Formulation of supersymmetry on a lattice as a representation of a deformed superalgebra

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The lattice superalgebra of the link approach is shown to satisfy a Hopf algebraic supersymmetry where the difference operators are introduced as momentum operators. The violation of the Leibniz rule for the lattice difference operators is interpreted as the coproduct structure of a (quasi)triangular Hopf algebra and the associated field theory is consistently defined as a braided quantum field theory. An algebraic formulation of the path integral is defined perturbatively and the corresponding Ward-Takahashi identities can be derived on the lattice. The claimed inconsistency of the link approach related to an ordering ambiguity for the product of fields is solved by introducing an almost trivial braiding structure corresponding to the triangular structure of the Hopf algebraic superalgebra. This can be seen as a generalization of the spin-statistics relation on the lattice. For the consistency of this braiding structure of fields a grading of the momentum operator is required.

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I. INTRODUCTION

We believe that a constructive definition of regularized supersymmetric field theories is becoming increasingly important. There are several reasons: first, from a phenomenological point of view, there is the possibility that superparticles might be discovered at LHC in the near future. If supersymmetry is experimentally confirmed, obviously we need a constructive formulation of supersymmetric field theories that can provide a basis for the numerical study of nonperturbative supersymmetry phenomenology. It is natural to expect that a lattice formulation of supersymmetry may play a crucial role just as lattice QCD plays an important role as the only numerical tool for strong interaction phenomenology. We expect that a fermionic counterpart of the QCD region may exist in the new energy scale.

Secondly, from a more theoretical point of view, it is not obvious that the lattice fermion problem [1,2] is well understood from the lattice regularization point of view. There is, however, a general consensus that the chiral fermion problem is solved for lattice QCD [3–5]. One may say that the species doublers of a chiral fermion on a lattice are lattice artifacts so that it would be better if they were not there. However it was also claimed that these extra species doubler degrees of freedom are needed as they correspond to the extra degrees of freedom of extended twisted supersymmetry [6–9]. It was shown that twisted supersymmetry can be derived by the Dirac-Kähler twisting procedure [10] in any dimensions: $\mathcal{N} = 2$ in two dimensions, $\mathcal{N} = 4$ in three dimensions, and $\mathcal{N} = 4$ in four dimensions which coincides with the twisted algebra derived by Marcus [11]. In these formulations the fermionic internal degrees of freedom can be defined semilocally on a lattice to be compatible with the differential form nature of the Dirac-Kähler fermions [6–9,12,13]. This type of correspondence had been anticipated in older papers [14]. However it turns out that the lattice Dirac-Kähler fermion formulation [15] is equivalent to the staggered fermion formulation [16], with the introduction of a mild noncommutativity between differential forms and fields to accommodate a modified Leibniz rule for the lattice difference operator. These results suggest that the regularization of fermions on a lattice naturally leads to the necessity of introducing supersymmetry in a fundamental way.

In the path integral formulation of field theory, fermionic fields are treated as odd Grassmann variables and thus have an anticommuting nature compatible with the spin and statistics theorem which originates from Lorentz invariance [17]. Since Lorentz invariance is broken on the lattice it is not obvious that the anticommuting nature of the fermion fields at the scale of the lattice constant is a mandatory requirement. In this paper we explore a possibility that the commuting and anticommuting nature of fields are modified with the introduction of a mild noncommutativity compatible with the Leibniz rule of the difference operator on the lattice. This may be interpreted as a generalization of the spin and statistics theorem on the lattice.

Attempts to formulate supersymmetric theories on a lattice have a long history. Since the lattice is not invariant under infinitesimal translations, it is difficult not to break supersymmetry which includes in its superalgebra the infinitesimal translation generator. To overcome this difficulty, various approaches and formulations have been proposed so far. If we focus on the treatment of the algebraic aspects of lattice supersymmetry, there are essentially three possible approaches:

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(1) superalgebra is kept on the lattice approximately, with corrections of the order of the lattice spacing;
(2) only one supersymmetry charge, which is nilpotent and does not contain the momentum operator, is preserved exactly on the lattice; or
(3) the continuum superalgebra is deformed into a lattice version of superalgebra, keeping exactly all supersymmetry charges.

A long list of attempts following the first approach, where supersymmetry is kept up to lattice corrections, is reviewed in [18–20] (see also previous references therein). There are also some later developments along this line [21–24]. In this approach one has to check if and how supersymmetry is restored in the continuum limit. In order to see the recovery of supersymmetry for the whole range of coupling constants, it is inevitable to find reliable methods for numerical analysis. Influenced by the developments of the renormalization group analysis and the Ginsparg-Wilson relation of chiral symmetry analysis for lattice QCD, there have been recent systematic applications of this method to supersymmetric models [25–27].

In the second approach [28–36], only one or at most two nilpotent supercharges are preserved exactly on the lattice. It is again particularly important to examine how the full continuum supersymmetry is recovered in the continuum limit. In this approach the recovery of supersymmetry in the infrared region is expected due to the suppression of supersymmetry breaking operators as a consequence of the partial preservation of the supersymmetry algebra [36,37]. In fact in some specific cases it can be shown that the part of supersymmetry which is realized exactly on the lattice is enough to avoid nonmanageable fine-tuning to preserve supersymmetry in the continuum limit [28,30–32,36,38]. One can then use such models as constructive definitions of the corresponding continuum supersymmetric models. We should, however, note that in extracting the sector of the superalgebra which can be preserved exactly on the lattice, extended supersymmetry and its twisting [10,39] play an important role.

In the third approach [6–8,12,13,34,40], one defines a lattice version of superalgebra in which the momentum operator of the continuum superalgebra is replaced by the finite difference operator on the lattice. This seems the most natural and naive deformation of the continuum algebra. Nevertheless, it is not straightforward due to the following obvious reason. In fact since finite difference operators are not infinitesimal operators, and hence they are not strictly speaking elements of an algebra, the deformed "superalgebra" is not strictly an algebra in the usual sense. This is actually not simply a question of terminology, but is crucial in the formulation. Namely, the finite difference operators do not obey the Leibniz rule, which just amounts to saying that they are not elements of an algebra. On the other hand, because of the nilpotency of Grassmann parameters, "normal" supercharges would always obey exactly the Leibniz rule. This mismatch of natures between finite difference operators and supercharges makes the naive realization of a lattice-deformed superalgebra in the above sense difficult. In this paper we follow a formulation, originally proposed in [6–8], that gives an answer to this difficult problem. We shall denote this formulation in general as the *link approach* but we shall use the term *DKKN formalism* (where DKKN is D'Adda, Kanamori, Kawamoto, and Nagata) when we refer strictly to the algebraic aspects of the formulation. In this approach, the notion of *modified Leibniz rules* is introduced to overcome the difficulty raised by the violation of the Leibniz rule. However, it is not clear whether this is sufficient to make the approach entirely consistent. In fact, it is first of all rather unclear whether a symmetry obeying the modified Leibniz rule, being different from the standard Lie algebraic symmetry and unsuitable to be used in a functional integral, is actually a symmetry of a quantum field theory. We will present an answer to this issue here, by showing that the deformed algebra forms a *Hopf algebra* which generalizes the Lie algebraic symmetry, and that a quantum field theory which has a Hopf algebraic symmetry can be constructed at least perturbatively within the framework known as *braided quantum field theory* (BQFT) [41]. It is important for this purpose to identify the DKKN superalgebra as a rigorous Hopf algebra, namely, to prove that it obeys the set of axioms defining a Hopf algebra. It is also crucial to determine a *braiding* structure on the representation space of the Hopf algebra. With the use of the BQFT formulation, we can then derive a series of Ward-Takahashi identities corresponding to the Hopf algebraic symmetry, thus giving to the deformed symmetry a clear interpretation as a physical symmetry of the quantum theory.

The second and most crucial aspect to be clarified in the link approach is about the claim of inconsistency raised in Ref. [42], where it is argued that the modified Leibniz rule inevitably leads to an ordering ambiguity caused essentially by the asymmetric nature of the deformation. In order to clarify the ordering problem, a matrix formulation for a one-dimensional model was explicitly analyzed. It was shown that there is no ambiguity at the superfield level but the problem remains at the component level [40]. Fortunately, our Hopf algebraic description also resolves this problem: given the Hopf algebraic symmetry together with an appropriate treatment of a braiding structure of BQFT, we will show that this difficulty no longer exists. The braiding structure, which could be interpreted as a kind of *generalized statistics* or a *mild noncommutativity*, is again the key ingredient for this argument. Gauge theories however will require a separate treatment, which is outside the scope of this paper.

Recently it has been pointed out that the orbifold construction, used to formulate supersymmetric theories on the lattice with one supersymmetry exactly realized, and the construction leading to the link approach are essentially equivalent [34,35]. It had been already remarked that a particular choice of the "shift" parameters appearing in the link approach can make the scalar supercharge in twisted supersymmetry shiftless and that with this choice the $\mathcal{N} = D = 2$ super Yang-Mills action of the link construction coincides with the one of the orbifold construction [7]. It was, however, argued in [34,35] that supercharges carrying shifts lead to a violation of supersymmetry, due to the nonstandard definition of the shifted (anti)commutators. Therefore in the so-called symmetric choice of the parameters in the link approach, as all supercharges carry shifts, no supersymmetry is exact, although the corresponding action has larger discrete chiral and spacetime symmetries than that of the orbifold construction [35]. This is exactly the point we want to stress in the present paper: all supercharges of the link construction preserve exactly a deformed lattice supersymmetry in which shifts play a crucial role.

Recently a no-go theorem has been proved [43], stating that under the conditions of translational invariance and locality on the lattice a proper definition of products of lattice fields naturally leads to a violation of the Leibniz rule for differential operators on the lattice. In a separate approach to the problem it was also shown that the extension to the supersymmetric case of a blocked symmetry transformation realizing the Ginsparg-Wilson relation leads to a SLAC-type derivative [44] as the only consistent solution [27], in agreement with the aforementioned no-go theorem. It is known, however, that the SLAC derivative is a highly nonlocal differential operator and it should be noted in this respect that the modified Leibniz rule or equivalently the deformation of the supersymmetry algebra is compatible with the results of those analyses.

This paper is organized as follows. In Sec. II we shall briefly review the formalism of the link approach in a somewhat general form, including the difficulties mentioned above. In Sec. III we will describe how to treat the superalgebra on the lattice as a deformed/modified algebra in the scheme of the Hopf algebra theory. We shall list all the necessary and sufficient formulas which form the basic structure of a Hopf algebra. We also derive the explicit form of the braiding which is necessary for the full consistency of the representation. Twisting of our Hopf algebra will be discussed, too, which naturally explains why our lattice theory should have a braiding or a mild noncommutativity. Using the general formulation of BQFT we shall illustrate how a quantum field theory endowed with this Hopf algebraic symmetry can be defined perturbatively. As a concrete example we show a two-dimensional $\mathcal{N} = 2$ Wess-Zumino model. Some conclusions and a brief discussion of some remaining issues are given in the last section. In the appendix we give a concise summary of the Hopf algebra to fix the notation and the terminology used in the text.

II. GENERAL FRAMEWORK OF THE LINK FORMULATION OF THE DIRAC-KÄHLER TWISTED SUPERSYMMETRY ON A LATTICE

A. General formalism

The link approach [6–8] to lattice supersymmetry is based on the simple assumption that the continuum superalgebra[1]

$$\{Q_A, Q_B\} = 2\tau^\mu_{AB} P_\mu, \qquad [Q_A, P_\mu] = [P_\mu, P_\nu] = 0 \tag{2.1}$$

has some natural counterpart on the lattice

$$\{Q^{\rm lat}_A, Q^{\rm lat}_B\} = 2\tau^\mu_{AB} P^{\rm lat}_\mu, \qquad [Q^{\rm lat}_A, P^{\rm lat}_\mu] = [P^{\rm lat}_\mu, P^{\rm lat}_\nu] = 0. \tag{2.2}$$

Here τ^μ_{AB} is just a constant coefficient, and $Q^{\rm lat}_A$ and $P^{\rm lat}_\mu$ are both understood as *deformed* operators on the lattice which reproduce Q_A and P_μ, respectively, in the naive continuum limit:

$$\lim_{a\to 0} = Q^{\rm lat}_A = Q_A, \qquad \lim_{a\to 0} = P^{\rm lat}_\mu = P_\mu. \tag{2.3}$$

We require that

$$\sum_x P^{\rm lat}_\mu \varphi(x) = 0 \tag{2.4}$$

for the "momentum" operator $P^{\rm lat}_\mu$ and any field $\varphi(x)$ on the lattice. This is because, in the continuum, superinvariance of the Lagrangian is up to a total divergence which vanishes under the integral, and the same thing should happen on the lattice to have exact supersymmetry, for which the above property is necessary. We shall also require translational invariance and (semi)locality for the operator $P^{\rm lat}_\mu$, so that the whole theory will satisfy these properties. Another possible requirement might be Hermiticity (or the reflection (Osterwalder-Schrader) positivity [45] of transfer matrices on the lattice [46]), but we do not force it here because it is related to the subtlety of the doubling phenomenon [1,2] for which we defer the discussion to later sections.

The simplest candidates for the momentum operator $P^{\rm lat}_\mu$ are the finite difference operators on the lattice,

$$P^{\rm lat}_\mu = i\partial_{\pm\mu}, \qquad i\partial^s_\mu, \qquad \text{etc.}, \tag{2.5}$$

where

$$\partial_{+\mu}\varphi(x) := \frac{1}{a}(\varphi(x + a\hat{\mu}) - \varphi(x)) \quad \text{(forward difference operator)}, \tag{2.6}$$

[1]The notation here is schematic; indices A and B could contain both spinor and internal d.o.f., and their conjugates as well.

$$\partial_{-\mu}\varphi(x) := \frac{1}{a}(\varphi(x) - \varphi(x - a\hat{\mu})) \tag{2.7}$$
(backward difference operator),

$$\partial^s_\mu\varphi(x) := \frac{1}{2}(\partial_{+\mu} + \partial_{-\mu})\varphi(x) = \frac{1}{2a}(\varphi(x + a\hat{\mu}) - \varphi(x - a\hat{\mu})) \tag{2.8}$$
(symmetric difference operator).

Here $\hat{\mu}$ is the unit vector in the direction of x^μ and a is the lattice constant.[2] The symmetric difference is self anti-Hermitian: $(\partial^s_\mu)^\dagger = -\partial^s_\mu$, while the others are anti-Hermitian conjugate to each other; $(\partial_{\pm\mu})^\dagger = -\partial_{\mp\mu}$. An immediate consequence of using these finite difference operators is that they break the Leibniz rule, or to put it mildly, obey the *modified* Leibniz rule as in

$$\begin{aligned}\partial_{\pm\mu}(\varphi_1\varphi_2)(x) &= \partial_{\pm\mu}\varphi_1(x)\varphi_2(x) + \varphi_1(x \pm a\hat{\mu})\partial_{\pm\mu}\varphi_2(x)\\ &= \partial_{\pm\mu}\varphi_1(x)\varphi_2(x \pm a\hat{\mu}) + \varphi_1(x)\partial_{\pm\mu}\varphi_2(x)\\ &= \partial_{\pm\mu}\varphi_1(x)\varphi_2(x) + \varphi_1(x)\partial_{\pm\mu}\varphi_2(x)\\ &\quad \pm a\partial_{\pm\mu}\varphi_1(x)\partial_{\pm\mu}\varphi_2(x),\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\partial^s_\mu(\varphi_1\varphi_2)(x) &= \partial^s_\mu\varphi_1(x)\varphi_2(x - a\hat{\mu}) + \varphi_1(x + a\hat{\mu})\partial^s_\mu\varphi_2(x)\\ &= \partial^s_\mu\varphi_1(x)\varphi_2(x + a\hat{\mu}) + \varphi_1(x - a\hat{\mu})\partial^s_\mu\varphi_2(x)\\ &= \partial^s_\mu\varphi_1(x)\varphi_2(x) + \varphi_1(x)\partial^s_\mu\varphi_2(x)\\ &\quad + \frac{a}{2}(\partial_{+\mu}\varphi_1(x)\partial_{+\mu}\varphi_2(x)\\ &\quad - \partial_{-\mu}\varphi_1(x)\partial_{-\mu}\varphi_2(x)).\end{aligned} \tag{2.10}$$

Although in each of these equations the term breaking the Leibniz rule is proportional to the lattice constant a, it is not granted in general that it will vanish in the continuum limit, due to the contribution of the momentum from the cutoff scale regions: $\partial_{\pm\mu}\varphi(x) \sim \mathcal{O}(1/a)$ [24]. Note also that the last term in (2.10) is proportional to a total difference $\partial_{-\mu}(\partial_{+\mu}\varphi_1(x)\partial_{+\mu}\varphi_2(x))$, so that one may think that this term is irrelevant when the summation over the lattice sites is taken. But this is true only for products of two fields, so that it might be a good property only in a free theory, not in an interacting case. (Even in the free case there is an associated doubler problem for the anti-Hermitian symmetric difference. We will see this later in more detail.) One might also try to impose constraints on fields to make the breaking terms vanish, but this would only result in a nonlocal formulation [18]. Thus, as long as we use the simple difference operators (2.5), we cannot naively neglect the breaking of the Leibniz rule. In fact, it is more generally shown [27,43] that we have to admit breaking of the Leibniz rule of any momentum operators on a lattice, unless we allow nonlocal operators like the so-called SLAC derivative [44] or, alternatively, many multi-flavors. These facts are already enough for the lattice counterpart of the superalgebra (2.2) to lose the nature of strict Lie superalgebra, which is the most evident and crucial obstacle to formulating supersymmetry on a lattice as based entirely on a superalgebra.

One possibility to overcome this difficulty is to interpret the superalgebra on the lattice (2.2) as a *deformed* Lie superalgebra with a deformation parameter that vanishes in the continuum limit. This is in fact the basic strategy in the link approach as we can see below.

Since, in the first equation of (2.2), the right-hand side obeys the modified Leibniz rule, it is natural to deform the algebra so that the generators in the left-hand side also obey a modified Leibniz rule. In the link approach, the central ansatz is that the supercharge $Q^{\rm lat}_A$ obeys a modified Leibniz rule of the form[3]

$$Q^{\rm lat}_A(\varphi_1\varphi_2)(x) = Q^{\rm lat}_A\varphi_1(x)\varphi_2(x) + (-1)^{|\varphi_1|}\varphi_1(x + a_A)Q^{\rm lat}_A\varphi_2(x), \tag{2.11}$$

where $x + a_A$ is to be interpreted as denoting the coordinate of an additional lattice site which merges with x in the naive continuum limit. Introducing a translation or shift operator T_{a_A} in the "fundamental" representation such that

$$T_{a_A}\varphi(x) = \varphi(x + a_A), \tag{2.12}$$

the Leibniz rule can be written as

$$\begin{aligned}Q^{\rm lat}_A(\varphi_1\varphi_2)(x) &= Q^{\rm lat}_A\varphi_1(x)\varphi_2(x) + (-1)^{|\varphi_1|}T_{a_A}(\varphi_1 T^{-1}_{a_A}Q^{\rm lat}_A\varphi_2)(x),\\ \text{i.e. } T^{-1}_{a_A}Q^{\rm lat}_A(\varphi_1\varphi_2)(x) &= (T^{-1}_{a_A}Q^{\rm lat}_A\varphi_1)(x)\varphi_2(x - a_A)\\ &\quad + (-1)^{|\varphi_1|}\varphi_1(x)(T^{-1}_{a_A}Q^{\rm lat}_A\varphi_2)(x),\end{aligned} \tag{2.13}$$

showing that the operator $T^{-1}_{a_A}Q^{\rm lat}_A$ obeys a slightly different modified Leibniz rule. We may also write it in a symmetric form as

$$\begin{aligned}T^{-1/2}_{a_A}Q^{\rm lat}_A(\varphi_1\varphi_2)(x) &= (T^{-1/2}_{a_A}Q^{\rm lat}_A\varphi_1)(x)\varphi_2(x - a_A/2)\\ &\quad + (-1)^{|\varphi_1|}\varphi_1(x + a_A/2)\\ &\quad \times (T^{-1/2}_{a_A}Q^{\rm lat}_A\varphi_2)(x),\end{aligned} \tag{2.14}$$

which is still a modified version of the Leibniz rule. We could have begun with a little more generalized modification such as

[2]We always write the lattice constant a explicitly in this paper unless otherwise specified.

[3]Here $|\varphi|$ is 0 or 1, depending on whether φ is bosonic or fermionic, respectively.

$$Q_A^{\rm lat}(\varphi_1\varphi_2)(x) = Q_A^{\rm lat}\varphi_1(x)\varphi_2(x+a_A^r) + (-1)^{|\varphi_1|}\varphi_1(x+a_A^l)Q_A^{\rm lat}\varphi_2(x), \quad (2.15)$$

but, with a redefinition of $Q_A^{\rm lat}$ to $T_{a_A^r}^{-1}Q_A^{\rm lat}$, this is always equivalent to the original one (2.11), which can be seen in a similar fashion as in the above. Such a redefinition only produces a total difference in the algebra (2.2), hence the original form (2.11) suffices without loss of generality.

Fields in the fundamental representation of the translation/shift operator (2.12) could be interpreted as normal functions on the lattice. If, by contrast, we introduce the "adjoint" representation of the translation/shift operator as in

$$T_{a_A}\varphi(x)T_{a_A}^{-1} = \varphi(x+a_A), \quad (2.16)$$

the Leibniz rule (2.11) can be written as

$$Q_A^{\rm lat}(\varphi_1\varphi_2)(x) = Q_A^{\rm lat}\varphi_1(x)\varphi_2(x) + (-1)^{|\varphi_1|}T_{a_A}\varphi_1(x)T_{a_A}^{-1}Q_A^{\rm lat}\varphi_2(x),$$

$$\text{i.e. } T_{a_A}^{-1}Q_A^{\rm lat}(\varphi_1\varphi_2)(x) = T_{a_A}^{-1}Q_A^{\rm lat}\varphi_1(x)\varphi_2(x) + (-1)^{|\varphi_1|}\varphi_1(x)T_{a_A}^{-1}Q_A^{\rm lat}\varphi_2(x). \quad (2.17)$$

Now we can see that the operator $T_{a_A}^{-1}Q_A^{\rm lat}$ obeys the usual exact Leibniz rule. We could write this further as

$$T_{a_A}^{-1}Q_A^{\rm lat}(\varphi_1\varphi_2)(x)T_{a_A'} = T_{a_A}^{-1}Q_A^{\rm lat}\varphi_1(x)T_{a_A'}\varphi_2(x-a_A') + (-1)^{|\varphi_1|}\varphi_1(x)T_{a_A}^{-1}Q_A^{\rm lat}\varphi_2(x)T_{a_A'}, \quad (2.18)$$

which shows that the operator $T_{a_A}^{-1}Q_A^{\rm lat}\overleftarrow{T}_{a_A'}$ (where the arrow denotes the multiplication from the right) follows a different Leibniz rule. We would thus again find that the original modified Leibniz rule (2.11) itself is equivalent to the more general form (2.15), and even to the usual Leibniz rule (2.17) with suitable redefinitions of the operator $Q_A^{\rm lat}$. Notice that in this adjoint representation such a redefinition would change the algebra (2.2) in a nontrivial way, except for the case $a_A^r = a_A^l$ in which the algebra, or more precisely the momentum operator, would remain unchanged up to a total difference so that the momentum operator still obeys the modified Leibniz rule. In other words, unless $a_A^r = a_A^l$, we have a possibility to redefine both the operators $Q_A^{\rm lat}$ and $P_\mu^{\rm lat}$ so as to follow the usual Leibniz rule for which the usual representation exists. This fact may play an important role for an explicit representation of the lattice superalgebra. Another point to observe is that, in the adjoint representation, the field itself should be identified as an operator or a matrix which formally belongs, together with the shift operator, to an algebra (which would be a universal enveloping algebra of a Lie superalgebra, just as in a canonical quantization scheme). Left/right multiplication of a field by the operator T_{a_A} produces a new field whose commutation properties with other fields are different from the original one. For instance, suppose $\varphi_1(x)$ and $\varphi_2(x)$ commute with each other: $\varphi_1(x)\varphi_2(x) = \varphi_2(x)\varphi_1(x)$. Then $T_{a_A}\varphi_1(x)$ and $\varphi_2(x)$ no longer commute strictly, but commute with a shift in the sense that $T_{a_A}\varphi_1(x)\varphi_2(x) = T_{a_A}\varphi_2(x)T_{a_A}^{-1}T_{a_A}\varphi_1(x) = \varphi_2(x+a_A)T_{a_A}\varphi_1(x)$. This type of mild noncommutativity does not simply appear in the fundamental case since $(T_{a_A}\varphi_1)\times(x)\varphi_2(x) = \varphi_2(x)(T_{a_A}\varphi_1)(x)$. We will see in the next section how the modified Leibniz rules in the fundamental representation (2.11), (2.13), and (2.14) can be more systematically treated in the framework of a Hopf algebraic symmetry. Here we continue the discussion on the adjoint representation case.

Suppose now that $Q_A^{\rm lat}$ also belongs to the same algebra formed by T_{a_A} and $\varphi(x)$. The fact that the combination $T_{a_A}^{-1}Q_A^{\rm lat}$ in (2.17) follows the usual Leibniz rule thus motivates us to write formally

$$T_{a_A}^{-1}Q_A^{\rm lat} =: \hat{Q}_A^{\rm lat} \equiv i\,{\rm ad}(\hat{\mathcal{Q}}_A^{\rm lat}), \quad (2.19)$$

which acts on a field as[4]

$$T_{a_A}^{-1}Q_A^{\rm lat}\varphi(x) = i\,{\rm ad}(\hat{\mathcal{Q}}_A^{\rm lat})\varphi(x) := i[\hat{\mathcal{Q}}_A^{\rm lat}, \varphi(x)]_{(-1)^{|\varphi|+1}},$$

$$\text{or} \quad Q_A^{\rm lat}\varphi(x) = iT_{a_A}[\hat{\mathcal{Q}}_A^{\rm lat}, \varphi(x)]_{(-1)^{|\varphi|+1}}. \quad (2.20)$$

It can also be written as

$$\begin{aligned} Q_A^{\rm lat}\varphi(x) &= iT_{a_A}\hat{\mathcal{Q}}_A^{\rm lat}\varphi(x) - (-1)^{|\varphi|}iT_{a_A}\varphi(x)T_{a_A}^{-1}T_{a_A}\hat{\mathcal{Q}}_A^{\rm lat} \\ &= i\mathcal{Q}_A^{\rm lat}\varphi(x) - (-1)^{|\varphi|}\varphi(x+a_A)i\mathcal{Q}_A^{\rm lat} \\ &=: i[\mathcal{Q}_A^{\rm lat}, \varphi(x)]^{\rm lat}_{(-1)^{|\varphi|+1}} =: i\,{\rm ad}^{\rm lat}(\mathcal{Q}_A^{\rm lat})\varphi(x), \end{aligned} \quad (2.21)$$

where $\mathcal{Q}_A^{\rm lat} := T_{a_A}\hat{\mathcal{Q}}_A^{\rm lat}$. In this last equation we have defined a kind of deformed adjoint operation ${\rm ad}^{\rm lat}$ which was referred to as the *shifted* (anti)commutator in the DKKN formalism. It illustrates the general fact that an operator which obeys a modified Leibniz rule could be expressed by a shifted (anti)commutator. We have, however, introduced objects like $\hat{\mathcal{Q}}_A^{\rm lat}$, $\mathcal{Q}_A^{\rm lat}$, and their (anti)commutator (${\rm ad}\,\hat{\mathcal{Q}}_A^{\rm lat}$) and shifted (anti)commutator ${\rm ad}^{\rm lat}(\mathcal{Q}_A^{\rm lat})$, respectively, only in a formal way; we have neither specified explicit forms nor even justified the existence of them. So far we have only found that $T_A^{-1}Q_A^{\rm lat} = \hat{Q}_A^{\rm lat}$ obeys the usual Leibniz rule, so that it would be regarded as a normal operator. The operator $Q_A^{\rm lat}$ would in turn be expressed as $Q_A^{\rm lat} = T_{a_A}\hat{Q}_A^{\rm lat}$. The point here is as follows: as we mentioned above, we assume that the shift parameter a_A reduces to zero in the naive continuum limit. Correspondingly the translation/shift operator T_{a_A} would go to unity in that limit: $T_{a_A} \rightarrow$

[4]Here in Eq. (2.20) $[A, B]_\pm := AB \pm BA$.

$\mathbb{1}$, and thus the formal expression $Q_A^{\rm lat} = iT_{a_A}{\rm ad}(\hat{\mathcal{Q}}_A^{\rm lat}) = i{\rm ad}^{\rm lat}(\mathcal{Q}_A^{\rm lat})$ reduces to the normal (anti)commutator $Q_A = i{\rm ad}(\mathcal{Q}_A)$. This implies that normal (anti)commutators in the continuum, if used in any algebraic expressions, should be simply replaced with the shifted (anti)commutators on the lattice to accommodate the modified Leibniz rule (2.11). This reminds us of the correspondence principle between the Poisson brackets in a classical theory and commutators in a quantum theory. We are motivated by this analogy to think of the lattice version of superalgebra of a "quantization" of the continuum superalgebra. This viewpoint for the formulation is discussed in the next section.

Let us move on to the algebra (2.2). Here, for generality, we consider the modified Leibniz rule (2.15). The left-hand side of the algebra acts on a product $\varphi_1\varphi_2$ as follows:

$$\begin{aligned}\{Q_A^{\rm lat}, Q_B^{\rm lat}\}(\varphi_1\varphi_2)(x) &= \{Q_A^{\rm lat}, Q_B^{\rm lat}\}\varphi_1(x)\varphi_2(x + a_A^r + a_B^r) \\ &\quad + \varphi_1(x + a_A^l + a_B^l)\{Q_A^{\rm lat}, Q_B^{\rm lat}\}\varphi_2(x) \\ &= \sum_\mu 2\tau_{AB}^\mu(P_\mu^{\rm lat}\varphi_1(x)\varphi_2(x + a_A^r + a_B^r) \\ &\quad + \varphi_1(x + a_A^l + a_B^l)P_\mu^{\rm lat}\varphi_2(x)),\end{aligned} \tag{2.22}$$

while the right-hand side gives

$$\begin{aligned}2\tau_{AB}^\mu P_\mu^{\rm lat}(\varphi_1\varphi_2)(x) &= \sum_\mu 2\tau_{AB}^\mu(P_\mu^{\rm lat}\varphi_1(x)\varphi_2(x + a^r\hat{\mu}) \\ &\quad + \varphi_1(x + a^l\hat{\mu})P_\mu^{\rm lat}\varphi_2(x)),\end{aligned} \tag{2.23}$$

where a^l and a^r are, depending on the choice of $P_\mu^{\rm lat}$,

$$(a^l, a^r) = \begin{cases}(\pm a, 0) \quad \text{or} \quad (0, \pm a) & \text{for } P_\mu^{\rm lat} = i\partial_{\pm\mu}, \\ (+a, -a) \quad \text{or} \quad (-a, +a) & \text{for } P_\mu^{\rm lat} = i\partial_\mu^s.\end{cases} \tag{2.24}$$

The algebra (2.2) requires these two equations to be equal. As a necessary condition, we find that the coefficient τ_{AB}^μ should have the form

$$\tau_{AB}^\mu = \tau_{AB}\delta^\mu{}_{\mu(A,B)} \tag{2.25}$$

for a certain vector index $\mu(A, B)$ uniquely determined by combinations of the spinor indices A and B. Namely, only one, at most, of D momenta[5] $P_1^{\rm lat}, \cdots, P_D^{\rm lat}$ could appear in the right-hand side of the algebra for each combination of A and B. Then the corresponding algebra in the continuum would be such as $\{Q_A, Q_B\} = 2\tau_{AB}P_{\mu(A,B)}$, which violates the Lorentz covariance of the algebra except in the case when A or B also have a "vector" index. It is known that the basis in which a supercharge has a vector index is the basis of *twisted* supersymmetry [10,39]. In fact in the link formalism and also in the other approaches to lattice supersymmetry the twisted basis for the spinor indices is commonly adopted, and it is in the twisted version of extended supersymmetry that lattice formulations are constructed in all these approaches. Here we see that the twist is necessary for the algebraic consistency in the link formalism, but the reason why the twist, or extended supersymmetry itself from a more general point of view, comes naturally into the lattice formulation is deeply connected to the doubling phenomenon on the lattice. Namely, the doubler's d.o.f., sometimes called "taste," is used as that of the R-symmetry "flavor" of the extended supersymmetry, and is put on the lattice in such a way that the theory becomes free from the mismatch of d.o.f. between fermions and bosons. In this way the theory meets the nonperturbative criterion for supersymmetry, namely, that the partition function must be one. We will come to this point again later.

At any rate suppose the coefficient τ_{AB}^μ satisfies the condition (2.25). The condition that (2.22) coincides with (2.23) leads in this case to

$$a_A^l + a_B^l = a^l\hat{\mu}(A, B), \qquad a_A^r + a_B^r = a^r\hat{\mu}(A, B). \tag{2.26}$$

For the link formalism to work, it is necessary that these equations have consistent solutions for $a_A^{l,r}$. Recalling that for the operator $Q_A^{\rm lat}$, which satisfies the modified Leibniz rule (2.15), the combination $T_{a_A^l}^{-1}Q_A^{\rm lat}\overleftarrow{T}_{a_A^r}$ follows the usual Leibniz rule in the adjoint representation, we may consider the corresponding algebra

$$\{T_{a_A^l}^{-1}Q_A^{\rm lat}\overleftarrow{T}_{a_A^r}, T_{a_B^l}^{-1}Q_B^{\rm lat}\overleftarrow{T}_{a_B^r}\} = 2\tau_{AB}T_{a_A^l}^{-1}T_{a_B^l}^{-1}P_{\hat{\mu}(A,B)}^{\rm lat}\overleftarrow{T}_{a_A^r}\overleftarrow{T}_{a_B^r}, \tag{2.27}$$

where we have assumed that the condition (2.26) is met. The relation (2.26) assures that the operator in the right-hand side of this equation also follows the usual Leibniz rule. In fact, we find

$$T_{a_A^l}T_{a_B^l} = T_{a^l\hat{\mu}(A,B)}, \qquad T_{a_A^r}T_{a_B^r} = T_{a^r\hat{\mu}(A,B)}, \tag{2.28}$$

and we may write $P_{\hat{\mu}(A,B)}^{\rm lat}$ as, up to the lattice constant and other constant factors,

$$\begin{aligned}P_{\hat{\mu}(A,B)}^{\rm lat} &= {\rm Ad}(T_{a^l\hat{\mu}(A,B)}) - {\rm Ad}(T_{a^r\hat{\mu}(A,B)}), \\ \text{i.e. } P_{\hat{\mu}(A,B)}^{\rm lat}\varphi(x) &= T_{a^l\hat{\mu}(A,B)}\varphi(x)T_{a^l\hat{\mu}(A,B)}^{-1} \\ &\quad - T_{a^r\hat{\mu}(A,B)}\varphi(x)T_{a^r\hat{\mu}(A,B)}^{-1},\end{aligned} \tag{2.29}$$

where we define ${\rm Ad}(T_{a^l\hat{\mu}(A,B)})\varphi(x) = T_{a^l\hat{\mu}(A,B)}\varphi(x)\times T_{a^l\hat{\mu}(A,B)}^{-1}$ which should be compared with the definition of $({\rm ad}\hat{\mathcal{Q}}_A^{\rm lat})$ in (2.20). Then

[5]We denote the spacetime dimension as D.

$$T_{a_A^l}^{-1} T_{a_B^l}^{-1} P^{\rm lat}_{\hat{\mu}(A,B)} \tilde{T}_{a_A^r} \tilde{T}_{a_B^r} = -{\rm ad}(T^{-1}_{a^l \hat{\mu}(A,B)} T_{a^r \hat{\mu}(A,B)}),$$
$$\text{i.e. } T_{a_A^l}^{-1} T_{a_B^l}^{-1} P^{\rm lat}_{\hat{\mu}(A,B)} \varphi(x) T_{a_A^r} T_{a_B^r} = -(T^{-1}_{a^l \hat{\mu}(A,B)} T_{a^r \hat{\mu}(A,B)} \varphi(x) - \varphi(x) T^{-1}_{a^l \hat{\mu}(A,B)} T_{a^r \hat{\mu}(A,B)}), \tag{2.30}$$

which is a normal commutator. Notice that, as mentioned before, this redefinition of the momentum operator is possible only for $a_A^r \neq a_A^l$, which is assured here by the requirement that (2.26) holds. We have seen that all "generators" in the algebra (2.27) follow the usual Leibniz rule, so that would give a basis for the construction of supersymmetry on the lattice in a manner quite parallel to that in the continuum.

B. Twisted basis and the doubling of chiral fermion

When one regularizes chiral fermions on the lattice species doublers of chiral fermions inevitably appear [1,2]. It was shown that the naive fermion formulation where the continuum differential operator in the Dirac action is simply replaced by the lattice difference operators can be spin diagonalized and leads to the staggered fermion formulation [47] which is shown to be essentially equivalent [48,49] to the Kogut-Susskind fermion formulation [50]. The equivalence of the staggered fermion formulation and that of Dirac-Kähler fermion has been proved exactly with the introduction of a mild noncommutativity between differential forms and fields [16]. This means that all these lattice fermion formulations are equivalent and mild noncommutativity seems to play an important role in this context. Among these fermion formulations the Dirac-Kähler fermion formulation has a clear geometrical correspondence with respect to the fields since the differential form and the lattice simplex have a one to one correspondence.

The main point of the Dirac-Kähler twisting procedure is that the species doublers are not just lattice artifacts but fundamental d.o.f. for the regularization of fermions [10]. In fact these d.o.f. are identified with the extra fermionic d.o.f. of the twisted extended supersymmetry: $\mathcal{N} = 2$ in two dimensions, $\mathcal{N} = 4$ in three and four dimensions. The four-dimensional Dirac-Kähler twisting procedure coincides with the twisting derived by Marcus [11]. These arguments apply in higher dimensions, too, requiring that in D dimensions, which has $2^{D/2}$ (on-shell) doubler's degeneracy, should be treated with $\mathcal{N} = 2^{D/2}$ extended supersymmetry. (In two dimensions $\mathcal{N} = 2$, for example, does not correspond to the number of total charges and thus it is sometimes denoted as $\mathcal{N} = (2, 2)$ instead.)

In the Dirac-Kähler twisting procedure the spinor index and the flavor index are both regarded as indices of the twisted algebra and combine into forming scalar, vector, tensor, ... supercharges. In other words the flavor d.o.f. which are originally the species doubler's d.o.f. are now identified as the extended supersymmetry d.o.f. The corresponding suffix can be rotated by the internal R-symmetry generator of extended supersymmetry. In this way the internal d.o.f. play the role of changing the spin of fields. The mechanism of how spin and internal flavor are related should be understood from the lattice point of view. This issue is fundamentally related to the spin and statistics problem on the lattice. Since the Lorentz invariance is broken on the lattice it is natural to expect that the (anti)commuting nature of fields will be modified.

Let us begin with the two-dimensional case. Here we only consider the simplest cases. Superalgebra in the Dirac-Kähler twisted basis on the lattice is given as

$$\{Q^{\rm lat}, Q^{\rm lat}_\mu\} = P^{\rm lat}_\mu, \qquad \{\tilde{Q}^{\rm lat}, Q^{\rm lat}_\mu\} = -\epsilon_{\mu\nu} P^{{\rm lat}'}{}_\nu, \qquad \{\text{others}\} = 0, \tag{2.31}$$

which is the twisted version of $\mathcal{N} = (2, 2)$ superalgebra in two dimensions. We have put a prime on the second momentum operator to distinguish from the first one, since there is an ambiguity for the lattice momentum operator as explained above. Note that in each commutator the right-hand side contains only one momentum operator for each given combination of indices, which is necessary for the algebraic consistency as claimed in the preceding section. The reason we specified $\mathcal{N} = (2, 2)$ is that the corresponding supermultiplet contains four fermions, which has the same (on-shell) d.o.f. as that of the Dirac-Kähler/staggered fermions which originate the doubler's d.o.f. on the lattice in two dimensions.

The shift variable condition (2.26) reads in this case

$$a^{l,r} + a^{l,r}_\mu = a^{l,r}\hat{\mu}, \qquad \tilde{a}^{l,r} + a^{l,r}_\mu = |\epsilon_{\mu\nu}| a'^{l,r}\hat{\mu}. \tag{2.32}$$

With the same argument given in the original link formalism these lead to $a^{l,r} + a^{l,r}_1 + a^{l,r}_2 + \tilde{a}^{l,r} = (a^{l,r} + a'^{l,r})\hat{1} = (a^{l,r} + a'^{l,r})\hat{2}$, which is only possible if $a^{l,r} = -a'^{l,r}$. In our simple choices of the momentum operators, it implies that, due to (2.24),

$$P^{\rm lat}_\mu = i\partial_{\pm\mu}, \qquad P^{{\rm lat}'}{}_\mu = i\partial_{\mp\mu} \quad \text{or} \quad P^{\rm lat}_\mu = P^{{\rm lat}'}{}_\mu = i\partial^s_\mu. \tag{2.33}$$

The former possibility was considered in the original link formulation, whereas the latter one, although a solution from the consistency point of view, might not be so good from the viewpoint of the doubling issue: it would create, if naively used, a doubling degeneracy again. In any of these cases, the shift conditions become

$$a^{l,r} + a^{l,r}_\mu = a^{l,r}\hat{\mu}, \qquad \tilde{a}^{l,r} + a^{l,r}_\mu = -|\epsilon_{\mu\nu}| a^{l,r}\hat{\mu}, \tag{2.34}$$

which are four conditions with one constraint, so three remaining conditions in total, for four shift variables. It thus seems that one shift variable could be free. In view of the lattice structure, however, this free parameter should not be irrational, otherwise it would lead to an uncountable number of "dual" lattice points, which spoils the lattice regularization! Although in principle any set of rational numbers is allowed, we would in general have unnecessary d.o.f. and/or an unnatural lattice structure, except for the case when the free shift parameter (for instance the scalar one) is fixed to zero or to half the lattice constant. These choices of the free parameter were referred to as the asymmetric and symmetric choices, respectively, in the link formalism.

Similarly in four dimensions, we take the superalgebra

$$\{Q^{\rm lat}, Q^{\rm lat}_{\mu}\} = P^{\rm lat}_{+\mu}, \qquad \{Q^{\rm lat}_{\mu\nu}, Q^{\rm lat}_{\rho}\} = \delta_{\mu\nu,\rho\sigma}P^{\rm lat}_{-\sigma},$$
$$\{\tilde{Q}^{\rm lat}, \tilde{Q}^{\rm lat}_{\mu}\} = P^{\rm lat}_{-\mu}, \qquad \{Q^{\rm lat}_{\mu\nu}, \tilde{Q}^{\rm lat}_{\rho}\} = \epsilon_{\mu\nu\rho\sigma}P^{\rm lat}_{+\sigma}, \tag{2.35}$$

and the other commutators all vanish. This is the Dirac-Kähler twisted superalgebra of $\mathcal{N} = 4$ which is required, as explained above, from the general argument on the fermionic d.o.f. We can show that these combinations of $P^{\rm lat}_{\pm\mu}$ indeed lead to the Leibniz rule conditions for the shift variables which have a nontrivial set of solutions [7].

C. The claimed inconsistency

What is intriguing in the link formalism is the algebraic structure based on the modified Leibniz rule for the symmetry operators. If a suitable representation of this algebra is unambiguously obtained, it seems at first sight that it gives a consistent formulation of supersymmetry on the lattice. It turns out, however, that such a representation would conflict with the conventional component field path integral formulation on the lattice. This problem can be seen as the fact that, although supertransformations of single component fields are well-defined, supertransformations of products of fields become sensitive to the order of the fields in the products. If such an order is uniquely determined, it is nothing harmful. However, we have no criteria to introduce such an order on the conventional lattice and thus we have a serious difficulty that supertransformations are not totally defined in a unique and consistent manner as transformations of path integral variables. In fact, this difficulty was claimed to result in an inconsistency of the link formalism in [42]. The criticism is twofold: one is for the nongauge theories [6], the other is, also investigated in a similar attitude in [34], for the case of gauge theories [7,8], and they both can be summarized by saying that supercharges in the link formalism add nontrivial link structure to the component fields changing the original link nature of the fields in an ordering sensitive way.

Let us see these arguments more explicitly. In the link formalism, scalar fields $\phi(x)$ defined on sites of the lattice are naturally assumed to be commutative:

$$\phi_1(x)\phi_2(x) = \phi_2(x)\phi_1(x). \tag{2.36}$$

Applying a supertransformation on the both sides of this equation, we have, from the left-hand side, that

$$Q^{\rm lat}_A(\phi_1(x)\phi_2(x)) = \psi_{1A}(x)\phi_2(x+a^r_A) + \phi_1(x+a^l_A)\psi_{2A}(x), \tag{2.37}$$

where $\psi_{1,2A}(x) := Q^{\rm lat}_A\phi_{1,2}(x)$, and from the right,

$$Q^{\rm lat}_A(\phi_2(x)\phi_1(x)) = \psi_{2A}(x)\phi_1(x+a^r_A) + \phi_2(x+a^l_A)\psi_{1A}(x). \tag{2.38}$$

These two equations must be the same as they are the transformations of one and the same quantity; otherwise supertransformations on products of fields are not uniquely defined. But actually these two conflict with each other if the fermions $\psi_{1,2A}$ are also assumed to be simple (anti) commuting objects: the term containing $\psi_{1A}(x)$ in the first equation has the factor $\phi_2(x+a^r_A)$, whereas in the second has $\phi_2(x+a^l_A)$, and they are different unless $a^l_A = a^r_A$, which, however, would not be a consistent solution for the shift variable conditions as already explained in the previous subsections. The discrepancy between these two equations cannot be expressed as a total difference, so that it gives an essential obstacle for the invariance of any possible action. It causes similar difficulties also in the gauge theory actions.

In the following sections, we will propose a possible solution to the above-mentioned first criticism for the nongauge theories by introducing the following mild noncommutativity [8]:

$$\varphi_A(x)\varphi_B(y) = (-1)^{|\varphi_A||\varphi_B|}\varphi_B(y+a_A)\varphi_A(x-a_B), \tag{2.39}$$

where $\varphi_A(x)$ and $\varphi_B(y)$ are assumed to carry the shifts a_A and a_B, respectively. In fact we can easily check that the expressions of (2.37) and (2.38) coincide if we assume that $\psi_{1,2A}$ carry a shift $a^l_A - a^r_A$ while $\phi_{1,2}$ carry no shift and that they satisfy the noncommutative relation (2.39). The key point is to treat each field as a noncommutative object, or an object with nontrivial statistics, to uniquely define the ordering which is necessary to avoid the conflict.

If we introduce the noncommutative nature for the fields as in (2.39), the formulation of field theory should be modified from the conventional definition in such a way that any algebraic manipulation of fields and operators should be compatible with the new deformed supersymmetry. In the following section we show that it is possible to define a new lattice field theory which has the exact deformed supersymmetry with Hopf algebraic nature. Addressing similar questions in gauge theories is out of the scope of this paper.

III. HOPF ALGEBRAIC STRUCTURE OF THE LATTICE SUPERALGEBRA

In this section, we investigate the "lattice superalgebra" from a yet different algebraic viewpoint, namely, in terms of Hopf algebra. As it has been shown in recent years, extending the notion of symmetry in a field theory to that of Hopf algebraic symmetry provides a framework which is useful in some specific cases, especially in noncommutative theories [41,51–54]. A slightly different application is found in [16]. In the present case, the superalgebra on the lattice is understood as a deformed algebra on the lattice and turns out to be a Hopf algebra. This identification guarantees that the deformed algebra is mathematically consistent. Using the general scheme called braided quantum field theory [41,53], we will show that the field theory whose symmetry is prescribed by the deformed algebra can be constructed at least perturbatively. The deformed symmetry leads to the corresponding Ward-Takahashi identities on the lattice, which provide a good physical interpretation of the deformed symmetry itself.

In Appendix A a brief description of the mathematical basis on Hopf algebra is given and our notation and terminology are summarized.

A. Lattice superalgebra as a Hopf algebra

To begin with we consider the lattice superalgebra in its original form (2.2), or in the twisted basis (2.31) and (2.35). Here we treat each of them as an abstract Lie superalgebra which we denote as $\mathcal{A}$, so that $P_\mu^{\rm lat}, Q_A^{\rm lat} \in \mathcal{A}$. We then introduce the space of fields on the lattice as $X = X_e \oplus X_o$, where X_e consists of all bosonic fields and X_o of all fermionic fields. We need a multiplication/product of fields to construct a field theory, which is in general noncommutative. We assume here this multiplication is associative for our current application. Then the space X is supposed to be an associative graded algebra. However in a quantum field theory products of fields, i.e. composite fields, are distinct from single fields, i.e. elementary fields. This occurs because the elementary fields are the integration variables of the path integral (if any are defined), and the ones obeying the canonical (anti)commutation relations, whose behavior is clearly different from that of the composite fields. We thus denote by X the elementary fields and extend the definition to the formal space of tensor products of the elementary fields to include every composite field:

$$\hat{X} := \bigoplus_{n=0}^{\infty} X^n, \qquad X^0 :== X_e^0 \oplus X_o^0, \qquad X^n := \underbrace{X \otimes \cdots \otimes X}_{n}, \tag{3.1}$$

where X_e^0 and X_o^0 are the space of bosonic and fermionic constant functions, respectively. Multiplications/products of fields are naturally defined in $\hat{X}$ as $m(\varphi \otimes \varphi') = \varphi \cdot \varphi' \in \hat{X}$ $(\varphi, \varphi' \in \hat{X})$.

We now consider the *action* (see Appendix A) of $\mathcal{A}$ on the space of fields $\hat{X}$. We denote the action of an operator $a \in \mathcal{A}$ as $a\triangleright$. With successive actions, we are naturally led to the notion of an (associative) universal enveloping algebra $\mathcal{U}(A)$ of $\mathcal{A}$, as in $(a \cdot b)\triangleright := a\triangleright \circ b\triangleright := a\triangleright(b\triangleright)$, for $a, b \in A$, and $a \cdot b \in \mathcal{U}(\mathcal{A})$. We also introduce the identity operator $\mathbb{1}$ as the unit element of the universal enveloping algebra. We may define the unit map by $\eta(c) := c\mathbb{1}$, $c \in \mathbb{C}$.

Even on the lattice, actions or representations of the operators $Q_A^{\rm lat}$ and $P_\mu^{\rm lat}$ on elementary fields would be well-defined without any difficulty. We denote these formally as

$$\begin{aligned} Q_A^{\rm lat}\triangleright\varphi(x) &= (Q_A^{\rm lat}\varphi)(x), \\ P_\mu^{\rm lat}\triangleright\varphi(x) &= (P_\mu^{\rm lat}\varphi)(x), \qquad \varphi \in X. \end{aligned} \tag{3.2}$$

The explicit form of $Q_A^{\rm lat}\varphi$ depends on the model we take. An example is listed in Appendix B. As for the expression $P_\mu^{\rm lat}\varphi$, we could essentially take some of the difference operators as in (2.5), but it turns out that the lattice momentum operator $P_\mu^{\rm lat}$ should carry a nontrivial grading structure, which is required from a Hopf algebraic consistency. We will see this point in the following subsection.

Actions on trivial/constant fields are also easily defined as in

$$Q_A^{\rm lat}\triangleright f = 0, \qquad P_\mu^{\rm lat}\triangleright f = 0, \qquad f \in X_e^0. \tag{3.3}$$

As a matter of convention, we write these equations in terms of a map ϵ called *counit* as in

$$\begin{aligned} Q_A^{\rm lat}\triangleright f &= \epsilon(Q_A^{\rm lat})f = 0, \quad \text{i.e. } \epsilon(Q_A^{\rm lat}) = 0, \\ P_\mu^{\rm lat}\triangleright f &= \epsilon(P_\mu^{\rm lat})f = 0, \quad \text{i.e. } \epsilon(P_\mu^{\rm lat}) = 0. \end{aligned} \tag{3.4}$$

The essential nontriviality comes in the actions of operators on composite fields, i.e. products of elementary fields, due to the failure of the usual Leibniz rule. The link formalism manages this difficulty with the introduction of appropriate deformation or modification of the Leibniz rule when operators act on composite fields. Mathematically, this is understood as equipping the universal enveloping algebra $\mathcal{U}(\mathcal{A})$ with an additional structure, *coproduct/comultiplication*, denoted by Δ. To be specific, consider the actions of $Q_A^{\rm lat}$ and $P_\mu^{\rm lat}$ on a product of two elementary fields $\varphi_1(x), \varphi_2(x) \in X$. Introducing the modified Leibniz rule (2.15) and (2.23) is equivalent to defining these actions to be

$$\begin{aligned} Q_A^{\rm lat}\triangleright(\varphi_1(x)\cdot\varphi_2(x)) &:= m(\Delta(Q_A^{\rm lat})\triangleright(\varphi_1(x)\otimes\varphi_2(x))), \\ P_A^{\rm lat}\triangleright(\varphi_1(x)\cdot\varphi_2(x)) &:= m(\Delta(P_A^{\rm lat})\triangleright(\varphi_1(x)\otimes\varphi_2(x))), \end{aligned} \tag{3.5}$$

together with the coproducts

$$\begin{aligned} \Delta(Q_A^{\rm lat}) &= Q_A^{\rm lat} \otimes T_{a_A^r} + (-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes Q_A^{\rm lat}, \\ \Delta(P_\mu^{\rm lat}) &= P_\mu^{\rm lat} \otimes T_{a^r\hat{\mu}} + T_{a^l\hat{\mu}} \otimes P_\mu^{\rm lat}, \end{aligned} \tag{3.6}$$

where $\mathcal{F}$ is the fermion number operator with which $(-1)^{\mathcal{F}}$ takes care of the statistics factors, and the shift operator T_b, which is also assumed to belong to $\mathcal{U}(\mathcal{A})$, acts as

$$T_b \triangleright \varphi(x) := \varphi(x+b). \tag{3.7}$$

For these operators we set

$$\epsilon(T_b) = 1, \qquad \Delta(T_b) = T_b \otimes T_b, \tag{3.8}$$

and

$$\epsilon((-1)^{\mathcal{F}}) = 1, \qquad \Delta((-1)^{\mathcal{F}}) = (-1)^{\mathcal{F}} \otimes (-1)^{\mathcal{F}}. \tag{3.9}$$

Note that these definitions are natural, since the counit essentially prescribes the action on constants, whereas the coproduct defines the action on products. We also list, though obvious, the action of the identity operator $\mathbb{1}$ on $\hat{X}$. It must be, by definition, such that $\mathbb{1} \triangleright \varphi = \varphi$, $\varphi \in X$. On a constant field, $f = \mathbb{1} \triangleright f = \epsilon(\mathbb{1}) f$, $f \in X^0$, so that

$$\epsilon(\mathbb{1}) = 1. \tag{3.10}$$

On a product of elementary fields, $\varphi_1 \cdot \varphi_2 = \mathbb{1} \triangleright (\varphi_1 \cdot \varphi_2) = m(\Delta(\mathbb{1}) \triangleright (\varphi_1 \otimes \varphi_2))$ so that

$$\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}. \tag{3.11}$$

The counit ϵ and coproduct Δ have to satisfy some consistency conditions. First, we note that any single elementary field φ can be written as the product of unity and itself: $\varphi = m(\mathbf{1} \otimes \varphi) = m(\varphi \otimes \mathbf{1})$. The action of an operator should be uniquely determined regardless of the way the field is written. More specifically, this requires that

$$\begin{aligned}(Q_A^{\rm lat}\varphi)(x) &= Q_A^{\rm lat} \triangleright \varphi(x) = Q_A^{\rm lat} \triangleright m(\mathbf{1} \otimes \varphi(x)) \\ &= m(\Delta(Q_A^{\rm lat}) \triangleright (\mathbf{1} \otimes \varphi(x))) \\ &= m((Q_A^{\rm lat} \triangleright \mathbf{1}) \otimes (T_{a_A^r} \triangleright \varphi(x)) \\ &\quad + (T_{a_A^l} \triangleright \mathbf{1}) \otimes (Q_A^{\rm lat} \triangleright \varphi(x))) \\ &= m((\epsilon(Q_A^{\rm lat})\mathbf{1}) \otimes (T_{a_A^r} \triangleright \varphi(x)) \\ &\quad + (\epsilon(T_{a_A^l})\mathbf{1}) \otimes (Q_A^{\rm lat} \triangleright \varphi(x))) \\ &= m(\mathbf{1} \otimes (Q_A^{\rm lat}\varphi)(x)) = (Q_A^{\rm lat}\varphi)(x),\end{aligned} \tag{3.12}$$

which is consistently realized. Similarly we have another consistency condition:

$$\begin{aligned}(Q_A^{\rm lat}\varphi)(x) &= Q_A^{\rm lat} \triangleright \varphi(x) = Q_A^{\rm lat} \triangleright m(\varphi(x) \otimes \mathbf{1}) \\ &= m(\Delta(Q_A^{\rm lat}) \triangleright (\varphi(x) \otimes \mathbf{1})) \\ &= m((Q_A^{\rm lat} \triangleright \varphi(x)) \otimes (T_{a_A^r} \triangleright \mathbf{1}) \\ &\quad + ((-1)^{\mathcal{F}} \cdot T_{a_A^l} \triangleright \varphi(x)) \otimes (Q_A^{\rm lat} \triangleright \mathbf{1})) \\ &= m((Q_A^{\rm lat} \triangleright \varphi(x)) \otimes (\epsilon(T_{a_A^r})\mathbf{1}) \\ &\quad + ((-1)^{|\varphi|} T_{a_A^l} \triangleright \varphi(x)) \otimes (Q_A^{\rm lat})\mathbf{1})) \\ &= m((Q_A^{\rm lat}\varphi)(x) \otimes \mathbf{1}) = (Q_A^{\rm lat}\varphi)(x).\end{aligned} \tag{3.13}$$

Similar results hold for $P_\mu^{\rm lat}$. As for T_b,

$$\begin{aligned}\varphi(x+b) &= T_b \triangleright \varphi(x) = T_b \triangleright m(\mathbf{1} \otimes \varphi(x)) \\ &= m(\Delta(T_b) \triangleright (\mathbf{1} \otimes \varphi(x))) \\ &= m((T_b \triangleright \mathbf{1}) \otimes (T_b \triangleright \varphi(x))) \\ &= m((\epsilon(T_b)\mathbf{1}) \otimes \varphi(x+b)) = m(\mathbf{1} \times \varphi(x+b)) \\ &= \varphi(x+b),\end{aligned} \tag{3.14}$$

which is again consistent. These results show that the definitions of counit and coproduct in (3.4), (3.6), and (3.8) are compatible with the trivial structure of the algebra $\hat{X}$ obtained by unit multiplication. A second consistency condition is the so-called *coassociativity.* Since the multiplication on $\hat{X}$ is associative, the action of an operator on the product of three elementary fields should respect this associativity. This requires coassociativity for the coproduct. It also means that the action on products of three elementary fields is defined in a natural way as in

$$\begin{aligned}&m \circ (m \otimes {\rm id}) \circ (\Delta \otimes {\rm id}) \circ \Delta(Q_A^{\rm lat}) \triangleright ((\varphi_1(x) \otimes \varphi_2(x)) \otimes \varphi_3(x)) \\ &\quad = Q_A^{\rm lat} \triangleright ((\varphi_1(x) \cdot \varphi_2(x)) \cdot \varphi_3(x)) \\ &\quad = Q_A^{\rm lat} \triangleright (\varphi_1(x) \cdot \varphi_2(x) \cdot \varphi_3(x))) \\ &\quad = Q_A^{\rm lat} \triangleright (\varphi_1(x) \cdot (\varphi_2(x) \cdot \varphi_3(x))) \\ &\quad = m \circ ({\rm id} \otimes m) \circ ({\rm id} \otimes \Delta) \circ \Delta(Q_A^{\rm lat}) \triangleright (\varphi_1(x) \\ &\qquad \otimes (\varphi_2(x) \otimes \varphi_3(x))).\end{aligned} \tag{3.15}$$

Since the product m is associative,

$$m \circ (m \otimes {\rm id}) = m \circ ({\rm id} \otimes m), \tag{3.16}$$

it requires that

$$(\Delta \otimes {\rm id}) \circ \Delta(Q_A^{\rm lat}) = ({\rm id} \otimes \Delta) \circ \Delta(Q_A^{\rm lat}). \tag{3.17}$$

The same condition should apply to $P_\mu^{\rm lat}$ and T_b. These conditions are indeed satisfied for the coproducts in the present case. Using (3.6), we compute[6]

$$\begin{aligned}&(\Delta \otimes {\rm id}) \circ \Delta(Q_A^{\rm lat}) \\ &\quad = (\Delta \otimes {\rm id})(Q_A^{\rm lat} \otimes T_{a_A^r} + (-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes Q_A^{\rm lat}) \\ &\quad = (Q_A^{\rm lat} \otimes T_{a_A^r} + (-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes Q_A^{\rm lat}) \otimes T_{a_A^r} \\ &\qquad + ((-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes (-1)^{\mathcal{F}} \cdot T_{a_A^l}) \otimes Q_A^{\rm lat} \\ &\quad = Q_A^{\rm lat} \otimes T_{a_A^r} \otimes T_{a_A^r} + (-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes Q_A^{\rm lat} \otimes T_{a_A^r} \\ &\qquad + (-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes (-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes Q_A^{\rm lat},\end{aligned} \tag{3.18}$$

and

[6] Here we use the relation $\Delta((-1)^{\mathcal{F}} \cdot T_b) = \Delta((-1)^{\mathcal{F}}) \cdot \Delta(T_b)$, which will be explained shortly.

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\circ\Delta(Q_A^{\mathrm{lat}}) &= (\mathrm{id}\otimes\Delta)(Q_A^{\mathrm{lat}}\otimes T_{a_A^r} + (-1)^{\mathcal{F}}\cdot T_{a_A^l}\otimes Q_A^{\mathrm{lat}})\\ &= Q_A^{\mathrm{lat}}\otimes(T_{a_A^r}\otimes T_{a_A^r}) + (-1)^{\mathcal{F}}\cdot T_{a_A^l}\\ &\quad\otimes(Q_A^{\mathrm{lat}}\otimes T_{a_A^r} + (-1)^{\mathcal{F}}\cdot T_{a_A^l}\otimes Q_A^{\mathrm{lat}})\\ &= Q_A^{\mathrm{lat}}\otimes T_{a_A^r}\otimes T_{a_A^r}\\ &\quad + (-1)^{\mathcal{F}}\cdot T_{a_A^l}\otimes Q_A^{\mathrm{lat}}\otimes T_{a_A^r}\\ &\quad + (-1)^{\mathcal{F}}\cdot T_{a_A^l}\otimes(-1)^{\mathcal{F}}\cdot T_{a_A^l}\otimes Q_A^{\mathrm{lat}},\end{aligned} \tag{3.19}$$

which shows that (3.17) holds for Q_A^{lat}. We have thus found unambiguously that

$$\begin{aligned}&Q_A^{\mathrm{lat}}\triangleright(\varphi_1(x)\cdot\varphi_2(x)\cdot\varphi_3(x))\\ &\quad= (Q_A^{\mathrm{lat}}\varphi_1)(x)\cdot\varphi_2(x+a_A^r)\cdot\varphi_3(x+a_A^r)\\ &\qquad + (-1)^{|\varphi_1|}\varphi_1(x+a_A^l)\cdot(Q_A^{\mathrm{lat}}\varphi_2)(x)\cdot\varphi_3(x+a_A^r)\\ &\qquad + (-1)^{|\varphi_1|+|\varphi_2|}\varphi_1(x+a_A^l)\cdot\varphi_2(x+a_A^l)\\ &\qquad \cdot(Q_A^{\mathrm{lat}}\varphi_3)(x).\end{aligned} \tag{3.20}$$

The same is true for P_μ^{lat}

$$\begin{aligned}&P_\mu^{\mathrm{lat}}\triangleright(\varphi_1(x)\cdot\varphi_2(x)\cdot\varphi_3(x))\\ &\quad= (P_\mu^{\mathrm{lat}}\varphi_1)(x)\cdot\varphi_2(x+a^r\hat\mu)\cdot\varphi_3(x+a^r\hat\mu)\\ &\qquad + \varphi_1(x+a^l\hat\mu)\cdot(P_\mu^{\mathrm{lat}}\varphi_2)(x)\cdot\varphi_3(x+a^r\hat\mu)\\ &\qquad + \varphi_1(x+a^l\hat\mu)\cdot\varphi_2(x+a^l\hat\mu)\cdot(P_\mu^{\mathrm{lat}}\varphi_3)(x).\end{aligned} \tag{3.21}$$

Similarly, T_b satisfies coassociativity, namely

$$\begin{aligned}(\Delta\otimes\mathrm{id})\circ\Delta(T_b) &= (\Delta\otimes\mathrm{id})(T_b\otimes T_b) = (T_b\otimes T_b)\otimes T_b\\ &= T_b\otimes(T_b\otimes T_b) = (\mathrm{id}\otimes\Delta)(T_b\otimes T_b)\\ &= (\mathrm{id}\otimes\Delta)\Delta(T_b),\end{aligned} \tag{3.22}$$

so that

$$T_b\triangleright(\varphi_1(x)\cdot\varphi_2(x)\cdot\varphi_3(x)) = \varphi_1(x+b)\cdot\varphi_2(x+b)\cdot\varphi_3(x+b). \tag{3.23}$$

The same result for $(-1)^{\mathcal{F}}$ is obvious.

Now that we have shown that the operators Q_A^{lat}, P_μ^{lat}, T_b are well-defined in regard to their action on the elementary fields, constants, and products of two or three elementary fields, we find that any other action of those operators is also consistently defined (needless to say actions as well as the maps introduced above are all linear). In particular their action on any number of elementary fields can be computed inductively using the coassociativity. We need now to define the action of products of operators. As stated above, a product of operators is defined as the operator obtained by successive applications of each operator in the product. On the elementary fields, this is easily understood, because it is nothing but the definition. On the trivial (i.e. constant) fields, this implies a consistency condition on the counit map, namely

$$\epsilon(a\cdot b)f = (a\cdot b)\triangleright f = a\triangleright\circ b\triangleright f = \epsilon(a)\epsilon(b)f, \tag{3.24}$$

i.e.

$$\epsilon(a\cdot b) = \epsilon(a)\epsilon(b). \tag{3.25}$$

Similarly, the product of operators should act on a product of elementary fields by acting successively with the operators of the product, that is

$$\begin{aligned}m(\Delta(a\cdot b)\triangleright(\varphi_1\otimes\varphi_2)) &= (a\cdot b)\triangleright(\varphi_1\cdot\varphi_2)\\ &= a\triangleright\circ b\triangleright(\varphi_1\cdot\varphi_2)\\ &= a\triangleright m(\Delta(b)\triangleright(\varphi_1\otimes\varphi_2))\\ &= m(\Delta(a)\triangleright\Delta(b)\triangleright(\varphi_1\otimes\varphi_2))\\ &= m((\Delta(a)\cdot\Delta(b))\triangleright(\varphi_1\otimes\varphi_2)),\end{aligned} \tag{3.26}$$

which implies

$$\Delta(a\cdot b) = \Delta(a)\cdot\Delta(b). \tag{3.27}$$

As an example, we compute

$$\begin{aligned}\Delta(Q_A^{\mathrm{lat}}\cdot Q_B^{\mathrm{lat}}) &= \Delta(Q_A^{\mathrm{lat}})\cdot\Delta(Q_B^{\mathrm{lat}})\\ &= (Q_A^{\mathrm{lat}}\otimes T_{a_A^r} + (-1)^{\mathcal{F}}\cdot T_{a_A^l}\otimes Q_A^{\mathrm{lat}})\\ &\quad\cdot(Q_B^{\mathrm{lat}}\otimes T_{a_B^r} + (-1)^{\mathcal{F}}\cdot T_{a_B^l}\otimes Q_B^{\mathrm{lat}})\\ &= Q_A^{\mathrm{lat}}\cdot Q_B^{\mathrm{lat}}\otimes T_{a_A^r}\cdot T_{a_B^r} + Q_A^{\mathrm{lat}}\cdot(-1)^{\mathcal{F}}\cdot T_{a_B^l}\\ &\quad\otimes T_{a_A^r}\cdot Q_B^{\mathrm{lat}} + (-1)^{\mathcal{F}}\cdot T_{a_A^l}\cdot Q_B^{\mathrm{lat}}\otimes Q_A^{\mathrm{lat}}\cdot T_{a_B^r}\\ &\quad + (-1)^{\mathcal{F}}\cdot T_{a_A^l}\cdot(-1)^{\mathcal{F}}\cdot T_{a_B^l}\otimes Q_A^{\mathrm{lat}}\cdot Q_B^{\mathrm{lat}}.\end{aligned} \tag{3.28}$$

Other simple examples are

$$\begin{aligned}\Delta(Q_A^{\mathrm{lat}}\cdot P_\mu^{\mathrm{lat}}) &= \Delta(Q_A^{\mathrm{lat}})\cdot\Delta(P_\mu^{\mathrm{lat}})\\ &= (Q_A^{\mathrm{lat}}\otimes T_{a_A^r} + (-1)^{\mathcal{F}}\cdot T_{a_A^l}\otimes Q_A^{\mathrm{lat}})\\ &\quad\cdot(P_\mu^{\mathrm{lat}}\otimes T_{a^r\hat\mu} + T_{a^l\hat\mu}\otimes P_\mu^{\mathrm{lat}})\\ &= Q_A^{\mathrm{lat}}\cdot P_\mu^{\mathrm{lat}}\otimes T_{a_A^r}\cdot T_{a^r\hat\mu} + Q_A^{\mathrm{lat}}\cdot T_{a^l\hat\mu}\otimes T_{a_A^r}\\ &\quad\cdot P_\mu^{\mathrm{lat}} + (-1)^{\mathcal{F}}\cdot T_{a_A^l}\cdot P_\mu^{\mathrm{lat}}\otimes Q_A^{\mathrm{lat}}\cdot T_{a^r\hat\mu}\\ &\quad + (-1)^{\mathcal{F}}\cdot T_{a_A^l}\cdot T_{a^l\hat\mu}\otimes Q_A^{\mathrm{lat}}\cdot P_\mu^{\mathrm{lat}},\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}\Delta(T_b\cdot Q_A^{\mathrm{lat}}) &= \Delta(T_b)\cdot\Delta(Q_A^{\mathrm{lat}})\\ &= (T_b\otimes T_b)\cdot(Q_A^{\mathrm{lat}}\otimes T_{a_A^r} + (-1)^{\mathcal{F}}\cdot T_{a_A^l}\otimes Q_A^{\mathrm{lat}})\\ &= T_b\cdot Q_A^{\mathrm{lat}}\otimes T_b\cdot T_{a_A^r}\\ &\quad + T_b\cdot(-1)^{\mathcal{F}}\cdot T_{a_A^l}\otimes T_b\cdot Q_A^{\mathrm{lat}},\end{aligned} \tag{3.30}$$

$$\Delta(T_b \cdot T_c) = \Delta(T_b) \cdot \Delta(T_c) = (T_b \otimes T_b) \cdot (T_c \otimes T_c) = T_b \cdot T_c \otimes T_b \cdot T_c. \quad (3.31)$$

Let us recall now the superalgebra (2.2), and introduce a natural algebra with respect to T_b as in

$$\{Q_A^{\rm lat}, Q_B^{\rm lat}\} = 2\tau^\mu_{AB} P_\mu^{\rm lat}, \qquad [Q_A^{\rm lat}, P_\mu^{\rm lat}] = [P_\mu^{\rm lat}, P_\nu^{\rm lat}] = 0,$$
$$[Q_A^{\rm lat}, T_b] = [P_\mu^{\rm lat}, T_b] = [T_b, T_c] = 0. \quad (3.32)$$

The last relations are in a way obvious, and state that

$$Q_A^{\rm lat}\varphi(x+b) = T_b(Q_A^{\rm lat}\varphi)(x),$$
$$T_b\varphi(x+c) = T_c\varphi(x+b) = \varphi(x+b+c) = T_{b+c}\varphi(x), \quad (3.33)$$

and similarly for $P_\mu^{\rm lat}$. We also list here the obvious algebra for $(-1)^{\mathcal{F}}$:

$$\{Q_A^{\rm lat}, (-1)^{\mathcal{F}}\} = [P_A^{\rm lat}, (-1)^{\mathcal{F}}] = [T_b, (-1)^{\mathcal{F}}] = 0,$$
$$(-1)^{\mathcal{F}} \cdot (-1)^{\mathcal{F}} = \mathbb{1}. \quad (3.34)$$

From these relations and using (3.28), we find that

$$\Delta(\{Q_A^{\rm lat}, Q_B^{\rm lat}\}) = \{Q_A^{\rm lat}, Q_B^{\rm lat}\} \otimes T_{a_A^r} \cdot T_{a_B^r} + T_{a_A^l} \cdot T_{a_B^l} \otimes \{Q_A^{\rm lat}, Q_B^{\rm lat}\}, \quad (3.35)$$

reproducing the general result found in (2.22). Just as an additional explicit check of the consistency, we compute the action of the product $Q_A^{\rm lat} \cdot Q_B^{\rm lat}$ on the product of three fields $\varphi_1 \cdot \varphi_2 \cdot \varphi_3$, which is given by the expression

$$\begin{aligned}(\Delta \otimes {\rm id}) \circ (Q_A^{\rm lat} \cdot Q_B^{\rm lat}) &= (\Delta \otimes {\rm id}) \circ (\Delta(Q_A^{\rm lat}) \cdot \Delta(Q_B^{\rm lat})) \\ &= \Delta(Q_A^{\rm lat} \cdot Q_B^{\rm lat}) \otimes T_{a_A^r} \cdot T_{a_B^r} + \Delta(Q_A^{\rm lat} \cdot (-1)^{\mathcal{F}} \cdot T_{a_B^l}) \\ &\quad \otimes T_{a_A^r} \cdot Q_B^{\rm lat} - \Delta(Q_B^{\rm lat} \cdot (-1)^{\mathcal{F}} \cdot T_{a_A^l}) \otimes Q_A^{\rm lat} \cdot T_{a_B^r} \\ &\quad + \Delta(T_{a_A^l} \cdot T_{a_B^l}) \otimes Q_A^{\rm lat} \cdot Q_B^{\rm lat}\end{aligned} \quad (3.36)$$

and then this can be computed using (3.28), (3.30), and (3.31). This of course leads to

$$\begin{aligned}(\Delta \otimes {\rm id}) \circ \Delta(\{Q_A^{\rm lat}, Q_B^{\rm lat}\}) &= (\Delta \otimes {\rm id}) \circ (\{\Delta(Q_A^{\rm lat}), \Delta(Q_B^{\rm lat})\}) \\ &= \Delta(\{Q_A^{\rm lat}, Q_B^{\rm lat}\} \otimes T_{a_A^r} \cdot T_{a_B^r} + \Delta \\ &\quad \times (T_{a_A^l} \cdot T_{a_B^l}) \otimes \{Q_A^{\rm lat}, Q_B^{\rm lat}\} \\ &= \{Q_A^{\rm lat}, Q_B^{\rm lat}\} \otimes T_{a_A^r} \cdot T_{a_B^r} \otimes T_{a_A^r} \\ &\quad \cdot T_{a_B^r} + T_{a_A^l} \cdot T_{a_B^l} \otimes \{Q_A^{\rm lat}, Q_B^{\rm lat}\} \\ &\quad \otimes T_{a_A^r} \cdot T_{a_B^r} + T_{a_A^l} \cdot T_{a_B^l} \otimes T_{a_A^l} \\ &\quad \cdot T_{a_B^l} \otimes \{Q_A^{\rm lat}, Q_B^{\rm lat}\}.\end{aligned} \quad (3.37)$$

Equations (3.10), (3.25), (3.11), and (3.27) naturally require that the counit and coproduct are both consistent with the structure of the algebra $\mathcal{U}(\mathcal{A})$, i.e. that both are algebra maps (algebra homomorphisms).[7] With these properties, we can compute the action of any operator on any field (fundamental or composite) in a consistent manner. Mathematically, all these features assure that our lattice superalgebra actually forms a *bialgebra*.

Notice that our bialgebra is a mixture of both algebralike elements, like $Q_A^{\rm lat}$ or $P_\mu^{\rm lat}$, and grouplike elements, like T_b. The latter elements have their inverse, like T_b^{-1}. The former would also have a sort of inverse, $-Q_A^{\rm lat}$ and $-P_\mu^{\rm lat}$, implying a naive connection between group and algebra. In fact, we need one more ingredient, namely, the *antipode*, to claim that the DKKN lattice superalgebra is a Hopf algebra. The antipode is essentially a map that gives the "inverse" element of each operator. It can be introduced as a linear map which satisfies the identity

$$\cdot \circ (S \otimes {\rm id}) \circ \Delta = \cdot \circ ({\rm id} \otimes S) \circ \Delta = \eta \circ \epsilon, \quad (3.38)$$

where we have used the notation $\cdot(a \otimes b) = a \cdot b$ for products of operators. We define it explicitly, on the single operators, as

$$\begin{aligned} S(Q_A^{\rm lat}) &= -T_{a_A^l}^{-1} \cdot (-1)^{\mathcal{F}} \cdot Q_A^{\rm lat} \cdot T_{a_A^r}^{-1}, \\ S(P_\mu^{\rm lat}) &= -T_{a^l \hat{\mu}}^{-1} \cdot P_\mu^{\rm lat} \cdot T_{a^r \hat{\mu}}^{-1}, \\ S(T_b) &= T_b^{-1}, \\ S((-1)^{\mathcal{F}}) &= (-1)^{-\mathcal{F}} = (-1)^{\mathcal{F}}, \end{aligned} \quad (3.39)$$

and extend it so that it becomes linear and *antialgebraic* namely $S(a \cdot b) = S(b) \cdot S(a)$, $S(\mathbb{1}) = \mathbb{1}$, $(a, b \in \mathcal{U}(\mathcal{A}))$. In fact it is shown that its antialgebraic nature automatically follows if the identity (3.38) holds for the antipode. Here we just see what this identity implies in our superalgebra, without going into details. Applying the first two terms of (3.38) on $Q_A^{\rm lat}$, we find[8]

$$\begin{aligned} \cdot \circ (S \otimes {\rm id}) \circ \Delta(Q_A^{\rm lat}) &= \cdot \circ (S \otimes {\rm id})(Q_A^{\rm lat} \otimes T_{a_A^r} \\ &\quad + (-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes Q_A^{\rm lat}) \\ &= \cdot(-T_{a_A^l}^{-1} \cdot (-1)^{\mathcal{F}} \cdot Q_A^{\rm lat} \cdot T_{a_A^r}^{-1} \otimes T_{a_A^r} \\ &\quad + T_{a_A^l}^{-1} \cdot (-1)^{\mathcal{F}} \otimes Q_A^{\rm lat}) \\ &= -T_{a_A^l}^{-1} \cdot (-1)^{\mathcal{F}} \cdot Q_A^{\rm lat} \\ &\quad + T_{a_A^l}^{-1} \cdot (-1)^{\mathcal{F}} \cdot Q_A^{\rm lat} = 0, \end{aligned} \quad (3.40)$$

and

[7]These conditions are equivalent to imposing that the product m and unit η are coalgebra maps.

[8]We use here $S((-1)^{\mathcal{F}} \cdot T_b) = S(T_b) \cdot S((-1)^{\mathcal{F}})$ as explicitly shown as (3.49).

$$\begin{aligned}\cdot \circ (\mathrm{id} \otimes S) \circ \Delta(Q_A^{\mathrm{lat}}) &= \cdot \circ (\mathrm{id} \otimes S)(Q_A^{\mathrm{lat}} \otimes T_{a_A^r} \\ &\quad + (-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes Q_A^{\mathrm{lat}}) \\ &= \cdot(Q_A^{\mathrm{lat}} \otimes T_{a_A^r}^{-1} - (-1)^{\mathcal{F}} \cdot T_{a_A^l} \\ &\quad \otimes T_{a_A^l}^{-1} \cdot (-1)^{\mathcal{F}} \cdot Q_A^{\mathrm{lat}} \cdot T_{a_A^r}^{-1}) \\ &= Q_A^{\mathrm{lat}} \cdot T_{a_A^r}^{-1} - Q_A^{\mathrm{lat}} \cdot T_{a_A^r}^{-1} = 0,\end{aligned} \tag{3.41}$$

while the last term gives

$$\eta \circ \epsilon(Q_A^{\mathrm{lat}}) = 0. \tag{3.42}$$

Thus the identity (3.38) holds for the operator Q_A^{lat} with the definition (3.39). Similar calculations show that P_μ^{lat} also obeys the same identity. As for T_b, we compute

$$\begin{aligned}\cdot \circ (S \otimes \mathrm{id}) \circ \Delta(T_b) &= \cdot \circ (S \otimes \mathrm{id})(T_b \otimes T_b) \\ &= \cdot(T_b^{-1} \otimes T_b) = \mathbb{1},\end{aligned} \tag{3.43}$$

and

$$\begin{aligned}\cdot \circ (\mathrm{id} \otimes S) \circ \Delta(T_b) &= \cdot \circ (\mathrm{id} \otimes S)(T_b \otimes T_b) \\ &= \cdot(T_b \otimes T_b^{-1}) = \mathbb{1},\end{aligned} \tag{3.44}$$

whereas

$$\eta \circ \epsilon(T_b) = \mathbb{1}, \tag{3.45}$$

again showing the consistency. Let us calculate antipode of a product of operators by using the identity (3.38). Applying the left-hand side of the identity to $T_b \cdot T_c$, we have

$$\begin{aligned}\cdot \circ (S \otimes \mathrm{id}) \circ \Delta(T_b \cdot T_c) &= \cdot \circ (S \otimes \mathrm{id})(T_b \cdot T_c \otimes T_b \cdot T_c) \\ &= \cdot(S(T_b \cdot T_c) \otimes T_b \cdot T_c) \\ &= S(T_b \cdot T_c) \cdot (T_b \cdot T_c),\end{aligned} \tag{3.46}$$

while the right-hand side gives

$$\eta \circ \epsilon(T_b \cdot T_c) = \eta(\epsilon(T_b)\epsilon(T_c)) = \eta(1) = \mathbb{1}, \tag{3.47}$$

so that the identity reads

$$S(T_b \cdot T_c) = (T_b \cdot T_c)^{-1} = T_c^{-1} \cdot T_b^{-1} = S(T_c) \cdot S(T_b), \tag{3.48}$$

showing the antialgebraic nature of the antipode. In a similar way can we show that in general

$$S(g_1 \cdots g_n) = S(g_n) \cdots S(g_1), \qquad g_i = T_b \quad \text{or} \quad (-1)^{\mathcal{F}}. \tag{3.49}$$

Applying the same identity to $T_b \cdot Q_A^{\mathrm{lat}}$, we get for the left-hand side

$$\begin{aligned}\cdot \circ (S \otimes \mathrm{id}) \circ \Delta(T_b \cdot Q_A^{\mathrm{lat}}) &= \cdot \circ (S \otimes \mathrm{id})(T_b \cdot Q_A^{\mathrm{lat}} \otimes T_b \cdot T_{a_A^r} \\ &\quad + T_b \cdot (-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes T_b \cdot Q_A^{\mathrm{lat}}) \\ &= S(T_b \cdot Q_A^{\mathrm{lat}}) \cdot T_b \cdot T_{a_A^r} \\ &\quad + S(T_b \cdot (-1)^{\mathcal{F}} \cdot T_{a_A^l}) \\ &\quad \cdot T_b \cdot Q_A^{\mathrm{lat}},\end{aligned} \tag{3.50}$$

while the right-hand side is

$$\eta \circ \epsilon(T_b \cdot Q_A^{\mathrm{lat}}) = \eta(\epsilon(T_b)\epsilon(Q_A^{\mathrm{lat}})) = \eta(0) = 0, \tag{3.51}$$

and thus the identity gives

$$\begin{aligned}S(T_b \cdot Q_A^{\mathrm{lat}}) &= -S(T_b \cdot (-1)^{\mathcal{F}} \cdot T_{a_A^l}) \cdot T_b \cdot Q_A^{\mathrm{lat}} \cdot T_{a_a^r}^{-1} \cdot T_b^{-1} \\ &= -T_{a_A^l}^{-1} \cdot (-1)^{\mathcal{F}} \cdot Q_A^{\mathrm{lat}} \cdot T_{a_a^r}^{-1} \cdot T_b^{-1} \\ &= S(Q_A^{\mathrm{lat}}) \cdot S(T_b).\end{aligned} \tag{3.52}$$

Now we proceed to do the same calculation for $Q_A^{\mathrm{lat}} \cdot Q_B^{\mathrm{lat}}$: the left-hand side reads

$$\begin{aligned}\cdot \circ (S \otimes \mathrm{id}) \circ \Delta(Q_A^{\mathrm{lat}} \cdot Q_B^{\mathrm{lat}}) &= \cdot \circ (S \otimes \mathrm{id})(Q_A^{\mathrm{lat}} \cdot Q_B^{\mathrm{lat}} \otimes T_{a_A^r} \cdot T_{a_B^r} + Q_A^{\mathrm{lat}} \cdot (-1)^{\mathcal{F}} \cdot T_{a_B^l} \otimes T_{a_A^r} \cdot Q_B^{\mathrm{lat}} \\ &\quad + (-1)^{\mathcal{F}} \cdot T_{a_A^l} \cdot Q_B^{\mathrm{lat}} \otimes Q_A^{\mathrm{lat}} \cdot T_{a_B^r} + (-1)^{\mathcal{F}} \cdot T_{a_A^l} \cdot (-1)^{\mathcal{F}} \cdot T_{a_B^l} \otimes Q_A^{\mathrm{lat}} \cdot Q_B^{\mathrm{lat}}) \\ &= S(Q_A^{\mathrm{lat}} \cdot Q_B^{\mathrm{lat}}) \cdot T_{a_A^r} \cdot T_{a_B^r} + S(Q_A^{\mathrm{lat}} \cdot (-1)^{\mathcal{F}} \cdot T_{a_B^l}) \cdot T_{a_A^r} \cdot Q_B^{\mathrm{lat}} \\ &\quad + S((-1)^{\mathcal{F}} \cdot T_{a_A^l} \cdot Q_B^{\mathrm{lat}}) \cdot Q_A^{\mathrm{lat}} \cdot T_{a_B^r} + S((-1)^{\mathcal{F}} \cdot T_{a_A^l} \cdot (-1)^{\mathcal{F}} \cdot T_{a_B^l}) \cdot Q_A^{\mathrm{lat}} \cdot Q_B^{\mathrm{lat}},\end{aligned} \tag{3.53}$$

and the right-hand side

$$\eta \circ \epsilon(Q_A^{\mathrm{lat}} \cdot Q_B^{\mathrm{lat}}) = \eta \circ (\epsilon(Q_A^{\mathrm{lat}})\epsilon(Q_B^{\mathrm{lat}})) = \eta(0) = 0, \tag{3.54}$$

so that the identity requires that

$$\begin{aligned}S(Q_A^{\mathrm{lat}} \cdot Q_B^{\mathrm{lat}}) \cdot T_{a_A^r} T_{a_B^r} &= T_{a_B^l}^{-1} \cdot (-1)^{\mathcal{F}} \cdot T_{a_A^l}^{-1} \cdot (-1)^{\mathcal{F}} \cdot Q_A^{\mathrm{lat}} \cdot T_{a_A^r}^{-1} \cdot T_{a_A^r} \cdot Q_B^{\mathrm{lat}} \\ &\quad + T_{a_B^l}^{-1} \cdot (-1)^{\mathcal{F}} \cdot Q_B^{\mathrm{lat}} \cdot T_{a_B^r}^{-1} \cdot T_{a_A^l}^{-1} \cdot (-1)^{\mathcal{F}} \cdot Q_A^{\mathrm{lat}} \cdot T_{a_B^r} - T_{a_B^l}^{-1} \cdot (-1)^{\mathcal{F}} \cdot T_{a_A^l}^{-1} \cdot (-1)^{\mathcal{F}} \cdot Q_A^{\mathrm{lat}} \cdot Q_B^{\mathrm{lat}}, \\ \text{i.e. } S(Q_A^{\mathrm{lat}} \cdot Q_B^{\mathrm{lat}}) &= T_{a_B^l}^{-1} \cdot (-1)^{\mathcal{F}} \cdot Q_B^{\mathrm{lat}} \cdot T_{a_B^r}^{-1} \cdot T_{a_A^l}^{-1} \cdot (-1)^{\mathcal{F}} \cdot Q_A^{\mathrm{lat}} \cdot T_{a_A^r}^{-1} = S(Q_B^{\mathrm{lat}}) \cdot S(Q_A^{\mathrm{lat}}),\end{aligned} \tag{3.55}$$

thus we find that the antialgebraic nature of the antipode map holds regardless of the fermionic nature of the supercharges.

The other identity $\cdot \circ (\mathrm{id} \otimes S) = \eta \circ \epsilon$ also gives the same result.

With all these algebraic consistencies satisfied, we conclude that the universal enveloping algebra, $\mathcal{U}(\mathcal{A})$, i.e. our lattice superalgebra, can be regarded as a Hopf algebra. We now move on to the construction of a field theory which has this Hopf algebraic symmetry. We follow the general scheme formulated by Oeckl [41] and called braided quantum field theory (BQFT). To this purpose, we have to specify the complete algebraic nature of the fields which should be consistent with the algebraic structure of the Hopf algebra. This requires one more nontrivial ingredient called braiding, or "shifted commutation" in our language. It is in a way a generalization of the statistics of fields. In short, covariance of the field theory under the Hopf algebraic symmetry forces us to introduce the braiding in a consistent way. We will see this below for our specific case of the lattice superalgebra and the corresponding field theory.

B. Shift structure as a braiding

Here we explain why we need braiding or shift structure in the space of fields, beginning with a simple illustration. Suppose we are considering normal supersymmetry with a bosonic field ϕ and a fermionic field ψ. Needless to say, bosonic fields commute with any other fields, while fermionic fields anticommute only with other fermions. Now take a supertransformation $Q\phi = \chi$ with a normal supercharge Q which is supposed to obey the Leibniz rule $Q(\varphi_1\varphi_2) = Q\varphi_1\varphi_2 + (-1)^{|\varphi_1|}\varphi_1 Q\varphi_2$. In the Hopf algebraic description we may say that it has the coproduct $\Delta(Q) = Q \otimes \mathbb{1} + (-1)^{\mathcal{F}} \mathbb{1} \otimes Q$ as before. We "know" that the field χ is fermionic, as a field resulting from a supersymmetry transformation of the boson ϕ. The point is that this fact is indeed inevitable; we are forced to choose χ to be fermionic for the algebraic consistency. In fact, note that the quantity $Q(\phi\psi) = \chi\psi + \phi(Q\psi)$ is equal to $Q(\psi\phi) = (Q\psi)\phi - \psi\chi$, because ϕ is defined as a boson, i.e. $\phi\psi = \psi\phi$. Comparing these two relations, we find $\chi\psi + \psi\chi = (Q\psi)\phi - \phi(Q\psi)$, which is zero again due to the fact that ϕ is bosonic. As a result we have that $\chi\psi = -\psi\chi$, "proving" that χ is a fermion. The essence of this proof is twofold: on one side the coproduct structure of the transformation operator Q, especially the factor $(-1)^{\mathcal{F}}$, and on the other the covariance of the transformation Q under the exchange of fields. Namely, when we exchange the order of the fields in a product and then apply the transformation Q, the result is the same as when we first apply the transformation Q on the product of fields and then exchange the order of the transformed objects. It is nothing but the statistics of fields that prescribes how the order of fields can be exchanged. Here we have just seen a natural and obvious fact that the statistics of fields should be consistent with the algebraic structure and the covariance of the transformations applied to the fields. It might still be worth stressing it, however, because it is the reason why we need braiding for our present application of a Hopf algebraic symmetry on fields. It is also the reason why we think of braiding as giving a generalized statistics. We are going to investigate these issues in detail in the following.

Let us introduce the general notion of exchanging the order of fields. We denote the exchanged object of $\varphi_1 \otimes \varphi_2$ as

$$\Psi_{X_1,X_2}(\varphi_1 \otimes \varphi_2), \qquad \varphi_1 \in X_1, \qquad \varphi_2 \in X_2. \tag{3.56}$$

The map Ψ is called a braid when it satisfies some natural consistency conditions (see Appendix A). The trivial braiding is given as the normal transposition and, in the application to the link formalism, we assume that the scalar fields on the sites of the lattice would have the trivial braiding nature:

$$\begin{aligned} &\Psi_{X_s,X_s}(\phi_1 \otimes \phi_2) = \phi_2 \otimes \phi_1, \\ &\phi_1, \phi_2 \in X_s\text{: scalar fields on sites.} \end{aligned} \tag{3.57}$$

Repeating the argument above, we may apply Q_A^{lat} on the product of scalar fields, or equivalently, take the action of coproduct of Q_A^{lat} as

$$\begin{aligned} \Delta(Q_A^{\mathrm{lat}}) \triangleright (\phi_1 \otimes \phi_2) = {}& (Q_A^{\mathrm{lat}}\phi_1) \otimes \phi_2(x + a_A^r) \\ &+ \phi_1(x + a_A^l) \otimes (Q_A^{\mathrm{lat}}\phi_2). \end{aligned} \tag{3.58}$$

Similarly on the exchanged product,

$$\begin{aligned} \Delta(Q_A^{\mathrm{lat}}) \triangleright (\phi_2 \otimes \phi_1) &= \Delta(Q_A^{\mathrm{lat}}) \triangleright \Psi_{X_s,X_s}(\phi_1 \otimes \phi_2) \\ &= (Q_A^{\mathrm{lat}}\phi_2) \otimes \phi_1(x + a_A^r) \\ &\quad + \phi_2(x + a_A^l) \otimes (Q_A^{\mathrm{lat}}\phi_1). \end{aligned} \tag{3.59}$$

We now assume the covariance of the braiding under symmetry transformations or, in other words, we assume the braiding to be an intertwiner of the transformations. In the present case, this requires that

$$\Delta(Q_A^{\mathrm{lat}}) \triangleright \Psi_{X_s,X_s}(\phi_1 \otimes \phi_2) = \Psi'(\Delta(Q_A^{\mathrm{lat}}) \triangleright (\phi_1 \otimes \phi_2)). \tag{3.60}$$

The left-hand side is given by (3.59), while the right-hand side is

$$\begin{aligned} \Psi'(\Delta(Q_A^{\mathrm{lat}}) \triangleright (\phi_1 \otimes \phi_2)) = {}& \Psi_{X_{fA},Xs}((Q_A^{\mathrm{lat}}\phi_1) \otimes \phi_2(x + a_A^r)) \\ &+ \Psi_{Xs,X_{fA}}(\phi_1(x + a_A^l) \\ &\otimes (Q_A^{\mathrm{lat}}\phi_2)), \end{aligned} \tag{3.61}$$

where we have denoted the space of fermionic fields of the index A as X_{fA} to which the transformed fields $Q_A^{\mathrm{lat}}\phi_{1,2}$ are to belong. Comparing these two equations, and noting that the fields ϕ_1 and ϕ_2 could be completely independent, we find

$$\begin{aligned}\Psi_{X_{fA},Xs}((Q_A^{\rm lat}\phi_1)\otimes\phi_2(x+a_A^r)) &= \phi_2(x+a_A^l)\otimes(Q_A^{\rm lat}\phi_1),\\ \Psi_{Xs,X_{fA}}(\phi_1(x+a_A^l)\otimes(Q_A^{\rm lat}\phi_2)) &= (Q_A^{\rm lat}\phi_2)\otimes\phi_1(x+a_A^r).\end{aligned}\tag{3.62}$$

This is not the trivial braiding as in (3.57). Instead, this braiding means that when we exchange the order of the fermion $Q_A^{\rm lat}\phi$ with the other field, it changes the argument of the other field by the amount $a_A^l - a_A^r$ under the exchange from the left to the right, and by the opposite amount under the exchange from the right to the left. Recalling that the scalar fields obey the trivial braiding, we might interpret this fact as that the transformed fields, which are fermions, inherited the nontrivial braiding nature from the supercharge, which, in a way, shows that the nontrivial braiding is already in the structure of coproduct. In fact, this kind of nontrivial braiding is referred to as the shifted commutation structure in the link formalism.

We have to emphasize here that the "claimed inconsistency" [42] explained in Sec. II no longer appears after incorporating this nontrivial braiding in the nongauged link formalism. Our approach which is purely based on the Hopf algebraic description clarifies the necessity of the braiding and shows how that claimed inconsistency can be resolved.

As a further example, to show how things work, let us consider

$$\Delta(Q_A^{\rm lat})\triangleright(\psi_{1A}(x)\otimes\phi_2(x)) = (Q_B^{\rm lat}\psi_{1A})(x)\otimes\phi_2(x+a_B^r) - \psi_{1A}(x+a_B^l)\otimes\psi_{2B}(x),\tag{3.63}$$

where $\psi_2 := Q_B^{\rm lat}\phi_2$, and thus

$$\begin{aligned}\Psi'(\Delta(Q_B^{\rm lat})\triangleright(\psi_{1A}(x)\otimes\phi_2(x))) &= \Psi_{X_{AB},Xs}((Q_B^{\rm lat}\psi_{1A})(x)\otimes\phi_2(x+a_B^r))\\ &\quad - \Psi_{X_{fA},X_{fB}}(\psi_{1A}(x+a_B^l)\psi_{2B}(x)),\end{aligned}\tag{3.64}$$

whereas

$$\begin{aligned}\Delta(Q_B^{\rm lat})\triangleright(\phi_2(x+a_A^l-a_A^r)\otimes\psi_{1A}(x)) &= \Delta(Q_B^{\rm lat})\triangleright\Psi_{X_{fA},X_s}(\psi_{1A}(x)\otimes\phi_2(x))\\ &= \psi_{2B}(x+a_A^l-a_A^r)\otimes\psi_{1A}(x+a_B^r)\\ &\quad + \phi_2(x+a_A^l-a_A^r+a_B^l)\otimes(Q_B^{\rm lat}\psi_{1A}(x)).\end{aligned}\tag{3.65}$$

Here X_{AB} is such that $Q_B^{\rm lat}\psi_A \in X_{AB}$. Assuming again the covariance

$$\Psi'(\Delta(Q_B^{\rm lat})\triangleright(\psi_{1A}(x)\otimes\phi_2(x))) = \Delta(Q_B^{\rm lat})\triangleright\Psi_{X_{fA},X_s}(\psi_{1A}(x)\otimes\phi_2(x)),\tag{3.66}$$

we obtain the following braiding relations:

$$\begin{aligned}\Psi_{X_{AB},Xs}((Q_B^{\rm lat}\psi_{1A})(x)\otimes\phi_2(x+a_B^r)) &= \phi_2(x+a_A^l-a_A^r+a_B^l)\otimes(Q_B^{\rm lat}\psi_{1A}(x)),\\ \Psi_{X_{fA},X_{fB}}(\psi_{1A}(x+a_B^l)\otimes\psi_{2B}(x)) &= -\psi_{2B}(x+a_A^l-a_A^r)\otimes\psi_{1A}(x+a_B^r).\end{aligned}\tag{3.67}$$

Notice, in passing, that from the first equation of (3.67), we have

$$\begin{aligned}\Psi_{X_{AB},Xs}((Q_B^{\rm lat}\psi_{1A})(x)\otimes\phi_2(x)) &= \phi_2(x+a_A^l-a_A^r+a_B^l-a_B^r)\otimes(Q_B^{\rm lat}\psi_{1A})(x),\\ \Psi_{X_{AB},Xs}((Q_A^{\rm lat}\psi_{1B})(x)\otimes\phi_2(x)) &= \phi_2(x+a_B^l-a_B^r+a_A^l-a_A^r)\otimes(Q_A^{\rm lat}\psi_{1B})(x),\end{aligned}\tag{3.68}$$

so that, summing up these two,

$$\begin{aligned}\Psi_{X_{AB},Xs}((\{Q_B^{\rm lat},Q_A^{\rm lat}\}\phi_1)(x)\otimes\phi_2(x)) &= 2\tau^\mu_{AB}\Psi_{X_{AB},Xs}((P_\mu^{\rm lat}\phi_1)(x)\otimes\phi_2(x))\\ &= \phi_2(x+a_A^l-a_A^r+a_B^l-a_B^r)\otimes(\{Q_B^{\rm lat},Q_A^{\rm lat}\}\phi_1)(x)\\ &= 2\tau^\mu_{AB}\phi_2(x+a_A^l-a_A^r+a_B^l-a_B^r)\otimes(P_\mu^{\rm lat}\phi_1)(x).\end{aligned}\tag{3.69}$$

The previous examples show how the braiding relation works in general. We can write it as

$$\begin{aligned}\Psi(_{A_0\cdots A_p}(x)\otimes\varphi'_{B_0\cdots B_q}(y)) &= (-1)^{pq}\varphi'_{B_0\cdots B_q}\Big(y+\sum_{i=1}^{p}(a^l_{A_i} -_{A_i} a^r_{A_i})\Big)\\ &\quad\otimes\varphi_{A_0\cdots A_p}\Big(x-\sum_{i=1}^{q}(a^l_{B_i}-a^r_{B_i})\Big),\end{aligned}\tag{3.70}$$

where we have used the abbreviation $\varphi_{A_0\cdots A_p} := \cdots Q_{A_p}^{\rm lat}\cdots Q_{A_1}^{\rm lat}\varphi_{A_0}$, (which could vanish) and $\varphi_{A_0} := \phi$. If we had introduced a scalar field with a nontrivial braiding/shift structure itself, we would have had to further generalize this relation.

The exchanging of a product of more than three fields should be naturally introduced. In the case of the trivial braiding, we have

$$\begin{aligned}\Psi_{X_1\otimes X_2,X_3}((\phi_1\otimes\phi_2)\otimes\phi_3) &= \phi_3\otimes(\phi_1\otimes\phi_2)\\ &= \phi_3\otimes\phi_1\otimes\phi_2\\ &= \Psi_{X_1,X_3}((\phi_1\otimes\phi_3)\otimes\phi_2)\\ &= \Psi_{X_1,X_3}\circ\Psi_{X_2,X_3}\\ &\quad\times((\phi_1\otimes\phi_2)\otimes\phi_3)),\end{aligned}\tag{3.71}$$

which can be extended in the general case to

$$\Psi_{X_1\otimes X_2,X_3} = \Psi_{X_1,X_3}\circ\Psi_{X_2,X_3},\qquad \Psi_{X_1,X_2\otimes X_3} = \Psi_{X_1,X_3}\circ\Psi_{X_1,X_2}. \tag{3.72}$$

For example,

$$\Psi_{X_1,X_2\otimes X_3}(\phi_1(x)\otimes\phi_2(x)\otimes\psi_{3A}(x)) = \phi_2(x)\otimes\psi_{3A}(x)\otimes\phi_1(x-a^l_A+a^r_A). \tag{3.73}$$

For the exchanging with the trivial or constant fields, we should impose

$$\Psi_{X^0_e,X} = \Psi_{X,X^0_e} = \mathrm{id}, \tag{3.74}$$

where X^0_e denotes the space of trivial bosonic fields. Using these rules, let us calculate one more example:

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\circ\Delta(Q^{\mathrm{lat}}_A)\triangleright(\phi_1\otimes\phi_2\otimes\psi_{3B}) &= \psi_{1A}(x)\otimes\phi_2(x+a^r_A)\psi_{3B}(x+_A a^r_A)\\ &\quad+\phi_1(x+a^l_A)\otimes\psi_{2A}(x)\otimes\psi_{3B}(x+a^r_A)\\ &\quad+\phi_1(x+a^l_A)\otimes\phi_2(x+a^l_A)\otimes(Q^{\mathrm{lat}}_A\psi_{3B})(x),\end{aligned} \tag{3.75}$$

where we have used the coassociativity (3.18). Applying $\Psi_{X_1,X_2\otimes X_3}$, and comparing it with

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\circ\Delta(Q^{\mathrm{lat}}_A)&\triangleright(\Psi_{X_1,X_2\otimes X_3}(\phi_1(x)\otimes\phi_2(x)\otimes\psi_{3B}(x)))\\ &=(\mathrm{id}\otimes\Delta)\circ\Delta(Q^{\mathrm{lat}}_A)\triangleright(\phi_2(x)\otimes\psi_{3B}(x)\\ &\quad\otimes\phi_1(x-a^l_B+a^r_B))\\ &=\psi_{2A}(x)\otimes\psi_{3B}(x+a^r_A)\otimes\phi_1(x+a^r_A-a^l_B+a^r_B)\\ &\quad+\phi_2(x+a^l_A)\otimes(Q^{\mathrm{lat}}_A\psi_{3B})(x)\otimes\phi_1(x+a^r_A-a^l_B+a^r_B)\\ &\quad-\phi_2(x+a^l_A)\otimes\psi_{3B}(x+a^l_A)\otimes\psi_{1A}(x-a^l_B+a^r_B),\end{aligned} \tag{3.76}$$

where we have used (3.73) and (3.18) again, we find that

$$\begin{aligned}&\Psi_{X_1,X_2\otimes X_3}(\psi_{1A}(x)\otimes(\phi_2(x+a^r_A)\otimes\psi_{3B}(x+a^r_A)))\\ &= -\phi_2(x+a^l_A)\otimes\psi_{3B}(x+a^l_A)\otimes\psi_{1A}(x-a^l_B+a^r_B),\\ &\Psi_{X_1,X_2\otimes X_3}(\phi_1(x+a^l_A)\otimes(\psi_{2A}(x)\otimes\psi_{3B}(x+a^r_A)))\\ &= \psi_{2A}(x)\otimes\psi_{3B}(x+a^r_A)\otimes\phi_1(x+a^r_A-a^l_B+a^r_B),\\ &\Psi_{X_1,X_2\otimes X_3}(\phi_1(x+a^l_A)\otimes(\phi_2(x+a^l_A)\otimes(Q^{\mathrm{lat}}_A\psi_{3B})(x)))\\ &= \phi_2(x+a^l_A)\otimes(Q^{\mathrm{lat}}_A\psi_{3B})(x)\otimes\phi_1(x+a^r_A-a^l_B+a^r_B).\end{aligned} \tag{3.77}$$

These examples show that the braiding, i.e. the amount of shifts of the arguments of fields induced under exchanging, is additive; for a field φ_1 with the shift a_1 and another φ_2 with the shift a_2, the product $\varphi_1\otimes\varphi_2$ has the shift a_1+a_2. This is a simple consequence of the natural braiding rule (3.72).

These observations motivate us to introduce the notion of shift structure of fields as a kind of an additive "grading" determined by which supercharges are acting on the scalar fields to produce the given field. We may thus introduce, in addition to the normal graded structure of fields, i.e. bosonic and fermionic statistics, a graded structure which we call the shift structure so that the space of elementary fields X is decomposed in general as

$$X = \bigoplus_{\mathrm{grading}} X_e\oplus X_o. \tag{3.78}$$

The space of whole fields, $\hat{X}$, is also decomposed with respect to the shift/grading structure the same way;

$$\hat{X} = \bigoplus_{n=0}^{\infty}\bigoplus_{\mathrm{grading}} X^n. \tag{3.79}$$

The field contents and their shift structure are determined in each model, mainly with the use of the Leibniz rule consistency conditions. We have to emphasize that this grading structure is especially crucial to define an explicit form of the momentum operator P^{lat}_μ. As mentioned at the beginning of the previous subsection, we might have started with taking a difference operator as its representation: $(P^{\mathrm{lat}}_\mu\phi)(x) = a^{-1}(\phi(x+a^l\hat{\mu})-\phi(x+a^r\hat{\mu}))$. This, however, does not satisfy the relation (3.69), since we have assumed that ϕ obeys the trivial braiding and thus $a^{-1}(\phi(x+a^l\hat{\mu})-\phi(x+a^r\hat{\mu}))$ has the same trivial braiding. We thus need an expression like $(P^{\mathrm{lat}}_\mu\phi)\times(x) = a^{-1}(\phi'(x+a^l\hat{\mu})-\phi'(x+a^r\hat{\mu}))$ for which ϕ' has an additional grading to satisfy the relation (3.69). To give its consistent representation is important for the formulation and will be treated elsewhere. Here our claim is that the algebraic description presented here can still formalize a field theory with the Hopf algebraic symmetry even if we do not have the explicit representation for these graded fields and the momentum operator, as is seen below.

Let us note also that our braiding satisfies the relation

$$\Psi_{X_1,X_2}\circ\Psi_{X_2,X_1} = \mathrm{id}, \tag{3.80}$$

or equivalently,

$$\Psi_{X_2,X_1} = \Psi^{-1}_{X_1,X_2}. \tag{3.81}$$

In a standard mathematical terminology this kind of exchanging map Ψ *is not* referred to as a braid, or one may distinguish it from the *strictly braided* case. Here we use the term braiding in a broader sense, including also the ones like (3.80) that have no real braiding structure. We emphasize that it is still nontrivial in the sense that $\Psi\neq\tau$, where τ is the simple transposition: $\tau(\varphi_1\otimes\varphi_2) = \varphi_2\otimes\varphi_1$. In fact, our braiding is a transposition plus some shifts of the arguments of the fields up to the statistics factors. This should be compared with the statistics of usual bosons and fermions; for that case the braiding is nothing but the simple exchange up to the statistics. We could therefore describe these facts as that the fields which represent our Hopf algebraic lattice superalgebra naturally obtain a braiding structure which expresses a slightly more general statistics than the usual one.[9]

According to the general discussion (see Appendix A), it seems that the simple braiding structure (3.80) might be given as an explicit formula (A20) when the corresponding Hopf algebra is *triangular*. We find that this is indeed the case at least formally; our symmetry algebra could be identified as a triangular Hopf algebra with an additional grading structure, and the braiding (3.70) be given with the corresponding *(quasi)triangular structure* $\mathcal{R}$. To see this, let us first introduce a formal expression for the shift operator T_b

$$T_b = \exp(b^\mu \partial_\mu). \tag{3.82}$$

We write this as if the continuum derivative operator ∂_μ were introduced on the lattice; however it must be understood as a formal operator and only well-defined when exponentiated to give the lattice proper operator T_b. We may impose

$$\Delta(\partial_\mu) = \partial_\mu \otimes \mathbb{1} + \mathbb{1} \otimes \partial_\mu, \quad \epsilon(\partial_\mu) = 0, \quad S(\partial_\mu) = -\partial_\mu, \tag{3.83}$$

which should be interpreted as formal equivalents of the relations (3.8) and (3.38) for T_b. We then recall that the generator Q_A^{lat} has a kind of grading corresponding to the shift $a_A := a_A^l - a_A^r$ induced under the exchange of $Q_A^{\mathrm{lat}}\varphi$ with other fields. We may express this fact by introducing another operator L^μ such that

$$a[L^\mu, Q_A^{\mathrm{lat}}] = (a_A)^\mu Q_A^{\mathrm{lat}}, \quad \text{i.e. } [L^\mu, Q_A^{\mathrm{lat}}] = l_A^\mu Q_A^{\mathrm{lat}}, \tag{3.84}$$

where $l_A^\mu = a^{-1}(a_A)^\mu$. Since P_μ^{lat} is given as $P_\mu^{\mathrm{lat}} \sim \{Q_A^{\mathrm{lat}}, Q_B^{\mathrm{lat}}\}$, it also has grading as in

$$\begin{aligned} a[L^\mu, P_\nu^{\mathrm{lat}}] &= a_P(\hat{\nu})^\mu P_\nu^{\mathrm{lat}} = a_P \delta_\nu^\mu P_\nu^{\mathrm{lat}}, \\ \text{i.e. } [L^\mu, P_\nu^{\mathrm{lat}}] &= l_P \delta_\nu^\mu P_\nu^{\mathrm{lat}}, \end{aligned} \tag{3.85}$$

where $a_P := a^l - a^r$ and $l_P := a^{-1}a_P$. We list the other relations

$$[L^\mu, T_b] = [L^\mu, (-1)^{\mathcal{F}}] = [L^\mu, L^\nu] = 0, \tag{3.86}$$

where the first two are due to the fact that neither T_b nor $(-1)^{\mathcal{F}}$ induces shift and the latter one is automatic because of the "Abelian" nature of (3.84) and (3.85) and the others. For completeness, we set

$$\begin{aligned} \Delta(L^\mu) &= L^\mu \otimes \mathbb{1} + \mathbb{1} \otimes L^\mu, \\ \epsilon(L^\mu) &= 0, \qquad S(L^\mu) = -L^\mu. \end{aligned} \tag{3.87}$$

Now let

$$\mathcal{R} := \exp(aL^\mu \otimes \partial_\mu - a\partial_\mu \otimes L^\mu + i\pi\mathcal{F} \otimes \mathcal{F}). \tag{3.88}$$

We can show that this formal operator $\mathcal{R} \in \mathcal{U}(\mathcal{A}) \otimes \mathcal{U}(\mathcal{A})$ is invertible and satisfies the relations

$$\begin{aligned} \tau \circ \Delta(h) = \mathcal{R} \cdot \Delta(h) \cdot \mathcal{R}^{-1}, \qquad (\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \\ (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}. \end{aligned} \tag{3.89}$$

(See Appendix A for the notation.) Notice first that $\mathcal{R}^{-1}$ is given as

$$\mathcal{R}^{-1} = \exp(-aL^\mu \otimes \partial_\mu + a\partial_\mu \otimes L^\mu + i\pi\mathcal{F} \otimes \mathcal{F}) \tag{3.90}$$

(recall that $\mathcal{F}$ only gives integer numbers), and so that

$$\begin{aligned} \mathcal{R}_{21} &= \exp(a\partial_\mu \otimes L^\mu - aL^\mu \otimes \partial_\mu + i\pi\mathcal{F} \otimes \mathcal{F}) \\ &= \mathcal{R}^{-1}. \end{aligned} \tag{3.91}$$

As for the first relation in (3.89), compute

$$\mathcal{R} . \Delta(h) \cdot \mathcal{R}^{-1} = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathrm{ad}(L))^n \Delta(h), \tag{3.92}$$

where we have written $L := aL^\mu \otimes \partial_\mu - a\partial_\mu \otimes L^\mu + i\pi\mathcal{F} \otimes \mathcal{F}$ just for simplicity, and used ad to denote the Lie derivative. For $h = Q_A^{\mathrm{lat}}$,

$$\begin{aligned} \mathrm{ad}(L)\Delta(Q_A^{\mathrm{lat}}) &= [aL^\mu \otimes \partial_\mu - a\partial_\mu \otimes L^\mu + i\pi\mathcal{F} \otimes \mathcal{F}, Q_A^{\mathrm{lat}} \otimes T_{a_A^r} + (-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes Q_A^{\mathrm{lat}}] \\ &= [aL^\mu \otimes \partial_\mu + i\pi\mathcal{F} \otimes \mathcal{F}, Q_A^{\mathrm{lat}} \otimes T_{a_A^r}] + [-a\partial_\mu \otimes L^\mu + i\pi\mathcal{F} \otimes \mathcal{F}, (-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes Q_A^{\mathrm{lat}}] \\ &= a[L^\mu, Q_A^{\mathrm{lat}}] \otimes \partial_\mu \cdot T_{a_A^r} + i\pi[\mathcal{F}, Q_A^{\mathrm{lat}}] \otimes \mathcal{F} \cdot T_{a_A^r} - a\partial_\mu \cdot (-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes [L^\mu, Q_A^{\mathrm{lat}}] \\ &\quad + i\pi\mathcal{F} \cdot (-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes [\mathcal{F}, Q_A^{\mathrm{lat}}] \\ &= Q_A^{\mathrm{lat}} \otimes ((a_A)^\mu \partial_\mu + i\pi\mathcal{F}) \cdot T_{a_A^r} + (-(a_A)^\mu \partial_\mu + i\pi\mathcal{F}) \cdot (-1)^{\mathcal{F}} T_{a_A^l} \otimes Q_A^{\mathrm{lat}}, \end{aligned} \tag{3.93}$$

so that

$$\begin{aligned} (\mathrm{ad}(L))^n \Delta(Q_A^{\mathrm{lat}}) = Q_A^{\mathrm{lat}} \otimes ((a_A)^\mu \partial_\mu + i\pi\mathcal{F})^n \cdot T_{a_A^r} \\ + (-(a_A)^\mu \partial_\mu + i\pi\mathcal{F})^n \\ \cdot (-1)^{\mathcal{F}} \cdot T_{a_A^l} \otimes Q_A^{\mathrm{lat}}. \end{aligned} \tag{3.94}$$

[9] A well-known example of generalized statistics is that of anyons, for which the exchanging map is strictly braided in general. Our statistics is thus more like the usual statistics than the anyonic one.

We therefore obtain

$$\begin{aligned}\mathcal{R}\cdot\Delta(Q_A^{\rm lat})\cdot\mathcal{R}^{-1} &= Q_A^{\rm lat}\otimes\exp((a_A)^\mu\partial_\mu+i\pi\mathcal{F})\cdot T_{a_A^r}\\ &\quad+\exp(-(a_A)^\mu\partial_\mu+i\pi\mathcal{F})\\ &\quad\cdot(-1)^{\mathcal{F}}\cdot T_{a_A^l}\otimes Q_A^{\rm lat}\\ &= Q_A^{\rm lat}\otimes(-1)^{\mathcal{F}}\cdot T_{a_A^l}+T_{a_A^r}\otimes Q_A^{\rm lat}\\ &= \tau\circ\Delta(Q_A^{\rm lat}),\end{aligned} \tag{3.95}$$

since

$$\begin{aligned}\exp(\pm(a_A)^\mu\partial_\mu+i\pi\mathcal{F}) &= \exp(\pm(a_A^l)^\mu\partial_\mu)\\ &\quad\cdot\exp(\mp(a_A^r)^\mu\partial_\mu)\cdot\exp(i\pi\mathcal{F})\\ &= T^{\pm}_{a_A^l}\cdot T^{\mp}_{a_A^r}.(-1)^{\mathcal{F}}.\end{aligned} \tag{3.96}$$

A simpler calculation leads to similar result for $h=P_\mu^{\rm lat}$ too. For $h=T_b$, $(-1)^{\mathcal{F}}$, L^μ it is rather clear that

$$\mathcal{R}\cdot\Delta(h)\cdot\mathcal{R}^{-1}=\Delta(h)=\tau\circ\Delta(h). \tag{3.97}$$

Thus the first equation in (3.89) indeed holds with the choice (3.88) for $\mathcal{R}$. The second relation follows as

$$\begin{aligned}(\Delta\otimes{\rm id})\mathcal{R} &= \exp(a\Delta(L^\mu)\otimes\partial_\mu-a\Delta(\partial_\mu)\otimes L^\mu+i\pi\Delta(\mathcal{F})\otimes\mathcal{F}\\ &= \exp(aL^\mu\otimes\mathbb{1}\otimes\partial_\mu-a\partial_\mu\otimes\mathbb{1}\otimes L^\mu+i\pi\mathcal{F}\otimes\mathbb{1}\otimes\mathcal{F}+a\mathbb{1}\otimes L^\mu\otimes\partial_\mu-a\mathbb{1}\otimes\partial_\mu\otimes L^\mu+i\pi\mathbb{1}\otimes\mathcal{F}\otimes\mathcal{F})\\ &= \exp(aL^\mu\otimes\mathbb{1}\otimes\partial_\mu-a\partial_\mu\otimes\mathbb{1}\otimes L^\mu+i\pi\mathcal{F}\otimes\mathbb{1}\otimes\mathcal{F})\cdot\exp(a\mathbb{1}\otimes L^\mu\otimes\partial_\mu-a\mathbb{1}\otimes\partial_\mu\otimes L^\mu+i\pi\mathbb{1}\otimes\mathcal{F}\otimes\mathcal{F})\\ &= \mathcal{R}_{13}\cdot\mathcal{R}_{23}.\end{aligned} \tag{3.98}$$

The third one is almost the same.

We have thus shown that the formal operator $\mathcal{R}$ given as (3.88) is a quasitriangular structure and, due to (3.91), our lattice superalgebra is identified as a triangular Hopf algebra. The whole space of fields, as a representation space of a triangular Hopf algebra, would be braided by $\mathcal{R}$ as in

$$\Psi=\tau\circ\mathcal{R}\triangleright, \tag{3.99}$$

which agrees with our formula (3.70) as now seen. We need the representation of L^μ on the elementary fields. First for the normal scalar fields $\{\phi,\cdots\}$ let

$$L^\mu\triangleright\phi=0\cdot\phi=0. \tag{3.100}$$

For the other fields in the irreducible supermultiplet to which the above bosonic fields belong, the actions of L^μ are automatically determined by the algebra (3.84). For instance, on $\psi_A:=Q_A^{\rm lat}\phi$, we find

$$\begin{aligned}L^\mu\triangleright\psi_A &= L^\mu\triangleright(Q_A^{\rm lat}\phi)=([L^\mu,Q_A^{\rm lat}]+Q_A^{\rm lat}\cdot L^\mu)\triangleright\phi\\ &= l_A^\mu Q_A^{\rm lat}\phi=l_A^\mu\psi_A.\end{aligned} \tag{3.101}$$

Then inductively, we find for $\varphi_{A_1\cdots A_n}=Q_{A_n}^{\rm lat}\cdots Q_{A_1}^{\rm lat}\phi$ that

$$L^\mu\triangleright\varphi_{A_1\cdots A_n}=(l_{A_1}+\cdots+l_{A_n})^\mu\varphi_{A_1\cdots A_n}. \tag{3.102}$$

These relations express explicitly the grading structure of fields explained above. We thus compute

$$\begin{aligned}&\mathcal{R}\triangleright(\varphi_{A_1\cdots A_p}(x)\otimes\varphi_{B_1\cdots B_q}(y))\\ &=\exp(\mathbb{1}\otimes((a_{A_1}+\cdots+a_{A_p})^\mu\partial_\mu)-((a_{B_1}+\cdots+a_{B_q})^\mu\partial_\mu)\\ &\quad\otimes\mathbb{1}+i\pi pq\mathbb{1}\otimes\mathbb{1})\triangleright(\varphi_{A_1\cdots A_p}(x)\otimes\varphi_{B_1\cdots B_q}(y))\\ &=(-1)^{pq}(\mathbb{1}\otimes T_{a_{A_1}+\cdots+a_{A_p}})\cdot(T^{-1}_{a_{B_1}+\cdots+a_{B_q}}\otimes\mathbb{1})\\ &\quad\times\triangleright(\varphi_{A_1\cdots A_p}(x)\otimes\varphi_{B_1\cdots B_q}(y))\\ &=\varphi_{A_1\cdots A_p}\Big(x-\sum_{i=1}^{p}a_{B_i}\Big)\otimes\varphi_{B_1\cdots B_q}\Big(y+\sum_{i=1}^{q}a_{A_i}\Big).\end{aligned} \tag{3.103}$$

Since here $a_{A_i}=a_{A_i}^l-a_{A_i}^r$ etc., we have shown that Eq. (3.99) does reproduce the general braiding rule (3.70).

It is worth pointing out that our quasitriangular structure $\mathcal{R}$ can be written as

$$\mathcal{R}=\chi_{21}\cdot\mathcal{R}_0\cdot\chi^{-1},\qquad\mathcal{R}_0:=\exp(i\pi\mathcal{F}\otimes\mathcal{F}), \tag{3.104}$$

with some invertible operator $\chi\in\mathcal{U}(\mathcal{A})\otimes\mathcal{U}(\mathcal{A})$ which satisfies the so-called *2-cocycle condition*

$$(\chi\otimes\mathbb{1})\cdot(\Delta\otimes{\rm id})\chi=(\mathbb{1}\otimes\chi)\cdot({\rm id}\otimes\Delta)\chi, \tag{3.105}$$

and the *counital condition*

$$(\epsilon\otimes{\rm id})\chi=({\rm id}\otimes\epsilon)\chi=\mathbb{1}. \tag{3.106}$$

Such an operator is not necessarily unique. We take one specific example to illustrate it:

$$\chi := \exp(a\partial_\mu \otimes L^{l\mu} + aL^{r\mu} \otimes \partial_\mu),$$
$$\chi_{21} = \exp(aL^{l\mu} \otimes \partial_\mu + a\partial_\mu \otimes L^{r\mu}), \tag{3.107}$$
$$\chi^{-1} = \exp(-a\partial_\mu \otimes L^{l\mu} - aL^{r\mu} \otimes \partial_\mu),$$

where we have introduced two more operators $L^{l\mu}$ and $L^{r\mu}$ such that $L^\mu = L^{l\mu} - L^{r\mu}$, namely

$$a[L^{l,r\mu}, Q_A^{\rm lat}] = (a_A^{l,r})^\mu Q_A^{\rm lat}, \quad \text{etc.}, \tag{3.108}$$

with coproduct, counit, and antipode formulas similar to those of L^μ. It is easy to see that (3.104) actually holds for this operator χ. The cocycle condition is fulfilled as

$$\begin{aligned}
(\text{l.h.s.}) &= \exp(a\partial_\mu \otimes L^{l\mu} \otimes \mathbb{1} + aL^{r\mu} \otimes \partial_\mu \otimes \mathbb{1}) \cdot \exp(a\Delta(\partial_\mu) \otimes L^{l\mu} + a\Delta(L^{r\mu}) \otimes \partial_\mu) \\
&= \exp(a\partial_\mu \otimes L^{l\mu} \otimes \mathbb{1} + aL^{r\mu} \otimes \partial_\mu \otimes \mathbb{1} + a\partial_\mu \otimes \mathbb{1} \otimes L^{l\mu} + a\mathbb{1} \otimes \partial_\mu \otimes L^{l\mu} + aL^{r\mu} \otimes \mathbb{1} \otimes \partial_\mu + a\mathbb{1} \otimes L^{r\mu} \otimes \partial_\mu) \\
&= \exp(\mathbb{1} \otimes a\partial_\mu \otimes L^{l\mu} + \mathbb{1} \otimes aL^{r\mu} \otimes \partial_\mu + a\partial_\mu \otimes L^{l\mu} \otimes \mathbb{1} + a\partial_\mu \otimes \mathbb{1} \otimes L^{l\mu} + aL^{r\mu} \otimes \partial_\mu \otimes \mathbb{1} + aL^{r\mu} \otimes \mathbb{1} \otimes \partial_\mu) \\
&= \exp(\mathbb{1} \otimes a\partial_\mu \otimes L^{l\mu} + \mathbb{1} \otimes aL^{r\mu} \otimes \partial_\mu) \cdot \exp(a\partial_\mu \otimes \Delta(L^{l\mu}) + aL^{r\mu} \otimes \Delta(\partial_\mu)) = (\text{r.h.s.}),
\end{aligned} \tag{3.109}$$

while counitality is clear because $\epsilon(\partial_\mu) = \epsilon(L^{l,r\mu}) = 0$. We thus conclude from these results that our lattice superalgebra $\mathcal{U}(\mathcal{A})$ with the quasitriangular structure $\mathcal{R}$ could be understood as the so-called *twist* by the *cocycle element* χ of some other Hopf algebra $\mathcal{U}(\mathcal{A})_0$ with the simple quasitriangular structure $\mathcal{R}_0$. The "untwisted" Hopf algebra $\mathcal{U}(\mathcal{A})_0$ has the same algebra and counit as those of $\mathcal{U}(\mathcal{A})$ but its coproduct and antipode are such that

$$\Delta(h) = \chi \cdot \Delta_0(h) \cdot \chi^{-1}, \qquad S(h) = U \cdot S_0(h) \cdot U^{-1},$$
$$U := \cdot(\text{id} \otimes S)\chi, \qquad U^{-1} = \cdot(S \otimes \text{id})\chi^{-1}. \tag{3.110}$$

Thus for $h = T_b$, $(-1)^{\mathcal{F}}$, L^μ we find $\Delta_0(h) = \Delta(h)$, whereas for $h = Q_A^{\rm lat}$, $P_\mu^{\rm lat}$, we can show that

$$\begin{aligned}
\Delta_0(Q_A^{\rm lat}) &= Q_A^{\rm lat} \otimes \mathbb{1} + (-1)^{\mathcal{F}} \otimes Q_A^{\rm lat}, \\
\Delta_0(P_\mu^{\rm lat}) &= P_\mu^{\rm lat} \otimes \mathbb{1} + \mathbb{1} \otimes P_\mu^{\rm lat}.
\end{aligned} \tag{3.111}$$

Since in the present case

$$U = \exp(-aL_+^\mu \cdot \partial_\mu), \qquad U^{-1} = \exp(aL_+^\mu \cdot \partial_\mu),$$
$$L_+^\mu := L^{l\mu} + L^{r\mu}, \tag{3.112}$$

the antipode as well remains unchanged for $h = T_b$, $(-1)^{\mathcal{F}}$, $L^{l,r\mu}$, but changed again for $h = Q_A^{\rm lat}$, $P_\mu^{\rm lat}$:

$$S_0(Q_A^{\rm lat}) = -(-1)^{\mathcal{F}} \cdot Q_A^{\rm lat}, \qquad S_0(P_\mu^{\rm lat}) = -P_\mu^{\rm lat}, \tag{3.113}$$

as it can be seen by using

$$\begin{aligned}
U \cdot Q_A^{\rm lat} \cdot U^{-1} &= \exp((a_A^l + a_A^r)^\mu \partial_\mu) \cdot Q_A^{\rm lat} \\
&= T_{a_A^l} \cdot T_{a_A^r} \cdot Q_A^{\rm lat}
\end{aligned} \tag{3.114}$$

and a similar equation for $P_\mu^{\rm lat}$.

We have found that the (un)twisted Hopf algebra $(\mathcal{U}(\mathcal{A})_0, \mathcal{R}_0)$ is much simpler and has the form of a normal universal enveloping superalgebra of normal supersymmetry. This result might seem confusing because under the twisting the algebraic structure of the original Hopf algebra remains the same and the operators themselves are not transformed; if such a simpler Hopf algebra exists, could we use that one to begin with without taking the deformed one $(\mathcal{U}(\mathcal{A}), \mathcal{R})$ into consideration? Actually we can equally formulate the whole story with the simpler Hopf algebra $(\mathcal{U}(\mathcal{A})_0, \mathcal{R}_0)$, but notice that this twisting transformation is only possible with the nontrivial "charge" or grading operators $L^{l,r\mu}$ at our disposal, and that the twisted Hopf algebra keeps them as well. On our original Hopf algebra $(\mathcal{U}(\mathcal{A}), \mathcal{R})$, these have a natural interpretation as those assigning how fields are geometrically put on the lattice and how operators affect such a geometrical structure. On the twisted Hopf algebra $(\mathcal{U}(\mathcal{A})_0, \mathcal{R}_0)$, this kind of interpretation is less clear since Δ_0, S_0, etc., just have normal structure and, nevertheless, these operators $L^{l,r\mu}$ must be included for the whole algebra to be represented exactly. This last observation would be quite crucial, particularly when compared with the no-go theorem presented in [43], since in the twisted algebra the momentum operator obeys the exact, not modified, Leibniz rule for which no local and translationally covariant representation is proved to exist. We expect that the nontrivial grading of the momentum operator may help resolve these difficulties.

One more aspect to be mentioned here is that the multiplication rule for the fields representing the algebra should be correspondingly modified under the twisting. Let us denote by $(\hat{X}, m)$ and $(\hat{X}_0, m_0)$ the spaces of fields which represent, respectively, the deformed algebra $(\mathcal{U}(\mathcal{A}), \mathcal{R})$ and the twisted algebra $(\mathcal{U}(\mathcal{A})_0, \mathcal{R}_0)$. Here m and m_0 are the multiplication maps on the spaces $\hat{X}$ and $\hat{X}_0$, respectively. On products in $\hat{X}$, $h \in \mathcal{U}(\mathcal{A})$ acts covariantly as we have seen in the previous subsection: $h \triangleright m(\varphi \otimes \varphi') = m(\Delta(h) \triangleright (\varphi \otimes \varphi'))$. According to the theory of twisting (see Appendix A), $h \in \mathcal{U}(\mathcal{A})_0$ can act covariantly on products of fields in $\hat{X}_0$ only with the product

$$m_0 := m \circ \chi \triangleright \tag{3.115}$$

[remember that the twisting from $(\mathcal{U}(\mathcal{A}), \mathcal{R})$ to $(\mathcal{U}(\mathcal{A})_0, \mathcal{R}_0)$ is given by χ^{-1}], as in

$$h \triangleright m_0(\varphi \otimes \varphi') = m_0(\Delta_0(h) \triangleright (\varphi \otimes \varphi')). \tag{3.116}$$

Suppose that this product m_0 is "commutative" in the sense that

$$m_0 \circ \Psi_0 = m_0, \quad \text{i.e. } m_0 \circ \tau \circ \mathcal{R}_0 \triangleright = m_0, \tag{3.117}$$

which means commutative up to the statistics factor induced by $\mathcal{R}_0$. This assumption would be natural because the twisted algebra $(\mathcal{U}(\mathcal{A})_0, \mathcal{R}_0)$ has the simple Hopf algebraic structure which is symmetric under the exchange of order of any objects. It turns out that then the multiplication m is again commutative up to the nontrivial statistics Ψ (thus noncommutative in the standard sense):

$$\begin{aligned} m \circ \Psi &= m_0 \circ \chi^{-1} \triangleright \circ \tau \circ \mathcal{R} \triangleright \\ &= m_0 \circ \tau \circ (\chi_{21}^{-1} \cdot \mathcal{R} \cdot \chi) \triangleright \circ \chi^{-1} \triangleright \\ &= m_0 \circ \Psi_0 \circ \chi^{-1} \triangleright = m_0 \circ \chi^{-1} \triangleright = m. \end{aligned} \tag{3.118}$$

This consequence in a way shows that the multiplication rule should incorporate the statistics in an obvious manner so that it becomes commutative up to the statistics. When the statistics is itself nontrivial, this notion of commutativity up to the statistics may be expressed as just a noncommutativity in the standard sense. In our case, we have

$$\varphi_{A_1 \cdots A_p}(x) \cdot \varphi_{B_1 \cdots B_q}(y) = (-1)^{pq} \varphi_{B_1 \cdots B_q}\Big(y + \sum_{i=1}^{p} a_{A_i}\Big) \cdot \varphi_{A_1 \cdots A_p}\Big(x - \sum_{i=1}^{q} a_{B_i}\Big). \tag{3.119}$$

We regard it as a consequence of either the lattice-deformed statistics, or the mild noncommutativity, and may use the notation $\varphi * \varphi'$ to emphasize its noncommutative nature.

We finally recall that the space of fields on the lattice $\hat{X}$, defined in (3.1), forms an algebra. It actually forms a Hopf algebra in a natural way [41,53]:

$$\begin{aligned} &m(\varphi_1 \otimes \varphi_2) = \varphi_1 \cdot \varphi_2 \quad \text{(product)}, \\ &\eta(1) = \mathbf{1} \quad \text{(unit)}, \\ &\Delta(\varphi) = \varphi \otimes \mathbf{1} + \mathbf{1} \otimes \varphi, \quad \Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \quad \text{(coproduct)}, \\ &\epsilon(\varphi) = 0, \quad \epsilon(\mathbf{1}) = 1 \quad \text{(counit)}, \\ &S(\varphi) = -\varphi, \quad S(\mathbf{1}) = \mathbf{1} \quad \text{(antipode)}, \end{aligned} \tag{3.120}$$

where $\varphi \in X$. This Hopf algebraic structure should not be confused with that of the symmetry operators $\mathcal{U}(\mathcal{A})$ acting on $\hat{X}$. In addition to this Hopf algebraic structure, the space $\hat{X}$ has the braiding/shift structure Ψ which obeys the consistency conditions (3.72) and (3.74). With the use of the braiding, the Hopf algebraic structure is extended to the whole field space $\hat{X}$; coproduct, counit, and antipode of a product of two elementary fields $\varphi_1, \varphi_2 \in X$ are defined by

$$\begin{aligned} &\Delta(\varphi_1 \cdot \varphi_2) := (m \otimes m) \circ (\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\Delta(\varphi_1) \otimes \Delta(\varphi_2)), \\ &\epsilon(\varphi_1 \cdot \varphi_2) := \epsilon(\varphi_1)\epsilon(\varphi_2), \\ &S(\varphi_1 \cdot \varphi_2) := m \circ \Psi(S(\varphi_1) \otimes S(\varphi_2)), \end{aligned} \tag{3.121}$$

and generalized inductively to any products in $\hat{X}$. One of the most crucial properties for this braiding structure is that it must be covariant under the symmetry operations. In fact we recall that the braiding structure is inevitable only for the covariant consistency under the Hopf algebraic symmetry: $a \triangleright \circ \Psi = \Psi \circ a \triangleright$, $a \in \mathcal{U}(\mathcal{A})$. With all these properties, the space $\hat{X}$ is called a braided Hopf algebra, or, more precisely, Hopf algebra in a braided category. We thus claim that the link formalism naturally treats the space of fields as a braided Hopf algebra with a Hopf algebraic symmetry, for which the general BQFT formalism can apply. We now see this application in the next subsection.

C. Perturbative definition of supersymmetry on the lattice as a braided quantum field theory

Following the general theory of BQFT given in [41], we can now construct a lattice theory which has the Hopf algebraic symmetry introduced in the previous subsections. Before giving concrete examples, let us here briefly review the general framework. The crucial ingredient to define a quantum field theory is the path integral. In order to define a perturbation theory it is enough to introduce a formal Gaussian integral, with the property that a total functional derivative under it vanishes. We therefore need to define the functional derivative.

Let us introduce the functional derivative with respect to $\varphi \in X$ as

$$\frac{\delta}{\delta\varphi(x)}\varphi(y) = \delta^D(x - y). \tag{3.122}$$

Following the more abstract definition given in [41], we write this as

$$\mathrm{ev}\Big(\frac{\delta}{\delta\varphi(x)} \otimes \varphi(y)\Big) := \frac{\delta}{\delta\varphi(x)}\varphi(y), \tag{3.123}$$

by introducing the evaluation map ev. It is a kind of natural contraction of X and X^*, where X^* is the dual space to X composed of $\delta/\delta\varphi$. Similarly we can introduce the opposite map, a kind of completeness relation, as

$$\mathrm{coev}\,(\lambda) := \lambda \sum_x \varphi(x) \otimes \frac{\delta}{\delta\varphi(x)}. \tag{3.124}$$

These maps are characterized by the identities

$$\begin{aligned} (\mathrm{ev} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{coev}) &= \mathrm{id}_{X^*}, \\ (\mathrm{id} \otimes \mathrm{ev})(\mathrm{coev} \otimes \mathrm{id}) &= \mathrm{id}_X. \end{aligned} \tag{3.125}$$

The functional derivative can be naturally extended to act on the whole space of fields $\hat{X}$ in the following way. On a product of two elementary fields $\varphi_1, \varphi_2 \in X$, the action of the functional derivative obeys a braided Leibniz rule, namely

$$\frac{\delta}{\delta\varphi(x)}(\varphi_1(x_1)\cdot\varphi_2(x_2)) = \frac{\delta}{\delta\varphi(x)}\varphi_1(x_1)\cdot\varphi_2(x_2) + \cdot\Big[\Psi^{-1}\Big(\frac{\delta}{\delta\varphi(x)}\otimes\varphi_1(x_1)\Big) \times (\mathbf{1}\otimes\varphi_2(x_2))\Big]. \tag{3.126}$$

On products of three or more fields the result can be found by induction. Needless to say, the derivative trivially commutes with a constant field [see (3.74)], and gives zero when it acts on a constant. More rigorous definition of the functional derivative is given in [41,53].

Now we can introduce Gaussian integration defined by the following property:

$$\int\frac{\delta}{\delta\varphi}(\mathcal{O}[\varphi]e^{-S_0}) = 0, \qquad \mathcal{O}[\varphi]\in\hat{X}, \qquad \frac{\delta}{\delta\varphi}\in X^*, \tag{3.127}$$

where $\exp(-S_0)\in\hat{X}$ is the corresponding Gaussian factor. In the application to field theory, S_0 is interpreted as the free part of an action. Notice that this integration is formally understood as the one which satisfies the property (3.127) without referring to its actual values. This way of abstract definition is already enough to define a perturbation theory and to compute correlation functions of arbitrary order, since for such computations only ratios of functional integrals are needed (the actual value of the partition function is not necessary as in standard field theory), and those ratios can be computed using just algebraic properties.

Let us now introduce a kind of propagator. Consider a map γ: $X^*\longrightarrow X$ defined by

$$\frac{\delta}{\delta\varphi(x)}e^{-S_0} = -\gamma\Big(\frac{\delta}{\delta\varphi(x)}\Big)e^{-S_0}, \tag{3.128}$$

or more specifically by

$$\gamma\Big(\frac{\delta}{\delta\varphi(x)}\Big) = \frac{\delta}{\delta\varphi(x)}S_0, \tag{3.129}$$

which roughly corresponds to the inverse propagator, so that the propagator is in a way given as γ^{-1}. This naive argument will be justified shortly.

The free n-point correlation function is now defined by

$$Z_n^{(0)}(\alpha_n) := \frac{\int\alpha_n e^{-S_0}}{\int e^{-S_0}}, \qquad \alpha_n\in X^n. \tag{3.130}$$

The superscript (0) stands for the free theory. In this definition, the denominator, denoted here tentatively as $Z^{(0)}$, might be interpreted as the free partition function, but in the general case we do not have any definition that allows one to compute it as mentioned above. Still this definition is enough to calculate the correlation functions of any order. To see this argument, notice first that

$$\alpha_n\varphi e^{-S_0} = \alpha_n\gamma(\gamma^{-1}(\varphi))e^{-S_0} = -\alpha_n\gamma^{-1}(\varphi)(e^{-S_0}), \tag{3.131}$$

where we have used the definition (3.128) and the fact that $\gamma^{-1}(\varphi)\in X^*$ and so is a functional derivative. We then find, using the braided Leibniz rule, that

$$-\alpha_n\gamma^{-1}(\varphi)(e^{-S_0}) = -(\gamma^{-1}(\varphi^{\alpha_n})(\alpha_n^{\varphi}e^{-S_0}) - \gamma^{-1}(\varphi^{\alpha_n})(\alpha_n^{\varphi})e^{-S_0}), \tag{3.132}$$

where we have denoted the "shifted" field as φ^{α_n} and α_n^{φ}, with the superscripts implying the amount of shifts.[10] We thus find

$$\int\alpha_n\varphi e^{-S_0} = -\int(\gamma^{-1}(\varphi^{\alpha_n})(\alpha_n^{-\varphi}e^{-S_0}) - \gamma^{-1}(\varphi^{\alpha_n})(\alpha_n^{\varphi})e^{-S_0}) = \int\gamma^{-1}(\varphi^{\alpha_n})(\alpha_n^{\varphi})e^{-S_0}. \tag{3.133}$$

In the second equality, the first term vanishes because it is a total derivative under the path integral. We therefore obtain a basic formula

$$Z^{(0)}(\alpha_n\varphi) = Z^{(0)}(\gamma^{-1}(\varphi^{\alpha_n})(\alpha_n^{\varphi})). \tag{3.134}$$

For example, putting $\alpha = \mathbb{1}$ ($n = 0$) in Eq. (3.134), it is clear that

$$Z_1^{(0)}(\varphi) = 0. \tag{3.135}$$

The simplest nontrivial example is given for $n = 1$ by taking $\alpha_1 = \varphi_1(x_1)\in X$ and $\varphi = \varphi_2(x_2)\in X$ in (3.134), so that

$$Z_2^{(0)}(\varphi_1(x_1)\varphi_2(x_2)) = Z_2^{(0)}(\gamma^{-1}(\varphi_2(x_2+a_{\varphi_1}))(\varphi_1(x+a_{\varphi_2}))) = \gamma^{-1}(\varphi_2(x_2+a_{\varphi_1}))(\varphi_1(x+a_{\varphi_2})). \tag{3.136}$$

The other formulas can be computed inductively using (3.134). The general results are summarized as follows:

$$Z_2^{(0)} = \mathrm{ev}\circ(\gamma^{-1}\otimes\mathrm{id})\circ\Psi, \tag{3.137}$$

$$Z_{2n}^{(0)} = (Z_2^{(0)})^n\circ[2n-1]'_{\Psi}!!, \tag{3.138}$$

$$Z_{2n+1}^{(0)} = 0, \tag{3.139}$$

where

[10]This notational simplicity can only apply to our present case for the specific braiding/shift structure. The general expression with general braiding Ψ is given in [41].

$$[2n-1]'_\Psi!! := ([1]'_\Psi \otimes \mathrm{id}^{2n-1}) \circ ([3]'_\Psi \otimes \mathrm{id}^{2n-3}) \circ \cdots \circ ([2n-1]'_\Psi \otimes \mathrm{id}),$$
$$[n]'_\Psi := \mathrm{id}^n + \mathrm{id}^{n-2} \otimes \Psi^{-1} + \cdots + \Psi^{-1}_{1,n-1}. \tag{3.140}$$

These formulas represent Wick's theorem in the BQFT formalism.

When an interaction is turned on, we can treat the theory perturbatively. Let the action be $S = S_0 + \lambda S_{\mathrm{int}}$. The n-point correlation function now reads

$$Z_n(\alpha_n) := \frac{\int \alpha_n e^{-S}}{\int e^{-S}} = \frac{\int \alpha_n(1 - \lambda S_{\mathrm{int}} + \cdots)e^{-S_0}}{\int(1 - \lambda S_{\mathrm{int}} + \cdots)e^{-S_0}}, \quad \alpha_n \in X^n. \tag{3.141}$$

Dividing both the numerator and denominator by the "partition function" $Z^{(0)}$, we find

$$Z_n = \frac{Z^{(0)}_n - \lambda Z^{(0)}_{n+k} \circ (\mathrm{id}^n \otimes S_{\mathrm{int}}) + \frac{1}{2}\lambda^2 Z^{(0)}_{n+2k} \circ (\mathrm{id}^n \otimes S_{\mathrm{int}} \otimes S_{\mathrm{int}}) + \cdots}{1 - \lambda Z^{(0)}_k \circ S_{\mathrm{int}} + \frac{1}{2}\lambda^2 Z^{(0)}_{2k} \circ (S_{\mathrm{int}} \otimes S_{\mathrm{int}}) + \cdots}, \tag{3.142}$$

where k is the order of the interaction S_{int}, i.e. $S_{\mathrm{int}} \in X^k$, and we have put a map $S_{\mathrm{int}}: \mathbb{C} \to X^k$ with the abuse of notation.

Let us give an example to see the formalism above more explicitly. Let us consider the $\mathcal{N} = (2,2)$ Wess-Zumino model in two dimensions in the Dirac-Kähler twisted basis. The superalgebra is given as before

$$\{Q^{\mathrm{lat}}, Q^{\mathrm{lat}}_\mu\} = i\partial_{+\mu}, \qquad \{\tilde{Q}^{\mathrm{lat}}, Q^{\mathrm{lat}}_\mu\} = -i\epsilon_{\mu\nu}\partial_{-\nu}. \tag{3.143}$$

The bosonic fields include scalars ϕ, σ and auxiliary fields $\tilde{\phi}$, $\tilde{\sigma}$, whereas the fermionic fields are ψ, $\tilde{\psi}$, ψ_μ. Supertransformations are given in Appendix B. The action is given as

$$S = \sum_x \Big[(\partial^{\mathrm{lat}}_{+\mu}\sigma)(x - a\hat{\mu}) \cdot (\partial^{\mathrm{lat}}_{-\mu}\phi)(x) - \tilde{\sigma}(x + a_1 + a_2) \cdot \tilde{\phi}(x) - i\psi(x - a) \cdot \partial^{\mathrm{lat}}_{-\mu}\psi_\mu(x) - i\epsilon_{\mu\nu}\tilde{\psi}(x - \tilde{a}) \cdot \partial^{\mathrm{lat}}_{+\mu}\psi_\nu(x) - W'(\phi(x + a_1 + a_2)) \cdot \tilde{\phi}(x) - V'(\sigma(x + a + \tilde{a})) \cdot \tilde{\sigma}(x) + V''(\sigma(x + a + \tilde{a})) \cdot \psi(x + \tilde{a}) \cdot \tilde{\psi}(x) + \frac{1}{2}\epsilon^{\mu\nu}W''(\phi(x + a_1 + a_2)) \cdot \psi_\mu(x + a_\nu) \cdot \psi_\nu(x) \Big], \tag{3.144}$$

where W and V are (super)potentials in the twisted basis. The invariance of the action can be unambiguously seen using the modified Leibniz rule taking care of the specific "staggered" configurations of arguments of the fields [6] as well as of the mildly generalized statistics (3.119).

D. Ward-Takahashi identities

Here we follow [53]. The invariance of the correlation functions can be written as

$$Z_n(a \triangleright \chi) = \epsilon(a)Z_n(\chi), \qquad a \in \mathcal{U}(\mathcal{A}), \qquad \chi \in X^n, \tag{3.145}$$

which is the Ward-Takahashi identity corresponding to the Hopf algebraic symmetry $\mathcal{U}(\mathcal{A})$. Just as in a usual field theory, the invariance of the correlation functions follows from the invariance of the action. One obvious difference from the usual case is that, with the nontrivial braiding, the symmetry operators must act on the fields in a manner consistent with the braiding structure. In fact it is shown that the identity (3.145) follows when the following four conditions are satisfied [53]:

(1) Invariance of the free action:

$$a \triangleright \gamma^{-1}(\varphi) = \gamma^{-1}(a \triangleright \varphi). \tag{3.146}$$

(2) Invariance of the interaction:

$$a \triangleright S_{\mathrm{int}} = \epsilon(a)S_{\mathrm{int}}. \tag{3.147}$$

(3) Covariance of the braiding:

$$\Psi(a \triangleright (X_1 \otimes X_2)) = a \triangleright \Psi(X_1 \otimes X_2). \tag{3.148}$$

(4) Invariance of the delta function:

$$\mathrm{ev}\,(a \triangleright (X^* \otimes X)) = \epsilon(a)\mathrm{ev}(X^* \otimes X). \tag{3.149}$$

In our current application, the general formula of Ward-Takahashi identity (3.145) naturally gives the correct identities on the lattice. It is important that the general formula (3.145) can be proved unambiguously using only algebraic relations.

E. Nonperturbative definition?

In this section, we first extracted the essential requirements for the symmetry operators in the link formalism, concluding that this symmetry is Hopf algebraic. Then we utilized the general framework of BQFT formulated in [41], showing that supersymmetric theory on a lattice in the link formalism can be treated with a formal definition of path integral. This path integral approach, however, only gives a perturbative formulation in general, due to the lack of an explicit definition of the path integral. As a field theory on a lattice, this situation would not be satisfactory at all, especially for the application to numerical simulations. It is known that in some cases one can define a "braided integral" explicitly [55]. We might be able to apply such an approach to the current problem and define a rigorous path integral on the lattice, which, if possible, should give a nonperturbative definition in this formulation based on the Hopf algebraic symmetry. As we have pointed in subsection III B, it is also crucial to accommodate an explicit representation of the lattice momentum operator endowed with the correct grading nature.

IV. CONCLUSION AND DISCUSSION

We have shown how the link formalism is treated as a field theory on the lattice with a deformed or modified algebraic symmetry. The deformation of the algebra is indeed the one naturally treated in the framework of Hopf algebra. We showed this argument explicitly, defining the corresponding Hopf algebraic structures of the supersymmetry algebra for the link formalism. The modified Leibniz rule, which is the crucial notion in the original link formalism, is incorporated as the coproduct structure of the Hopf algebra, whose consistency is assured with the other relevant structures of the algebra. The Hopf algebra introduced this way in fact turns out to be a (quasi)triangular Hopf algebra, which has a nontrivial universal R-matrix. When represented on the space of fields, this quasitriangular structure inevitably induces a nontrivial statistics, or a noncommutativity, which is the key ingredient for a consistent representation. With these algebraic descriptions, we could identify the link formalism as a representation theory of a quasitriangular Hopf algebra. On the other hand, it is known that there is a general scheme to construct a quantum field theory which has a Hopf algebraic symmetry, called braided quantum field theory. We applied this general formulation to the link formalism. The construction is purely algebraic. In particular, it defines a path integral using only algebraic properties. One can show that it still gives a well-defined perturbative description of the theory, providing full methods for calculating correlation functions in any order. It also gives a consistent way to derive the Ward-Takahashi identities corresponding to the Hopf algebraic symmetry. We therefore realized the link formalism as a quantum field theory which has the quasitriangular Hopf algebraic symmetry at least in the perturbative sense.

From the consistency of the Hopf algebraic structure, it is required that the lattice momentum operator which is proportional to the difference operator should carry grading compatible with the shifting nature of the difference operator. In this paper we have not given a concrete representation of this grading structure which may be needed to give an explicit nonperturbative definition of this formulation. We leave this issue for the future investigation.

The algebraic inconsistency pointed out in [42], which is connected with an ordering ambiguity of the component fields when applying a supersymmetry transformation, is solved by the introduction of the braiding structure according to the notion of coproduct for the lattice supercharges and the momentum operator in Hopf algebra.

It is then important to ask the question about how the continuum limit of this formulation is realized. If one can formulate the braided quantum field theory which respects the Hopf algebraic structure as a concrete representation for modified path integral, the twisted lattice supersymmetry will be kept in the continuum limit since the lattice twisted supersymmetry is exactly kept. As we have shown the lattice supersymmetry is kept in the perturbative level of a braided quantum field theory. It is still a nontrivial question how the symmetry is recovered even in the nonperturbative level. In any case we expect that fine-tuning is not needed to keep supersymmetry in the continuum limit if the formulation of deformed supersymmetry algebra is concretely constructed.

In the formulation of the orbifold construction of lattice field theories only a subset of lattice supercharges, in particular, the nilpotent scalar supercharge which corresponds to the shiftless charge in the link construction is exactly preserved on the lattice [28–36]. The lattice superalgebra in this case is the same as the continuum twisted supersymmetry algebra. It was stressed that the supercharges carrying shifts break lattice supersymmetry in the sense of the continuum twisted superalgebra [34,35]. Our claim in this paper is that these supercharges carrying shifts may break the continuum twisted supersymmetry but preserve exactly the Hopf algebraic supersymmetry. Thus in the link approach all the lattice supercharges are claimed to be preserved exactly in the framework of Hopf algebraic supersymmetry. The supersymmetry algebra is deformed from the continuum twisted superalgebra to the Hopf algebraic one.

We have not considered the gauge extension of the deformed supersymmetry in this paper. It was pointed out that there is a similar ordering ambiguity for the lattice super Yang-Mills formulation of the link approach [42]. We consider that this problem can be solved similarly as in the nongauge case by identifying the lattice supersymmetry with gauge symmetry in the link approach as a Hopf algebraic symmetry. There is, however, yet another problem in the gauge extension: the loss of the gauge invariance due to the link nature of the lattice supercharges. A possible solution was proposed by introducing covariantly constant superparameters η_A [8]:

$$\{\nabla_B, \eta_A\} = 0,$$

where ∇_B is the supercovariant derivative. This is a highly nontrivial relation in the sense that the fermionic parameter η_A carrying a shift should carry a spacetime dependence caused by the supercovariant derivative ∇_B to keep the covariant constant nature. Here we may consider that the fermionic link variables are defined on the links of internal spacetime. In other words the spacetime distortion of internal spacetime may compensate the required dependence of the fermionic parameter. There is a possibility that gravity may play a role in these questions.

It has been pointed out that the breakdown of the Leibniz rule for the lattice difference operator is inevitable under reasonable assumptions for algebraic properties on the lattice [43]. Recent renormalization group analysis confirm this statement from a different point of view [27]. In order to realize a supersymmetry algebra which includes the momentum operator on the lattice it is most natural to introduce the difference operator in the lattice superalgebra while, so far, the exact continuum supersymmetry was realized only for the nilpotent supercharge which is the scalar part of the twisted supersymmetry but does not include the crucial momentum dependence. We claim that the deformation of the Lie algebraic continuum supersymmetry to a Hopf algebraic supersymmetry on the lattice is inevitable to accommodate the difference operator in the algebra.

It is obviously very important to find a concrete representation of the Hopf algebraic superalgebra on the lattice to obtain a modified path integral definition of QFT with this particular braiding structure. As we have already shown this type of mild noncommutativity with shifting nature may be well accommodated by a matrix formulation of lattice noncommutativity [40,56]. This part of the concrete proposal with the necessary formulation of the lattice graded momentum operator will be given elsewhere. It would also be interesting to compare the formulation of the link approach with other noncommutative approaches [57] and the nonlattice formulations [58,59].

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APPENDIX A: BRIEF SUMMARY OF HOPF ALGEBRA

Here we briefly list the axioms of Hopf algebra and some related notions which are used in this article. For rigorous and complete descriptions, see, for example, [60–62].

1. Hopf algebra

A *Hopf algebra* over a field k ($=\mathbb{C}$ or $\mathbb{R}$) is a vector space H over k which has the following properties 1, 2, 3, and 4.

(1) H is a *unital associative algebra*, so that

(a) it has a k-linear multiplication (or product) map[11]

$$\cdot: H\otimes H \to H, \qquad \cdot(h_1\otimes h_2) = h_1\cdot h_2, \quad \text{(A1)}$$

which is associative

$$\cdot\circ(\cdot\otimes \mathrm{id}) = \cdot\circ(\mathrm{id}\otimes\cdot), \quad \text{i.e. } (h_1\cdot h_2)\cdot h_3 = h_1\cdot(h_2\cdot h_3); \quad \text{(A2)}$$

(b) it has unit element $\mathbb{1}$ which satisfies $\mathbb{1}\cdot h = h\cdot\mathbb{1} = h$, whose existence can be formally expressed as the existence of a k-linear map

$$\eta: k\to H, \qquad \eta(\lambda) = \lambda\mathbb{1}, \qquad \lambda\in k. \quad \text{(A3)}$$

(2) H is a *coalgebra*. Namely,

(a) it has a k-linear map called *coproduct*:

$$\Delta: H\to H\otimes H, \qquad \Delta(h) = \sum_i h_{i(1)}\otimes h_{i(2)}, \qquad h_{i(1)}, \quad h_{i(2)}\in H, \quad \text{(A4)}$$

which satisfies *coassociativity*[12]

$$(\Delta\otimes\mathrm{id})\circ\Delta = (\mathrm{id}\otimes\Delta)\circ\Delta, \quad \text{i.e. } h_{(1)(1)}\otimes h_{(1)(2)}\otimes h_{(2)} = h_{(1)}\otimes h_{(2)(1)}\otimes h_{(2)(2)}; \quad \text{(A5)}$$

(b) it has also another k-linear map called *counit*

$$\epsilon: H\to k, \quad \text{(A6)}$$

which obeys the relation

$$(\epsilon\otimes\mathrm{id})\circ\Delta = (\mathrm{id}\otimes\epsilon)\circ\Delta = \mathrm{id}, \quad \text{i.e. } \epsilon(h_{(1)})h_{(2)} = \epsilon(h_{(2)})h_{(1)} = h. \quad \text{(A7)}$$

(3) These structures of algebra and coalgebra are compatible with each other. Namely,

(a) the coproduct and the counit are both algebra maps:

$$\Delta(h_1\cdot h_2) = \Delta(h_1)\cdot\Delta(h_2), \qquad \epsilon(h_1\cdot h_2) = \epsilon(h_1)\epsilon(h_2). \quad \text{(A8)}$$

(4) H has one more map called *antipode*:

[11] In what follows we take, unless otherwise specified, $h, h_1, h_2, \cdots$, to be arbitrary elements of H.

[12] We use below a much simpler abbreviation $\Delta(h) = h_{(1)}\otimes h_{(2)}$ known as the Sweedler's notation.

(a) it has a k-linear map

$$S: H \to H, \tag{A9}$$

which obeys the identity

$$\cdot(S \otimes \mathrm{id}) \circ \Delta = \cdot(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon,$$
$$\text{i.e. } S(h_{(1)}) \cdot h_{(2)} = h_{(1)} \cdot S(h_{(2)}) = \epsilon(h)\mathbb{1}. \tag{A10}$$

If the k-linear space H satisfies these properties 1, 2, 3, but not 4, it is called a *bialgebra*.

2. Quasitriangular structure

A Hopf algebra H is said to be *quasitriangular* if there exists an invertible element $\mathcal{R} \in H \otimes H$ which satisfies

$$\tau \circ \Delta h = \mathcal{R} \cdot (\Delta h) \cdot \mathcal{R}^{-1}, \qquad (\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23},$$
$$(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{A11}$$

where

$$\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}, \qquad \mathcal{R}_{12} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \otimes \mathbb{1},$$
$$\mathcal{R}_{13} = \sum \mathcal{R}^{(1)} \otimes \mathbb{1} \otimes \mathcal{R}^{(2)}, \qquad \mathcal{R}_{23} = \sum \mathbb{1} \otimes \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}, \tag{A12}$$

and $\tau: H \otimes H \to H \otimes H$ is the transposition map

$$\tau(h_1 \otimes h_2) = h_2 \otimes h_1, \qquad h_1, h_2 \in H. \tag{A13}$$

The element $\mathcal{R}$, if it exists, is called a *quasitriangular structure* or *universal R-matrix*.

If a quasitriangular structure $\mathcal{R}$ of a quasitriangular Hopf algebra H obeys further the following condition, the Hopf algebra is said to be *triangular*:

$$\mathcal{R}_{21}\mathcal{R} = \mathbb{1} \otimes \mathbb{1}, \quad \text{i.e. } \mathcal{R}_{21} = \mathcal{R}^{-1},$$
$$\text{where } \mathcal{R}_{21} = \sum \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)}. \tag{A14}$$

3. Action on algebras

A (left) *action* of a Hopf algebra H on an associative algebra X is a representation $\rho: H \to \mathrm{Lin}(X)$, where $\mathrm{Lin}(X)$ is the algebra of linear maps on X, which satisfies the covariance in the following sense:

$$h \triangleright (\varphi \cdot \varphi') = m(\Delta(h) \triangleright (\varphi \otimes \varphi')),$$
$$h \triangleright \mathbf{1} = \epsilon(h)\mathbf{1}, \qquad \varphi, \varphi' \in X. \tag{A15}$$

We have here introduced the notation $h \triangleright \varphi := \rho(h)(\varphi)$, the product m of X with the abbreviation $\varphi \cdot \varphi' := m(\varphi \otimes \varphi')$, and the unit $\mathbf{1} \in X$.

4. Braiding

Let us consider a formal collection[13] of representation spaces $(\mathbf{1}, X, Y, Z, \cdots)$ of a Hopf algebra H together with the collection of tensor products of the representation spaces $(\mathbf{1} \otimes X \cong X \otimes \mathbf{1} \cong X, X \otimes Y, (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \cdots)$ on which H acts with the coproduct structure $((\Delta h) \triangleright (\varphi \otimes \chi), h \in H, \varphi \in X, \chi \in Y)$. If there exists an invertible intertwiner (isomorphism)

$$\Psi_{X,Y}: X \otimes Y \to Y \otimes X,$$
$$\Psi_{X,Y}(\Delta(h) \triangleright (\varphi \otimes \chi)) = \Delta(h) \triangleright \Psi_{X,Y}(\varphi \otimes \chi) \tag{A16}$$

with the properties

$$\Psi_{X\otimes Y,Z} = \Psi_{X,Z} \circ \Psi_{Y,Z}, \qquad \Psi_{X,Y\otimes Z} = \Psi_{X,Z} \circ \Psi_{X,Y}, \tag{A17}$$

it unambiguously relates the two representations on $X \otimes Y$ and $Y \otimes X$. It should be compatible with any maps which intertwine the representation spaces as in

$$\Psi_{Z,W} \circ (g_{XZ} \otimes g_{YW}) = (g_{YW} \otimes g_{XZ}) \circ \Psi_{X,Y},$$
$$g_{XZ}: X \to Z, \qquad g_{YW}: Y \to W. \tag{A18}$$

We call this isomorphism Ψ a *braid*. Strictly speaking, a braid should be such that

$$\Psi \circ \Psi \neq \mathrm{id}, \quad \text{or} \quad \Psi_{X,Y} \neq \Psi_{Y,X}^{-1}, \tag{A19}$$

which means there are two distinct ways in relating $X \otimes Y$ to $Y \otimes X$. It gives a nontrivial rule of exchanging factors of a tensor product, and generalizes the statistics of the representation spaces. If, on the other hand, it satisfies $\Psi \circ \Psi = \mathrm{id}$, the isomorphism is more like a simple transposition and said to be symmetric.

When the Hopf algebra H is quasitriangular, we can express braiding more explicitly using a quasitriangular structure $\mathcal{R}$ of H and the transposition map τ (A13) as in

$$\Psi_{X,X'}(\varphi \otimes \varphi') = \tau \circ \mathcal{R} \triangleright (\varphi \otimes \varphi'), \qquad \varphi \in X, \qquad \varphi' \in X'. \tag{A20}$$

This indeed becomes an invertible intertwiner (A16) and satisfies the conditions (A17) and (A18). We find that

$$\Psi_{X,X'} \circ \Psi_{X',X} = \tau \circ \mathcal{R} \triangleright (\tau \circ \mathcal{R} \triangleright)$$
$$= \tau((\mathcal{R} \cdot \mathcal{R}_{21}) \triangleright \circ \tau).$$

Thus the condition (A19), that is for Ψ to be strictly braided, is equivalent to

$$\mathcal{R} \cdot \mathcal{R}_{21} \neq \mathbb{1} \otimes \mathbb{1}, \tag{A21}$$

namely that the universal R-matrix is really quasitriangu-

[13]The notion of braiding would be most suitably defined in terms of category theory. Here instead we just give a simple and intuitive description.

lar. Equivalently, a symmetric isomorphism Ψ corresponds to a triangular structure $\mathcal{R} \cdot \mathcal{R}_{21} \neq \mathbb{1} \otimes \mathbb{1}$.

5. Twist

Let H be a Hopf algebra. An invertible element $\chi \in H \otimes H$ is called a *(2-)cocycle* when it satisfies the following condition:

$$(\chi \otimes \mathbb{1})(\Delta \otimes \mathrm{id})\chi = (\mathbb{1} \otimes \chi)(\mathrm{id} \otimes \Delta)\chi \quad ((2-)\text{cocycle condition}). \tag{A22}$$

A cocycle χ is said to be *counital* if[14]

$$(\epsilon \otimes \mathrm{id})\chi = \mathbb{1} \quad \text{and} \quad (\mathrm{id} \otimes \epsilon)\chi = \mathbb{1} \quad (\text{counital condition}). \tag{A23}$$

For a quasitriangular Hopf algebra $(H, \mathcal{R})$ and a counital 2-cocycle χ, there exists a new Hopf algebra $(H_\chi, \mathcal{R}_\chi)$ which has

(i) the same algebra and counit as those for $(H, \mathcal{R})$,
(ii) coproduct: $\Delta_\chi h = \chi(\Delta h)\chi^{-1}$,
(iii) antipode: $S_\chi h = U(Sh)U^{-1}$, where $U = \cdot(\mathrm{id} \otimes S)\chi$, $U^{-1} = \cdot(S \otimes \mathrm{id})\chi^{-1}$,
(iv) quasitriangular structure: $\mathcal{R}_\chi = \chi_{21}\mathcal{R}\chi^{-1}$, where $\chi_{21} = \tau(\chi)$ with τ given in (A13).

The process of obtaining the Hopf algebra $(H_\chi, \mathcal{R}_\chi)$ from the original one $(H, \mathcal{R})$ is called *twisting* with the element χ called a twist element. If $(H, \mathcal{R})$ is triangular, so is $(H_\chi, \mathcal{R}_\chi)$.

When a Hopf algebra H acts on an associative algebra X covariantly as in (A15), the twisted Hopf algebra H_χ with a twist element χ acts covariantly on a new algebra X_χ with a new product

$$\varphi * \varphi' := m \circ \chi^{-1} \triangleright (\varphi \otimes \varphi') \tag{A24}$$

and with the same unit. The new product $*$ is associative and in general noncommutative even if the original product $\cdot$ is commutative.

[14] Here actually only one of the two conditions suffices.

APPENDIX B: $N = (2, 2)$ WESS-ZUMINO MODEL IN TWO DIMENSIONS

We list here the explicit supertransformation formulas for the $\mathcal{N} = (2, 2)$ Wess-Zumino model in two dimensions. The superalgebra is

$$\{Q^{\mathrm{lat}}, Q^{\mathrm{lat}}_\mu\} = P^{\mathrm{lat}}_{+\mu}, \qquad \{\tilde{Q}^{\mathrm{lat}}, Q^{\mathrm{lat}}_\mu\} = -\epsilon_{\mu\nu} P^{\mathrm{lat}}_{-\nu}, \qquad (P^{\mathrm{lat}}_{\pm\mu} := i\partial_{\pm\mu}),$$

with the other commutators just vanishing. The field contents are $\{\phi, \sigma, \psi, \psi_\mu, \tilde{\psi}, \tilde{\phi}, \tilde{\sigma}\}$, for which supertransformations are as follows:

$$\begin{aligned}
&Q^{\mathrm{lat}}\phi = 0, \qquad Q^{\mathrm{lat}}_\mu \phi = \psi_\mu, \qquad \tilde{Q}^{\mathrm{lat}}\phi = 0,\\
&Q^{\mathrm{lat}}\psi_\nu = i\partial_{+\nu}\phi, \qquad Q^{\mathrm{lat}}_\mu \psi_\nu = -\epsilon_{\mu\nu}\tilde{\phi},\\
&\tilde{Q}^{\mathrm{lat}}\psi_\nu = -i\epsilon_{\nu\mu}\partial_{-\mu}\phi, \qquad Q^{\mathrm{lat}}\tilde{\phi} = -i\epsilon_{\mu\nu}\partial_{+\mu}\psi_\nu,\\
&Q^{\mathrm{lat}}_\mu\tilde{\phi} = 0, \qquad \tilde{Q}^{\mathrm{lat}}\tilde{\phi} = i\partial_{-\mu}\psi_\mu, \qquad Q^{\mathrm{lat}}\sigma = -\psi,\\
&Q^{\mathrm{lat}}_\mu\sigma = 0, \qquad \tilde{Q}^{\mathrm{lat}}\sigma = -\tilde{\psi}, \qquad Q^{\mathrm{lat}}\psi = 0,\\
&Q^{\mathrm{lat}}_\mu\psi = -i\partial_{+\mu}\sigma, \qquad \tilde{Q}^{\mathrm{lat}}\psi = -\tilde{\sigma}, \qquad Q^{\mathrm{lat}}\tilde{\psi} = \tilde{\sigma},\\
&Q^{\mathrm{lat}}_\mu\tilde{\psi} = i\epsilon_{\mu\nu}\partial_{-\nu}\sigma, \qquad \tilde{Q}^{\mathrm{lat}}\tilde{\psi} = 0, \qquad Q^{\mathrm{lat}}\tilde{\sigma} = 0,\\
&Q^{\mathrm{lat}}_\mu\tilde{\sigma} = i\epsilon_{\mu\nu}\partial_{-\nu}\psi + i\partial_{+\mu}\tilde{\psi}, \qquad \tilde{Q}^{\mathrm{lat}}\tilde{\sigma} = 0.
\end{aligned}$$

[1] H. B. Nielsen and M. Ninomiya, Nucl. Phys. **B185**, 20 (1981); **B195**, 541 (1982); **B193**, 173 (1981).
[2] L. H. Karsten and J. Smit, Nucl. Phys. **B183**, 103 (1981).
[3] P. H. Ginsparg and K. G. Wilson, Phys. Rev. D **25**, 2649 (1982).
[4] D. B. Kaplan, Phys. Lett. B **288**, 342 (1992).
[5] P. Hasenfratz, Nucl. Phys. B, Proc. Suppl. **63**, 53 (1998); H. Neuberger, Phys. Lett. B **427**, 353 (1998); M. Lüscher, Phys. Lett. B **428**, 342 (1998).
[6] A. D'Adda, I. Kanamori, N. Kawamoto, and K. Nagata, Nucl. Phys. **B707**, 100 (2005); Nucl. Phys. B, Proc. Suppl. **140**, 754 (2005); **140**, 757 (2005).
[7] A. D'Adda, I. Kanamori, N. Kawamoto, and K. Nagata, Phys. Lett. B **633**, 645 (2006).
[8] A. D'Adda, I. Kanamori, N. Kawamoto, and K. Nagata, Nucl. Phys. **B798**, 168 (2008); Proc. Sci., LAT2007 (2007) 271 [arXiv:0709.0722].
[9] S. Catterall, J. High Energy Phys. 11 (2004) 006; Proc. Sci., LAT2005 (2006) 006 [arXiv:0509136].
[10] N. Kawamoto and T. Tsukioka, Phys. Rev. D **61**, 105009 (2000); J. Kato, N. Kawamoto, and Y. Uchida, Int. J. Mod. Phys. A **19**, 2149 (2004); J. Kato, N. Kawamoto, and A. Miyake, Nucl. Phys. **B721**, 229 (2005); J. Kato and A. Miyake, Mod. Phys. Lett. A **21**, 2569 (2006); J. High Energy Phys. 03 (2009) 087; J. Saito, Soryushiron kenkyu **111**, 117 (2005).
[11] N. Marcus, Nucl. Phys. **B452**, 331 (1995).
[12] K. Nagata, J. High Energy Phys. 01 (2008) 041; K. Nagata and Y. S. Wu, Phys. Rev. D **78**, 065002 (2008).
[13] K. Nagata, J. High Energy Phys. 10 (2008) 036.
[14] S. Elitzur, E. Rabinovici, and A. Schwimmer, Phys. Lett. **119B**, 165 (1982); N. Sakai and M. Sakamoto, Nucl. Phys. **B229**, 173 (1983); V. A. Kostelecky and J. M. Rabin, J. Math. Phys. (N.Y.) **25**, 2744 (1984); D. M. Scott, J. Phys.

A **17**, 1123 (1984); H. Aratyn and A. H. Zimerman, J. Phys. A **18**, L487 (1985).
[15] P. Becher and H. Joos, Z. Phys. C **15**, 343 (1982); T. Banks and P. Windy, Nucl. Phys. **B198**, 226 (1982); J. M. Rabin, Nucl. Phys. **B201**, 315 (1982).
[16] I. Kanamori and N. Kawamoto, Int. J. Mod. Phys. A **19**, 695 (2004); Nucl. Phys. B, Proc. Suppl. **129**, 877 (2004).
[17] W. Pauli, Phys. Rev. **58**, 716 (1940).
[18] P. H. Dondi and H. Nicolai, Nuovo Cimento Soc. Ital. Fis. A **41**, 1 (1977).
[19] I. Montvay, Int. J. Mod. Phys. A **17**, 2377 (2002).
[20] A. Feo, Nucl. Phys. B, Proc. Suppl. **119**, 198 (2003); arXiv:heplat/0311037, and references therein; Mod. Phys. Lett. A **19**, 2387 (2004).
[21] K. Fujikawa and M. Ishibashi, Nucl. Phys. **B622**, 115 (2002); Phys. Lett. B **528**, 295 (2002); Y. Kikukawa and Y. Nakayama, Phys. Rev. D **66**, 094508 (2002); K. Fujikawa, Phys. Rev. D **66**, 074510 (2002); M. Bonini and A. Feo, J. High Energy Phys. 09 (2004) 011; J. W. Elliott and G. D. Moore, J. High Energy Phys. 11 (2005) 010.
[22] K. Itoh, M. Kato, H. Sawanaka, H. So, and N. Ukita, J. High Energy Phys. 02 (2003) 033; Prog. Theor. Phys. **108**, 363 (2002).
[23] H. Suzuki and Y. Taniguchi, J. High Energy Phys. 10 (2005) 082; H. Suzuki, J. High Energy Phys. 09 (2007) 052; Y. Kikukawa and H. Suzuki, J. High Energy Phys. 02 (2005) 012; M. Harada and S. Pinsky, Phys. Rev. D **71**, 065013 (2005).
[24] K. Fujikawa, Nucl. Phys. **B636**, 80 (2002).
[25] J. Nishimura, Phys. Lett. B **406**, 215 (1997); N. Maru and J. Nishimura, Int. J. Mod. Phys. A **13**, 2841 (1998); H. Neuberger, Phys. Rev. D **57**, 5417 (1998); D. B. Kaplan and M. Schmaltz, Chin. J. Phys. (Taipei) **38**, 543 (2000); G. T. Fleming, J. B. Kogut, and P. M. Vranas, Phys. Rev. D **64**, 034510 (2001).
[26] Y. Igarashi, H. So, and N. Ukita, Phys. Lett. B **535**, 363 (2002); G. Bergner, T. Kaestner, S. Uhlmann, and A. Wipf, Ann. Phys. (N.Y.) **323**, 946 (2008); T. Kastner, G. Bergner, S. Uhlmann, A. Wipf, and C. Wozar, Phys. Rev. D **78**, 095001 (2008); F. Synatschke, G. Bergner, H. Gies, and A. Wipf, J. High Energy Phys. 03 (2009) 028.
[27] G. Bergner, F. Bruckmann, and J. M. Pawlowski, Phys. Rev. D **79**, 115007 (2009).
[28] D. B. Kaplan, E. Katz, and M. Unsal, J. High Energy Phys. 05 (2003) 037; D. B. Kaplan, Nucl. Phys. B, Proc. Suppl. **119**, 900 (2003); A. G. Cohen, D. B. Kaplan, E. Katz, and M. Unsal, J. High Energy Phys. 08 (2003) 024; 12 (2003) 031; D. B. Kaplan, Nucl. Phys. B, Proc. Suppl. **129**, 109 (2004); D. B. Kaplan and M. Unsal, J. High Energy Phys. 09 (2005) 042; M. G. Endres and D. B. Kaplan, J. High Energy Phys. 10 (2006) 076.
[29] M. Unsal, J. High Energy Phys. 11 (2005) 013; 04 (2006) 002; 10 (2006) 089.
[30] S. Catterall and S. Karamov, Phys. Rev. D **65**, 094501 (2002); **68**, 014503 (2003); S. Catterall, J. High Energy Phys. 05 (2003) 038; S. Catterall and S. Ghadab, J. High Energy Phys. 05 (2004) 044; 10 (2006) 063; S. Catterall, J. High Energy Phys. 11 (2004) 006; 06 (2005) 027; 03 (2006) 032; 04 (2007) 015; S. Catterall and T. Wiseman, J. High Energy Phys. 12 (2007) 104; S. Catterall and A. Joseph, Phys. Rev. D **77**, 094504 (2008).
[31] F. Sugino, J. High Energy Phys. 01 (2004) 015; 03 (2004) 067; 01 (2005) 016; Phys. Lett. B **635**, 218 (2006).
[32] T. Onogi and T. Takimi, Phys. Rev. D **72**, 074504 (2005); K. Ohta and T. Takimi, Prog. Theor. Phys. **117**, 317 (2007); T. Takimi, J. High Energy Phys. 07 (2007) 010.
[33] J. Giedt, Proc. Sci., LAT2006 (2006) 008 [arXiv:hep-lat/0701006]; Nucl. Phys. **B726**, 210 (2005); Int. J. Mod. Phys. A **21**, 3039 (2006).
[34] P. H. Damgaard and S. Matsuura, J. High Energy Phys. 07 (2007) 051; 08 (2007) 087; 09 (2007) 097; Phys. Lett. B **661**, 52 (2008).
[35] M. Unsal, J. High Energy Phys. 05 (2009) 082.
[36] S. Catterall, D. Kaplan, and M. Unsal, Phys. Rep. **484**, 71 (2009).
[37] D. B. Kaplan, Phys. Lett. **136B**, 162 (1984).
[38] H. Fukaya, I. Kanamori, H. Suzuki, M. Hayakawa, and T. Takimi, Prog. Theor. Phys. **116**, 1117 (2006); H. Fukaya, I. Kanamori, H. Suzuki, and T. Takimi, Proc. Sci., LAT2007 (2007) 264 [arXiv:0709.4076]; I. Kanamori, H. Suzuki, and F. Sugino, Phys. Rev. D **77**, 091502(R) (2008); Prog. Theor. Phys. **119**, 797 (2008); I. Kanamori, AIP Conf. Proc. **1078**, 423 (2008); Proc. Sci., LAT2008 (2008) 232 [arXiv:0809.0655]; I. Kanamori and H. Suzuki, Nucl. Phys. **B811**, 420 (2009); Phys. Lett. B **672**, 307 (2009); I. Kanamori, Phys. Rev. D **79**, 115015 (2009).
[39] E. Witten, Commun. Math. Phys. **117**, 353 (1988); **118**, 411 (1988).
[40] S. Arianos, A. D'Adda, N. Kawamoto, and J. Saito, Proc. Sci., LAT2007 (2006) 259 [arXiv:0710.0487]; S. Arianos, A. D'Adda, A. Feo, N. Kawamoto, and J. Saito, Int. J. Mod. Phys. A **24**, 4737 (2009).
[41] R. Oeckl, Commun. Math. Phys. **217**, 451 (2001); Int. J. Mod. Phys. B **14**, 2461 (2000).
[42] F. Bruckmann and M. de Kok, Phys. Rev. D **73**, 074511 (2006); F. Bruckmann, S. Catterall, and M. de Kok, Phys. Rev. D **75**, 045016 (2007).
[43] M. Kato, M. Sakamoto, and H. So, J. High Energy Phys. 05 (2008) 057; Proc. Sci., LAT2005 (2006) 274 [arXiv: hep-lat/0509149]; Proc. Sci. LAT2008 (2008) 233 [arXiv:0810.2360].
[44] S. D. Drell, M. Weinstein, and S. Yankielowicz, Phys. Rev. D **14**, 1627 (1976).
[45] K. Osterwalder and R. Schrader, Commun. Math. Phys. **31**, 83 (1973); **42**, 281 (1975).
[46] K. Osterwalder and E. Seiler, Ann. Phys. (N.Y.) **110**, 440 (1978).
[47] N. Kawamoto and J. Smit, Nucl. Phys. **B192**, 100 (1981).
[48] F. Gliozzi, Nucl. Phys. **B204**, 419 (1982).
[49] H. Kluberg-Stern, A. Morel, O. Napoly, and B. Petersson, Nucl. Phys. **B220**, 447 (1983).
[50] J. B. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975); L. Susskind, Phys. Rev. D **16**, 3031 (1977).
[51] M. Chaichian, P. P. Kulish, K. Nishijima, and A. Tureanu, Phys. Lett. B **604**, 98 (2004); J. Wess, arXiv:hep-th/0408080; F. Koch and E. Tsouchnika, Nucl. Phys. **B717**, 387 (2005); R. Oeckl, Nucl. Phys. **B581**, 559 (2000); M. Chaichian, P. Presnajder, and A. Tureanu, Phys. Rev. Lett. **94**, 151602 (2005); G. Fiore and J. Wess, Phys. Rev. D **75**, 105022 (2007); A. P. Balachandran, A. R. Queiroz, A. M.

Marques, and P. Teotonio-Sobrinho, Phys. Rev. D **77**, 105032 (2008); E. Akofor, A. P. Balachandran, and A. Joseph, Int. J. Mod. Phys. A **23**, 1637 (2008).
[52] T. Asakawa, M. Mori, and S. Watamura, Prog. Theor. Phys. **120**, 659 (2008); J. High Energy Phys. 04 (2009) 117.
[53] Y. Sasai and N. Sasakura, Prog. Theor. Phys. **118**, 785 (2007).
[54] M. Riccardi and R. J. Szabo, J. High Energy Phys. 01 (2008) 016.
[55] S. Majid, J. Math. Phys. (N.Y.) **32**, 3246 (1991); **34**, 1176 (1993); **34**, 2045 (1993); **34**, 4843 (1993); arXiv:hep-th/9212151; A. Kempf and S. Majid, J. Math. Phys. (N.Y.) **35**, 6802 (1994).
[56] I. Bars and D. Minic, Phys. Rev. D **62**, 105018 (2000).
[57] J. Ambjorn, Y. M. Makeenko, J. Nishimura, and R. J. Szabo, J. High Energy Phys. 11 (1999) 029; Phys. Lett. B **480**, 399 (2000); J. High Energy Phys. 05 (2000) 023.
[58] M. Hanada, J. Nishimura, and S. Takeuchi, Phys. Rev. Lett. **99**, 161602 (2007); J. W. Elliott, J. Giedt, and G. D. Moore, Phys. Rev. D **78**, 081701(R) (2008).
[59] G. Ishiki, Y. Takayama, and A. Tsuchiya, J. High Energy Phys. 10 (2006) 007; G. Ishiki, S. Shimasaki, Y. Takayama, and A. Tsuchiya, J. High Energy Phys. 11 (2006) 089; T. Ishii, G. Ishiki, K. Ohta, S. Shimasaki, and A. Tsuchiya, Prog. Theor. Phys. **119**, 863 (2008); T. Ishii, G. Ishiki, S. Shimasaki, and A. Tsuchiya, Phys. Rev. D **77**, 126015 (2008); **78**, 106001 (2008); G. Ishiki, S. W. Kim, J. Nishimura, and A. Tsuchiya, Phys. Rev. Lett. **102**, 111601 (2009); G. Ishiki, K. Ohta, S. Shimasaki, and A. Tsuchiya, Phys. Lett. B **672**, 289 (2009).
[60] M. Chaichian and A. P. Demichev, *Introduction to Quantum Groups* (World Scientific, Singapore, 1996).
[61] S. Majid, *Foundations of Quantum Group Theory* (Cambridge University Press, Cambridge, England, 1995).
[62] V. Chari and A. Pressley, *A Guide to Quantum Groups* (Cambridge University Press, Cambridge, England, 1994).