

Gravitational duality and rotating solutions

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(Received 4 January 2010; published 8 March 2010)

We study how gravitational duality acts on rotating solutions, using the Kerr-NUT black hole as an example. After properly reconsidering how to take into account both electric (i.e. masslike) and magnetic (i.e. NUT-like) sources in the equations of general relativity, we propose a set of definitions for the dual Lorentz charges. We then show that the Kerr-NUT solution has nontrivial such charges. Further, we clarify in which respect Kerr's source can be seen as a mass M with a dipole of NUT charges.

DOI: 10.1103/PhysRevD.81.064010

PACS numbers: 04.20.-q

I. INTRODUCTION

The theory of general relativity, when linearized, shares many similarities with the much simpler theory of electromagnetism. The latter has the particular feature of having vacuum equations which are invariant under a duality transformation, which exchanges electric and magnetic fields. This invariance, extended to the case when sources are present, implies the existence not only of electric charges, but also of magnetic monopoles. The Dirac monopole [1], dual of the Coulomb charge, is then interpreted as a source in the Bianchi identities of the electromagnetic field strength rather than in its equations of motion.

Gravitational duality is the transposition to general relativity of the same idea of duality [2] (see [3] for a Hamiltonian proof of the duality). The vacuum equations are invariant under a duality which is defined on the Riemann tensor. Extending this duality in the presence of sources, see [4,5], implies the existence not only of matter giving rise to the ordinary stress-energy tensor, but also of “magnetic” matter giving rise to a dual stress-energy tensor. In particular, the Schwarzschild solution must have, at least at the linearized level, a dual solution, which was long ago identified with the Lorentzian Taub-NUT solution. The literature on the subject is vast; we list here a few references for the interested reader: for generalizations of the duality to (A)dS space, see [6–9]; for duality of higher-spin field theories, see, for example, [10,11] and references therein; for considerations about extending the duality to the full theory, see [9–13].

One feature of the Taub-NUT solution is that it has a stringlike singularity [14], sometimes called the Misner string, very similar to the Dirac string of the magnetic monopole. Then, similarly, the Misner string can be considered as a gauge artifact in the metric as soon as one is ready to accept the presence of a magnetic source in the right-hand side (r.h.s.) of the cyclic identity for the Riemann tensor, or in other words, a magnetic stress-energy tensor.

When the string singularity is properly taken into account in this way, it becomes possible in general relativity

to define a surface integral that computes precisely this magnetic NUT charge [15] (see also [16] for an approach using Komar charges).

It is the purpose of this paper to extend these ideas to rotating solutions. The simplest occurrence, which will be our main focus below, is the solution obtained when performing a duality rotation on the familiar Kerr black hole. The Kerr-NUT black hole [17] is a subgroup of the general Petrov type D metrics obtained by Plebanski and Demianski in [18], and it was shown to be consistent with gravitational duality in [19]. A global analysis of this solution can be found in [20].

It is then legitimate to ask what is the singularity structure, or what are the sources, for such a solution, and what are its charges and how does one compute them by surface integrals at infinity.¹

In the course of this investigation, we will see that there is also a “physical” choice involved. As the Dirac monopole could be considered as a semi-infinite solenoid, one could provide a similar physical interpretation of the Taub-NUT solution. This was first reported by Bonnor in [21] (see also [22]). The choice becomes more tricky when dealing with solutions like the linearized Kerr black hole and its dual (see, however, [20] for some considerations along the lines of [21]). In fact, we show that one could also consider the source of the Kerr metric as made of an electric mass M and a pair of NUT sources, which we will refer to as a di-NUT, in some appropriate limit.

The paper is organized as follows. In Sec. II, we review the way Einstein equations were written in a duality invariant way in [4] and we single out the fact that dualization looks more natural when realized on Lorentz indices (see also [23]). In Sec. III, we derive the Arnowitt-Deser-Misner (ADM) and dual ADM charges when string contributions are included. We notice that there exists, in our formalism, no way of expressing the generalized Lorentz charges as surface integrals in a

¹We will focus here and below only on solutions which are locally asymptotically flat.

gauge-invariant way. In Sec. IV, we study the particular case of the Kerr-NUT solution and show that the usual description of the Kerr metric as a rotating point source of mass M could also be interpreted as a point source M with a monopole-antimonopole pair in the limit where the monopole mass goes to infinity and the distance in between them goes to zero while keeping the orbital momentum fixed. Appendix A recalls the duality between the linearized Schwarzschild and NUT solutions, along with their respective sources. Appendix B details the calculations we need for the interpretation of the sources of the Kerr-NUT metric.

II. GRAVITATIONAL DUALITY ON LORENTZ INDICES

In this section, we review how gravitational duality works in linearized general relativity by rederiving the duality invariant form of the Einstein equations, and cyclic and Bianchi identities. We will argue that gravitational duality is best understood when dualization is performed on Lorentz indices. We show that this choice permits one to lower the duality relation to a duality between spin connections. We eventually give an expression of the spin connection in terms of the vielbein and a three-index object, first introduced in [4], that contains the magnetic information of the solution. Since we linearize around flat Minkowski space in Cartesian coordinates, there will be no distinction between curved and flat indices in the following.

When there are no magnetic charges, the Einstein equations, and cyclic and Bianchi identities are

$$\begin{aligned} G_{\mu\nu} &= 8\pi GT_{\mu\nu}, \\ R_{\mu[\nu\alpha\beta]} &= \frac{1}{3}(R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu}) = 0, \\ \partial_{[\alpha}R_{|\rho\sigma|\beta\gamma]} &= \frac{1}{3}(\partial_{\alpha}R_{\rho\sigma\beta\gamma} + \partial_{\gamma}R_{\rho\sigma\alpha\beta} + \partial_{\beta}R_{\rho\sigma\gamma\alpha}) = 0. \end{aligned} \quad (1)$$

The Bianchi identities are solved by expressing the Riemann tensor in terms of a spin connection. In turn, the cyclic identity is solved when the spin connection is expressed in terms of a vielbein or, when the local Lorentz gauge freedom is fixed, in terms of a (linearized) metric.

Gravitational duality tells us that for every metric, there exists a dual metric such that their respective Riemann tensors are dual to each other. One important difference with electromagnetism and its twoform field strength is that here the Riemann tensor has two pairs of antisymmetric indices (the Lorentz and the form indices, respectively, in reference to the spin connection) and a choice for the duality relation should be made. We will prefer here, for reasons to be explained later, a dualization on the Lorentz indices (the first two indices in our conventions, as is clear from the Bianchi identities above):

$$\tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma}, \quad R_{\mu\nu\rho\sigma} = -\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\tilde{R}^{\alpha\beta}{}_{\rho\sigma}, \quad (2)$$

where $\tilde{R}_{\mu\nu\rho\sigma}$ denotes the magnetic or dual Riemann tensor.

Looking now at the magnetic cyclic identity, we have

$$\begin{aligned} (\tilde{R}_{\mu\nu\alpha\beta} + \tilde{R}_{\mu\beta\nu\alpha} + \tilde{R}_{\mu\alpha\beta\nu}) &= 3\delta_{[\nu\alpha\beta]}^{\rho\sigma\kappa}\tilde{R}_{\mu\rho\sigma\kappa} \\ &= -\frac{1}{2}\varepsilon_{\gamma\nu\alpha\beta}(\varepsilon^{\gamma\rho\sigma\kappa}\tilde{R}_{\mu\rho\sigma\kappa}) \\ &= -\frac{1}{2}\varepsilon_{\gamma\nu\alpha\beta}(2R^{\gamma}{}_{\mu} - \delta^{\gamma}{}_{\mu}R) \\ &= 8\pi G\varepsilon_{\nu\alpha\beta\gamma}T^{\gamma}{}_{\mu}, \end{aligned} \quad (3)$$

and we see that the duality makes the electric stress-energy tensor appear on the r.h.s. of the equation. However, under a gravitational duality rotation

$$\begin{aligned} R_{\mu\nu\rho\sigma} &\rightarrow \tilde{R}_{\mu\nu\rho\sigma}, & \tilde{R}_{\mu\nu\rho\sigma} &\rightarrow -R_{\mu\nu\rho\sigma}, \\ T_{\mu\nu} &\rightarrow \Theta_{\mu\nu}, & \Theta_{\mu\nu} &\rightarrow -T_{\mu\nu}, \end{aligned} \quad (4)$$

meaning that the electric cyclic identity can be generalized such as to include a magnetic stress-energy tensor $\Theta_{\mu\nu}$. We write the full set of electric and magnetic equations, respectively, as

$$\begin{aligned} G_{\mu\nu} &= 8\pi GT_{\mu\nu}, \\ R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} &= -8\pi G\varepsilon_{\nu\alpha\beta\gamma}\Theta^{\gamma}{}_{\mu}, \\ \partial_{\epsilon}R_{\gamma\delta\alpha\beta} + \partial_{\alpha}R_{\gamma\delta\beta\epsilon} + \partial_{\beta}R_{\gamma\delta\epsilon\alpha} &= 0, \\ \tilde{G}_{\mu\nu} &= 8\pi G\Theta_{\mu\nu}, \\ \tilde{R}_{\mu\nu\alpha\beta} + \tilde{R}_{\mu\beta\nu\alpha} + \tilde{R}_{\mu\alpha\beta\nu} &= 8\pi G\varepsilon_{\nu\alpha\beta\gamma}T^{\gamma}{}_{\mu}, \\ \partial_{\epsilon}\tilde{R}_{\gamma\delta\alpha\beta} + \partial_{\alpha}\tilde{R}_{\gamma\delta\beta\epsilon} + \partial_{\beta}\tilde{R}_{\gamma\delta\epsilon\alpha} &= 0, \end{aligned} \quad (5)$$

where the electric and magnetic cyclic identities can also be written by means of (3) as

$$\tilde{G}_{\mu\nu} = 8\pi G\Theta_{\mu\nu}, \quad G_{\mu\nu} = 8\pi GT_{\mu\nu}, \quad (6)$$

thus showing the invariance of the equations under gravitational duality rotation. One advantage of dualizing on Lorentz indices, as compared to a dualization on form indices, is that we do not need to modify the Bianchi identity because

$$\partial_{[\alpha}\tilde{R}_{|\mu\nu|\beta\gamma]} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}\partial_{[\alpha}R_{|\rho\sigma|\beta\gamma]}. \quad (7)$$

Note that the vanishing of the Bianchi identity is consistent with the cyclic identity having a nontrivial source term if and only if the magnetic stress-energy tensor is conserved, $\partial_{\mu}\Theta^{\mu\nu} = 0$, just as the ordinary stress-energy tensor.

As already mentioned previously, the Riemann tensor can only be defined in terms of a metric when both the cyclic and Bianchi identities have a trivial right-hand side. To deal with the introduction of magnetic sources we introduce, as in [4], a three-index object $\Phi^{\mu\nu}{}_{\rho}$ such that

$$\partial_{\alpha}\Phi^{\alpha\beta}{}_{\gamma} = -16\pi G\Theta^{\beta}{}_{\gamma}, \quad \Phi^{\alpha\beta}{}_{\gamma} = -\Phi^{\beta\alpha}{}_{\gamma}. \quad (8)$$

Further we define

$$\begin{aligned}\bar{\Phi}^{\rho\sigma}{}_{\alpha} &= \Phi^{\rho\sigma}{}_{\alpha} + \frac{1}{2}(\delta^{\rho}{}_{\alpha}\Phi^{\sigma} - \delta^{\sigma}{}_{\alpha}\Phi^{\rho}), \\ \Phi^{\rho} &= \Phi^{\rho\alpha}{}_{\alpha}.\end{aligned}\quad (9)$$

The Riemann tensor that will be the solution of the set of equations (5) when making use of (8) is

$$R_{\alpha\beta\lambda\mu} = r_{\alpha\beta\lambda\mu} + \frac{1}{4}\epsilon_{\alpha\beta\rho\sigma}(\partial_{\lambda}\bar{\Phi}^{\rho\sigma}{}_{\mu} - \partial_{\mu}\bar{\Phi}^{\rho\sigma}{}_{\lambda}), \quad (10)$$

where $r_{\alpha\beta\lambda\mu}$ is the usual Riemann tensor verifying the usual cyclic and Bianchi identities with no magnetic stress-energy tensor. This means that $r_{\alpha\beta\lambda\mu} = r_{\lambda\mu\alpha\beta}$ and that it can be derived from a potential: $r_{\alpha\beta\lambda\mu} = 2\partial_{[\alpha}h_{\beta][\lambda,\mu]}$.

Another advantage of the dualization on Lorentz indices comes directly from the vanishing r.h.s. of the Bianchi identity, which gives us the right to express the linearized Riemann tensor in terms of a spin connection by

$$R_{\mu\nu\rho\sigma} = \partial_{\rho}\omega_{\mu\nu\sigma} - \partial_{\sigma}\omega_{\mu\nu\rho}, \quad (11)$$

and thus allows us to lower the duality relation between Riemann tensors to a duality between spin connections. With the help of (2) and (11) the gravitational duality relation becomes

$$\tilde{\omega}_{\mu\nu\sigma} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta}{}_{\sigma}, \quad (12)$$

where this relation is true up to a gauge transformation, as the spin connection is a gauge-variant object.

The linearized vielbein and spin connection for the Riemann tensor $r_{\mu\nu\alpha\beta}$ are

$$\begin{aligned}r_{\mu\nu\rho\sigma} &= \partial_{\rho}\Omega_{\mu\nu\sigma} - \partial_{\sigma}\Omega_{\mu\nu\rho}, \\ e^{\mu} &= dx^{\mu} + \frac{1}{2}\eta^{\mu\nu}(h_{\nu\rho} + v_{\nu\rho})dx^{\rho}, \\ \Omega_{\mu\nu} &= \Omega_{\mu\nu\rho}e^{\rho}, \\ \Omega_{\mu\nu\rho} &= \frac{1}{2}(\partial_{\nu}h_{\mu\rho} - \partial_{\mu}h_{\nu\rho} + \partial_{\rho}v_{\nu\mu}),\end{aligned}\quad (13)$$

where $h_{\mu\nu} = h_{\nu\mu}$ is the linearized metric and $v_{\mu\nu} = -v_{\nu\mu}$. Using this together with relations (11) and (12) gives us the spin connection in terms of the vielbein and the three-index object $\Phi_{\mu\nu\rho}$:

$$\begin{aligned}\omega_{\mu\nu\rho} &= \Omega_{\mu\nu\rho} + \frac{1}{4}\epsilon_{\mu\nu\gamma\delta}\bar{\Phi}^{\gamma\delta}{}_{\rho} \\ &= \frac{1}{2}(\partial_{\nu}h_{\mu\rho} - \partial_{\mu}h_{\nu\rho} + \partial_{\rho}v_{\nu\mu}) + \frac{1}{4}\epsilon_{\mu\nu\gamma\delta}\bar{\Phi}^{\gamma\delta}{}_{\rho}.\end{aligned}\quad (14)$$

In [15] it was realized that by means of (12) there always exists a ‘‘regular’’ spin connection even when magnetic sources are present.² From the expression above this can be achieved for a specific choice of $v_{\mu\nu}$ that cancels string contributions coming from $\Phi_{\mu\nu\rho}$.

²By regular we should stress that we only refer to (string) singularities on the two-sphere at spatial infinity.

One also easily sees that

$$\begin{aligned}\tilde{\omega}_{\mu\nu\sigma} &= \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta}{}_{\sigma} \\ &= -\frac{1}{4}[\epsilon_{\mu\nu\alpha\beta}(2\partial^{\alpha}h^{\beta}{}_{\sigma} + \partial_{\sigma}v^{\alpha\beta}) + 2\bar{\Phi}_{\mu\nu\sigma}].\end{aligned}\quad (15)$$

III. THE DUAL POINCARÉ CHARGES

In this section, we give the generalized expressions for the ADM momenta and dual ADM momenta in the presence of the NUT charge. These were first established in [15] for a specific gauge choice of the vielbein. Here, we give a full treatment of the singular string contributions, obtaining gauge-independent expressions for the surface integrals. This is also a proof of the validity of the gauge choice of [15]. We eventually apply the same idea to derive general expressions for the Lorentz charges and their duals, though we will show that there is no way in this formalism to express the charges as surface integrals without partially fixing the gauge.

The generalized ADM momenta and dual ADM momenta are

$$\begin{aligned}P_{\mu} &= \int T_{0\mu}d^3x = \frac{1}{8\pi G} \int G_{0\mu}d^3x, \\ K_{\mu} &= \int \Theta_{0\mu}d^3x = \frac{1}{8\pi G} \int \tilde{G}_{0\mu}d^3x.\end{aligned}\quad (16)$$

However, the electric and magnetic Einstein tensors can be expressed as [15]

$$\begin{aligned}G_{0\mu} &= \partial_i(\omega^{0i}{}_{\mu} + \delta_{\mu}^0\omega^{i\rho}{}_{\rho} - \delta_{\mu}^i\omega^{0\rho}{}_{\rho}), \\ \tilde{G}_{0\mu} &= \epsilon^{ijk}\partial_i\omega_{\mu jk},\end{aligned}\quad (17)$$

where by convention $\epsilon^{ijk} = -\epsilon^{0ijk}$. This enables us to formulate the momenta as surface integrals:

$$P_{\mu} = \frac{1}{8\pi G} \oint [\omega^{0l}{}_{\mu} + \delta_{\mu}^0\omega^{l\rho}{}_{\rho} - \delta_{\mu}^l\omega^{0\rho}{}_{\rho}]d\Sigma_l, \quad (18)$$

$$K_{\mu} = \frac{1}{8\pi G} \oint \epsilon^{ljk}\omega_{\mu jk}d\Sigma_l. \quad (19)$$

With the help of (14), we have

$$P_0 = \frac{1}{16\pi G} \oint [\partial_i h^{li} - \partial^l h^i{}_i + \partial_i v^{il} + \epsilon^{ljk}\Phi_{0jk}]d\Sigma_l, \quad (20)$$

$$\begin{aligned}P_k &= \frac{1}{16\pi G} \oint [\partial_0 h^l{}_k - \partial^l h_{0k} + \delta_k^l \partial^i h_{0i} - \delta_k^l \partial_0 h^i{}_i \\ &\quad + \partial_k v_0{}^l + \delta_k^l \partial^i v_{i0} - \frac{1}{2}\epsilon^{lij}[\Phi_{ijk} + \delta_{ik}\Phi_{j0}{}^0 \\ &\quad + \delta_{ik}\Phi_{jm}{}^m] + \frac{1}{2}\delta_k^l \epsilon^{ijm}\Phi_{ijm}]d\Sigma_l,\end{aligned}\quad (21)$$

$$K_0 = \frac{1}{16\pi G} \oint [\varepsilon^{lij} [\partial_i h_{0j} + \partial_j v_{i0}] + \Phi^0] d\Sigma_l, \quad (22)$$

$$K_k = \frac{1}{16\pi G} \oint [\varepsilon^{lij} [\partial_i h_{kj} + \partial_j v_{ik}] + \Phi^0_k] d\Sigma_l. \quad (23)$$

When there are no magnetic charges, $\Theta_{\mu\nu}$ is zero and thus K_μ also by definition. Setting ourselves in the gauge where $v_{\mu\nu} = 0$, one easily recognizes the ADM momenta P_μ . The important difference with electromagnetism is that here the surface integrals for calculating the charges depend on the spin connection, a gauge-variant object. In electromagnetism the contribution of the Dirac string is always equal to the opposite of the string contribution coming from the regularized connection. Here, if we want to cancel the string contributions we need the additional gauge freedom of the vielbein to be fixed in the right gauge. By duality arguments we showed that such a choice is always possible. This completes the proof of the validity of the expressions used in [15]. Details of calculations can be found in Appendix A.

In the same spirit, the general expressions for the Lorentz charges and their duals are as follows³:

$$\begin{aligned} L^{\mu\nu} &= \int (x^\mu T^{0\nu} - x^\nu T^{0\mu}) d^3x \\ &= \frac{1}{8\pi G} \int (x^\mu G^{0\nu} - x^\nu G^{0\mu}) d^3x, \\ \tilde{L}^{\mu\nu} &= \int (x^\mu \Theta^{0\nu} - x^\nu \Theta^{0\mu}) d^3x \\ &= \frac{1}{8\pi G} \int (x^\mu \tilde{G}^{0\nu} - x^\nu \tilde{G}^{0\mu}) d^3x. \end{aligned} \quad (24)$$

Plugging expression (17) into the definition of the electric Lorentz charges leads us to

$$\begin{aligned} L^{ij} &= \frac{1}{8\pi G} \int (x^i G^{0j} - x^j G^{0i}) d^3x \\ &= \frac{1}{8\pi G} \oint [x^j [\omega^{0li} - \delta^{li} \omega^{0k}_k] \\ &\quad - x^i [\omega^{0lj} - \delta^{lj} \omega^{0k}_k]] d\Sigma_l \\ &\quad + \frac{1}{8\pi G} \int [\omega^{0ij} - \omega^{0ji}] d^3x, \\ L^{0i} &= \frac{1}{8\pi G} \int (t G^{0i} - x^i G^{00}) d^3x \\ &= \frac{1}{8\pi G} \oint [-t [\omega^{0li} - \delta^{li} \omega^{0k}_k] - x^i \omega^{lj}_j] d\Sigma_l \\ &\quad + \frac{1}{8\pi G} \int \omega^{ij}_j d^3x. \end{aligned} \quad (25)$$

³Note that the fixed timelike index is now upstairs, contrary to the definitions of the momenta. We hope that this (arbitrary but innocuous) switch in the convention will not upset the reader too much.

We see that in the presence of nontrivial $\Phi_{\mu\nu\rho}$, we have *a priori* no way to express the charges as surface integrals. However, we know that the charges are independent of the choice of $v_{\mu\nu}$; one could then always try to choose a gauge such as to cancel the $\Phi_{\mu\nu\rho}$ contributions present in the volume integrals by choosing an appropriate $v_{\mu\nu}$. Expanding the volume integrals in the above expressions,

$$\begin{aligned} \int 2\omega^{ij}_j d^3x &= \int [\partial_j h^{ij} - \partial^i h^j_j + \partial_j v^{ji} \\ &\quad + \varepsilon^{ijk} \Phi_{0jk}] d^3x, \\ \int 2[\omega^{0ij} - \omega^{0ji}] d^3x &= \int [\partial^i h^{0j} - \partial^j h^{0i} + \partial^j v^{i0} - \partial^i v^{j0} \\ &\quad - \varepsilon^{ijk} \Phi_{k0}^0] d^3x, \end{aligned} \quad (26)$$

where we simplified the last equation using the relation $\varepsilon^{lk[i} \bar{\Phi}_{lk}^{j]} = \varepsilon^{ijk} \Phi_{k0}^0$, we see that we can absorb the $\Phi_{\mu\nu\rho}$ by choosing the v_{ij} and the v_{0i} such that

$$\int \partial_j v^{ij} d^3x = \int \varepsilon^{ijk} \Phi_{0jk} d^3x, \quad (27)$$

$$\int [\partial^j v^{i0} - \partial^i v^{j0}] d^3x = \int \varepsilon^{ijk} \Phi_{k0}^0 d^3x. \quad (28)$$

Actually, these gauge choices do not fix completely the local Lorentz gauge, and hence $v_{\mu\nu}$. Rather, they restrict the gauge to a choice satisfying the above integral relations. Of course this can be done in the simplest way by choosing a $v_{\mu\nu}$ that locally compensates the singularity contained in $\Phi_{\mu\nu\rho}$.

In the gauge choice of expressions (27) and (28), we now have

$$\begin{aligned} L^{ij} &= \frac{1}{8\pi G} \oint [x^j [\omega^{0li} - \delta^{li} \omega^{0k}_k] - x^i [\omega^{0lj} - \delta^{lj} \omega^{0k}_k] \\ &\quad + \frac{1}{2} [\delta^{il} h^{0j} - \delta^{jl} h^{0i}]] d\Sigma_l, \\ L^{0i} &= \frac{1}{8\pi G} \oint [-t [\omega^{0li} - \delta^{li} \omega^{0k}_k] - x^i \omega^{lj}_j \\ &\quad + \frac{1}{2} [h^{il} - \delta^{il} h]] d\Sigma_l. \end{aligned} \quad (29)$$

If we now look at the dual Lorentz charges, we have

$$\begin{aligned}
\tilde{L}^{0i} &= \frac{1}{8\pi G} \int (t\tilde{G}^{0i} - x^i\tilde{G}^{00})d^3x \\
&= \frac{1}{8\pi G} \oint -\varepsilon^{ljk}[t\omega^i_{jk} + x^i\omega_{0jk}]d\Sigma_l \\
&\quad + \frac{1}{16\pi G} \int \varepsilon^{ikl}[\omega_{0kl} - \omega_{0lk}]d^3x \\
\tilde{L}^{ij} &= \frac{1}{8\pi G} \int (x^i\tilde{G}^{0j} - x^j\tilde{G}^{0i})d^3x \\
&= \frac{1}{8\pi G} \oint \varepsilon^{ikm}[x^j\omega^i_{km} - x^i\omega^j_{km}]d\Sigma_l \\
&\quad + \frac{1}{8\pi G} \int \varepsilon^{ijk}\omega_k^l d^3x,
\end{aligned} \tag{30}$$

where in the last equality we used $\varepsilon^{ikm}\omega^j_{km} - \varepsilon^{jkm}\omega^i_{km} = \varepsilon^{ijk}\omega_k^l$.

It is amusing to observe that the pieces in $L_{\mu\nu}$ and $\tilde{L}_{\mu\nu}$ that cannot be expressed as surface integrals actually enjoy a duality relation, $\tilde{L}_{\mu\nu}^{\text{bulk}} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}L_{\text{bulk}}^{\rho\sigma}$. This surprising property cannot of course be extended to the full charges, as is obvious from their definition in terms of the stress-energy tensor and its dual, respectively. However, a consequence of this observation is that with the previous choice of gauge, we can also express the dual charges as surface integrals:

$$\begin{aligned}
\tilde{L}^{0i} &= \frac{1}{8\pi G} \oint \left[-\varepsilon^{ljk}[t\omega^i_{jk} + x^i\omega_{0jk}] + \frac{1}{2}\varepsilon^{ilk}h_{0k} \right] d\Sigma_l, \\
\tilde{L}^{ij} &= \frac{1}{8\pi G} \oint \left[\varepsilon^{ikm}[x^j\omega^i_{km} - x^i\omega^j_{km}] \right. \\
&\quad \left. + \frac{1}{2}\varepsilon^{ijk}[h^l_k - \delta^l_k] \right] d\Sigma_l.
\end{aligned} \tag{31}$$

The expressions derived here for the electric and magnetic Lorentz charges are thus valid in any gauge when expressed as volume integrals like in (25) and (30). Moreover, we have shown that there exists a gauge choice valid for the Lorentz charges and their dual that permits us to eliminate the $\Phi_{\mu\nu\rho}$ and simplify the expressions to surface integrals. Note that in the case where all the $\Phi_{\mu\nu\rho}$ would be zero, any gauge is obviously fine, and we recover the ADM expressions.

We are now prepared to apply those formulas to the Kerr-NUT solution. Actually, it will prove more efficient to work out the sources of the solution, encoded in $T_{\mu\nu}$ and $\Theta_{\mu\nu}$, and compute the charges from their original definition. The above arguments ensure that the surface integrals, with a correct choice of gauge, will yield the same result.

IV. KERR AND DI-NUT SOURCES

There exists in the literature a generalization of the Taub-NUT metric with three parameters, the ADM mass M , the NUT charge N , and a rotation parameter a . This

solution is known as the Kerr-NUT metric. It is a particular case of the general Petrov type D solution found in [18]. It was shown in [19] that this metric is consistent with gravitational duality. What we want to study here are the different possible sources for the linearized metric. We will see that to obtain a magnetic stress-energy tensor such as the one for Kerr, we will need to introduce the Misner string contribution in $\Phi_{\mu\nu\rho}$ which appears in the surface integrals for P_μ and K_μ , but also pointlike (Dirac delta) contributions.

The Kerr-NUT metric reads

$$\begin{aligned}
ds^2 &= -\frac{\lambda^2}{R^2}[dt - (a\sin^2\theta - 2N\cos\theta)d\phi]^2 \\
&\quad + \frac{\sin^2\theta}{R^2}[(r^2 + a^2 + N^2)d\phi -adt]^2 + \frac{R^2}{\lambda^2}dr^2 \\
&\quad + R^2d\theta^2,
\end{aligned} \tag{32}$$

where $\lambda^2 = r^2 - 2Mr + a^2 - N^2$ and $R^2 = r^2 + (N + a\cos\theta)^2$. We now consider some specific cases.

Taub-NUT ($a = 0$)

If we set $a = 0$ in the above solution, we recover the Taub-NUT solution

$$\begin{aligned}
ds^2 &= -\frac{\lambda^2}{R^2}[dt + 2N\cos\theta d\phi]^2 + \frac{R^2}{\lambda^2}dr^2 \\
&\quad + R^2(d\theta^2 + \sin^2\theta d\phi^2),
\end{aligned} \tag{33}$$

where $\lambda^2 = r^2 - 2Mr - N^2$ and $R^2 = r^2 + N^2$. We review, in Appendix A, the well-known duality that brings the linearized Schwarzschild ($N = 0$) to the linearized NUT solution ($M = 0$). We also see that the linearized NUT metric is actually to be supplemented with the term $\Phi^{0z}_0 = -16\pi N\delta(x)\delta(y)\vartheta(z)$ to describe a source that is a point of magnetic mass N .⁴ If we do not add this string contribution, the singularity is physical (as considered in [21]) and can be interpreted as a semi-infinite source of angular momentum $\Delta L^{xy} = N\Delta z$.

Kerr ($N = 0$)

If we set $N = 0$ in the metric (32), we recover the Kerr metric in Boyer-Lindquist coordinates:

$$\begin{aligned}
ds^2 &= -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar}{\Sigma}\sin^2\theta dt d\phi + \frac{\Sigma}{\Delta}dr^2 \\
&\quad + \Sigma d\theta^2 + \frac{B}{\Sigma}\sin^2\theta d\phi^2,
\end{aligned} \tag{34}$$

where $\Delta \equiv \lambda^2(N = 0) = r^2 - 2Mr + a^2$, $\Sigma \equiv R^2(N = 0) = r^2 + a^2\cos^2\theta$, and $B = (r^2 + a^2)^2 - \Delta a^2\sin^2\theta$. The charges of this metric are easily calculated. If we linearize this metric at first order in the charges, meaning we only

⁴Actually, in order for the string to be along the positive z axis, we need to implement the change of coordinates $t \rightarrow t + 2N\phi$ in the above metrics. This will always be assumed when referring to singularities. We refrain from implementing it on the explicit metrics to avoid unnecessary complications.

keep terms in M and Ma , we obtain

$$h_{00} = \frac{2M}{r}, \quad h_{ij} = \frac{2M}{r^3} x_i x_j, \quad h_{0i} = \frac{2Ma}{r^3} \varepsilon_{zij} x^j. \quad (35)$$

It is then shown in Appendix B 1 that, starting from the information about the metric, the source for this solution is a rotating mass M with angular momentum $J_z = L^{xy} = Ma$.

Rotating NUT ($M = 0$)

A more interesting metric is the one where we set M to zero in (32). This is the rotating NUT metric. Again, linearizing as before gives us

$$\begin{aligned} \tilde{h}_{tx} &= \frac{2Nyz}{r(x^2 + y^2)}, & \tilde{h}_{ty} &= \frac{-2Nxz}{r(x^2 + y^2)}, \\ \tilde{h}_{\mu\mu} &= \frac{2Naz}{r^3}. \end{aligned} \quad (36)$$

It is shown in Appendix B 2 that this linearized metric (after we set the string along the positive z axis) supplemented with the $\Phi_{\mu\nu\rho}$ contributions,

$$\begin{aligned} \Phi^{0z}_0 &= -16\pi N \delta(x) \delta(y) \vartheta(z), \\ \Phi^{0y}_x &= -\Phi^{0x}_y = -\Phi^{xy}_0 = \Phi^{yx}_0 = 8\pi Na \delta(\mathbf{x}), \end{aligned} \quad (37)$$

where ϑ is the usual Heaviside function, describes the dual solution to the linearized Kerr. This means it describes a point of magnetic mass N and a magnetic angular momentum $\tilde{L}^{xy} = Na$.

Let us recall now that the choice for the $\Phi_{\mu\nu\rho}$ in the case of the Taub-NUT solution found its meaning in the existence of a string singularity in the linearized metric. This is also justified by considering the Schwarzschild metric as electric and imposing gravitational duality. Here, for the Kerr-NUT solution, one should note that some $\Phi_{\mu\nu\rho}$ terms are only singular in $r = 0$. Besides duality, we do not have any *a priori* argument in favor of adding these delta contributions to the rotating NUT solution. One could think of the linearized rotating NUT with only the string contribution Φ^{0z}_0 as another physical solution. As shown in Appendix B 3, this would imply the presence of singular terms in the electric stress-energy tensor corresponding to a dipole of a positive and a negative mass at infinitesimal distance. This interpretation is to be rejected on physical grounds because of the presence of a negative mass in the compound.

It is, on the other hand, amusing to contemplate the dual situation, i.e. the usual Kerr solution, where, however, we insert a nontrivial magnetic stress-energy tensor so that the nontrivial charges become $P_0 = M$ and $\tilde{L}_{0z} = Ma$. The sources for this solution are

$$T_{00} = M\delta(\mathbf{x}), \quad \Theta_{00} = Ma\delta(x)\delta(y)\delta'(z), \quad (38)$$

an electric point of mass M and a di-NUT, a dipole of NUT charges $+N$ and $-N$, separated by a distance ϵ when we

take the limit $\epsilon \rightarrow 0$ and $N \rightarrow \infty$ but with the product $N\epsilon$ constant and equal to $\tilde{L}_{0z} = N\epsilon = Ma$:

$$\begin{aligned} \Theta_{00} &= \lim_{\epsilon \rightarrow 0} [N\delta(x)\delta(y)\delta(z + \epsilon/2) - N\delta(x)\delta(y)\delta(z - \epsilon/2)] \\ &= Ma\delta(x)\delta(y)\delta'(z). \end{aligned} \quad (39)$$

This situation is physical since there is no obstruction in having negative NUT charges. Indeed, the Taub-NUT metrics with opposite signs of N are just related by a flip of the sign of the ϕ variable. We should, however, note that this leads seemingly to a clash between the statement of gravitational duality and positivity of the mass for the Schwarzschild solution. In other words, according to the above arguments the gravitational dual of a physical situation is not necessarily physical. It would be nice to better understand this issue, with the use, for instance, of positive energy theorems.

Concerning the Euclidean Kerr black hole, this interpretation had already been noticed a long time ago in [24]. For the Lorentzian signature, it has recently been observed in [25] that the Kerr metric could be reproduced by a nonlinear superposition of two Taub-NUT black holes of opposite NUT charges.⁵ Here, we have clarified that if this is indeed true from the perspective of the metrics, there is nevertheless a difference depending on whether the δ' singularities find themselves in the T_{0i} components of the ordinary stress-energy tensor or in the Θ_{00} component of the magnetic dual. The difference is encoded in the tensor $\Phi_{\mu\nu\rho}$ and is reflected on which Lorentz charges are nontrivial, the electric or the magnetic ones. We suggest to identify the Kerr metric as a di-NUT only in the case where there is a nontrivial Θ_{00} .

ACKNOWLEDGMENTS

We would like to thank G. Barnich, A. Kleinschmidt, L. Houart, C. Troessaert, A. Virmani, and V. Wens for interesting discussions. This work was supported in part by IISN-Belgium (Convention No. 4.4511.06, No. 4.4505.86, and No. 4.4514.08) and by the Belgian Federal Science Policy Office through the Interuniversity Attraction Pole P5/27. R. A. is a Research Associate of the Fonds de la Recherche Scientifique–F.N.R.S. (Belgium).

APPENDIX A: TAUB-NUT SOLUTION

In this appendix, we review the duality between the linearized Schwarzschild and the linearized NUT solution with the conventions set in Sec. II. We recover both the ideas of Misner [14] and Bonnor [21] for how to interpret the Taub-NUT solution. To deal with the Taub-NUT metric, Misner noticed in [14] the presence of a string singularity. Considering it as nonphysical, he identifies time to

⁵We would like to thank A. Virmani and R. Emparan for pointing out this reference to us.

get rid of it. We show in Appendix A 2 that by gravitational duality the string singularity in fact determines a magnetic stress-energy tensor and is thus nonphysical in an “electric” theory. We do not discuss the identification, as this is really a feature that should be treated in the full theory. If we drop this contribution, the magnetic stress-energy tensor is zero and we end up with a massless source of angular momentum N at every point along the physical singularity at $\theta = 0$. This is Bonnor’s interpretation of the Taub-NUT solution. The string is considered as a physical singularity in the electric theory. This is presented in Appendix A 3.

1. The linearized Schwarzschild solution

Considering the Schwarzschild solution, the nontrivial components of the linearized metric and spin connection are

$$\begin{aligned} h_{tt} &= \frac{2M}{r}, \\ h_{ij} &= \frac{2M}{r^3} x_i x_j, \\ \omega_{0i0} &= \frac{1}{2} \partial_i h_{00} = -M \frac{x_i}{r^3}, \\ \omega_{ijk} &= \frac{1}{2} (\partial_j h_{ik} - \partial_i h_{jk}) = \frac{M}{r^3} (\delta_{jk} x_i - \delta_{ik} x_j). \end{aligned} \quad (\text{A1})$$

The nontrivial components of the linearized Riemann tensor are

$$\begin{aligned} R_{0i0j} &= -\partial_j \omega_{0i0} = M \left(-\frac{3x_i x_j}{r^5} + \frac{\delta_{ij}}{r^3} + \frac{4\pi}{3} \delta_{ij} \delta(\mathbf{x}) \right), \\ R_{ijkl} &= \partial_k \omega_{ijl} - \partial_l \omega_{ijk} \\ &= \left(\frac{2M}{r^3} + \frac{8\pi M}{3} \delta(\mathbf{x}) \right) (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \\ &\quad - \frac{3M}{r^5} (\delta_{ik} x_j x_l - \delta_{jk} x_i x_l - \delta_{il} x_j x_k + \delta_{jl} x_i x_k), \end{aligned} \quad (\text{A2})$$

where we used

$$\partial_j \frac{x_k}{r^3} = \frac{\delta_{jk}}{r^3} - \frac{3x_k x_j}{r^5} + \frac{4\pi}{3} \delta_{jk} \delta(\mathbf{x}). \quad (\text{A3})$$

We finally obtain $R_{00} = 4\pi M \delta(\mathbf{x})$, $R_{ij} = 4\pi M \delta_{ij} \delta(\mathbf{x})$, and $R = 8\pi M \delta(\mathbf{x})$, and also $G_{00} = 8\pi T_{00} = 8\pi M \delta(\mathbf{x})$, $G_{ij} = T_{ij} = 0$, and $G_{0j} = T_{0j} = 0$. The source for the linearized Schwarzschild solution is thus a point of mass M .

2. The NUT solution from the dual Schwarzschild solution

To obtain the electric NUT spin connection, we use the duality relation $\omega_{\mu\nu\sigma} = -\frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \tilde{\omega}^{\alpha\beta}{}_{\sigma}$, where $\tilde{\omega}$ is the spin connection for the linearized Schwarzschild solution after we applied the duality rotation $\omega \rightarrow \tilde{\omega}$ and $M \rightarrow N$. We thus obtain the regular spin connection for the NUT solution:

$$\begin{aligned} \omega_{ij0} &= \varepsilon_{ijk} \tilde{\omega}_{0k0} = -N \varepsilon_{ijk} \frac{x_k}{r^3}, \\ \omega_{0ij} &= -\frac{1}{2} \varepsilon_{ikl} \tilde{\omega}_{klj} = N \varepsilon_{ijk} \frac{x_k}{r^3}. \end{aligned} \quad (\text{A4})$$

This gives the nontrivial components of the Riemann tensor:

$$\begin{aligned} R_{0i0j} &= 0, \quad R_{ijkl} = 0, \\ R_{ij0k} &= N \varepsilon_{ijl} \partial_k \left(\frac{x^l}{r^3} \right) = N \varepsilon_{ijl} \left(\frac{\delta_{kl}}{r^3} - \frac{3x_k x_l}{r^5} + \frac{4\pi}{3} \delta_{kl} \delta(\mathbf{x}) \right), \\ R_{0ijk} &= \partial_j \omega_{0ik} - \partial_k \omega_{0ij} \\ &= -2N \varepsilon_{ijk} \left(\frac{1}{r^3} + \frac{4\pi}{3} \delta(\mathbf{x}) \right) \\ &\quad + 3N \left(\varepsilon_{ijl} \frac{x_k x_l}{r^5} - \varepsilon_{ikl} \frac{x_j x_l}{r^5} \right). \end{aligned} \quad (\text{A5})$$

For the Einstein equation, we have trivially $R_{00} = R_{ij} = 0$. From the expressions above, one easily sees that $R_{0i} = R_{i0} = 0$. This means that $T_{\mu\nu} = 0$. Plugging the expressions into the cyclic identity, we obtain

$$\begin{aligned} R_{0ijk} + R_{0kij} + R_{0jki} &= -8\pi \varepsilon_{ijk} \Theta^{00} \\ &= -2N \varepsilon_{ijk} \left(\frac{3}{r^3} + 4\pi \delta(\mathbf{x}) \right) \\ &\quad + 6N \left(\varepsilon_{ijl} \frac{x_k x_l}{r^5} - \varepsilon_{ikl} \frac{x_j x_l}{r^5} \right. \\ &\quad \left. - \varepsilon_{kjl} \frac{x_i x_l}{r^5} \right), \\ R_{00ij} + R_{0j0i} + R_{0i0j} &= -8\pi \varepsilon_{ijk} \Theta^k{}_0, \\ R_{i0jk} + R_{ik0j} + R_{ijk0} &= -\partial_j (\omega_{0ik} + \omega_{ik0}) \\ &\quad + \partial_k (\omega_{0ij} + \omega_{ij0}) \\ &= -8\pi \varepsilon_{jkl} \Theta^l{}_i. \end{aligned} \quad (\text{A6})$$

This gives us

$$\Theta^{00} = N \delta(\mathbf{x}), \quad \Theta^{0k} = 0, \quad \Theta^{li} = 0. \quad (\text{A7})$$

For a solution describing a magnetic particle of mass N , and thus a magnetic stress-energy tensor $\Theta^{00} = N \delta(\mathbf{x})$, we need, using relation (8),

$$\Phi^{0z}{}_0 = -16\pi N \delta(x) \delta(y) \vartheta(z). \quad (\text{A8})$$

The previous nontrivial spin connections are expressed as

$$\begin{aligned} \omega_{ij0} &= \frac{1}{2} (\partial_j h_{0i} - \partial_i h_{0j}) + \frac{1}{4} \varepsilon_{ij0k} \Phi^{0k}{}_0, \\ \omega_{0ij} &= \frac{1}{2} (\partial_i h_{0j} + \partial_j v_{i0}) - \frac{1}{4} \varepsilon_{0ijk} \Phi^{0k}{}_0, \end{aligned} \quad (\text{A9})$$

where we only assumed that the linearized vielbein is time independent. As we have established that the regular spin connection is such that $\omega_{ij0} = -\omega_{0ij}$, we immediately see that the right gauge fixing will be $h_{0i} = -v_{i0}$. The previous spin connections are recovered with

$$\begin{aligned} h_{0x} &= v_{0x} = 2N \frac{y}{r(r-z)}, \\ v_{0y} &= h_{0y} = -2N \frac{x}{r(r-z)}, \end{aligned} \quad (\text{A10})$$

where the metric has a singularity on the positive z axis, in agreement with the form of the Φ_{z00} term. To check that this is the right result, we use a standard regularization procedure (also used in the context of the Dirac monopole; see e.g. [26]):

$$\begin{aligned} \vec{A} &= (h_{0x}, h_{0y}, h_{0z}), \\ \vec{B} &= \vec{\nabla} \times \vec{A} = 2N \frac{\vec{r}}{r^3} - 8\pi N \delta(x)\delta(y)\vartheta(z)\hat{z}, \end{aligned} \quad (\text{A11})$$

where \hat{z} is the unit vector along the z axis, and then

$$\partial_j h_{0i} - \partial_i h_{0j} = -2N \varepsilon_{ijk} \frac{x^k}{r^3} + \varepsilon_{zij} 8\pi N \delta(x)\delta(y)\vartheta(z). \quad (\text{A12})$$

Eventually, note that the nontrivial contribution to the linearized metric in spherical coordinates is

$$h_{0\phi} = -2N(1 + \cos\theta), \quad (\text{A13})$$

which is also the only nontrivial component for the linearized NUT metric.

As previously stated, this partially meets up with Misner's interpretation of the Taub-NUT metric. Here, we interpret the singularity at $\theta = 0$ as nonphysical in an electric way, but it contributes to the magnetic stress-energy tensor. The solution describes a particle of magnetic mass N .

3. The NUT solution without the string

To recover Bonnor's interpretation, we set $\Phi^{\mu\nu}{}_{\rho}$ to zero. Then, we obviously have $\Theta_{\mu\nu} = 0$. With the previous choice of $v_{\mu\nu}$, the nontrivial components of the spin connections are now

$$\begin{aligned} \omega_{ij0} &= -N \varepsilon_{ijk} \frac{x_k}{r^3} + \varepsilon_{zij} 4\pi N \delta(x)\delta(y)\vartheta(z), \\ \omega_{0ij} &= N \varepsilon_{ijk} \frac{x_k}{r^3} - \varepsilon_{zij} 4\pi N \delta(x)\delta(y)\vartheta(z). \end{aligned} \quad (\text{A14})$$

Note that we still have $\omega_{ij0} = -\omega_{0ij}$ so that from (A6) we still immediately see that $\Theta^{il} = \Theta^{0i} = 0$. We can check that $\Theta^{00} = 0$, as it should be.

The nontrivial components for the Einstein tensor are

$$G_{i0} = -\partial_j (\varepsilon_{zij} 4\pi N \delta(x)\delta(y)\vartheta(z)), \quad (\text{A15})$$

giving the nontrivial components of $T_{\mu\nu}$:

$$T_{x0} = -\frac{N}{2} \delta(x)\delta'(y)\vartheta(z), \quad T_{y0} = \frac{N}{2} \delta'(x)\delta(y)\vartheta(z). \quad (\text{A16})$$

Note that such $T_{\mu\nu}$ are conserved.

This shows that $P_\mu = 0$ and $\Delta L^{xy}/\Delta z = N$ for every value along the singularity. This agrees with Bonnor's interpretation of the NUT solution as a massless source of angular momentum at the singularity $\theta = 0$.

APPENDIX B: KERR-NUT METRIC

We now want to generalize the analysis of Appendix A to the case of the Kerr-NUT solution presented in Sec. IV. We will see here that the dual Kerr solution possesses the usual Misner string but also additional delta contributions to the $\Phi_{\mu\nu\rho}$. If we include these contributions, we get by gravitational duality a magnetic mass N with a magnetic angular momentum $J_z = Na$. If we do not, we see that it corresponds to a dipole of electric masses M separated by a distance ϵ in the limit where $M \rightarrow \infty$, $\epsilon \rightarrow 0$ but $L_{0z} = M\epsilon = Na$ is constant. We only present the additional information not contained in the previous Taub-NUT example as the nontrivial contributions of the Kerr-NUT metric split into contributions that were already present in the Taub-NUT case and additional contributions in Ma or Na .

1. Kerr metric

The additional nontrivial components of the linearized metric and linearized spin connection are

$$\begin{aligned} h_{0i} &= \frac{2Ma}{r^3} \varepsilon_{zij} x^j, \\ \omega_{0ij} &= \frac{1}{2} \partial_i h_{0j} = -Ma \varepsilon_{zij} \left(\frac{1}{r^3} + \frac{4\pi}{3} \delta(\mathbf{x}) \right) - \frac{3Ma}{r^5} \varepsilon_{zjl} x_i x^l, \\ \omega_{ij0} &= \frac{1}{2} (\partial_j h_{i0} - \partial_i h_{j0}) = \omega_{0ji} - \omega_{0ij} \\ &= Ma \varepsilon_{zij} \left(\frac{2}{r^3} + \frac{8\pi}{3} \delta(\mathbf{x}) \right) - \frac{3Max^l}{r^5} (\varepsilon_{zil} x_j - \varepsilon_{zjl} x_i). \end{aligned} \quad (\text{B1})$$

The additional nontrivial components of the linearized Riemann tensor are

$$\begin{aligned} R_{0ijk} &= -Ma \varepsilon_{zkl} \left(\partial_j \partial_i \partial_l \frac{1}{r} \right) + Ma \varepsilon_{zjl} \left(\partial_k \partial_i \partial_l \frac{1}{r} \right), \\ R_{ij0k} &= -Ma \varepsilon_{zjl} \left(\partial_k \partial_i \partial_l \frac{1}{r} \right) + Ma \varepsilon_{zil} \left(\partial_k \partial_j \partial_l \frac{1}{r} \right), \end{aligned} \quad (\text{B2})$$

where one can show that

$$\begin{aligned} \partial_i \partial_j \partial_k \frac{1}{r} &= -15 \frac{x_i x_j x_k}{r^7} + \frac{3}{r^5} (\delta_{ij} x_k + \delta_{ki} x_j + \delta_{jk} x_i) \\ &\quad - \frac{4\pi}{5} (\delta_{ij} \partial_k \delta(\mathbf{r}) + \delta_{ki} \partial_j \delta(\mathbf{r}) + \delta_{jk} \partial_i \delta(\mathbf{r})). \end{aligned} \quad (\text{B3})$$

Combining these results with the ones from Appendix A, we easily obtain $R_{j0} = R_{0j} = R_{0ij}{}^i = Ma \varepsilon_{zjl} (\partial_l \Delta \frac{1}{r}) = -4\pi Ma \varepsilon_{zjl} \partial_l \delta(\mathbf{x})$. This also gives us $R_{00} = 4\pi M \delta(\mathbf{x})$,

$R_{ij} = 4\pi M \delta_{ij} \delta(\mathbf{x})$, $R = 4\pi M \delta(\mathbf{x})$. Eventually, we find $G_{00} = 8\pi T_{00} = 8\pi M \delta(\mathbf{x})$, $G_{ij} = T_{ij} = 0$, and $G_{0j} = R_{0j} = 8\pi T_{0j} = -4\pi M a \varepsilon_{zjl} \partial_l \delta(\mathbf{x})$. This solution describes a point of electric mass M with an electric angular momentum $L^{xy} = Ma$.

2. The rotating NUT solution from the dual Kerr

As for the dual of the linearized Schwarzschild solution, by duality rotation we obtain the additional spin connections of the dual Kerr metric:

$$\begin{aligned} \omega_{0i0} &= -\frac{1}{2} \varepsilon_{ijk} \tilde{\omega}_{jk0} = Na \frac{\delta_{zi}}{r^3} - \frac{8}{3} \pi Na \delta_{zi} \delta(\mathbf{x}) - 3Na \frac{zx_i}{r^5}, \\ \omega_{ijk} &= \varepsilon_{ijl} \tilde{\omega}_{0lk} = Na (\delta_{zi} \delta_{kj} - \delta_{zj} \delta_{ki}) \left(-\frac{2}{r^3} + \frac{4\pi}{3} \delta(\mathbf{x}) \right) \\ &\quad + \frac{3Na}{r^5} (x_k (x_j \delta_{zi} - x_i \delta_{zj}) + z (x_i \delta_{kj} - x_j \delta_{ki})), \end{aligned} \quad (\text{B4})$$

where we used $\varepsilon_{ijk} \varepsilon^{zjk} = 2\delta_i^z$ and $\varepsilon_{ijk} \varepsilon^{zjl} = \delta_z^i \delta_l^k - \delta_z^k \delta_l^i$. One can easily derive the Einstein tensor and find that this solution corresponds to a magnetic point of mass N with a magnetic angular momentum $\tilde{L}^{xy} = Na$. This is the gravitational dual of the Kerr solution with a $\Theta_{\mu\nu}$ with a structure equal to the stress-energy tensor for the Kerr solution, meaning

$$\begin{aligned} \Theta^{00} &= N \delta(\mathbf{x}), & \Theta^{0x} &= \frac{Na}{2} \partial_y \delta(\mathbf{x}), \\ \Theta^{0y} &= -\frac{Na}{2} \partial_x \delta(\mathbf{x}). \end{aligned} \quad (\text{B5})$$

The nontrivial components for $\Phi_{\mu\nu\rho}$ are

$$\begin{aligned} \Phi^{0z}_0 &= -16\pi N \delta(x) \delta(y) \vartheta(z), \\ \Phi^{0y}_x &= -\Phi^{0x}_y = -\Phi^{xy}_0 = \Phi^{yx}_0 = 8\pi Na \delta(\mathbf{x}). \end{aligned} \quad (\text{B6})$$

We have

$$\begin{aligned} \omega_{0i0} &= \frac{1}{2} \partial_i h_{00} + \frac{1}{4} \varepsilon_{0ijk} \Phi^{jk}_0 = \frac{1}{2} \partial_i h_{00} + \frac{1}{2} \delta_{iz} \Phi^{xy}_0, \\ \omega_{ijk} &= \frac{1}{2} (\partial_j h_{ik} - \partial_i h_{jk} + \partial_k v_{ji}) + \frac{1}{2} \varepsilon_{ij0l} \bar{\Phi}^{0l}_k, \end{aligned} \quad (\text{B7})$$

where for our choice of $\Phi_{\mu\nu\rho}$,

$$\frac{1}{2} \varepsilon_{ij0l} \bar{\Phi}^{0l}_k = \frac{1}{2} \varepsilon_{ij0l} \Phi^{0l}_k = (\delta_{iz} \delta_{jk} - \delta_{jz} \delta_{ik}) \Phi^{0y}_x. \quad (\text{B8})$$

We then easily obtain⁶

$$\begin{aligned} h_{00} &= \frac{2Naz}{r^3}, & h_{ij} &= \frac{2Naz}{r^3} \delta_{ij}, \\ v_{ij} &= \frac{2Na}{r^3} (\delta_{zi} x_j - \delta_{zj} x_i). \end{aligned} \quad (\text{B9})$$

⁶Note that the $v_{\mu\nu}$ obtained here, and which lead to a regular spin connection, do not satisfy the gauge fixing proposed in Sec. III, where the aim was rather to define surface integrals.

The nontrivial components of the linearized metric in spherical coordinates are then

$$h_{\mu\mu} = \frac{2Naz}{r^3}, \quad h_{0\phi} = 2N(1 + \cos\theta). \quad (\text{B10})$$

These are the nontrivial components of the linearized rotating NUT metric.

3. The rotating NUT metric without the delta contributions

If we set $\Phi^{0y}_x = -\Phi^{0x}_y = -\Phi^{xy}_0 = \Phi^{yx}_0 = 0$, the difference from the previous case appears for

$$\begin{aligned} \omega_{0i0} &= -Na \partial_i \partial_z \left(\frac{1}{r} \right), \\ \omega_{ijk} &= -Na \left[\delta_{ik} \partial_j \partial_z \left(\frac{1}{r} \right) - \delta_{jk} \partial_i \partial_z \left(\frac{1}{r} \right) + \frac{1}{2} \delta_{zi} \partial_k \partial_j \left(\frac{1}{r} \right) \right. \\ &\quad \left. - \frac{1}{2} \delta_{zj} \partial_k \partial_i \left(\frac{1}{r} \right) \right]. \end{aligned} \quad (\text{B11})$$

This means that

$$\begin{aligned} R_{00} &= -4\pi Na \delta(x) \delta(y) \delta'(z), \\ R_{ij} &= -4\pi Na \delta_{ij} \delta(x) \delta(y) \delta'(z). \end{aligned} \quad (\text{B12})$$

The electric Einstein tensor now has a nontrivial component:

$$G_{00} = -8\pi Na \delta(x) \delta(y) \delta'(z).$$

The charges for the solution are thus $K_0 = N$ and $L^{0z} = -Na$. This is thus a solution describing a point magnetic mass N with, in addition, a ‘‘boost mass’’ $-Na$ which can be understood as a dipole of electric masses M and $-M$ separated by a distance ϵ in the limit where $\epsilon \rightarrow 0$ and $L_{0z} = Na = M\epsilon$ is kept constant. Positivity of energy in general relativity tells us that this interpretation should be discarded. We present in Sec. IV the dual version of this calculation.

The interested reader could eventually wonder about different combinations of the previous considerations. One could, for example, try to interpret the rotating NUT solution with only the delta contributions and no string contribution (or, respectively, no $\Phi_{\mu\nu\rho}$ contributions at all). Following our analysis this only partially matches the proposal of Miller in [20] to interpret the Kerr-NUT metric as a Schwarzschild black hole and an infinite source of angular momentum along the singularity. Our calculations show that it should also be supplemented with a magnetic angular momentum when delta contributions are included (respectively, with a dipole of electric masses in the same limit as previously discussed when no contributions are taken into account). Dual considerations can also be implemented following the same ideas presented at the end of Sec. IV.

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