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Accelerated cosmic expansion in a scalar-field universe

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We consider here a spherically symmetric but inhomogeneous universe filled with a massless scalar field. The model obeys two constraints. The first one is that the gradient of the scalar field is timelike everywhere. The second constraint is that the radial coordinate basis vector is a unit vector field in the comoving coordinate system. We find that the resultant dynamical solutions compose a one-parameter family of self-similar models which is known as the Roberts solution. The solutions are divided into three classes. The first class consists of solutions with only one spacelike singularity in the synchronous-comoving chart. The second class consists of solutions with two singularities which are null and spacelike, respectively. The third class consists of solutions with two spacelike singularities which correspond to the big bang and big crunch, respectively. We see that, in the first case, a comoving volume exponentially expands as in an inflationary period; the fluid elements are accelerated outwards from the symmetry center, even though the strong energy condition is satisfied. This behavior is very different from that observed in the homogeneous and isotropic universe in which the fluid elements would move outwards with deceleration, if the strong energy conditions are satisfied. We are thus able to achieve the accelerated expansion of the universe for the models considered here, without a need to violate the energy conditions. The cosmological features of the models are examined in some detail.

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I. INTRODUCTION

The big bang universe model has succeeded in explaining three prominent observational facts, i.e., the Hubble's law, the abundance of light elements, and the 2.7 K cosmic microwave background radiation [1]. Further, within this model, the formations of various structures in the universe, i.e., stars, galaxies, clusters of galaxies, etc., seem to be naturally explained if we assume the existence of dark matter [2]. In the original big bang model, there are three basic unnatural aspects, which are known as the horizon problem, flatness problem, and monopole problem. Now it is widely believed that these three problems are the evidence for the existence of the so-called inflationary period in the past of our universe [3–5]. In this period, by virtue of the accelerated cosmic volume expansion, initial inhomogeneities are stretched, the space is made sufficiently flat, and, at the same time, moderate quantum instabilities generate the seeds of the present structures in the universe. The theoretical prediction of the inflationary scenario basically agrees well with the WMAP data on cosmic microwave background radiation [6].

Recent observational results also imply that our universe has again entered into another inflationary period at the redshifts of about $z \sim 0.3$. This is known as the dark energy problem. The WMAP data and the observations of the distance-redshift relations of supernovae [7–10] mean that the speed of the cosmic volume expansion is necessarily accelerated if the universe is homogeneous and isotropic. Further, if general relativity is the correct theory of gravity, then the accelerated cosmic expansion implies the existence of a dark energy component which does not satisfy the strong energy condition, i.e., effectively it has to have an equation of state given as $P = w\rho$ with $w < -1/3$, where ρ is the energy density and P is the pressure of the dark energy component. It is far from clear today what such a dark energy component could be. The simplest answer to this conundrum could be the existence of a nonvanishing cosmological constant in the universe, since in that case, only one constant can explain all of the observational data. However, it does not at all appear to be easy to explain the origin of such a cosmological term. Therefore, the issue is under an active investigation and several approaches to resolve the problem are being pursued simultaneously. These include the existence of a quintessence field, using an alternative theory of gravity (see, for example, Ref. [11]), and also inhomogeneous universe models, as we shall discuss below.

In this paper, we consider a self-similar model for the spherically symmetric but inhomogeneous universe, filled with a massless scalar field which is minimally coupled to gravity. This solution to Einstein equations has been discussed in the literature by Roberts [12], Brady [13], and Oshiro, Nakamura, and Tomimatsu [14]. However, these authors used this solution only to study the gravitational collapse of a massless scalar field and used it as a model for the collapsing matter. In this paper we will focus on the cosmological significance of this solution. All of the physically reasonable energy conditions hold for this model universe, such as the weak, strong, and dominant energy

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conditions [15]. Although this solution does not directly relate to our present universe, there are certain interesting properties that the models exhibit as we shall point out here. In the region where the gradient of the scalar field is timelike, the system is equivalent to a stiff perfect fluid, given by the equation of state $P = \rho$, and we can consider motions of the fluid elements. As we show here, in this solution the fluid elements are accelerated outwards, even though the strong energy condition holds.

Here we should mention the models for a dust-filled inhomogeneous universe, which have been considered to resolve the dark energy problem without introducing any exotic matter components, or without invoking any alternative theories of gravity. Mainly two approaches have been used in this connection. The first one invokes the backreaction effect of inhomogeneities in the universe, which causes the acceleration of the cosmic volume expansion in an average sense [16–22]. In the other approach, one invokes a spherically symmetric but inhomogeneous velocity field which causes the apparent acceleration of the cosmic volume expansion [23–29]. At present, it seems to be difficult to resolve the dark energy problem using the first approach (see, e.g., [30–33] for a discussion on this issue), and also there are some arguments which discuss the viability or otherwise of the second approach [34,35].

We would like to emphasize that the present solution that we discuss and consider here is different from both of the above possible alternatives that have been used for dust-filled inhomogeneous universe models. In the present solution, each fluid element is accelerated outwards in the real and exact sense from the observer at the center of symmetry.

The paper is organized as follows. In Sec. II, we briefly review the massless scalar field minimally coupled to gravity, and then, in Sec. III, the basic equations in a comoving reference frame are presented. In Sec. IV, by imposing a constraint that the radial coordinate basis vector is a unit vector field, we obtain the solution. We believe our method gives some insights into the properties of the solution we derived this way. In Sec. V, we show that the charts of the solutions obtained in some of the classes we considered do not cover the whole of the spacetime. Therefore, we perform two kinds of extensions, respectively, in Secs. VI and VII. In Sec. VI, we also discuss and analyze the global structure of the solutions. Finally, Sec. VIII is devoted to a summary and discussion.

In this paper, we adopt the abstract index notation [36]; the Latin indices denote the type of the tensor, whereas the Greek indices mean the components with respect to the coordinate basis. We use the units $8\pi G = c = 1$.

II. MASSLESS SCALAR FIELD

The massless scalar field $\phi(x^a)$ minimally coupled to gravity on a spacetime (M, g_{ab}) is described by the Lagrangian,

$$\mathcal{L} = -\tfrac{1}{2}\phi_{;a}\phi_{;b}g^{ab}, \tag{1}$$

where $\phi_{;\alpha}$ denotes the covariant derivative of ϕ. The corresponding Euler-Lagrange equation is $\phi_{;ab}g^{ab} = 0$, and the stress-energy tensor for the scalar field, as calculated from this Lagrangian, is

$$T_{ab} = \phi_{;a}\phi_{;b} - \tfrac{1}{2}g_{ab}(\phi_{;c}\phi_{;d}g^{cd}). \tag{2}$$

In the case where the gradient of the massless scalar field $\phi_{;a}$ is timelike or spacelike, we can introduce a unit vector field

$$u_a = \frac{\phi_{;a}}{N}, \tag{3}$$

where $N = \sqrt{|\phi^{;b}\phi_{;b}|}$. Then the stress-energy tensor of the massless scalar field is rewritten in the form

$$T_{ab} = \frac{N^2}{2}(2u_a u_b + \varepsilon g_{ab}), \tag{4}$$

where $\varepsilon = -1$ for spacelike $\phi_{;a}$, while $\varepsilon = +1$ for timelike $\phi_{;a}$. Clearly, u^a is one of eigenvectors of T_{ab}. In this case, the massless scalar field is categorized into a *Type I* matter field [15]; i.e., it admits one timelike and three spacelike eigenvectors. So, at each point $q \in M$, we can then express the tensor T^{ab} in terms of an orthonormal basis $(E_0^a, E_1^a, E_2^a, E_3^a)$, where E_0^a is a timelike eigenvector with an eigenvalue ρ and E_A^a $(A = 1, 2, 3)$ are three spacelike eigenvectors with eigenvalues p_A. Here ρ represents the energy density of the scalar field as measured by an observer with a 4-velocity E_0^a at q, and the eigenvalues p_A represent the principal pressures in three spacelike directions E_A^a.

In the case of spacelike $\phi_{;a}$, we can put $E_1^a = u^a$. Then the stress-energy tensor (4) is written in the form

$$T^{ab} = \frac{N^2}{2}(E_0^a E_0^b + E_1^a E_1^b - E_2^a E_2^b - E_3^a E_3^b). \tag{5}$$

It is seen from the above form that $\rho = p_1 = N^2/2 \geq 0$, whereas $p_2 = p_3 = -N^2/2 \leq 0$.

In the case of timelike $\phi_{;a}$, we can put $E_0^a = u^a$. Then the stress-energy tensor (4) is written in the form

$$T^{ab} = \frac{N^2}{2}(E_0^a E_0^b + E_1^a E_1^b + E_2^a E_2^b + E_3^a E_3^b). \tag{6}$$

It is seen from the above form that $\rho = p_1 = p_2 = p_3 = N^2/2 \geq 0$. This is equivalent to the stress-energy tensor of the stiff perfect fluid.

In the case that $\phi_{;a}$ is null, the stress-energy tensor takes the following form,

$$T_{ab} = \sigma k_a k_b, \tag{7}$$

where k_a is a null vector field and σ is a non-negative function. This is equivalent to the stress-energy tensor of the null dust, and is categorized into a *Type II* matter field.

III. BASIC EQUATIONS

Hereafter, we focus on the spherically symmetric system.

A. Comoving coordinate system

If the gradient of the scalar field is not null, as mentioned in the previous section, the stress-energy tensor has four eigenvectors (E_0^a, E_A^a). In this case, we can choose the spherically symmetric coordinates (t, r, θ, φ) such that the coordinate basis vectors are parallel to these eigenvectors. This is the comoving coordinate system in the sense that there is no energy flux; i.e., the time-space components of the stress-energy-tensor are vanishing. The line element in this coordinate system is then written in the following form,

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R^2(t,r)d\Omega^2, \tag{8}$$

where $d\Omega^2$ is the line element on a unit two-sphere. It is noted that there are still two scaling freedoms of one variable left in t and r. In the models considered here, R is a monotonically increasing function of r.

In the spherically symmetric system, by using Eq. (2), a nontrivial time-space component is given by

$$T_{tr} = \dot{\phi}\phi', \tag{9}$$

where $(\dot{})$ denotes the partial derivative with respect to t and $(')$ with respect to r. Thus, in the comoving system, one of the $\dot{\phi}$ and ϕ' must vanish.

Hereafter, as mentioned in Sec. I, we assume that $\phi_{;a}$ is everywhere timelike, i.e., $\phi = \phi(t)$, and thus this system is equivalent to that of the stiff perfect fluid. Here it is worth noting that if the norm of $\phi_{;a}$ does not have a definite sign, the comoving coordinate system cannot cover the whole of the spacetime.

The components of the stress-energy tensor are then given by

$$T^\mu{}_\nu = \mathrm{diag}[-\rho, \rho, \rho, \rho], \tag{10}$$

where

$$\rho = \tfrac{1}{2}e^{-2\nu}\dot{\phi}^2. \tag{11}$$

B. Einstein equations

The dynamic evolution of the initial data, as specified on a spacelike surface of constant time is determined by the Einstein equations. For the metric (8), using the definitions

$$G(t, r) = e^{-2\psi}(R')^2, \tag{12}$$

$$H(t, r) = e^{-2\nu}(\dot{R})^2, \tag{13}$$

and

$$F = R(1 - G + H), \tag{14}$$

the independent Einstein equations for the massless scalar field are

$$F' = \tfrac{1}{2}e^{-2\nu}\dot{\phi}^2R^2R', \tag{15}$$

$$\dot{F} = -\tfrac{1}{2}e^{-2\nu}\dot{\phi}^2R^2\dot{R}, \tag{16}$$

$$\partial_t(R^2e^{\psi-\nu}\dot{\phi}) = 0, \tag{17}$$

$$-2\dot{R}' + R'\frac{\dot{G}}{G} + \dot{R}\frac{H'}{H} = 0. \tag{18}$$

Here the function $F = F(t, r)$ has an interpretation of the mass function, and it gives the total mass in a shell of comoving radius r on any spacelike slice $t = \mathrm{const}$.

IV. SOLVING THE EINSTEIN EQUATIONS

The function $R(t, r)$ is the area radius of a shell labeled "r" at an epoch "t." For the sake of definiteness, let us consider the situation of an expanding universe, so we have $\dot{R} > 0$ because we are considering here the expanding branch of the solutions. If $\dot{R}$ changes sign then that corresponds to the recollapse of the field during the dynamical evolution of the universe. We note that Eq. (17) is the Klein-Gordon equation for the scalar field, which is a part here of the Einstein equations via the Bianchi identities. We can integrate this equation to get

$$R^2e^{\psi-\nu}\dot{\phi} = r^2f(r), \tag{19}$$

where $f(r)$ is an arbitrary function of integration. The solution of these equations, subject to the initial data and energy conditions, would determine the time evolution of the system.

We now construct a class of solutions with $\psi = 0$. This is the second assumption that we make here, namely, that the radial coordinate basis vector is a unit vector field. From (18), we have

$$H = H(t). \tag{20}$$

Also, using Eq. (19), Eq. (16) can be integrated to give

$$F = \frac{1}{2}\frac{r^4f^2(r)}{R} + h(r), \tag{21}$$

where $h(r)$ is an arbitrary function of integration. Together with the above equation, Eq. (15) leads to

$$\left(\frac{1}{R}\right)' + \frac{(r^2f)'}{r^2f}\frac{1}{R} + \frac{h'}{r^4f^2} = 0. \tag{22}$$

The solution of the above equation is given by

$$R = \frac{r^2f(r)}{g(t) + p(r)}, \tag{23}$$

where $g(t)$ is an arbitrary function of integration, and

$$p(r) = -\int \frac{h'(r)}{r^2 f(r)} dr. \tag{24}$$

Here, by a scaling freedom of t, we set

$$\phi(t) = \varphi t, \tag{25}$$

where φ is a constant with a dimension of length inverse (note that the scalar field is dimensionless in our unit). Then we have $\varphi e^{-\nu} R^2 = r^2 f(r)$ from Eq. (19). Further, using Eqs. (13) and (20), we have

$$\left(\frac{dR^{-1}}{dt}\right)^2 = \frac{\varphi^2 H(t)}{r^4 f^2(r)}. \tag{26}$$

Using Eq. (23), the above equation implies

$$\dot{g}^2(t) = \varphi^2 H(t). \tag{27}$$

Substituting the above results into Eq. (14), we have

$$H + 1 = \frac{1}{2}(g+p)^2 + \frac{h}{r^2 f}(g+p) + \left[\frac{(r^2 f)'(g+p) + h'}{(g+p)^2}\right]^2. \tag{28}$$

The left-hand side of the above equation depends on only t, and thus the right-hand side should also depend on only t. This equation gives a constraint on the functions f and h.

As long as we focus on the dynamical situations, we can see from Eq. (28) that the most general allowed form of $r^2 f(r)$ is

$$r f(r) = \alpha, \tag{29}$$

where α is a positive dimensionless constant, and further $h = 0$ (see Appendix A). Then, Eq. (28) leads to

$$\varphi^{-1}\dot{g}(t) = \pm\left[\frac{1}{2} g^2(t) + \frac{\alpha^2}{g^2(t)} - 1\right]^{1/2}. \tag{30}$$

We take the negative sign in the equation so that $\dot{R} > 0$. This can be easily solved, and we have

$$g = \sqrt{1 + \frac{1}{2}\left(Ce^{-\sqrt{2}\varphi t} + \frac{1-2\alpha^2}{Ce^{-\sqrt{2}\varphi t}}\right)}, \tag{31}$$

where C is an integration constant. So the metric we get is

$$ds^2 = -\frac{(\varphi\alpha r)^2}{g^4(t)} dt^2 + dr^2 + \frac{(\alpha r)^2}{g^2(t)} d\Omega^2 \tag{32}$$

Also, we have

$$F = \frac{1}{2}\alpha r g(t) \quad \text{and} \quad \rho = \frac{1}{2}\frac{g^4(t)}{(\alpha r)^2}. \tag{33}$$

From Eq. (33), we see that the energy density ρ diverges at $r = 0$ if g does not vanishes. Moreover, even for $r > 0$, ρ diverges if g diverges. Since the Ricci scalar is proportional to ρ, $g = \infty$ corresponds to the scalar polynomial singularity [15].

Here, we choose C so that $C = \sqrt{2\alpha^2 - 1}$ for $\alpha^2 > 1/2$, $C = 1$ for $\alpha^2 = 1/2$, and $C = \sqrt{1 - 2\alpha^2}$ for $\alpha^2 < 1/2$. Then, we have the following three distinct classes of solutions.

A. The case of $\alpha^2 > 1/2$

In this case, we obtain

$$g(t) = [1 - \sqrt{2\alpha^2 - 1}\sinh(\sqrt{2}\varphi t)]^{1/2}. \tag{34}$$

The domain of time t is $-\infty < t < t_b$, where

$$t_b := \frac{1}{2\sqrt{2}\varphi} \ln\left|\frac{\sqrt{2}\alpha + 1}{\sqrt{2}\alpha - 1}\right|. \tag{35}$$

For $t \to -\infty$, g diverges, and thus the energy density and the Ricci scalar also diverge for $r > 0$. As will be shown in the next section, $t \to -\infty$ is not infinity, and hence this should be regarded as an initial singularity (big bang). At $t = t_b$, g vanishes, and thus any two-dimensional spheres with positive comoving radii have infinite area, since the area A with a comoving radius r is given by

$$A = 4\pi(\alpha r)^2/g^2. \tag{36}$$

As will be shown later, $t = t_b$ is on the future null infinity except at $r = 0$; a "point" $(t, r) = (t_b, 0)$ is a sphere with finite area. As will be shown later, the extension over this sphere is possible.

B. The case of $\alpha^2 = 1/2$

In this case, we obtain

$$g(t) = [1 + \exp(-\sqrt{2}\varphi t)]^{1/2}. \tag{37}$$

The domain of time is $-\infty < t < \infty$. As in the case of $\alpha^2 > 1/2$, $t = -\infty$ corresponds to an initial singularity at which g diverges. However, in contrast with the case of $\alpha^2 > 1/2$, g is always larger than unity and finite except on $t = -\infty$. This chart is inextendible.

C. The case of $\alpha^2 < 1/2$

In this case, we obtain

$$g(t) = [1 + \sqrt{1 - 2\alpha^2}\cosh(\sqrt{2}\varphi t)]^{1/2}. \tag{38}$$

The domain of time is also $-\infty < t < \infty$. As in the previous cases of $\alpha^2 \geq 1/2$, $t \to -\infty$ corresponds to an initial singularity at which g diverges. However, in contrast with the case of $\alpha^2 = 1/2$, g diverges also in the limit of $t \to \infty$. As will be shown later, the limit of $t \to \infty$ is not infinity, and thus this is a final singularity (big crunch). This chart is also inextendible.

V. SOME PROPERTIES OF SOLUTIONS

In this section, we shall analyze a few properties of this solution, including its extendibility. For this purpose, we introduce a new time coordinate τ defined by

$$d\tau = \frac{dt}{g^2(t)}. \tag{39}$$

The line element with this new time coordinate is then given by

$$ds^2 = -(\varphi\alpha r)^2 d\tau^2 + dr^2 + \frac{(\alpha r)^2}{g^2} d\Omega^2. \tag{40}$$

This new time coordinate is proportional to the proper time for a comoving observer along the $r =$ constant line. Thus, this is a geometrically and physically meaningful quantity. We discuss the cases of $\alpha^2 > 1/2$, $\alpha^2 = 1/2$, and $\alpha^2 < 1/2$, separately.

A. The case of $\alpha^2 > 1/2$

By integration of Eq. (39) with an appropriate integration constant, we have

$$\tau = \frac{1}{2\alpha\varphi} \ln \left| \frac{1 + \sqrt{(\sqrt{2}\alpha + 1)/(\sqrt{2}\alpha - 1)} e^{\sqrt{2}\varphi t}}{1 - \sqrt{(\sqrt{2}\alpha - 1)/(\sqrt{2}\alpha + 1)} e^{\sqrt{2}\varphi t}} \right|. \tag{41}$$

We can easily see from the above equation that τ vanishes for $t \to -\infty$. This means that $t \to -\infty$ is not infinity. As mentioned, since the energy density ρ and Ricci scalar diverge for $t \to -\infty$, this is the initial singularity. By contrast, τ becomes infinite in the limit of $t \to t_b$. As will be shown later, $t = t_b$ corresponds to the future null infinity.

B. The case of $\alpha^2 = 1/2$

In this case, we obtain

$$\tau = \frac{1}{\sqrt{2}\varphi} \ln |e^{\sqrt{2}\varphi t} + 1|. \tag{42}$$

We can easily see that τ vanishes in the limit of $t \to -\infty$, whereas τ becomes infinite in the limit of $t \to \infty$. Thus, $t \to -\infty$ is the initial singularity also in this case, whereas, as will be shown later, $t \to \infty$ corresponds to the future timelike infinity.

C. The case of $\alpha^2 < 1/2$

In this case, we have

$$\tau = \frac{1}{2\alpha\varphi} \ln \left| \frac{1 + \sqrt{(1 + \sqrt{2}\alpha)/(1 - \sqrt{2}\alpha)} e^{\sqrt{2}\varphi t}}{1 + \sqrt{(1 - \sqrt{2}\alpha)/(1 + \sqrt{2}\alpha)} e^{\sqrt{2}\varphi t}} \right|. \tag{43}$$

Also in this case, τ vanishes in the limit of $t \to -\infty$, and this "moment" corresponds to the initial singularity. In contrast to the case of $\alpha^2 = 1/2$, we have, for $t \to \infty$,

$$\tau \to \tau_c := \frac{1}{2\alpha\varphi} \ln \left| \frac{1 + \sqrt{2}\alpha}{1 - \sqrt{2}\alpha} \right|. \tag{44}$$

Thus, the limit of $t \to \infty$ does not correspond to an infinity, but the final singularity.

Using the above results, we obtain

$$g^2 = \frac{4\alpha^2 e^{2\alpha\varphi\tau}}{(\sqrt{2}\alpha - 1)e^{4\alpha\varphi\tau} + 2e^{2\alpha\varphi\tau} - \sqrt{2}\alpha - 1} \tag{45}$$

for arbitrary positive α. If the denominator in the right-hand side of Eq. (45) vanishes, then g diverges. The denominator vanishes if

$$e^{2\alpha\varphi\tau} = 1 \tag{46}$$

or

$$e^{2\alpha\varphi\tau} = \frac{1 + \sqrt{2}\alpha}{1 - \sqrt{2}\alpha} \tag{47}$$

is satisfied. The second root is meaningful only if $\alpha^2 < 1/2$, since τ should be a real number. The first root is $\tau = 0$ which corresponds to the big bang, whereas the second one is $\tau = \tau_c$ which corresponds to the big crunch.

It might be a remarkable fact that, in the case of $\alpha^2 > 1/2$, the areal radius R behaves as

$$R^2 \to \frac{\sqrt{2}\alpha - 1}{4\alpha^2} r^2 e^{2\alpha\varphi\tau} \tag{48}$$

for $\tau \gg (\alpha\varphi)^{-1}$. Thus it is seen that the comoving volume exponentially expands as in the inflationary period, and that the acceleration of the cosmic volume expansion is realized, even though the strong energy condition holds.

The world interval of the submanifold (τ, r) in Eq. (40) takes the same form as that of the Rindler spacetime. Since the Rindler spacetime is extendible, we are led to infer that this solution might also be extendible. We discuss the maximal extension of this solution in the following two sections.

VI. ANALYTIC EXTENSION

Following the prescription of the maximal extension for the Rindler spacetime, we introduce following new coordinates,

$$T = r\sinh(\varphi\alpha\tau) \quad \text{and} \quad X = r\cosh(\varphi\alpha\tau); \tag{49}$$

then we have

$$ds^2 = -dT^2 + dX^2 + R^2 d\Omega^2. \tag{50}$$

From Eq. (49), we have

$$X^2 - T^2 = r^2 \quad \text{and} \quad \frac{T}{X} = \tanh(\varphi\alpha\tau). \tag{51}$$

The first equation implies that r-constant curves are timelike hyperbolic curves, whereas τ-constant curves are spacelike straight lines, in the (T, X) plane. The singularity $\tau = 0$ corresponds to $T = 0$, whereas $r = 0$ is null $T = \pm X$, and thus $T/X = \pm 1$. This implies that $r = 0$ also corresponds to $\tau = \pm\infty$ except at $T = X = 0$.

The square of the areal radius R^2 is written as a function of T and X in the form

$$R^2 = T(\sqrt{2}\alpha X - T). \tag{52}$$

The above equation explicitly shows the self-similarity of the present solution, since all the dimensionless variables are written as functions of a self-similar variable $\xi := X/T$. Further, there is only one parameter α. Thus, the solutions obtained here compose a one-parameter family of self-similar solutions which is known as the Roberts solution [12].

In order that R^2 is non-negative, both $T \geq 0$ and $T \leq \sqrt{2}\alpha X$ have to be satisfied, or both $T \leq 0$ and $T \geq \sqrt{2}\alpha X$ have to be satisfied. The boundary of these two regions in the (T, X) plane is the singularity $T = 0$ ($\tau = 0$). Thus we cannot regard these regions in (T, X) as two regions of one spacetime manifold: each region should be regarded as an independent spacetime manifold. Here the former, $T \geq 0$ and $T \leq \sqrt{2}\alpha X$, is of interest to us.

The energy density ρ is written in the form

$$\rho = \frac{\alpha^2(X^2 - T^2)}{2T^2(\sqrt{2}\alpha X - T)^2}. \tag{53}$$

Thus the spacetime singularities are located at $T = 0$ and $T = \sqrt{2}\alpha X$.

It is useful for understanding the physical situation of this spacetime to examine the expansions of future-directed null geodesic congruences, and also to consider the Misner-Sharp mass [37], since the Misner-Sharp mass gives a measure of the amount of energy within a sphere labeled by X at time T (see Appendix B). In the coordinate system (50), the expansions of outgoing and ingoing null are given, respectively, by

$$\vartheta_+ = \frac{1}{R^2}[\alpha X - (\sqrt{2} - \alpha)T], \tag{54}$$

$$\vartheta_- = \frac{1}{R^2}[\alpha X - (\sqrt{2} + \alpha)T]. \tag{55}$$

In accordance with Hayward's definition [38], a region or a surface is said to be *trapped* if $\vartheta_+\vartheta_- > 0$, *marginal* if $\vartheta_+\vartheta_- = 0$, and *untrapped* if $\vartheta_+\vartheta_- < 0$. A trapped region or a trapped surface of $\vartheta_- > 0$ is said to be past trapped, whereas a trapped region or a trapped surface of $\vartheta_+ < 0$ is said to be future trapped. Similarly, a region or surface of $\vartheta_- = 0$ is said to be past marginal, whereas a region or surface of $\vartheta_+ = 0$ is said to be future marginal.

From Eq. (B2), we have the Misner-Sharp mass $M_{\rm MS}$ of Eq. (50) as

$$M_{\rm MS} = \frac{\alpha^2}{4R}(X^2 - T^2). \tag{56}$$

It should be noted that the Misner-Sharp mass is given by $M_{\rm MS} = F/2$, where F has been defined by Eq. (14).

A. The case of $\alpha^2 > 1/2$

The domain covered by the original chart is $0 < T < X$. It is easily seen from Eqs. (32) and (52) that $T = X$ is regular (see also Appendix C). Thus we can analytically extend the region $0 < T < X$ covered by the original chart (32) over the region $0 < T < \sqrt{2}\alpha X$ where the positivity of R^2 is guaranteed. This extension has also been discussed in Oshiro, Nakamura, and Tomimatsu [14]. R^2 vanishes on $T = \sqrt{2}\alpha X$. Here it should be noted that $T = \sqrt{2}\alpha X$ is a spacetime singularity where the energy density ρ and scalar polynomials diverge there: see Eq. (53) and Appendix C. This singularity is a timelike naked singularity.

It is seen from Eq. (54) that the expansion of outgoing null ϑ_+ is positive everywhere in the maximally extended chart, whereas the expansion of ingoing null ϑ_- is negative for $\alpha X/(\sqrt{2} + \alpha) < T < \sqrt{2}\alpha$, vanishes on $T = \alpha X/(\sqrt{2} + \alpha)$, and is positive $0 < T < \alpha X/(\sqrt{2} + \alpha)$. Thus, the region of $0 < T < \alpha X/(\sqrt{2} + \alpha)$ is past trapped, $T = \alpha X/(\sqrt{2} + \alpha)$ is a past marginal surface, and $\alpha X/(\sqrt{2} + \alpha) < T < \sqrt{2}\alpha X$ is untrapped. The past marginal surface $T = \alpha X/(\sqrt{2} + \alpha)$ corresponds to the Hubble horizon. The singularities and the Hubble horizon in the (T, X) plane are depicted in Fig. 1, while the conformal diagram is given in Fig. 2. Here it should be noted that the Hubble horizon is spacelike; this is an important difference between the present solution and inflationary solutions.

We can easily see from Eq. (56) that $M_{\rm MS}$ is positive for $0 < T < X$, $M_{\rm MS}$ vanishes at $T = X$ ($r = 0$ in the original coordinate chart), and $M_{\rm MS}$ is negative for $T > X$. Further, $M_{\rm MS} = -\infty$ at $R = 0$, i.e., $T \to \sqrt{2}\alpha X_{+0}$. The contribution of the scalar fields to $M_{\rm MS}$ is positive and thus $M_{\rm MS}$ is a monotonically increasing function with respect to X. Thus the negativity of Misner-Sharp mass in the region added by the extension comes from the negative infinite Misner-

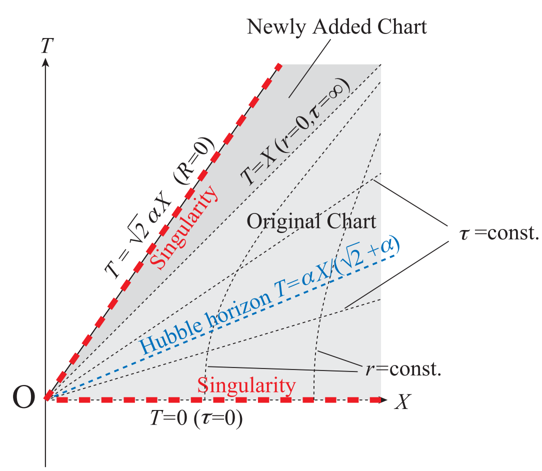

FIG. 1 (color online). The singularities and Hubble horizon of the analytically extended solution are depicted in (T, X) plane.

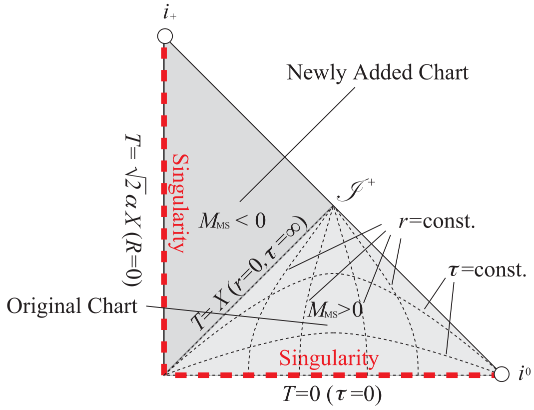

FIG. 2 (color online). The conformal diagram of the analytically extended solution is depicted. This is similar to the upper half of the Minkowski spacetime. Note that the comoving lines, r = constant, enter into the future null infinity. This means that the elements of stiff fluid are accelerated to the speed of light asymptotically.

Sharp mass concentrated on the central timelike singularity. It seems difficult to get physical interpretation of this analytically extended solution, since the effect of the central naked singularity on its surrounding region must not be negligible.

The scalar field now takes the following form:

$$\phi = \frac{1}{\sqrt{2}} \ln \left| \frac{T\sqrt{2\alpha^2 - 1}}{\sqrt{2}\alpha X - T} \right|. \tag{57}$$

At the singularities $T = 0$ and $T = \sqrt{2}\alpha X$, the scalar field diverges. Here, it is interesting to see the norm of the gradient of the scalar field. We have

$$g^{ab}\phi_{;a}\phi_{;b} = \frac{\alpha^2}{R^4}(T^2 - X^2). \tag{58}$$

We see from the above equation that $\phi_{;a}$ is timelike in the

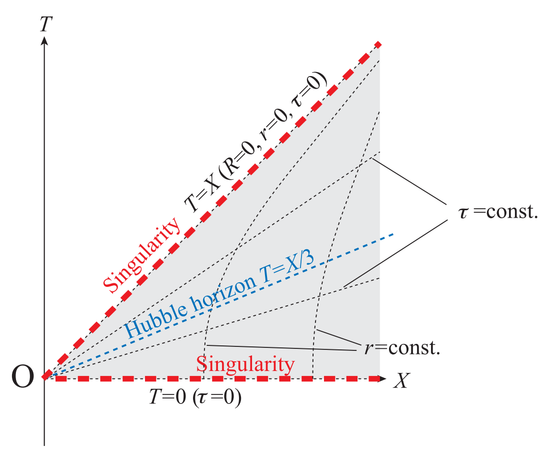

FIG. 3 (color online). The singularities and Hubble horizon for $\alpha^2 = 1/2$ are depicted in the (T, X) plane. There is a central null singularity which is not naked.

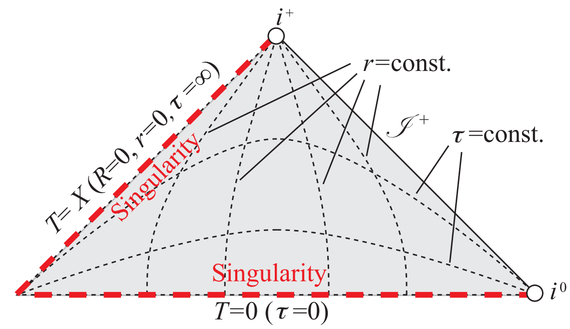

FIG. 4 (color online). The conformal diagram of the analytically extended solution of $\alpha^2 = 1/2$ is depicted. The comoving lines, r = constant, enter into the timelike infinity.

domain $0 < T < X$, null on $T = X$, and spacelike in $X < T < \sqrt{2}\alpha X$. Therefore, this analytically extended solution is not equivalent to the system of a stiff perfect fluid.

B. The case of $\alpha^2 = 1/2$

In this case, from Eq. (53), we find that the energy density ρ diverges at $T = X$, and thus $T = X$ is a null spacetime singularity which is not naked, and its Misner-Sharp mass vanishes. As a result, the original chart is inextendible. Also in this case, $T = X/3$ is a past marginal surface which corresponds to the Hubble horizon. The singularities and the Hubble horizon in the (T, X) plane are depicted in Fig. 3. The conformal diagram is given in Fig. 4. By contrast to the case of $\alpha^2 > 1/2$, comoving lines of constant r enter into a future timelike infinity. There is a future null and future timelike and spacelike infinities.

C. The case of $\alpha^2 < 1/2$

As mentioned earlier, in this case, there are two kinds of singularities in the universe. Both of them are spacelike: one is the big bang and the other is the big crunch. It is remarkable that even though this is not a closed universe in the usual sense, the big crunch exists. There are the past

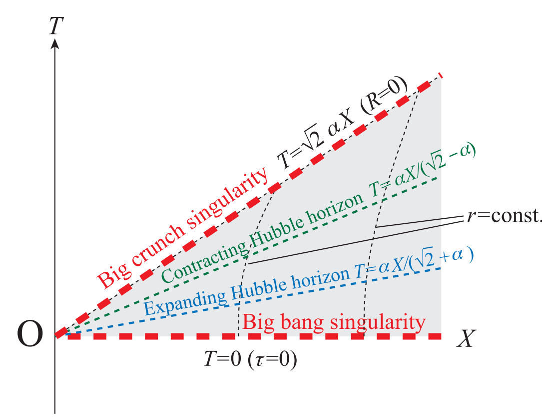

FIG. 5 (color online). The singularities and Hubble horizon for $0 < \alpha^2 < 1/2$ are depicted in the (T, X) plane.

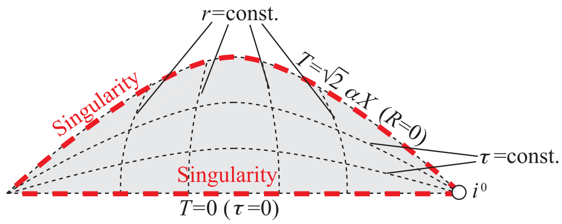

FIG. 6 (color online). The conformal diagram of the analytically extended solution of $\alpha^2 < 1/2$ is depicted. There are both the big bang and big crunch singularities, although the topology of the $t =$ constant hypersurface is $\mathbf{R}^3$.

and future marginal surfaces. Both of them correspond to the Hubble horizon, but the past one is in the expanding phase, whereas the future one is in the contracting phase. There is no null infinity. This type of solution is called the ‘‘universal’’ black hole by Carr and Gundlach [39]. The same solution as this solution has also been obtained by Carr, Harada and Maeda [40]. The singularities and past marginal surface are depicted in the (T, X) plane in Fig. 5. The conformal diagram is also given in Fig. 6.

VII. C^1 EXTENSION FOR $\alpha^2 > 1/2$: SPHERICAL VOID SOLUTION

Here, we consider another extension for the case of $\alpha^2 > 1/2$ without any serious singularities except for the big bang. For this purpose, it is very important to note that the Misner-Sharp mass vanishes on $T - X = 0$, i.e., $(\tau, r) = (\infty, 0)$ in the original chart. This implies that the extended region $T > X$ can be Minkowski spacetime where the Misner-Sharp mass vanishes. In order to make clear whether such an extension is possible, we consider the following Lorentz boost in the subspace (T, X),

$$T = \frac{\bar{T} + v\bar{X}}{\sqrt{1 - v^2}} \quad \text{and} \quad X = \frac{\bar{X} + v\bar{T}}{\sqrt{1 - v^2}}, \tag{59}$$

where

$$v = \frac{\sqrt{2}}{\alpha} - 1. \tag{60}$$

Then we have the metric, for $T - X \leq 0$, as

$$ds^2 = -d\bar{T}^2 + d\bar{X}^2 + R^2 d\Omega^2, \tag{61}$$

where

$$R^2 = \bar{X}^2 - \frac{\alpha^2}{2}(\bar{X} - \bar{T})^2. \tag{62}$$

By the Lorentz boost (59), the null hypersurface $T - X = 0$ is mapped to $\bar{T} - \bar{X} = 0$, and thus we have $R^2 = \bar{X}^2$ at $(\tau, r) = (\infty, 0)$. Further, we have, at $(\tau, r) = (\infty, 0)$, or equivalently $\bar{T} - \bar{X} = 0$,

$$\partial_{\bar{T}} R^2 = 0 \quad \text{and} \quad \partial_{\bar{X}} R^2 = 2\bar{X}. \tag{63}$$

Thus, if we put, for $\bar{T} + \bar{X} < 0$, the Minkowski spacetime

$$ds^2 = -d\bar{T}^2 + d\bar{X}^2 + \bar{X}^2 d\Omega^2, \tag{64}$$

we can easily see that this is a C^1 extension of the original solution. Note that the Ricci tensor of this solution does not vanish on $T = X$ and hence the second order derivatives of the metric tensor are discontinuous. This extension has also been discussed in Oshiro, Nakamura, and Tomimatsu [14].

The region newly added by this extension is regarded as an expanding void. The big bang singularity and cosmological horizon of this C^1 extended spacetime are depicted in the $(\bar{T}, X)$ plane in Fig. 7. The conformal diagram of this C^1 extended spacetime is depicted in Fig. 8.

This universe model has a few very peculiar properties. Since the gradient of the scalar field is null on $\bar{T} = \bar{X}$, this solution cannot be regarded as the universe filled with a stiff perfect fluid only. However, we can regard the matter field in $\bar{T} < \bar{X}$ as the stiff perfect fluid. The world lines of fluid elements are the curves with constant r. These are expressed in the $(\bar{T}, \bar{X})$-coordinate system by hyperbolic curves

$$\bar{X}^2 - \bar{T}^2 = r^2. \tag{65}$$

We see from the above equation that the world lines of fluid elements become asymptotically null. This fact might be rather surprising, since the fluid elements are accelerated outward even though there is a positive gravitational mass (Misner-Sharp mass) inside the mass shell on which the fluid elements stay.

The outward acceleration causes the temporal growth of the redshift of the light ray emitted from a fluid element in $\bar{T} < \bar{X}$ to the observer at the symmetry center $\bar{X} = 0$. (See Fig. 8.) The ingoing null geodesics are given by

$$\bar{T} = \omega_0 \lambda + \bar{T}_0 \quad \text{and} \quad \bar{X} = -\omega_0 \lambda, \tag{66}$$

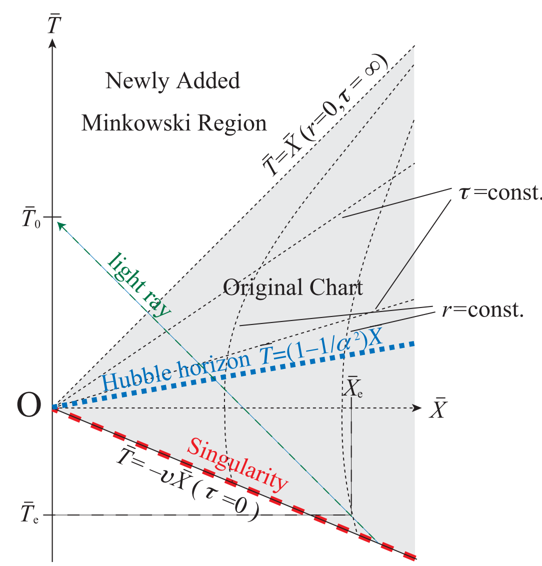

FIG. 7 (color online). In the case of the C^1-extended solution, the big bang singularity, cosmological horizon, and light ray are depicted in the (T, X) plane.

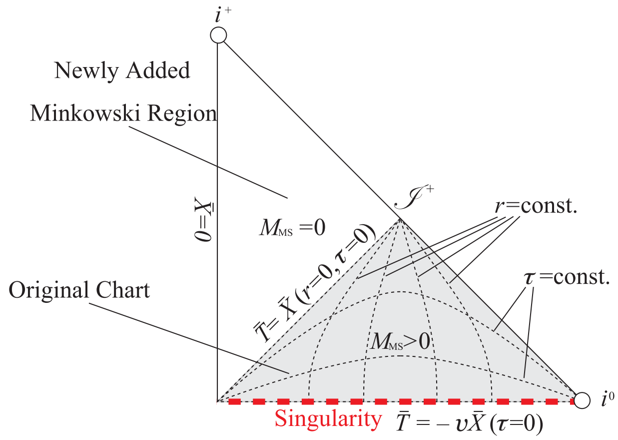

FIG. 8 (color online). The conformal diagram of the C^1 maximally extended solution is depicted. This solution describes an expanding spherical void in the universe filled with a massless scalar-field.

where λ is the affine parameter, and ω_0 and $\bar{T}_0$ are positive constants which correspond to the angular frequency detected by the observer at $(\bar{T}, \bar{X}) = (\bar{T}_0, 0)$. The big bang singularity is given by $\bar{T} = -\upsilon\bar{X}$ in this coordinate system, and thus the affine parameter takes the values in $-T_0/\sqrt{2}\omega_0(1-\upsilon) < \lambda \leq 0$. The tangent vector of this null geodesic is

$$k^\mu = (\omega_0, -\omega_0, 0, 0). \tag{67}$$

The 4-velocity of the fluid element labeled by r is given by

$$u^\mu = \frac{1}{r}(\sqrt{\bar{T}^2 + r^2}, \bar{T}, 0, 0). \tag{68}$$

The light ray detected by the observer at $(\bar{T}, \bar{X}) = (\bar{T}_0, 0)$ should be emitted from the fluid element of r at

$$(\bar{T}, \bar{X}) = (\bar{T}_e, \bar{X}_e) := \left(\frac{\bar{T}_0^2 - r^2}{2\bar{T}_0}, \frac{\bar{T}_0^2 + r^2}{2\bar{T}_0}\right). \tag{69}$$

Thus the redshift z of this light ray is given by

$$z = \frac{-k^a u_a|_{(\bar{T},\bar{X})=(\bar{T}_e,\bar{X}_e)}}{\omega_0} - 1 = \frac{\bar{T}_0 - r}{r}. \tag{70}$$

The important and peculiar feature is that the redshift z is a monotonically decreasing function of r. For a fluid element of $\bar{T}_0 < r$, the redshift z is negative; i.e., the light suffers the blueshift. The fluid element of $\bar{T}_0 < r$ emits the light ray at $\bar{T} = \bar{T}_e < 0$, and at this moment, the radial component of the 4-velocity is negative. Thus the reason for the blueshift for $\bar{T}_0 < r$ is recognized as a result of the ingoing motion of the fluid element. By contrast, if $\bar{T}_0 > r$, i.e., $\bar{T}_e > 0$, the redshift z is positive. If $\bar{T}_0 = r$, i.e., $\bar{T}_e = 0$, the redshift vanishes. The temporal variation rate of z is

$$\frac{dz}{d\bar{T}_0} = \frac{1}{r} > 0. \tag{71}$$

The redshift temporally increases. This is also a very different property from the dust-filled inhomogeneous universe model [41,42].

Since the cosmological horizon $\bar{T} = (1 - 1/\alpha^2)\bar{X}$, or equivalently, $T = \alpha X/(\sqrt{2} + \alpha)$, is spacelike, the fluid elements necessarily enter into the inside of the cosmological horizon from its outside. Thus even though the fluid elements are accelerated outward, the inflation does not occur in the usual sense. (We note that in the inflationary period, comoving world lines go outside the cosmological horizon from its inside, and such behaviors of comoving world lines are essential to resolve the horizon problem.)

Finally, we investigate the distance-redshift relation. Observationally useful distance is the luminosity distance d_L [1] which is given by

$$\begin{aligned} d_L &= (1+z)^2 R|_{(\bar{T},\bar{X})=(\bar{T}_e,\bar{X}_e)} \\ &= \frac{\bar{T}_0}{2}\sqrt{(1+z)^4 + 2(1+z)^2 - 2\alpha^2 + 1}. \end{aligned} \tag{72}$$

Since d_L is a monotonically increasing function of z, it is a monotonically decreasing function with respect to r. The redshift of the light ray from the big bang $z = z_b$ is given by

$$z_b = \sqrt{\sqrt{2}\alpha - 1} - 1. \tag{73}$$

We can easily see that the luminosity distance of the big bang vanishes. Since the light ray emitted from a comoving source at the big bang suffers finite blueshift, the flux from the big bang $\mathcal{F}_b = L_b/4\pi d_L^2$ diverges at the observer, if the luminosity of the big bang singularity L_b is finite. This feature implies that comoving sources of radiation behave as white holes at this big bang singularity [43,44]. In the limit $r \to 0_+$, $z \to \infty$, and thus d_L of the boundary of void $r = 0$ is infinite. It is hard to observe the vicinity of the boundary of the void, while it is easy to observe the vicinity of the big bang, in this universe.

VIII. SUMMARY AND DISCUSSION

We have rederived and investigated here the solution obtained by Roberts, which is a spherically symmetric but inhomogeneous universe filled with a massless scalar field minimally coupled to gravity, from a cosmological perspective. The solutions obtained compose a one-parameter family, which is divided into three distinct classes. The first class consists of solutions with only one spacelike singularity in the comoving chart. The second class consists of solutions with two singularities which are null and spacelike, respectively. The third class consists of solutions with two spacelike singularities which correspond to the big bang and big crunch, respectively.

In the case of the first class, the comoving chart does not cover the whole spacetime. Hence, we constructed two maximally extended solutions from the solution in this class. The analytic extension leads to the solution which contains a timelike singularity at the symmetry center and thus seems to be unphysical. By another extension, we obtained a solution which has no singularity other than the big bang but one that contains a spherical void. The latter one has very peculiar but interesting properties. If the gradient of the massless scalar field is timelike, this is equivalent to the stiff perfect fluid case. Then in the region where the scalar field has a timelike norm, we can naturally define fluid elements and consider their motions. In this solution, the fluid elements move outwards, and further, their outward speeds are accelerated, even though the strong energy condition holds. This is a feature which is significantly different from the homogeneous and isotropic universe models filled with the matter which satisfies the strong energy condition, and also from a spherically symmetric but inhomogeneous universe filled with the dust matter.

The physical reason for the outward acceleration of the universe that we have deduced here appears to lie essentially in the inherent physical nature itself of the massless scalar field. To try to understand this better, we note that we can construct a similar and somewhat parallel situation in a Minkowski background. Using the same coordinate system as Eq. (64), let us consider a following solution,

$$\phi(\bar{T},\bar{X}) = \frac{A}{\bar{X}}(\bar{X}-\bar{T})^2\theta_{\rm H}(\bar{X}-\bar{T}), \qquad (74)$$

where A is constant and $\theta_{\rm H}$ is the Heaviside's step function. We focus on the domain of $\bar{T} > 0$, since ϕ is everywhere finite for $\bar{T} \geq 0$, but is infinite at $\bar{X} = 0$ for $T < 0$. Then we have

$$g^{ab}\phi_{;a}\phi_{;b} = -\frac{A^2}{\bar{X}^4}(\bar{X}-\bar{T})^3(3\bar{X}+\bar{T})\theta_{\rm H}(\bar{X}-\bar{T}). \qquad (75)$$

We can easily see from the above equation that $\phi_{;a}$ is timelike for $\bar{T} < \bar{X}$, whereas it is null at $\bar{T} = \bar{X}$. Further, we see that in the limit of $\bar{T} \to \infty$ with $\bar{U} := \bar{T} - \bar{X}$ fixed, the norm of $\phi_{;a}$ becomes

$$g^{ab}\phi_{;a}\phi_{;b} \to \lim_{\bar{T}\to\infty}\frac{A^2}{(\bar{T}-\bar{U})^4}\bar{U}^3(4\bar{T}+3\bar{U}) = 0. \qquad (76)$$

The above results show that $\phi_{;a}$ becomes null asymptotically. This behavior is very similar to the present solution. This seems to indicate that the outward acceleration comes from the inherent nature of the massless scalar field. As it turns out, in the present solution, the gravity produced by the scalar field itself is too small to decelerate the outward motion of the scalar field, even though the total (Misner-Sharp) mass for the universe is infinite.

ACKNOWLEDGMENTS

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APPENDIX A: ON THE FUNCTION $f(r)$

For notational simplicity, we define the following quantities

$$\chi := r^2 f(r) \quad \text{and} \quad u := g(t) + p(r), \qquad (A1)$$

where f and p are defined by Eqs. (19) and (24), respectively. Then, Eq. (28) is written in the form

$$H + 1 = \frac{u^2}{2} + \frac{hu}{\chi} + \left(\frac{\chi'}{u} + \frac{h'}{u^2}\right)^2. \qquad (A2)$$

By differentiating the above equation with respect to r, we have

$$c_6u^6 + c_5u^5 + c_4u^3 + c_3u^2 + c_2u + c_1 = 0, \qquad (A3)$$

where

$$c_1 = -4h'^2p', \qquad (A4)$$

$$c_2 = 2(h'h'' - 3\chi'h'p'), \qquad (A5)$$

$$c_3 = 2[(\chi'h')' - \chi'^2p'], \qquad (A6)$$

$$c_4 = 2\chi'\chi'', \qquad (A7)$$

$$c_5 = \frac{hp'}{\chi}, \qquad (A8)$$

$$c_6 = \left(p + \frac{h}{\chi}\right)'. \qquad (A9)$$

Here note that all of the coefficients c_n do not depend on t. This implies that if there is a nontrivial real root $u = U$ of Eq. (A3), U must depend on only r. If so, we have

$$g(t) = U(r) - p(r). \qquad (A10)$$

The above equation holds only if both g and $U - p$ are constants. Thus, there are two possibilities; one is $g(t) =$ constant, and the other is that all of the coefficients c_n ($n = 1, \ldots, 6$) identically vanish so that Eq. (A3) is trivial. In the latter case, g may depend on t.

In the former case, without loss of generality, we can put $u = p$. Then, by definition of p, we find

$$\chi = -\frac{h'}{p'}. \qquad (A11)$$

Further, substituting $\dot{g} = 0$ into Eq. (27), we have $H = 0$. By using this fact and Eq. (A11), χ can be eliminated from Eq. (A3), and we have

$$\left(p'' - \frac{h''}{h'}p' + \frac{p'^2}{p}\right)^2 = \frac{p^2 p'^4}{h'^2}\left(1 + \frac{hp}{h'}p' - \frac{p^2}{2}\right). \quad \text{(A12)}$$

The above differential equation gives a relation between $h(r)$ and $p(r)$. The above equation, however, corresponds to the static case, in which we are not interested here in the present work. The static solutions will be discussed elsewhere.

As for the dynamical case, the conditions $c_n = 0$ give differential equations for χ and h. From $c_6 = 0$, we have

$$\frac{h}{\chi^2}\chi' = 0. \quad \text{(A13)}$$

Thus, we have $h = 0$ or $\chi' = 0$. From $c_5 = 0$, we have $hh' = 0$, and thus

$$h = \text{constant}. \quad \text{(A14)}$$

From $c_4 = 0$, we have $\chi' = \text{constant}$, and thus

$$\chi = \alpha r + \beta, \quad \text{(A15)}$$

where α and β are integration constants. Equations (A14) and (A15) lead to $c_1 = c_2 = c_3 = 0$. Since we assume that $r = 0$ is a nonsingular point at least initially, R vanishes at $r = 0$. Thus, we can see from Eq. (19) that β should vanish. Further, it is seen from Eq. (14) that F should vanish at the regular origin $r = 0$. Hence, from Eq. (21), h should vanish at $r = 0$, and thus, we have $h = 0$.

APPENDIX B: MISNER-SHARP MASS

In general, the metric of the spherically symmetric spacetime is given by

$$ds^2 = A^2(t, r)dt^2 - B^2(t, r)dr^2 - R^2(t, r)d\Omega^2. \quad \text{(B1)}$$

In this coordinate system, the Misner-Sharp mass is given by [37]

$$M_{\rm MS} = \frac{R}{2}\left(1 + \frac{R^2}{2}\vartheta_+\vartheta_-\right), \quad \text{(B2)}$$

where

$$\vartheta_\pm = \frac{1}{\sqrt{2}}\left(\frac{1}{A}\partial_t \pm \frac{1}{B}\partial_r\right)\ln R^2. \quad \text{(B3)}$$

Here note that ϑ_+ is the expansion of the outgoing radial null geodesic congruences, while ϑ_- is the expansion of the ingoing null geodesic congruences, which are not necessarily with the affine parametrization.

APPENDIX C: RICCI TENSOR

Nonvanishing components of the Ricci tensors with respect to the analytically extended chart are

$$\mathrm{Ric}\left(\frac{\partial}{\partial T}, \frac{\partial}{\partial T}\right) = \frac{\alpha^2 X^2}{R^4}, \quad \text{(C1)}$$

$$\mathrm{Ric}\left(\frac{\partial}{\partial T}, \frac{\partial}{\partial X}\right) = -\frac{\alpha^2 TX}{R^4}, \quad \text{(C2)}$$

$$\mathrm{Ric}\left(\frac{\partial}{\partial X}, \frac{\partial}{\partial X}\right) = \frac{\alpha^2 T^2}{R^4}. \quad \text{(C3)}$$

The Ricci scalar is then given by

$$\mathrm{trRic} = \frac{\alpha^2}{R^4}(T^2 - X^2). \quad \text{(C4)}$$

Since R vanishes on $T = \sqrt{2}\alpha X$, the Ricci scalar trRic diverges there.

[1] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[2] P.J.E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, Princeton, NJ, 1993).
[3] K. Sato, Mon. Not. R. Astron. Soc. **195**, 467 (1981).
[4] A. H. Guth, Phys. Rev. D **23**, 347 (1981).
[5] A. D. Linde, *Particle Physics and Inflationary Cosmology* (Harwood Academic, Chur, Switzerland, 1990).
[6] D. N. Spergel *et al.*, Astrophys. J. Suppl. Ser. **148**, 175 (2003).
[7] A. G. Riess *et al.* (Supernova Search Team Collaboration), Astron. J. **116**, 1009 (1998).
[8] S. Perlmutter *et al.* (Supernova Cosmology Project Collaboration), Astrophys. J. **517**, 565 (1999).
[9] R. A. Knop *et al.*, Astrophys. J. **598**, 102 (2003).
[10] A. G. Riess *et al.*, Astrophys. J. **607**, 665 (2004).
[11] R. R. Caldwell and M. Kamionkowski, Annu. Rev. Nucl. Part. Sci. **59**, 397 (2009).
[12] M. D. Roberts, Gen. Relativ. Gravit. **21**, 907 (1989).
[13] P. R. Brady, Classical Quantum Gravity **11**, 1255 (1994).
[14] Y. Oshiro, K. Nakamura, and A. Tomimatsu, Prog. Theor. Phys. **91**, 1265 (1994).
[15] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
[16] T. Futamase, Phys. Rev. Lett. **61**, 2175 (1988).
[17] H. Russ, M. H. Soffel, M. Kasai, and G. Borner, Phys. Rev. D **56**, 2044 (1997).
[18] Y. Nambu, Phys. Rev. D **62**, 104010 (2000).
[19] H. Kozaki and K. Nakao, Phys. Rev. D **66**, 104008 (2002).
[20] E. W. Kolb, S. Matarrese, A. Notari, and A. Riotto, Phys. Rev. D **71**, 023524 (2005).
[21] Y. Nambu and N. Tanimoto, arXiv:gr-qc/0507057.
[22] T. Kai, H. Kozaki, K. Nakao, Y. Nambu, and C. M. Yoo,

Prog. Theor. Phys. **117**, 229 (2007).
[23] I. Zehavi, A. G. Riess, R. P. Kirshner, and A. Dekel, Astrophys. J. **503**, 483 (1998).
[24] K. Tomita, Astrophys. J. **529**, 38 (2000).
[25] M. N. Celerier, Astron. Astrophys. **353**, 63 (2000).
[26] H. Iguchi, T. Nakamura, and K. Nakao, Prog. Theor. Phys. **108**, 809 (2002).
[27] R. A. Vanderveld, E. E. Flanagan, and I. Wasserman, Phys. Rev. D **74**, 023506 (2006).
[28] H. Alnes, M. Amarzguioui, and O. Gron, Phys. Rev. D **73**, 083519 (2006).
[29] J. Garcia-Bellido and T. Haugboelle, J. Cosmol. Astropart. Phys. 04 (2008) 003.
[30] E. E. Flanagan, Phys. Rev. D **71**, 103521 (2005).
[31] C. M. Hirata and U. Seljak, Phys. Rev. D **72**, 083501 (2005).
[32] A. Ishibashi and R. M. Wald, Classical Quantum Gravity **23**, 235 (2006).
[33] A. Paranjape and T. P. Singh, Phys. Rev. Lett. **101**, 181101 (2008).
[34] J. P. Zibin, A. Moss, and D. Scott, Phys. Rev. Lett. **101**, 251303 (2008).
[35] J. Garcia-Bellido and T. Haugbolle, J. Cosmol. Astropart. Phys. 09 (2008) 016.
[36] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[37] C. W. Misner and D. H. Shrap, Phys. Rev. **136**, B571 (1964).
[38] S. A. Hayward, Phys. Rev. D **53**, 1938 (1996).
[39] B. J. Carr and C. Gundlach, Phys. Rev. D **67**, 024035 (2003).
[40] B. J. Carr, T. Harada, and H. Maeda (unpublished).
[41] J. P. Uzan, C. Clarkson, and G. F. R. Ellis, Phys. Rev. Lett. **100**, 191303 (2008).
[42] C. M. Yoo, T. Kai, and K. Nakao, Prog. Theor. Phys. **120**, 937 (2008).
[43] I. D. Novikov, Astron. Zh. **41**, 1075 (1964) [Sov. Astron. **8**, 857 (1965)].
[44] Y. Ne'eman, Astrophys. J. **141**, 1303 (1965).