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Entropy of quasiblack holes

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We trace the origin of the black hole entropy S, replacing a black hole by a quasiblack hole. Let the boundary of a static body approach its own gravitational radius, in such a way that a quasihorizon forms. We show that if the body is thermal with the temperature taking the Hawking value at the quasihorizon limit, it follows, in the nonextremal case, from the first law of thermodynamics that the entropy approaches the Bekenstein-Hawking value $S = A/4$. In this setup, the key role is played by the surface stresses on the quasihorizon and one finds that the entropy comes from the quasihorizon surface. Any distribution of matter inside the surface leads to the same universal value for the entropy in the quasihorizon limit. This can be of some help in the understanding of black hole entropy. Other similarities between black holes and quasiblack holes such as the mass formulas for both objects had been found previously. We also discuss the entropy for extremal quasiblack holes, a more subtle issue.

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I. INTRODUCTION

As it is known, the entropy S of a nonextremal black hole is equal to the Bekenstein-Hawking value, $S = A/4$, where A is the area of the black hole horizon (we use units such that Newton's constant, Planck's constant, and the speed of light are put to one). Its formal appearance is especially transparent in a Euclidean action approach where this term stems entirely due to the presence of the horizon [1–4]. A development on these issues was performed by Brown and York [5], where from a quasilocal energy formalism one can deduce, among other things, black hole thermodynamics itself.

Imagine now, a collapsing body. When the surface of the body is close to its own horizon r_+, but does not coincide with it, there is no obvious reason for the presence of such an $S = A/4$ term for the entropy of the body. Therefore, at first glance, the entropy $A/4$ appears as a jump, when the black hole forms. Nonetheless, we will see below that we can restore the continuity and trace the origin of the entropy under discussion, if instead of a black hole we will consider a quasiblack hole. Roughly speaking, a quasiblack hole is an object in which the boundary gets as close as one likes to the horizon. However, an event horizon does not form (see [6–8] and references therein for more on the definition and properties of quasiblack holes, and [9–15] for examples of quasiblack holes themselves).

We will study a distribution of matter constrained inside a boundary, at some temperature T, in the vicinity of being a quasiblack hole and find the entropy S of such a system. The steps to find the universal entropy formula consist of using the first law together with the quasilocal formalism of [5]. We will see that the matter entropy helps in the generation of the $A/4$ term when the boundary approaches the horizon. Instead of giving some examples of calculating the entropy for some spherically symmetric shell (see, e.g., the very interesting studies in [16,17]), we proceed in a model-independent way and exploit essentially the fact that the boundary almost coincides with the would-be horizon, i.e., with the quasihorizon. In the procedure, it is essential that some components of surface stresses diverge in the quasihorizon limit, in the case of nonextremal quasiblack holes. It is the price paid for keeping a shell (which is inevitable near the nonextremal quasihorizon [6]) in equilibrium without collapsing. Although by themselves, such stresses that grow unbounded look unphysical, the whole quasiblack hole picture turns out to be at least useful methodically since it enables us to trace some features of black holes. In particular, in previous works [7,8] we managed to derive the black hole mass formula using a quasiblack hole approach in which both the meaning of terms and their derivation are different from the standard black hole case. Thus, the concept of a quasiblack hole has two sides: it simultaneously mimics some features of black holes but also shows those features in a quite different setting. To some extent, it can be compared with the membrane paradigm [18,19]. However, the key role that the diverging stresses play here was not exploited there (a more detailed general comparison of both approaches would be important, but is beyond the scope of our paper). In the present work, we extend the approach to the entropy of nonextremal quasiblack holes. We also discuss the issue

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of the entropy for extremal quasiblack holes. In some papers [20–22] it was argued that $S = 0$ for extremal black holes, while in other works there were arguments to recover the value $S = A/4$ (see [23] for works within general relativity, and [24] for additional arguments within string theory). Thus, it is useful to examine the issue of the entropy of extremal quasiblack holes at the classical level.

A point worth raising is that the entropy of a pure black hole can in part be recovered from entanglement arguments [25,26]. Quasiblack holes appeared first in the context of self-gravitating magnetic monopoles [9,10] and this prompted Lue and Weinberg [27] to argue that an observer in the outer region describes the quasiblack hole inner region in terms of a statistical density matrix ρ defined upon taking the trace of the degrees of freedom in this inner region. Then, defining as usual the entropy S as $S = -\mathrm{Tr}(\rho \ln\rho)$, one can argue that the emergence of the black hole entropy, or quasiblack hole entropy, can be consistently ascribed to the entanglement of the outer and inner fields [27]. This entanglement entropy has some drawbacks, one of which is that, although it gives a quantity proportional to the horizon area A, the proportionality constant is infinite, unless there is a ultraviolet cutoff presumably at the Planck scale, which somehow would give in addition the required $1/4$ value. Here we do not touch on the physical statistical approach; we rather use the thermodynamic approach to the entropy of quasiblack holes and find precisely the value $S = A/4$.

The issue discussed by us is also relevant in the context of [28] (see also [29–31]), where it was argued that the Einstein equation can be derived from the first law of thermodynamics $TdS = \delta Q$ and the proportionality between the entropy and the area A of some causal horizon. In this setting, T and δQ are the temperature and the heat flux seen by an accelerated local Rindler observer on a surface which is close to the horizon but does not coincide with it and remains timelike. Since this setup is based on a timelike surface, rather than on a null surface, strictly speaking there is a gap in the derivation, as the $S = A/4$ value follows from the space-time structure for the horizon, a null surface. In the present work we show how this $S = A/4$ term is indeed recovered in the quasihorizon limit and, thus, fill this gap.

II. ENTROPY AND FIRST LAW OF THERMODYNAMICS FOR QUASIBLACK HOLES

A. Entropy in the nonextremal case

1. Entropy formula

Consider a static metric, not necessarily spherically symmetric. We assume that there is a compact body. Then, at least in some vicinity of its boundary the line element can be written in Gaussian coordinates as

$$ds^2 = -N^2 dt^2 + dl^2 + g_{ab} dx^a dx^b, \tag{1}$$

where (t, l) are the time and radial coordinates, respectively, and x^a, x^b represent the angular part. We suppose that the boundary of the compact body is at $l = \text{const}$. The metric functions N and g_{ab} generically have different forms for the inner and outer parts. Now also assume that the system is at a local Tolman temperature T given by

$$T = \frac{T_0}{N}, \tag{2}$$

where $T_0 = \text{const}$. T_0 should be considered as the temperature at asymptotically flat infinity. Assuming the validity of the first law of thermodynamics we can write it in terms of boundary values

$$Td(s\sqrt{g}) = d(\sqrt{g}\epsilon) + \frac{\Theta^{ab}}{2}\sqrt{g}dg_{ab} + \varphi d(\sqrt{g}\rho_e). \tag{3}$$

One should carefully define each term appearing in Eq. (3). The quantity g is defined as $g \equiv \det g_{ab}$. The quantity s is the entropy density entering the expression for the total entropy

$$S = \int d^2x \sqrt{g} s. \tag{4}$$

The quantity ϵ is the quasilocal energy density, defined as [5]

$$\epsilon = \epsilon^g - \epsilon^0, \tag{5}$$

where

$$\epsilon^g = \frac{K}{8\pi} \tag{6}$$

and $\epsilon^0 = K^0/8\pi$,. So, $\epsilon = \frac{K - K_0}{8\pi}$. Here K is the trace of the two-dimensional extrinsic curvature K_{ab} of the boundary surface embedded in the three-dimensional manifold $t = \text{const}$, and K_0 is a similar term for the reference background manifold, e.g., flat space-time, but in our context the precise background is unimportant. In more detail,

$$K_{ab} = -\tfrac{1}{2} {g_{ab}}', \tag{7}$$

is the extrinsic curvature, where a prime denotes differentiation with respect to l, i.e., $' \equiv \frac{\partial}{\partial l}$, and

$$K = -\frac{1}{\sqrt{g}}\sqrt{g}' \tag{8}$$

its trace, $K = K_{ab} g^{ab}$. Finally, the spatial energy-momentum tensor Θ_{ab} is equal to [5] (see also [32–34] for the more traditional approach)

$$\Theta_{ab} = \Theta^g_{ab} - \Theta^0_{ab}, \tag{9}$$

where

$$8\pi\Theta^{g}_{ab} = K_{ab} + \left(\frac{N'}{N} - K\right)g_{ab}, \qquad (10)$$

and Θ^0_{ab} is the corresponding background tensor, with a form similar to Eq. (10). Finally, the quantity φ is the electric potential and ρ_e is the electric charge density of the matter.

Our strategy consists of integrating the first law (3) to obtain the entropy. In general, for an arbitrary boundary, if one is far from the quasihorizon, the integration procedure requires knowledge of the equation of state of the matter, as has been worked out by Martinez for some specific models of a shell in vacuum [17]. However, we will now see that if we choose a sequence of configurations such that all its members remain on the threshold of the formation of a horizon, and integrate just over this very subset, the answer turns out to be model independent, and there is no need to specify an equation of state. To this end, we must simultaneously change the size and the proper mass M of the configuration, to keep it near its gravitational radius r_+, and in such a way that $N \to 0$ for all such configurations. This will allow us to integrate the first law along such a sequence and obtain the value of the entropy for a shell near the quasihorizon.

We point out some subtleties. We use the definitions of quasiblack holes done in [6–8] for general and spherical systems. Consider, for simplicity, spherical configurations. Then, there are two relevant quantities, the system radius R and its gravitational radius r_+. For r_+ fixed and $R \to r_+$, we deal with the situation of [6–8]. In doing $N \to 0$ the small parameter is $\varepsilon = \frac{R-r_+}{R} \ll 1$. But, now, actually there are two small parameters ε and $\delta = \frac{(\delta r_+)}{r_+}$, since we want to consider small variations of thermodynamic quantities for two close systems which differ in r_+. Then, we must first send $\varepsilon \to 0$ and only afterward consider $\delta \ll 1$. It is this approach that ensures that we are dealing with quasiblack holes having slightly different radii r_+. If the system is kept near the quasihorizon, this allows us to integrate the first law along such a sequence counting different members of the same family of states and obtain the value of S for the shell layer at the quasihorizon. It is worth dwelling on this point, and in order to understand it, consider, for instance, the simplest configuration with a Schwarzschild exterior solution and a Minkowski metric inside. Then, the Arnowitt-Deser-Misner (ADM) mass m of the solution, the radius of the shell R, and its proper mass M are connected by the relation $m = M - \frac{M^2}{2R}$ [2,16]. The horizon, if there is one, is at radius $r_+ \equiv 2m$. We can characterize the system by two independent parameters, R and m, say. Then, in the whole space of parameters, we must choose the curve lying slightly above the straight line $R = 2m$. Then, in the process of integration along this curve all three quantities R, m, and M change but in such a way that the approximate equality $R \approx 2m$ holds.

Thus, in the outer region, neglecting the difference between quasiblack hole and black hole metrics (which can be done), we can write [35,36] for the system near the formation of a quasihorizon,

$$K_{ab} = K^{(1)}_{ab} l + O(l^2). \qquad (11)$$

Equation (11) follows from regularity conditions. It is then seen from (5) that the quasilocal energy density ϵ remains finite, and from (10) that the spatial stresses Θ_{ab} diverge due to the term $\frac{1}{N}(\frac{\partial N}{\partial l})_+$. In the outer region $(\frac{\partial N}{\partial l})_+ \to \kappa$, where κ is the surface gravity of the body. Leaving in Eq. (3) only the dominant contribution, we obtain that

$$d(s\sqrt{g}) = \frac{\kappa}{16\pi T_0}\sqrt{g} g^{ab} d g_{ab}. \qquad (12)$$

Up to now, the quantity T_0 is arbitrary. However, we should take into account that near the quasihorizon quantum fields are inevitably present. For an arbitrary temperature, their backreaction becomes divergent and only the choice

$$T_0 = T_{\rm H} = \frac{\kappa}{2\pi}, \qquad (13)$$

where $T_{\rm H}$ is the Hawking temperature, enables us to obtain a finite result [see, e.g., [23] and references therein, for the proof that the stress-energy tensor and other quantities diverge strongly on the horizon unless the corresponding fields are in a state with a temperature equal to the natural black hole temperature $T_{\rm H}$; we list the corresponding expression for the stress-energy tensor in Eq. (21)]. If, thus, neglecting again the difference between a black hole and quasiblack hole, we substitute this equality in (12), we obtain

$$d(s\sqrt{g}) = \tfrac{1}{4} d\sqrt{g}. \qquad (14)$$

Upon integration over an area A, we reproduce the famous result

$$S = \tfrac{1}{4}A, \qquad (15)$$

up to a constant c. In these considerations, we took into account the leading term while integrating the first law (3). It follows from the Tolman formula (2) that the corrections which come from the first and last terms are of the order $O(\sqrt{-g_{00}})$ and vanish when the quasihorizon is approached. In general, there are also corrections which stem from the second term. They are model dependent and depend on how rapidly the temperature T_0 approaches the Hawking value $T_{\rm H}$. In the limit $T_0 \to T_{\rm H}$ they vanish by construction.

It is interesting to note that in the quasihorizon limit an analog of the Euler relation is found. Indeed, the Euler relation has the form

$$Ts = \sigma + p, \tag{16}$$

where σ is the energy density and p plays the role of a pressure. It is easy to check that the relation (16) does not hold in general. However, one can check directly that near the quasihorizon Eq. (16) with the mean pressure given by $p = \Theta^{ab} g_{ab}/2$ holds approximately. In doing so, in the main approximation, one finds $p \approx \kappa/8\pi N$ and $s \approx \frac{1}{4}$, in agreement with (15). One sees that σ does not enter this equation at all, being thus negligible.

2. Choice of the constant

We obtained that in the quasiblack hole limit $S = A/4 + c$, where $c = \text{const}$. To substantiate our choice $c = 0$, we can require that $S \to 0$ when the quasiblack hole disappears, so $A \to 0$. Then, indeed $c = 0$ and the continuity of the quasiblack hole entropy ensues.

3. Layer and boundary stresses

(i) Universality of the entropy formula, independently of the specific layer or boundary stresses adopted for the model: Our derivation is essentially based on the quasilocal approach [5] which also admits an interpretation in terms of the formalism of thin shells [32,33] (see [34] for a discussion of this point and a more general setup, where mass and quasilocal energy are defined for a naked black hole, a relative of a quasiblack hole). In our context this is especially important although it does not show up quite explicitly.

Equation (9) for the surface stresses refers to the difference between quantities defined in the given metric and those coming from the background metric, say, a flat one. So Eq. (9) does not necessarily require a thin shell as a model for the matter; it is irrelevant for the calculation of quasilocal quantities in the outer region, so this equation is valid for any inner region joined through the boundary to an outer region (say, to vacuum) with fixed boundary data. This is because information about the inner region is encoded in the boundary values [2–4]. For example, denoting by r the radial coordinate, if there is a spherically symmetric body with radius R, such that $0 \le r \le R$, and with a distribution of matter with a mass function $m(r)$, the quasilocal energy at some r, $E(r)$, is $E(r) = r(1 - \sqrt{1 - [2m(r)/r]})$. It only depends on quantities defined at r. In particular, at the surface, $r = R$, $E(R) = R(1 - \sqrt{1 - [2m(R)/R]})$.

However, in any model, be it thin shell or distributed matter joined by a boundary to the outer space, a crust in the form of a thin layer arises inevitably when some surface behaving as the boundary of the matter approaches the quasihorizon. Moreover, the amplitude of such stresses becomes infinite in this limit for nonextremal quasiblack holes [6].

Thus, whatever distribution of entropy a body would have, in the quasihorizon limit all the material distributions (including a disperse distribution or even other shells of matter) give the same universal result $S \to \frac{A}{4}$. So the total entropy agrees then with the Bekenstein-Hawking value. Here, in the quasiblack hole approach we see a manifestation of the universality inherent to black hole physics in general and black hole entropy, in particular. One reservation is in order. If we try to take into account the thermal radiation from the boundary toward the inside region, we encounter the difficulty that the local temperature $T = T_0/N$ grows unbound, and so the mass of the radiation and the entropy also explode. As a result, collapse ensues with the appearance of a true black hole inside the shell instead of a quasiblack hole. However, because of the infinite redshift due to the factor N, any typical time t_0 connected with emission of photons inside will grow unbounded as $\frac{t_0}{N}$ for an external observer as well, so the concept of a quasiblack hole remains valid and self-consistent, up to an almost infinite time in the limit under discussion.

(ii) Nonessentiality of the choice of the background stresses: In the above derivation of the entropy it is essential that the boundary approaches a quasihorizon. If we have some distribution of matter, two shells, say, in general different cases of forming a quasihorizon are possible [11–13]: in one case it can appear at the outer shell, in the other the horizon forms at the inner one. In the latter case the derivation of the entropy formula follows the same lines as before but with the change that the role of boundary is now played by the inner shell, or, more specifically, by a surface on the inner shell. Thus, in any case the term Θ^g_{ab} is to be calculated near the quasihorizon from the outside.

As far as the subtraction term Θ^0_{ab} is concerned, one can choose among several different possibilities. In the present work we have chosen flat space-time to find Θ^0_{ab}; in previous works [7,8] we have chosen it differently. However, this difference is irrelevant in the given context. Indeed, one can also calculate the stresses given in Eq. (9) using a modified version of Eq. (9) itself, in which Θ^0_{ab} is replaced by the term Θ^-_{ab} determined from the inside. In general both quantities $\Theta^g_{ab} - \Theta^0_{ab}$ and $\Theta^g_{ab} - \Theta^-_{ab}$ are different and even refer to different boundaries. To clarify, let us suppose the following example: one has a thick shell with an inner radius $R_{\rm in}$ and an outer radius $R_{\rm out}$, with $R_{\rm in} < R_{\rm out}$. Let us also assume that when the system approaches its own gravitational radius r_+, one has $R_{\rm out} \to R_{\rm in} \to r_+$. (This is not a generic behavior, it is an example, see [11–13] for other manners in which the system can approach r_+; in such cases one should redefine the boundary.) In this example Θ^0_{ab} refers to the background (say, flat space-time) energy-momentum tensor at the outer radius

R_{out} (which is the radius of the outer boundary and, thus, the radius of the system as a whole), whereas Θ^{-}_{ab} refers to the energy-momentum tensor at the radius R_{in}. But now note that, in our example, in the quasihorizon limit, when $R_{out} \to R_{in} \to r_+$, the difference, $\Theta^{-}_{ab} - \Theta^{0}_{ab}$, between both subtraction terms becomes inessential for our purposes. This is because both Θ^{-}_{ab} and Θ^{0}_{ab} remain finite. Indeed, Θ^{0}_{ab} is finite by its very meaning, and the only potentially dangerous term in Θ^{-}_{ab}, namely, $(N^{-1}\frac{\partial N}{\partial l})^{-}$, is also finite since in the inner region $N = \epsilon f(l)$ where, by definition of a quasiblack hole, ϵ is a small parameter and $f(l)$ is a regular function (see [6] and especially [7] for more detailed explanations). In the calculation of the entropy described above all these finite terms are multiplied by the factor $N \to 0$, and do not contribute. The nonzero contribution to the entropy comes from the leading divergent term $\frac{1}{N}\frac{\partial N}{\partial l} \approx \frac{\kappa}{N}$ in Θ^{g}_{ab}. In addition, it is worth noting that it is this term which ensures the existence of a mass formula for quasiblack holes similar to the mass formula for black holes [7,8]. Now we see that, actually, this term plays a crucial role also in the derivation of the entropy of quasiblack holes.

The issue of the influence of boundary stresses at infinity (see, e.g., [37]) has been in great focus recently. Although important, we are mainly interested in local boundary stresses.

4. Spherically symmetric configurations: Entropy issues

(i) Entropy of spherically symmetric thin shells in vacuum: The simplest example of a spherically symmetric configuration is given by a thin shell of radius R surrounded by vacuum, with a Schwarzschild metric outside and a Minkowski one inside. Its thermal properties were discussed by Martinez in [17] but all the examples studied there do not consider the formation of a quasihorizon. Although this model of [17] looks simple, it is of interest in our context, as it enables one to compare the exact results that can be extracted from the thin shell model with our results, providing thus a convenient test of our method. Following [17], let us write down the first law as

$$TdS = dE + pdA. \tag{17}$$

Here $E = R(1 - \sqrt{V})$ is the quasilocal energy [5], which in this case coincides also with the proper mass of the shell, $V = 1 - \frac{r_+}{R}$, and $r_+ = 2m$ is the horizon radius, with m being the ADM mass. $A = 4\pi R^2$ is the surface area of the shell of radius R, and p is the gravitational pressure given by

$$p = \frac{(1 - \sqrt{V})^2}{16\pi R\sqrt{V}}, \tag{18}$$

see [17] for details. One can now take two routes. The one followed by Martinez [17] and push the thin shell up to the horizon, and the one advocated by us here. We will see that both routes give the same result, as they should.

To start, we follow [17]. If one takes into account the integrability conditions of Eq. (17), and changes variables from (E, R) to (m, R), it turns out that [17]

$$T = \frac{T_0(m)}{\sqrt{V}}, \tag{19}$$

and Eq. (17) is reduced to [17]

$$dS = \frac{dm}{T_0(m)}. \tag{20}$$

Hence the entropy can be found by direct integration. Here, T_0 has the usual meaning of the temperature measured by an observer at infinity. It is seen from (20) that the entropy in this example does not depend on R. This is a consequence of the fact that matter is absent inside, so $\frac{\partial S}{\partial R} = 0$ everywhere. Formally, Eq. (20) is valid everywhere including the near-horizon region with an arbitrary temperature $T_0(m)$. However, near the horizon another factor becomes important, which was not taken into account in [17] since in it this region was avoided altogether. Outside the shell there is a backreaction of quantum fields, and the stress-energy tensor T^{ν}_{μ} can be written as

$$T^{\nu}_{\mu} = \frac{T_0^4 - T_H^4}{g_{00}^2} f^{\nu}_{\mu} + h^{\nu}_{\mu}, \tag{21}$$

where f^{ν}_{μ} and h^{ν}_{μ} are finite quantities (see, e.g., [23] and references therein). At the horizon g_{00} vanishes. So, if $T_0 \neq T_H$, inevitable divergences destroy the horizon of a black hole or the quasihorizon of a quasiblack hole. Therefore, we must assume that T_0 is equal to the Hawking temperature, $T_0 = T_H = \frac{1}{8\pi m}$. Then, the assumption of negligible backreaction becomes evident and in the main zero-loop approximation we still may continue to use Eq. (20). Substituting $T_0 = \frac{1}{8\pi m}$ in Eq. (20), and integrating it, we obtain $S = 4\pi m^2$, i.e., $S = \pi r_+^2$, yielding

$$S = \tfrac{1}{4}A, \tag{22}$$

where A now is the quasihorizon area.

Now we follow our formalism and proceed directly from Eq. (17). This is less convenient in this simple model, but enables us to check the general approach. Let us consider variations of the system parameters for which the shell remains in equilibrium near the would-be horizon, $R = r_+(1 + \delta)$ with δ small and fixed, $0 < \delta \ll 1$. This means that we have to change simultaneously the radius of the shell R and its ADM mass $m = \frac{r_+}{2}$ when we pass from one equilibrium configuration to another. As a result, the quantity $V = 1 - \frac{r_+}{R}$ appearing in the expression (20) for the energy $E = R(1 - \sqrt{V})$ is fixed and small. So the first term in (17) is $dE \approx dR \approx dr_+$. The second term in (17) is huge since $p \sim \frac{1}{\sqrt{V}} \rightarrow \infty$. Thus, the first term in (17) is negligible as compared to the second one. Then, writing $p \approx \frac{1}{16\pi R\sqrt{V}}$, $T = \frac{T_0}{\sqrt{V}}$, $R \approx r_+$ and integrating over r_+, we reobtain the result (22), where again we have omitted the integration constant to make sure that $S = 0$ when $r_+ = 0$. This example clearly demonstrates the validity of our approach and of the key role played by the surfaces stresses, which are described by the quantity p in this example.

An important and interesting feature can be taken from this model of Martinez [17] due to the simplicity of the configuration. The fact that the entropy S depends on the ADM mass m only, means that the entropy does not change when the shell with a given m is displaced radially. In particular, one cannot say that the entropy (22) was generated in the process of a quasistatic collapse since the entropy itself remains constant. Its value is due to the value of the Hawking temperature $T_{\rm H}$. This value for S is enormous when $T_{\rm H}$ is small, which is the case for a configuration with a relatively large gravitational radius r_+.

Another remark of importance is that it follows from the derivation given above that no gravitational entropy was assigned to the system *a priori*, in accord with some previous observations on the study of thin shells [16]. Indeed, the Bekenstein-Hawking value for the entropy (22) was obtained without invoking special additional assumptions; it is a direct consequence of the first law for matter only. One can say that ordinary matter in a nontrivial way mimics the thermal properties of the horizon when its size approaches the gravitational radius. In doing so, the requirement $T = T_{\rm H}$ is essential. One can take $T_0 \neq T_{\rm H}$ as it was done in [17] but only when one considers shells far from the would-be horizon. To relate both situations, we can consider $T_0 = T_{\rm H}[1 + \varepsilon\chi(m)]$, where $\varepsilon \ll 1$. Then, the backreaction described by Eq. (21) is still bounded, and even small due to the smallness of f_μ^ν and h_μ^ν, if the numerator in the first term is of the same order of the denominator, which includes the square of the Schwarzschild metric coefficient, i.e., $g_{00}^2 = (1 - \frac{r_+}{r})^2$. Then, we obtain that the admissible minimum radius for the shell is $R = r_+(1 + \delta)$ where $\delta \sim \sqrt{\varepsilon}$. The sign of the correction term to the entropy as compared to (22) depends then on that of χ.

(ii) Entropy of spherically symmetric continuous distributions of matter: To make the example more realistic, we consider a continuous distribution of matter, rather than a thin shell. Then, the arguments of [17] do not apply and both the entropy and temperature may depend not only on the mass m but also on radius R of the boundary.

Consider then a general metric for a spherically symmetric distribution of matter. The metric potentials in Eq. (1) can be chosen such that $N^2(r) = V(r)e^{2\psi(r)}$, $dl = dr/\sqrt{V(r)}$, with $V(r) \equiv 1 - 2m(r)/r$, and $g_{\theta\theta} = g_{\phi\phi}/\sin^2\theta = r^2$, with $m(r)$ and $\psi(r)$ being the new functions. The metric is then written in the convenient form

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right)e^{2\psi(r)}dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (23)$$

where $m(r)$ and $\psi(r)$ are the relevant metric functions which depend on the coordinate r alone. Defining ρ and p_r as the matter energy density and radial pressure, respectively, Einstein's equations yield

$$m(r) = 4\pi \int_0^r d\bar{r}\bar{r}^2\rho, \quad (24)$$

$$\psi(r) = 4\pi \int_{r_0}^r d\bar{r} \frac{(\rho + p_r)\bar{r}}{1 - \frac{2m(\bar{r})}{\bar{r}}}, \quad (25)$$

and a third equation for the tangential pressures that we do not need in our discussion. Here the constant r_0 defines the low limit of integration. In general, it is arbitrary, but there are convenient choices. If the matter is constrained to the region $r \leq R$, with $\rho + p_r = 0$ for $r \geq R$, then one can choose $r_0 = R$ and $\psi = 0$ for $r \geq R$. For example, for an electrically charged object, one has the Reissner-Nordström metric outside in which case indeed $\rho + p_r = 0$ [35], for all $r \geq R$, so $\psi = 0$ and $m(r) = m - \frac{Q^2}{2r}$ where m is the ADM mass and Q is the electric charge. For an electrical neutral object $\rho + p_r = 0$ trivially and the choice $\psi = 0$, $m(r) = m$ for the outside also follows, yielding the Schwarzschild metric. Of course, other long-range fields, like dilatonic and other fields, can also be present. If one requires that the metric is asymptotically flat at infinity, then a good choice is always $m(r) \rightarrow m$ and $\psi \rightarrow 0$ there, with $r_0 = \infty$ in (25). We will see that the final physical results do not depend on the choice of r_0. Moreover, we recall that, actually, this approach works even without the requirement of spherical symmetry or asymptotic flatness as it follows from Sec. II A 1, where only local properties of the quasihorizon were used.

Now, for spherically symmetric systems, the first law in terms of the ADM mass m and boundary radius R has the form, alternative to the form given in Eq. (3),

$$T_0 dS = \exp\psi(R)(dm + 4\pi p_r R^2 dR), \quad (26)$$

with p_r being the radial pressure. See the end of the section for the proof of this nontrivial result. Without knowing the system details, one cannot find in general $S(m, R)$. However, at the quasihorizon limit one can bypass this restriction. Indeed from Eq. (26) we are able to deduce the Bekenstein-Hawking law for spherical quasiblack holes. The steps are as follows: (1) Since we want $R \rightarrow r_+$ we also have to put T_0 as $T_0 \rightarrow T_{\rm H}$. Now, $T_{\rm H}$ is given by $T_{\rm H} = (e^{\psi(r_+)}/4\pi)[d(1 - \frac{2m(r)}{r})/dr](r_+)$, i.e., at the quasihorizon,

$$T_0 = T_H = \frac{e^{\psi(r_+)}}{4\pi r_+}(1 - 8\pi\rho(r_+)r_+^2). \tag{27}$$

Thus, substituting Eq. (27) in Eq. (26) we obtain

$$dS = \exp(\psi(R) - \psi(r_+))\frac{dm + 4\pi p_r R dR}{(1 - 8\pi\rho_+ r_+^2)}, \tag{28}$$

giving the change of the entropy in terms of the changes of the ADM mass and the boundary radius R. (2) Since we are in the quasihorizon limit $R \to r_+$, the factor $\exp(\psi(R) - \psi(r_+))$ in Eq. (28) drops out and one has

$$dS = \frac{dm + 4\pi p_r R dR}{(1 - 8\pi\rho_+ r_+^2)}, \tag{29}$$

a simplified version of Eq. (28). Note that the entropy is a function of R since now $\frac{\partial S}{\partial R} \neq 0$, unlike the case studied previously; see Eq. (20). (3)-(a) Neglecting the difference between a quasihorizon and a horizon, it follows

$$p_r(r_+) = -\rho(r_+), \tag{30}$$

from the regularity conditions on the horizon itself [36] (see also [24]). (3)-(b) In general, the variations dm and dR are independent. However, as we are interested in the quasihorizon limit, we want to move along the line

$$R \approx 2m = r_+, \tag{31}$$

in the space of parameters, so that

$$dm \approx \frac{dr_+}{2}, \qquad dR \approx dr_+. \tag{32}$$

Thus, putting (30)–(32) into (29) yields

$$dS = 2\pi r_+ dr_+, \tag{33}$$

immediately. (4) Then upon integration one recovers, up to a constant c (which one can put to zero), the Bekenstein-Hawking value,

$$S = \tfrac{1}{4}A, \tag{34}$$

as promised.

Now, we discuss how the two forms of the first law Eqs. (3) and (26) are equivalent. Equation (3) involves the tangential pressures, whereas Eq. (26) the radial pressure. One relates to quasilocal energy E and the other to the ADM mass m, one to the local temperature T and the other to the temperature at infinity T_0. Consider a thermal compact body at temperature T_0 at infinity, with matter with density ρ distributed up to radius R, and vacuum outside. For simplicity, we assume that the shell is massless, so the mass is continuous on the boundary, $m(R) = m$. Then, Eq. (3) reduces after integration over angles to

$$TdS = dE + 8\pi\Theta_\theta^\theta R dR. \tag{35}$$

From Eqs. (9) and (10), with Θ_{ab}^0 corresponding to a flat metric, it follows

$$8\pi\Theta_\theta^\theta = \frac{1}{R}\left(\sqrt{1 - \frac{2m}{R}} - 1\right) + \left(\frac{1}{2}\frac{1}{\sqrt{1-\frac{2m}{R}}}\frac{d(1-\frac{2m}{R})}{dr} + \sqrt{1-\frac{2m}{R}}\frac{d\psi}{dr}\right)_{r=R_-}, \tag{36}$$

and the same for $8\pi\Theta_\phi^\phi$, where $r = R_-$ means the quantities are evaluated at R from the inside, and $\psi(R)$ is as in (25). Now we use the quasilocal energy formula for the spherically symmetric case [2–4],

$$E(R) = R\left(1 - \sqrt{1 - \frac{2m}{R}}\right). \tag{37}$$

Performing dE in (37) and putting it together with (36) and $T = T_0/(\sqrt{V}e^\psi)$ [see Eq. (2)] in Eq. (35) yields

$$T_0 dS = \exp\psi(R)(dm + 4\pi p_r R^2 dR), \tag{38}$$

which is precisely Eq. (26). We have now completed the derivation of the equivalence between the two different formulas of the first law of thermodynamics, Eqs. (3) and (26), in a thorough manner, leaving no doubts about its validity. Note that Eq. (26) is valid if, in addition to matter, there is also a true black hole horizon at some radius r_{bh}, in this case the formula for the mass being slightly changed,

$$m(r) = \frac{r_{bh}}{2} + 4\pi\int_{r_+}^{r} d\bar{r}\bar{r}^2\rho. \tag{39}$$

(iii) Continuity of the entropy function as the radius approaches the quasihorizon: In the above consideration, we were interested in obtaining the asymptotic form of the entropy when the boundary of a body approaches the quasihorizon, so that we considered the change of the system configuration along the curve that is approaching the line $R = 2m$. On the other hand, it is also important to trace what happens in the physically relevant situation when the boundary of the body with fixed ADM mass m changes slowly its position from infinity toward the horizon, while the radius of the body and its proper mass are being changed. In the space of parameters (R, m), this corresponds now to a vertical line $m = \text{const}$. One may ask what happens to the entropy function in this process near the quasihorizon, whether a jump in $S(R)$ can occur or not. It follows from the first law (28) that $(\frac{\partial S}{\partial R})_m = 4\pi R^2 p_r(R)$ is finite. Thus, on the quasihorizon in the process of slowly compressing the shell toward its own gravitational radius there is no jump in the entropy. This no jump can be generalized to metrics not necessarily spherically symmetric.

B. Entropy in the extremal case

Here we discuss the issue of the entropy for extremal quasiblack holes. It was argued that $S = 0$ in [20]; see also

[21,22]. In [23] it was shown that one has to take into account that one-loop consideration may change the picture drastically. So, the issue remains contradictory even in general relativity. It was also demonstrated within string theory that $S = A/4$ (see [24] for a concise review). Because of these contradictory results, we find it useful to examine the issue of the entropy of extremal quasiblack holes at the classical level, hoping to give more insight into it. However, we do not intend to find a definitive conclusion about the true value of S in this extremal case. Rather, we only examine which consequences follow from the assumptions of [20] when one uses the quasiblack hole picture.

By the definition of the extremal case, $N \sim \exp(Bl)$ where B is a constant and $l \rightarrow -\infty$. As a result, $\frac{\partial N}{\partial l} \sim N$ and we have an additional factor $N \rightarrow 0$ in the numerator in Eq. (10). Therefore $d(s\sqrt{g}) = 0$ and, again omitting a constant, we obtain $S = 0$. Thus, using the picture of a thermal body with the boundary approaching its own quasihorizon we obtain the value $S = 0$, with an arbitrary temperature T_0, thus confirming the conclusions of [20]; see also [21,22]. However, considering that T_0 is not arbitrary might lead to another result. The discussion above for the choice of the constant also holds for extremal quasiblack holes, so $c = 0$.

III. CONCLUSIONS

We have considered the entropy for a system in which a black hole event horizon never forms, instead a quasihorizon appears. From this we can draw some remarks:

(i) The crucial difference between the usual way of obtaining the entropy of black holes by integration of the first law and our version of obtaining the entropy of quasiblack holes, also by integration of the first law, consists of the fact that we are dealing with systems which do not have a horizon. Quasiblack holes do not have horizons as black holes do. The would-be horizon appears only asymptotically. Thus, as it is exhaustively shown in our paper, it was not obvious in advance how to get the universal term $A/4$, which is intimately connected with a horizon, from matter configurations with timelike boundaries, instead of a lightlike surface as is the case for a black hole. Our work provides the bridge between thermal matter configurations and black holes in what concerns entropy. In our view, this is a very important point.

(ii) The entropy comes from the quasihorizon surface alone, i.e., the entropy of a quasiblack hole stems from the contribution of the states living in a thin layer. That the entropy comes from the quasihorizon surface alone automatically emphasizes that the properties of matter inside the quasihorizon are irrelevant, and the final answer for the entropy is insensitive to them. So, a quasiblack hole deletes information revealing its similarity to what happens in black hole physics. Thus, the present work, along with our previous papers on the mass formula for quasiblack holes [7,8], confirms that, for outer observers, quasiblack holes are objects that yield a smooth transition to black holes. In particular, there is the special interesting issue of a detailed comparison of the quasiblack hole picture with the membrane paradigm [18,19].

(iii) Another important point consists of the role of the huge surface stresses appearing due to the presence of a quasiblack hole. We showed that the fact that these stresses are infinite leads to the Bekenstein-Hawking value $S = A/4$ for the entropy of a nonextremal quasiblack hole. In doing so, we obtained this result in a model-independent way and showed that all corrections to the $A/4$ term vanish in the limit under discussion. For extremal quasiblack holes, assuming a finite arbitrary temperature at infinity leads to $S = 0$ at the classical level, the fact that the stresses are finite playing a key role in this derivation. However, considering that T_0 is not arbitrary might lead to another result.

(iv) The fact that our approach reveals the key role played by the surface layer near the quasihorizon (which is inevitable there in the nonextremal case and may appear in the extremal one) supports the viewpoint according to which the quantum states which generate entropy live on the quasihorizon of a quasiblack hole. We did not consider here quantum properties of the system explicitly. However, there is one important implicit exception. We have assumed that the temperature of the environment tends to the Hawking value T_H. This is a separate problem that requires further discussion. In particular, quantum backreaction drastically changes the whole picture in the extremal case since a nonzero temperature due to backreaction effects makes the stress-energy tensor of the quantum fields diverge on the horizon.

(v) Attempts to place the degrees of freedom that yield the entropy of a pure black hole on the vicinity of the horizon are not new. One of those first tries, where the degrees of freedom are on the matter, was developed in [38], while attempts to place the degrees of freedom on the horizon properties, yielding an entropy coming from the gravitational field alone, have also been performed, e.g., in [39] (see [40] for a review). Our approach for the entropy of quasiblack holes shows that their entropy, although in the matter before its boundary achieves the quasihorizon, comes ultimately from both the space-time geometry and the fields in the local neighborhood at the Hawking temperature. Thus our approach gives a tie between space-time and matter in what concerns the origin of a quasiblack hole entropy. Pushing the analogy between quasiblack holes and black holes to the end, our approach hints that the black hole's degrees of freedom appear as nontrivial interplay between gravitational and matter fields.

(vi) It would certainly be of further interest to trace the dynamical process of entropy formation in quasiblack hole scenarios [41].

(vii) Another important task is the generalization of the present results to the rotating case.

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[1] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).
[2] J. W. York, Phys. Rev. D **33**, 2092 (1986).
[3] E. A. Martinez and J. W. York, Phys. Rev. D **40**, 2124 (1989).
[4] O. B. Zaslavskii, Phys. Lett. A **152**, 463 (1991).
[5] J. D. Brown and J. W. York, Phys. Rev. D **47**, 1407 (1993).
[6] J. P. S. Lemos and O. B. Zaslavskii, Phys. Rev. D **76**, 084030 (2007).
[7] J. P. S. Lemos and O. B. Zaslavskii, Phys. Rev. D **78**, 124013 (2008).
[8] J. P. S. Lemos and O. B. Zaslavskii, Phys. Rev. D **79**, 044020 (2009).
[9] A. Lue and E. J. Weinberg, Phys. Rev. D **60**, 084025 (1999).
[10] A. Lue and E. J. Weinberg, Phys. Rev. D **61**, 124003 (2000).
[11] J. P. S. Lemos and E. J. Weinberg, Phys. Rev. D **69**, 104004 (2004).
[12] A. Kleber, J. P. S. Lemos, and V. T. Zanchin, Gravitation Cosmol. **11**, 269 (2005).
[13] J. P. S. Lemos and V. T. Zanchin, J. Math. Phys. (N.Y.) **47**, 042504 (2006).
[14] O. B. Zaslavskii, Phys. Rev. D **70**, 104017 (2004).
[15] O. B. Zaslavskii, Phys. Lett. B **634**, 111 (2006).
[16] P. C. W. Davies, L. H. Ford, and D. N. Page, Phys. Rev. D **34**, 1700 (1986).
[17] E. A. Martinez, Phys. Rev. D **53**, 7062 (1996).
[18] K. S. Thorne, R. H. Price, and D. A. Macdonald, *Black Holes: The Membrane Paradigm* (Yale University Press, New Haven, CT, 1986).
[19] M. K. Parikh and F. Wilczek, Phys. Rev. D **58**, 064011 (1998).
[20] S. W. Hawking, G. T. Horowitz, and S. F. Ross, Phys. Rev. D **51**, 4302 (1995).
[21] C. Teitelboim, Phys. Rev. D **51**, 4315 (1995).
[22] G. W. Gibbons and R. E. Kallosh, Phys. Rev. D **51**, 2839 (1995).
[23] D. J. Loranz, W. A. Hiscock, and P. R. Anderson, Phys. Rev. D **52**, 4554 (1995).
[24] V. P. Frolov and I. D. Novikov, *Black Hole Physics: Basic Concepts and New Developments* (Kluwer Academic, Amsterdam, 1998), Sec. 12.6.4.
[25] L. Bombelli, R. Koul, J. Lee, and R. D. Sorkin, Phys. Rev. D **34**, 373 (1986).
[26] M. Srednicki, Phys. Rev. Lett. **71**, 666 (1993).
[27] A. Lue and E. J. Weinberg, Gen. Relativ. Gravit. **32**, 2113 (2000).
[28] T. Jacobson, Phys. Rev. Lett. **75**, 1260 (1995).
[29] R. Brustein and M. Hadad, Phys. Rev. Lett. **103**, 101301 (2009).
[30] M. Parikh and S. Sarkar, arXiv:0903.1176.
[31] T. Padmanabhan, arXiv:0903.1254.
[32] W. Israel, Nuovo Cimento B **44**, 1 (1966).
[33] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Sec. 21.13.
[34] I. S. Booth and R. B. Mann, Phys. Rev. D **60**, 124009 (1999).
[35] M. Visser, Phys. Rev. D **46**, 2445 (1992).
[36] A. J. M. Medved, D. Martin, and M. Visser, Classical Quantum Gravity **21**, 3111 (2004).
[37] R. B. Mann, D. Marolf, R. McNees, and A. Virmani, Classical Quantum Gravity **25**, 225019 (2008).
[38] W. H. Zurek and K. S. Thorne, Phys. Rev. Lett. **54**, 2171 (1985).
[39] S. Carlip, Phys. Rev. D **55**, 878 (1997).
[40] J. P. S. Lemos, in *Proceedings of the Fifth International Workshop on New Worlds in Astroparticle Physics*, edited by Ana M. Mourão *et al.* (World Scientific, Singapore, 2006), p. 71.
[41] C. Barcelo, S. Liberati, S. Sonego, and M. Visser, Phys. Rev. D **77**, 044032 (2008).