \frac{(\pi\theta)^{d/2}}{\sqrt{\theta}}\sum_{\substack{\vec{m}\,\vec{n}\\ \vec{p}\,\vec{q}}}\phi_{\vec{m}\,\vec{n}}\phi_{\vec{p}\,\vec{q}}\sum_i(\delta_{m_iq_i}\delta_{n_ip_i}(-m_i-n_i-1).$$
$$+ \delta_{m_i+1q_i}\delta_{n_ip_i-1}\sqrt{q_ip_i} + \delta_{n_i-1p_i}\delta_{m_iq_i+1}\sqrt{(q_i+1)(p_i+1)})\Pi_{j\neq i}\delta_{m_jq_j}\delta_{n_jp_j}$$
$$+ \sum_{\substack{\vec{m}\,\vec{n}\\ \vec{r}\,\vec{s}\,\vec{p}}}\Pi_i\frac{(p_i+m_i+1)!(p_i+n_i+1)!\theta^{r_i-m_i-p_i}}{(m_i+1)(n_i+1)(r_i+1)(s_i+1)p_i!(p_i+n_i-r_i)!}$$
$$\times \delta_{m_i+r_i,n_i+s_i}\sum_{\vec{l}=0}^{\min\{\vec{m},\vec{n}\}}\sum_{\vec{k}=0}^{\min\{\vec{r},\vec{s}\}}\Pi_j\frac{(-1)^{l_j+k_j}\theta^{-m_j-r_j}\phi_{\vec{m}-\vec{l}\,\vec{n}-\vec{l}}\phi_{\vec{r}-\vec{k}\,\vec{s}-\vec{k}}}{l_j!k_j!\sqrt{(m_j-l_j)!(n_j-l_j)!(r_j-k_j)!(s_j-k_j)!}}. \tag{4.43}$$

V. CONCLUSION AND PERSPECTIVES

In this paper we have proposed a solution for curing the UV/IR mixing which appears when implementing field theories using a translation-invariant product on $\mathbb{R}^4$. This solution generalizes the one proposed in [14] for curing the UV/IR mixing on Moyal space. We explicitly compute the one-loop Feynman amplitudes of the proposed model, as well as higher loop amplitudes of some graphs obtained by an insertion of nonplanar tadpoles. Our result is mainly due to the cocycle condition (2.13).

An immediate perspective is to obtain a proof of the perturbative renormalization of the proposed model at any order in perturbation theory. This could be achieved by investigating the general form of the factor generalizing the Moyal oscillating phase of some Feynman amplitude.

As already stated in this paper, the noncommutative products that we have worked here are equivalent in the formal series sense of Kontsevich. We thus explicitly show that an important field theoretical result—curing the UV/IR mixing—can be obtained applying the same recipe as in Moyal field theory. It is interesting to further understand the explicit relation between this formal series equivalence of the noncommutative products and the equivalence of the Euclidean field theories thus implemented.

Nevertheless, as already argued in [33], when doing Minkowskian field theory, the situation is manifestly different, because the different factors on the external propagator can lead to a different S matrix form, unless properly implementing the quantum symmetry of the model as a twisted symmetry.

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