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Casimir energy, dispersion, and the Lifshitz formula

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Despite suggestions to the contrary, we show in this paper that the usual dispersive form of the electromagnetic energy density must be used to derive the Lifshitz force between parallel dielectric media. This conclusion follows from the general form of the quantum vacuum energy, which is the basis of the multiple-scattering formalism. As an illustration, we explicitly derive the Lifshitz formula for the interaction between parallel dielectric semispaces, including dispersion, starting from the expression for the total energy of the system. The issues of constancy of the energy between parallel plates and of the observability of electrostrictive forces are briefly addressed.

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I. INTRODUCTION

Recent years have yielded considerable progress in understanding quantum vacuum or Casimir energies, both theoretically and experimentally. For a very recent review, see Ref. [1]. However, there are controversial aspects, both having to do with the concept of zero-point energy applied to a single system, or to the Universe as a whole [2], and with including thermal corrections and their observability in experiment [1]. The latter question refers to how the electric properties of materials depend on (imaginary) frequencies, that is, upon dispersion.

In this paper, we address the latter issue. In a recent paper [3], we had proposed, following a suggestion of Lifshitz [4], that the usual dispersive term in the electromagnetic energy for a given frequency [5],

$$U = \frac{1}{2}\int (d\mathbf{r}) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\left[\frac{d(\omega\varepsilon)}{d\omega} E^2(\mathbf{r}) + H^2(\mathbf{r})\right] \qquad (1.1)$$

(we ignore the magnetic susceptibility; that is, we set $\mu = 1$), should not be included. However, the usual derivations of the Lifshitz interaction between dielectric slabs are not based on the total energy. For example, in Ref. [6] the Lifshitz formula is derived from the pressure, or equivalently the spatial components of the stress tensor, and also from the variational principle enunciated in Ref. [7]. It is also easy to obtain this same result using the recently repopularized multiple-scattering approach to Casimir energies [8,9]. Equivalently, the multiple-reflection expansion yields the Lifshitz formula immediately [10].

In this article, we derive the Lifshitz energy directly from Eq. (1.1). We first see, in Sec. II, how dispersion is incorporated in a general formulation. This demonstrates that the dispersive form of the energy is required. Then, after giving the form of the Green's dyadic in Sec. III, in Sec. IV we will explicitly derive the Lifshitz formula from Eq. (1.1), and will see manifestly that the dispersive term provides the Jacobian of the required transformation of coordinates necessary to obtain the necessary log det form. In the conclusions, we also bring up the related possibility of measuring electrostrictive effects in liquids. The Appendix points out that the well-known constancy of the energy density between parallel perfectly conducting plates does not hold for dielectric plates (that is, if regions 1 and 2, defined in Sec. III, are constituted of dielectric material), or even if a dispersive medium exists between metallic plates.

II. GENERAL FORMULATION

Let us start from Eq. (1.1), and consider the quantum vacuum energy associated with electromagnetic field fluctuations:

$$\mathfrak{E} = \frac{1}{2}\int (d\mathbf{r}) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\left[\frac{d(\varepsilon\omega)}{d\omega}\langle E^2\rangle + \langle H^2\rangle\right]. \qquad (2.1)$$

The expectation values appearing here are given by the electromagnetic Green's dyadic,

$$\langle \mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}')\rangle = \frac{1}{i}\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}'), \qquad (2.2a)$$

$$\langle \mathbf{H}(\mathbf{r})\mathbf{H}(\mathbf{r}')\rangle = -\frac{1}{i}\frac{1}{\omega^2}\boldsymbol{\nabla}\times\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}')\times\overleftarrow{\boldsymbol{\nabla}}', \qquad (2.2b)$$

and so inserting these into the energy expression (2.1), integrating by parts, and using the differential equation satisfied by the Green's dyadic,

$$-\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\Gamma} + \omega^2\varepsilon\boldsymbol{\Gamma} = -\omega^2\mathbf{1}, \qquad (2.3)$$

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we obtain the expression for the energy

$$\mathfrak{E} = -\frac{i}{2}\int(d\mathbf{r})\int\frac{d\omega}{2\pi}\left[2\varepsilon\,\mathrm{tr}\mathbf{\Gamma} + \omega\frac{d\varepsilon}{d\omega}\,\mathrm{tr}\mathbf{\Gamma}\right]. \tag{2.4}$$

We can obtain this same result starting from the standard trace-log formula:

$$\begin{aligned}\mathfrak{E} &= \frac{i}{2}\int\frac{d\omega}{2\pi}\,\mathrm{Tr}\ln\mathbf{\Gamma} = -\frac{i}{2}\int\frac{d\omega}{2\pi}\omega\frac{d}{d\omega}\,\mathrm{Tr}\ln\mathbf{\Gamma}\\ &= -\frac{i}{2}\int\frac{d\omega}{2\pi}\omega\,\mathrm{Tr}\mathbf{\Gamma}^{-1}\frac{d}{d\omega}\mathbf{\Gamma} = \frac{i}{2}\int\frac{d\omega}{2\pi}\omega\,\mathrm{Tr}\mathbf{\Gamma}\frac{d}{d\omega}\mathbf{\Gamma}^{-1}\\ &= \frac{i}{2}\int\frac{d\omega}{2\pi}\omega\,\mathrm{Tr}\left(-\frac{2}{\omega^3}\nabla\times\nabla\times - \frac{d\varepsilon}{d\omega}\right)\mathbf{\Gamma}\\ &= -\frac{i}{2}\int\frac{d\omega}{2\pi}\,\mathrm{Tr}\left(2\varepsilon\mathbf{\Gamma} + \omega\frac{d\varepsilon}{d\omega}\mathbf{\Gamma}\right),\end{aligned} \tag{2.5}$$

where from Eq. (2.3)

$$\mathbf{\Gamma}^{-1} = \frac{1}{\omega^2}\nabla\times\nabla\times - \varepsilon, \tag{2.6}$$

and where Tr includes the trace over spatial coordinates. The final form in Eq. (2.5) is exactly the result (2.4) derived from the expectation value of the classical electromagnetic energy (1.1).

To conclusively demonstrate that the dispersive term must be included, we derive the variational principle used to obtain the Lifshitz formula in Ref. [7]. This depends upon the variational statement

$$\delta\mathbf{\Gamma} = -\mathbf{\Gamma}\delta\mathbf{\Gamma}^{-1}\mathbf{\Gamma} = \mathbf{\Gamma}\delta\varepsilon\mathbf{\Gamma}. \tag{2.7}$$

Also, using the differential equation for the Green's dyadic, we find

$$\frac{d\mathbf{\Gamma}}{d\omega} = -\mathbf{\Gamma}\frac{d\mathbf{\Gamma}^{-1}}{d\omega}\mathbf{\Gamma} = \frac{2}{\omega}\mathbf{\Gamma}\varepsilon\mathbf{\Gamma} + \mathbf{\Gamma}\frac{d\varepsilon}{d\omega}\mathbf{\Gamma} + \frac{2}{\omega}\mathbf{\Gamma}. \tag{2.8}$$

Therefore, the ε-variation of Eq. (2.4) yields

$$\begin{aligned}\delta\mathfrak{E} &= -\frac{i}{2}\int\frac{d\omega}{2\pi}\,\mathrm{Tr}\left(2\delta\varepsilon\mathbf{\Gamma} + 2\varepsilon\mathbf{\Gamma}\delta\varepsilon\mathbf{\Gamma} + \omega\frac{d\delta\varepsilon}{d\omega}\mathbf{\Gamma}\right.\\ &\quad\left. + \omega\frac{d\varepsilon}{d\omega}\mathbf{\Gamma}\delta\varepsilon\mathbf{\Gamma}\right)\\ &= -\frac{i}{2}\int\frac{d\omega}{2\pi}\omega\frac{d}{d\omega}\,\mathrm{Tr}\delta\varepsilon\mathbf{\Gamma},\end{aligned} \tag{2.9}$$

which, upon integration by parts, yields the variational principle used in Refs. [6,7]:

$$\delta\mathfrak{E} = \frac{i}{2}\int\frac{d\omega}{2\pi}\,\mathrm{Tr}\delta\varepsilon\mathbf{\Gamma}. \tag{2.10}$$

See also Ref. [11].

III. GREEN'S DYADIC FOR PARALLEL SLABS

In this and the following section, we supply an explicit derivation of the Casimir-Lifshitz interaction between parallel dielectric slabs (of infinite thickness). Specifically, consider a dielectric function in the following form:

$$\varepsilon(\mathbf{r}) = \begin{cases}\epsilon_1, & z<0,\\ \epsilon_3, & 0<z<a,\\ \epsilon_2, & a<z.\end{cases} \tag{3.1}$$

Then, the Green's dyadic can be written as a transverse Fourier transform,

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{r}') = \int\frac{(d\mathbf{k}_\perp)}{(2\pi)^2}e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')_\perp}\mathbf{g}(z,z';\mathbf{k}_\perp,\omega), \tag{3.2}$$

where the reduced Green's dyadic has the form [6,7],

$$\mathbf{g}(z,z') = \begin{pmatrix}\frac{1}{\varepsilon}\frac{\partial}{\partial z}\frac{1}{\varepsilon'}\frac{\partial}{\partial z'}g^E & 0 & \frac{ik}{\varepsilon\varepsilon'}\frac{\partial}{\partial z}g^E\\ 0 & \omega^2 g^H & 0\\ -\frac{ik}{\varepsilon\varepsilon'}\frac{\partial}{\partial z'}g^E & 0 & \frac{k^2}{\varepsilon\varepsilon'}g^E\end{pmatrix}. \tag{3.3}$$

Here, we have dropped δ-function terms, we have denoted $\varepsilon = \varepsilon(z)$, $\varepsilon' = \varepsilon(z')$, and we have chosen the coordinate system so that $\mathbf{k}_\perp = k\hat{x}$. Here, the transverse electric (TE or H) and transverse magnetic (TM or E) (relative to the z axis) Green's functions satisfy the differential equations

$$\left(-\frac{\partial^2}{\partial z^2} + \kappa^2\right)g^H = \delta(z-z'), \tag{3.4a}$$

$$\left(-\frac{\partial}{\partial z}\frac{1}{\varepsilon}\frac{\partial}{\partial z} + \frac{1}{\varepsilon}\kappa^2\right)g^E = \delta(z-z'). \tag{3.4b}$$

We will solve these equations in each of the three regions given in Eq. (3.1), subject to boundary conditions between the regions that g^H and $\partial_z g^H$ are continuous, and that g^E and $(1/\varepsilon)\partial_z g^E$ are continuous. These boundary conditions reflect the underlying requirement that the transverse parts of $\mathbf{E}$ and $\mathbf{H}$ are continuous, while the normal component of $\mathbf{D} = \varepsilon\mathbf{E}$ is continuous (there are no surface charges or currents). It is a straightforward calculation to find the Green's functions in each region. We display the results for the only situation we need in the following, when z and z' are both in the same regions. Below the first interface, z, $z' < 0$,

$$g^H(z,z') = \frac{1}{2\kappa_1}[e^{-\kappa_1|z-z'|} + r_1 e^{\kappa_1(z+z')}], \tag{3.5a}$$

$$r_1 = \frac{\kappa_1-\kappa_3}{\kappa_1+\kappa_3} + \frac{4\kappa_1\kappa_3}{\kappa_3^2-\kappa_1^2}\frac{1}{d}, \tag{3.5b}$$

where $\kappa_a^2 = k^2 + \zeta^2\varepsilon_a(i\zeta)$, $a = 1, 2, 3$, and we have made a Euclidean rotation, $\omega = i\zeta$. Here, we have introduced the abbreviation

$$d = \frac{\kappa_3+\kappa_2}{\kappa_3-\kappa_2}\frac{\kappa_3+\kappa_1}{\kappa_3-\kappa_1}e^{2\kappa_3 a} - 1. \tag{3.6}$$

Similarly, above the second interface, z, $z' > a$,

$$g^H(z,z') = \frac{1}{2\kappa_2}[e^{-\kappa_2|z-z'|} + r_2 e^{-\kappa_2(z+z'-2a)}], \tag{3.7a}$$

$$r_2 = \frac{\kappa_2-\kappa_3}{\kappa_2+\kappa_3} + \frac{4\kappa_2\kappa_3}{\kappa_3^2-\kappa_2^2}\frac{1}{d}. \tag{3.7b}$$

In the intermediate region, $a > z$, $z' > 0$,

$$g^H(z,z') = \frac{1}{2\kappa_3}\bigg[e^{-\kappa_3|z-z'|} + \frac{2}{d}\cosh\kappa_3(z-z') + \frac{\kappa_3+\kappa_1}{\kappa_3-\kappa_1} \times \frac{1}{d}e^{\kappa_3(z+z')} + \frac{\kappa_3+\kappa_2}{\kappa_3-\kappa_2}\frac{1}{d}e^{-\kappa_3(z+z'-2a)}\bigg]. \tag{3.8}$$

The transverse magnetic Green's function g^E is obtained from the above by replacing $\kappa_a \to \kappa_a/\epsilon_a$ except in the exponents.

The fact that $g(z,z)$ depends on z implies, in general, that the mean-squared electric and magnetic fields also depend on position, as does the energy density. This seems to contradict the fact that for parallel conducting plates the energy density is constant in each region [3,12]. We shall show, in fact, in the Appendix that the electromagnetic energy density between perfectly conducting plates is indeed constant, provided the intervening medium is nondispersive.

IV. LIFSHITZ ENERGY

The Casimir-Lifshitz energy per unit area for the situation of parallel slabs described by the dielectric function (3.1) is [1,6]

$$\mathcal{E} = \frac{1}{4\pi^2}\int_0^\infty d\zeta \int_0^\infty dkk[\ln(1 - r_{\rm TE}r'_{\rm TE}e^{-2\kappa_3 a}) + \ln(1 - r_{\rm TM}r'_{\rm TM}e^{-2\kappa_3 a})], \tag{4.1}$$

where

$$r_{\rm TE} = \frac{\kappa_3 - \kappa_1}{\kappa_3+\kappa_1}, \qquad r'_{\rm TE} = \frac{\kappa_3-\kappa_2}{\kappa_3+\kappa_2}, \tag{4.2}$$

and the TM reflection coefficients are obtained by replacing $\kappa_a \to \kappa'_a = \kappa_a/\epsilon_a$. In this section, we rederive this result from Eq. (2.4).

A. TE contribution to the energy

The TE part of the energy can be written as

$$\mathcal{E}^{\rm TE} = \frac{1}{4\pi^2}\int_0^\infty d\zeta \int_0^\infty kdk \int_{-\infty}^\infty dz \zeta \frac{d}{d\zeta}(-\kappa^2) g^H(z,z). \tag{4.3}$$

In each region, the dispersive term is necessary to change variables from ζ to κ_a. In the first region, omitting infinite terms which contain no reference to the separation a between the regions, we find rather immediately (we assume, as usual, $\zeta^2\epsilon_a(\zeta) \to 0$ as $\zeta \to 0$)

$$\mathcal{E}^{\rm TE,1} = -\frac{1}{2\pi^2}\int_0^\infty kdk \int_k^\infty d\kappa_1 \frac{\zeta}{2}\frac{\partial}{\partial\kappa_1}\ln\Delta_{\rm TE}, \tag{4.4}$$

where the partial derivative means that κ_2 and κ_3 are not altered, and

$$\Delta_{\rm TE} = 1 - r_{\rm TE}r'_{\rm TE}e^{-2\kappa_3 a}. \tag{4.5}$$

Similarly, in region 2,

$$\mathcal{E}^{\rm TE,2} = -\frac{1}{2\pi^2}\int_0^\infty kdk \int_k^\infty d\kappa_2 \frac{\zeta}{2}\frac{\partial}{\partial\kappa_2}\ln\Delta_{\rm TE}. \tag{4.6}$$

The intermediate region involves a slightly more involved calculation, but the result has the same form (after we omit a constant term in the force):

$$\mathcal{E}^{\rm TE,3} = -\frac{1}{2\pi^2}\int_0^\infty kdk \int_k^\infty d\kappa_3 \frac{\zeta}{2}\frac{\partial}{\partial\kappa_3}\ln\Delta_{\rm TE}, \tag{4.7}$$

where now the derivative acts also on the exponent in $\Delta_{\rm TE}$. In this way, we obtain exactly the expected TE contribution:

$$\mathcal{E}^{\rm TE} = -\frac{1}{2\pi^2}\int_0^\infty kdk\int_k^\infty d\kappa_3\frac{\zeta}{2} \times\left(\frac{\partial}{\partial\kappa_3} + \frac{d\kappa_2}{d\kappa_3}\frac{\partial}{\partial\kappa_2} + \frac{d\kappa_1}{d\kappa_3}\frac{\partial}{\partial\kappa_1}\right)\ln\Delta_{\rm TE} = \frac{1}{4\pi^2}\int_0^\infty dkk\int_0^\infty d\zeta \ln\Delta_{\rm TE}, \tag{4.8}$$

which is just the first term in Eq. (4.1).

B. TM contribution to energy

The TM contribution requires a somewhat more elaborate calculation. The TM contribution to the trace of the Green's dyadic is

$$\mathrm{tr}\,\mathbf{g}^E = \frac{1}{\varepsilon}\frac{\partial}{\partial z}\frac{1}{\varepsilon'}\frac{\partial}{\partial z'}g^E + \frac{k^2}{\varepsilon\varepsilon'}g^E. \tag{4.9}$$

This differential structure has different forms depending on whether it acts on the pure exponential terms in g^E, or on the hyperbolic cosine in Eq. (3.8), namely, in the first case,

$$\frac{1}{\varepsilon}\frac{\partial}{\partial z}\frac{1}{\varepsilon'}\frac{\partial}{\partial z'} + \frac{k^2}{\varepsilon\varepsilon'} \to \frac{1}{\epsilon_a^2}(2\kappa_a^2 - \zeta^2\epsilon_a), \tag{4.10}$$

and in the second case,

$$\frac{1}{\varepsilon}\frac{\partial}{\partial z}\frac{1}{\varepsilon'}\frac{\partial}{\partial z'} + \frac{k^2}{\varepsilon\varepsilon'} \to -\frac{\zeta^2}{\epsilon_3}. \tag{4.11}$$

Except for that last exceptional case, combining this trace term with the dispersive term in the energy gives

$$\left(\epsilon_a + \frac{\zeta}{2}\frac{d\epsilon_a}{d\zeta}\right)\frac{1}{\epsilon_a^2}(2\kappa_a^2 - \zeta^2\epsilon_a) = 2\kappa_a\left(\kappa'_a - \frac{\zeta}{2}\frac{d\kappa'_a}{d\zeta}\right). \tag{4.12}$$

Thus, in region 1, the contribution to the TM energy is

$$\mathcal{E}^{\rm TM,1} = \frac{1}{4\pi^2}\int_0^\infty kdk\int_0^\infty d\zeta\left(\kappa'_1 - \frac{\zeta}{2}\frac{d\kappa'_1}{d\zeta}\right)\frac{\partial}{\partial\kappa'_1}\ln\Delta_{\rm TM}, \tag{4.13}$$

where $\Delta_{\rm TM}$ differs from $\Delta_{\rm TE}$ by replacing κ_a by $\kappa'_a = \kappa_a/\epsilon_a$ except in the exponents, that is, $r_{\rm TE} \to r_{\rm TM}$. Similarly, in region 2,

$$\mathcal{E}^{\mathrm{TM},2} = \frac{1}{4\pi^2}\int_0^\infty k dk \int_0^\infty d\zeta\left(\kappa_2' - \frac{\zeta}{2}\frac{d\kappa_2'}{d\zeta}\right)\frac{\partial}{\partial \kappa_2'}\ln\Delta_{\mathrm{TM}}. \tag{4.14}$$

In region 3, however, we have to take into account the special case (4.11). It is most convenient then to regard κ_3 and κ_3' as independent, in which case we can write

$$\mathcal{E}^{\mathrm{TM},3} = \frac{1}{4\pi^2}\int_0^\infty k dk \int_0^\infty d\zeta\left[\left(\kappa_3' - \frac{\zeta}{2}\frac{d\kappa_3'}{d\zeta}\right)\frac{\partial}{\partial \kappa_3'} - \frac{\zeta}{2}\frac{d\kappa_3}{d\zeta}\frac{\partial}{\partial\kappa_3}\right]\ln\Delta_{\mathrm{TM}}. \tag{4.15}$$

Thus, the total TM contribution is

$$\mathcal{E}^{\mathrm{TM}} = \frac{1}{4\pi^2}\int_0^\infty dk k\left\{\int_0^\infty d\zeta\left[\kappa_1'\frac{\partial}{\partial\kappa_1'} + \kappa_2'\frac{\partial}{\partial\kappa_2'} + \kappa_3'\frac{\partial}{\partial\kappa_3'}\right]\times\ln\Delta_{\mathrm{TM}} - \int_k^\infty d\kappa_3\frac{\zeta}{2}\left(\frac{\partial}{\partial\kappa_3} + \frac{d\kappa_3'}{d\kappa_3}\frac{\partial}{\partial\kappa_3'} + \frac{d\kappa_2'}{d\kappa_3}\frac{\partial}{\partial\kappa_2'} + \frac{d\kappa_1'}{d\kappa_3}\frac{\partial}{\partial\kappa_1'}\right)\ln\Delta_{\mathrm{TM}}\right\}. \tag{4.16}$$

The first term here is actually zero, because the differential operator annihilates Δ_{TM}, since

$$\kappa_1'\frac{\partial}{\partial\kappa_1'}r_{\mathrm{TM}} = -\kappa_3'\frac{\partial}{\partial\kappa_3'}r_{\mathrm{TM}}, \tag{4.17}$$

and so we obtain exactly the Lifshitz result

$$\mathcal{E}^{\mathrm{TM}} = \frac{1}{4\pi^2}\int_0^\infty k dk\int_0^\infty d\zeta\ln\Delta_{\mathrm{TM}}. \tag{4.18}$$

V. CONCLUSIONS

Ordinarily one calculates the Casimir-Lifshitz free energy directly from the pressure, or from an equivalent variational approach. Therefore, it was not obvious how the dispersive term present in the energy in order to have the required balance between energy and momentum, as in the electromagnetic energy-momentum tensor, plays a role. Earlier we had suggested [3] that such a term simply be omitted. However, we now see that the dispersive term is precisely what is needed to achieve agreement between the different formulations of the energy, and that the dispersive term provides the Jacobian factor necessary to derive the Lifshitz free energy from the expectation value of the electromagnetic energy.

The following point, related to the possibilities of experimental observations, should be noticed: As we have seen, a characteristic property of dispersion is that the factor $d(\varepsilon\omega)/d\omega$ occurs in the energy and not in the pressure or the stress. This has a bearing on the famous Abraham-Minkowski energy-momentum problem. As it is known, an important experiment in this area is the Jones-Richards radiation pressure experiment [13], showing how the effective pressure against a mirror immersed in a liquid varies with respect to the refractive index (cf. also the follow-up experiment of Jones and Leslie [14]). The book by Jones [15] contains a nice exposition of these very accurate experiments. The electrostrictive forces do not contribute to the radiation pressure.

Does this imply that electrostrictive forces in a liquid are generally nonobservable? Not quite so, although a difficulty is that at thermal equilibrium, the electrostrictive forces give rise to elastic pressures in the liquid, acting in the opposite direction. There are ways to overcome this difficulty, however. One option is to proceed as in the Goetz-Zahn nonequilibrium experiment [16,17]; cf. also the detailed discussion on this experiment in Ref. [18], p. 149. One applies an electric field with high frequency ω between two condenser plates in a liquid, and measures the attractive force between the plates for instance by means of a piezoelectric transducer. The point is that ω must be so high that the elastic pressure does not have time to built itself up. The critical parameter here is thus the velocity of sound in the liquid.

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APPENDIX: CONSTANCY OF ENERGY FOR CONDUCTING PLATES

Consider the case of parallel perfect conducting plates separated by a nondispersive medium with dielectric constant ϵ. The TE Green's function is [obtained by taking $\kappa_{1,2}\to\infty$ in Eq. (3.8)]

$$g^H = \frac{1}{2\kappa}\left\{e^{-\kappa|z-z'|} + \frac{2\cosh\kappa(z-z') - e^{\kappa(z+z')} - e^{-\kappa(z+z'-2a)}}{e^{2\kappa a}-1}\right\}, \tag{A1}$$

where $\kappa^2 = k^2 + \zeta^2\epsilon$. The TE energy density is given by

$$u^{\mathrm{TE}} = \tfrac{1}{2}\epsilon\langle E_y^2\rangle + \tfrac{1}{2}\langle H_x^2 + H_z^2\rangle, \tag{A2}$$

where

$$\langle E_y^2\rangle = -\int_{-\infty}^\infty\frac{d\zeta}{2\pi}\int\frac{(d\mathbf{k}_\perp)}{(2\pi)^2}\zeta^2 g^H(z,z), \tag{A3a}$$

$$\langle H_x^2 + H_z^2\rangle = \int_{-\infty}^\infty\frac{d\zeta}{2\pi}\int\frac{(d\mathbf{k}_\perp)}{(2\pi)^2}(\partial_z\partial_{z'} + k^2)g^H(z,z')|_{z'=z}. \tag{A3b}$$

Then, we easily see that

$$u^{\mathrm{TE}} = \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} \int_0^{\infty} \frac{dkk}{2\pi} \frac{1}{\kappa} \frac{1}{e^{2\kappa a} - 1} \times \left[-\epsilon\zeta^2 - \frac{k^2}{2}(e^{2\kappa z} + e^{-2\kappa(z-a)}) \right]. \quad \text{(A4)}$$

The TM Green's function for perfectly conducting plates has a similar form,

$$g^E = \frac{1}{2\kappa'} \left\{ e^{-\kappa|z-z'|} + \frac{2\cosh\kappa(z-z') + e^{\kappa(z+z')} + e^{-\kappa(z+z'-2a)}}{e^{2\kappa a} - 1} \right\}, \quad \text{(A5)}$$

and the corresponding energy density is

$$\begin{aligned} u^{\mathrm{TM}} &= \frac{1}{2}\epsilon\langle E_x^2 + E_z^2 \rangle + \frac{1}{2}\langle H_y^2 \rangle \\ &= \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} \int \frac{(d\mathbf{k}_\perp)}{(2\pi)^2} \left[\frac{1}{2\epsilon}(\partial_z \partial_{z'} + k^2) g^E(z, z')|_{z'=z} - \frac{1}{2\zeta^2\epsilon^2}(\partial_z^2 - k^2)(\partial_{z'}^2 - k^2) g^E(z, z')|_{z'=z} \right] \\ &= \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} \int_0^{\infty} \frac{dkk}{2\pi} \frac{1}{\kappa} \frac{1}{e^{2\kappa a} - 1} \times \left[-\zeta^2\epsilon + \frac{k^2}{2}(e^{2\kappa z} + e^{-2\kappa(z-a)}) \right]. \quad \text{(A6)} \end{aligned}$$

The z-dependent terms exactly cancel between Eqs. (A4) and (A6) and the remaining terms are equal, and sum to the usual Casimir energy density,

$$u = -\frac{1}{3\pi^2\sqrt{\epsilon}} \int_0^{\infty} d\kappa\kappa^3 \frac{1}{e^{2\kappa a} - 1} = -\frac{\pi^2}{720\sqrt{\epsilon}a^4}. \quad \text{(A7)}$$

This cancellation, resulting in the constancy of the energy density, is rather special, however. It does not occur if dispersion is present, $d\epsilon/d\zeta \neq 0$, in which case the local energy density has the nonconstant form

$$u = \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} \int \frac{(d\mathbf{k}_\perp)}{(2\pi)^2} \frac{1}{\kappa} \frac{1}{e^{2\kappa a} - 1} \left\{ -\zeta^2\epsilon \left(2 + \frac{\zeta}{\epsilon}\frac{d\epsilon}{d\zeta} \right) + \frac{k^2}{2}\frac{\zeta}{\epsilon}\frac{d\epsilon}{d\zeta}(e^{2\kappa z} + e^{-2\kappa(z-a)}) \right\}. \quad \text{(A8)}$$

The cancellation also cannot occur for dielectric media constituting regions 1 and 2, since the TE and TM reflection coefficients are then different. Nevertheless, we note that the z integral of the spatially varying part of the energy density is a constant, independent of a, and so does not contribute to the force on the plates. This is just as it occurs for a nonconformally coupled massless scalar field confined between Dirichlet plates.

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