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Analytic treatment of the black-hole bomb

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A bosonic field impinging on a rotating black hole can be amplified as it scatters off the hole, a phenomenon known as superradiant scattering. If in addition the field has a nonzero rest mass μ, the mass term effectively works as a mirror, reflecting the scattered wave back towards the black hole. In this physical system, known as a black-hole bomb, the wave may bounce back and forth between the black hole and some turning point, amplifying itself each time. Consequently, the field grows exponentially over time and is unstable. In this paper we study *analytically* for the first time the phenomenon of superradiant instability (the black-hole bomb mechanism) in the regime $M\mu = O(1)$ of greatest instability. We find a maximal instability growth rate of $\tau^{-1} = 1.7 \times 10^{-3} M^{-1}$. This instability is 4 orders of magnitude *stronger* than has been previously estimated.

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Black holes are the fundamental "atoms" of general relativity. They also play a central role in high energy physics, astrophysics, and even in condensed matter physics. The fundamental role of black holes makes it highly important to study the nature of their stability: If a black hole is perturbed in some small way, will the perturbation die away over time? Or will it grow exponentially until it can no longer be considered a perturbation and hence demonstrate the instability of the black hole?

The issue of black-hole stability was first addressed by Regge and Wheeler [1] who demonstrated the stability of the spherically symmetric Schwarzschild black hole. If a Schwarzschild black hole is perturbed, then the perturbation will oscillate and damp out over time [2]. This implies that perturbation fields would either be radiated away to infinity or swallowed by the black hole.

The stability question of rotating Kerr black holes is a bit more involved. Press and Teukolsky [3,4] have shown that rotating black holes are stable under free gravitational perturbations (see also [5] and references therein). However, the superradiance effect may change this conclusion. Superradiant scattering is a well-known phenomenon in quantum systems [6,7] as well as in classical ones [8,9]. Considering a wave of the form $e^{im\phi}e^{-i\omega t}$ incident upon a rotating object whose angular velocity is Ω, one finds that if the frequency ω of the incident wave satisfies the relation

$$\omega < m\Omega, \tag{1}$$

then the scattered wave is amplified.

A bosonic field impinging upon a rotating Kerr black hole can be amplified if the superradiance condition (1) is satisfied, where in this case $\Omega = \frac{a}{r_+^2+a^2}$ is the angular velocity of the black-hole horizon. Here r_+ and a are the horizon radius and the angular momentum per unit mass of the black hole, respectively. The energy radiated away to infinity may actually exceed the energy present in the initial perturbation. Feeding back the amplified scattered wave, one can gradually extract the rotational energy of the black hole. Press and Teukolsky suggested to use this mechanism to build a *black-hole bomb* [10]: If one surrounds the black hole by a reflecting mirror, the wave will bounce back and forth between the black hole and the mirror, amplifying itself each time. Thus, the total energy extracted from the black hole will gradually grow.

Remarkably, nature sometimes provides its own mirror [9]: If one considers a *massive* scalar field with mass $\mathcal{M}$ scattered off a rotating black hole, then for $\omega < \mu \equiv \mathcal{M}G/\hbar c$ the mass term effectively works as a mirror [11–15]. The physical idea is to consider a wave packet of the massive field in a bound orbit around the black hole [12,13]. The gravitational force binds the field and keeps it from escaping to infinity. At the event horizon some of the field goes down the black hole, and if the frequency of the wave is in the superradiance regime (1), then the field is amplified. In this way the field is amplified at the horizon while being bound away from infinity. Consequently, the massive field grows exponentially over time and is unstable [13].

The nature of the superradiant instability (the black-hole bomb) depends on two parameters:

(i) The rotation rate a of the black hole.

(ii) The dimensionless product of the black-hole mass M and the field mass μ. The product $M\mu$ is actually the ratio of the black-hole size to the Compton wavelength associated with the rest mass of the field. (We shall henceforth use natural units in which $G = c = \hbar = 1$. In these units μ has the dimensions of 1/length.)

Former analytical estimates of the instability time scale associated with the dynamics of a massive scalar field in the rotating Kerr spacetime were restricted to the regimes $M\mu \gg 1$ [12] and $M\mu \ll 1$ [13,14]. In these two limits the growth rate of the field (the imaginary part ω_I of the mode's frequency) was found to be very weak, scaling like $M^{-1}e^{-1.84M\mu} \ll 1$ for $M\mu \gg 1$ [12] and like $M^{-1}(M\mu)^9 \ll 1$ for the $M\mu \ll 1$ case [13,14]. We note, however, that the former analytical approximations [12–14] fail in the regime $M\mu = O(1)$. Thus, direct *numerical* integration of the perturbation equations seemed necessary to find the actual growth rate of the perturbations in this regime [12,14,15]. These numerical investigations have indicated that the superradiant instability is in fact greatest in the regime $M\mu = O(1)$ [12,14–16]. A new *analytical* study of the superradiant instability in the regime $M\mu = O(1)$ is therefore physically desirable.

It is worth noting that previous numerical investigations [12,14,15] of the black-hole bomb have indicated that the instability is most effective under the following conditions:

(i) The black hole is maximally rotating with $a \simeq M$.

(ii) The dimensionless product $M\mu$ satisfies the relation $M\mu \sim \frac{1}{2}$.

(iii) The frequency of the unstable mode satisfies the relations $\omega \simeq m\Omega$ and $\omega \simeq \mu$. (Of course, for the mode to be in the superradiant regime one should have $\omega < m\Omega$. In addition, for the mode to be in a bound state it should satisfy $\omega < \mu$.)

As we shall show below, the black-hole bomb and the associated instability time scale can be studied *analytically* in the above-mentioned regime of physical interest (the regime of the greatest instability). The physical system we consider consists of a massive scalar field coupled to a rotating Kerr black hole. The dynamics of a scalar field Ψ of mass μ in the Kerr spacetime [17] is governed by the Klein-Gordon equation

$$(\nabla^a\nabla_a - \mu^2)\Psi = 0. \tag{2}$$

One may decompose the field as

$$\Psi_{lm}(t, r, \theta, \phi) = e^{im\phi}S_{lm}(\theta; a\omega)R_{lm}(r; a\omega)e^{-i\omega t}, \tag{3}$$

where (t, r, θ, ϕ) are the Boyer-Lindquist coordinates [17], ω is the (conserved) frequency of the mode, l is the spheroidal harmonic index, and m is the azimuthal harmonic index with $-l \le m \le l$. (We shall henceforth omit the indices l and m for brevity.) With the decomposition (3), R and S obey radial and angular equations, both of confluent Heun type, coupled by a separation constant $A(a\omega)$ [18,19]. The sign of ω_I determines whether the solution is decaying ($\omega_I < 0$) or growing ($\omega_I > 0$) in time.

The angular functions $S(\theta; a\omega)$ are the spheroidal harmonics which are solutions of the angular equation [4,19]

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \left[a^2(\omega^2 - \mu^2)\cos^2\theta - \frac{m^2}{\sin^2\theta} + A\right]S = 0. \tag{4}$$

The angular functions are required to be regular at the poles $\theta = 0$ and $\theta = \pi$. These boundary conditions pick out a discrete set of eigenvalues $\{A_l\}$ labeled by an integer l. For $\omega \simeq \mu$ one can treat $a^2(\omega^2 - \mu^2)\cos^2\theta$ in Eq. (4) as a perturbation term on the generalized Legendre equation. We can then expand the separation constants in powers of $a^2(\mu^2 - \omega^2)$ to find [20]

$$A = l(l + 1) + \sum_{k=1}^{\infty} c_k a^{2k}(\mu^2 - \omega^2)^k. \tag{5}$$

The expansion coefficients $\{c_k\}$ are given in [20]. [For example, for $l = m = 1$, the case of physical interest (see below), we find $c_1 = 1/5$, $c_2 = -4/875$, $c_3 = 8/65\,625, \ldots$.]

The radial Teukolsky equation is given by [21–23]

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) + [K^2 - \Delta(a^2\omega^2 - 2ma\omega + \mu^2r^2 + A)]R = 0, \tag{6}$$

where $\Delta \equiv r^2 - 2Mr + a^2$ and $K \equiv (r^2 + a^2)\omega - am$. The zeroes of Δ, $r_\pm = M \pm (M^2 - a^2)^{1/2}$, are the black-hole (event and inner) horizons.

We are interested in solutions of the radial equation (6) with the physical boundary conditions of purely ingoing waves at the black-hole horizon (as measured by a comoving observer) and a decaying (bounded) solution at spatial infinity [12,15]. That is,

$$R \sim \begin{cases} \frac{1}{r}e^{-\sqrt{\mu^2-\omega^2}y} & \text{as } r \to \infty (y \to \infty) \\ e^{-i(\omega - m\Omega)y} & \text{as } r \to r_+ (y \to -\infty), \end{cases} \tag{7}$$

where the "tortoise" radial coordinate y is defined by $dy = [(r^2 + a^2)/\Delta]dr$. These boundary conditions single out a discrete set of resonances $\{\omega_n\}$ which correspond to bound states of the massive field [12,15].

It is convenient to define new dimensionless variables

$$x \equiv \frac{r - r_+}{r_+}; \qquad \tau \equiv 8\pi MT_{\rm BH} = \frac{r_+ - r_-}{r_+}; \qquad \varpi \equiv \frac{\omega - m\Omega}{2\pi T_{\rm BH}}; \qquad k \equiv 2\omega r_+, \tag{8}$$

in terms of which the radial equation becomes

$$x(x + \tau)\frac{d^2R}{dx^2} + (2x + \tau)\frac{dR}{dx} + VR = 0, \tag{9}$$

where $V \equiv K^2/r_+^2x(x + \tau) - [a^2\omega^2 - 2ma\omega + \mu^2r_+^2(x + 1)^2 + A]$ and $K = r_+^2\omega x^2 + r_+kx + r_+\varpi\tau/2$.

As discussed above, previous numerical investigations [12,14,15] have indicated that the black-hole instability is most pronounced in the regime $\tau \ll 1$ with $M(m\Omega - \omega) \ll 1$. As we shall now show, the radial equation is amenable to an analytic treatment in this regime of physical interest.

We first consider the radial equation (9) in the far region $x \gg \max\{\tau, M(m\Omega - \omega)\}$. Then Eq. (9) is well approximated by

$$x^2\frac{d^2R}{dx^2}+2x\frac{dR}{dx}+V_{\rm far}R=0, \tag{10}$$

where $V_{\rm far}=(\omega^2-\mu^2)r_+^2x^2+2(\omega k-\mu^2r_+)r_+x-(a^2\omega^2-2ma\omega+\mu^2r_+^2+A-k^2)$. A solution of Eq. (10) that satisfies the boundary condition (7) can be expressed in terms of the confluent hypergeometric functions $M(a,b,z)$ [20,24],

$$R=C_1(2\sqrt{\mu^2-\omega^2}r_+)^{(1/2)+\beta}x^{-(1/2)+\beta}e^{-\sqrt{\mu^2-\omega^2}r_+x}\times M\Big(\frac{1}{2}+\beta-\kappa,1+2\beta,2\sqrt{\mu^2-\omega^2}r_+x\Big)+C_2(\beta\to-\beta), \tag{11}$$

where C_1 and C_2 are constants and

$$\beta^2\equiv a^2\omega^2-2ma\omega+\mu^2r_+^2+A-k^2+\frac{1}{4};\qquad \kappa\equiv\frac{\omega k-\mu^2r_+}{\sqrt{\mu^2-\omega^2}}. \tag{12}$$

The notation $(\beta\to-\beta)$ means "replace β by $-\beta$ in the preceding term."

We next consider the near horizon region $x\ll1$. The radial equation is given by Eq. (9) with $V\to V_{\rm near}\equiv-(a^2\omega^2-2ma\omega+\mu^2r_+^2+A)+(kx+\varpi\tau/2)^2/x(x+\tau)$. The physical solution obeying the ingoing boundary conditions at the horizon is given by [20,24]

$$R=x^{-(i/2)\varpi}\Big(\frac{x}{\tau}+1\Big)^{i((1/2)\varpi-k)}{}_2F_1\Big(\frac{1}{2}+\beta-ik,\frac{1}{2}-\beta-ik;1-i\varpi;-x/\tau\Big), \tag{13}$$

where $2F_1(a,b;c;z)$ is the hypergeometric function.

The solutions (11) and (13) can be matched in the overlap region $\max\{\tau,M(m\Omega-\omega)\}\ll x\ll1$. The $x\ll1$ limit of Eq. (11) yields [20,24]

$$R\to C_1(2\sqrt{\mu^2-\omega^2}r_+)^{(1/2)+\beta}x^{-(1/2)+\beta}+C_2(\beta\to-\beta). \tag{14}$$

The $x\gg\tau$ limit of Eq. (13) yields [20,24]

$$R\to\tau^{(1/2)-\beta-i\varpi/2}\frac{\Gamma(2\beta)\Gamma(1-i\varpi)}{\Gamma(\frac{1}{2}+\beta-ik)\Gamma(\frac{1}{2}+\beta-i\varpi+ik)}\times x^{-(1/2)+\beta}+(\beta\to-\beta). \tag{15}$$

By matching the two solutions in the overlap region one finds

$$C_1=\tau^{(1/2)-\beta-i\varpi/2}\frac{\Gamma(2\beta)\Gamma(1-i\varpi)}{\Gamma(\frac{1}{2}+\beta-ik)\Gamma(\frac{1}{2}+\beta-i\varpi+ik)}\times(2\sqrt{\mu^2-\omega^2}r_+)^{-(1/2)-\beta}, \tag{16}$$

$$C_2=\tau^{(1/2)+\beta-i\varpi/2}\frac{\Gamma(-2\beta)\Gamma(1-i\varpi)}{\Gamma(\frac{1}{2}-\beta-ik)\Gamma(\frac{1}{2}-\beta-i\varpi+ik)}\times(2\sqrt{\mu^2-\omega^2}r_+)^{-(1/2)+\beta}. \tag{17}$$

Approximating Eq. (11) for $x\to\infty$, one gets [20,24]

$$R\to\Big[C_1(2\sqrt{\mu^2-\omega^2}r_+)^{-\kappa}\frac{\Gamma(1+2\beta)}{\Gamma(\frac{1}{2}+\beta-\kappa)}x^{-1-\kappa}+C_2(\beta\to-\beta)\Big]e^{\sqrt{\mu^2-\omega^2}r_+x}+\Big[C_1(2\sqrt{\mu^2-\omega^2}r_+)^{\kappa}\frac{\Gamma(1+2\beta)}{\Gamma(\frac{1}{2}+\beta+\kappa)}\times x^{-1+\kappa}(-1)^{-(1/2)-\beta+\kappa}+C_2(\beta\to-\beta)\Big]\times e^{-\sqrt{\mu^2-\omega^2}r_+x}. \tag{18}$$

A bound state is characterized by a decaying field at spatial infinity. The coefficient of the growing exponent $e^{\sqrt{\mu^2-\omega^2}r_+x}$ in Eq. (18) should therefore vanish. Taking cognizance of Eqs. (16)–(18) for $\varpi\gg1$ [here we use Stirling's formula [20] for $\Gamma(\frac{1}{2}\pm\beta-i\varpi+ik)$], one finds the resonance condition for the bound states of the field,

$$\frac{1}{\Gamma(\frac{1}{2}+\beta-\kappa)}=(8i)^{2\beta}\Big[\frac{\Gamma(-2\beta)}{\Gamma(2\beta)}\Big]^2\times\frac{\Gamma(\frac{1}{2}+\beta-ik)}{\Gamma(\frac{1}{2}-\beta-ik)\Gamma(\frac{1}{2}-\beta-\kappa)}\times\Big[Mr_+\sqrt{\mu^2-\omega^2}(m\Omega-\omega)\Big]^{2\beta}. \tag{19}$$

The growing resonances of the field can be estimated analytically in the regime of physical interest $\omega\simeq\mu\simeq m\Omega$: We first note that in this regime the right-hand side of Eq. (19) is small [due to the factors $r_+\sqrt{\mu^2-\omega^2}$ and $M(m\Omega-\omega)$]. One may therefore write a zeroth-order approximation for the resonance condition: LHS $\equiv 1/\Gamma(\frac{1}{2}+\beta-\kappa)\simeq0$. Using the well-known pole structure of the Gamma functions [20], one finds the approximated resonance condition

$$\tfrac{1}{2}+\beta-\kappa=-n, \tag{20}$$

where $n\geq0$ is a non-negative integer. Taking cognizance of Eqs. (5) and (12), one realizes that Eq. (20) is a simple polynomial equation for the variable $M^2(\mu^2-\omega^2)$, whose solutions we denote by $\omega_R^{(0)}$. [Note that for the zeroth-order approximation one has $\omega_I^{(0)}=0$.] One can then use $\omega_R^{(0)}$ in the equation $\mathrm{Im\,LHS}(\omega_R^{(0)},\omega_I^{(1)})=\mathrm{Im\,RHS}(\omega_R^{(0)},\omega_I^{(1)})$ to obtain a simple polynomial equation for the first-order solution $\omega_I^{(1)}$.

In Fig. 1 we depict results for the most unstable (fastest growing) mode with $l = m = 1$. We present results of the direct solutions of both the exact resonance condition (19) and the polynomial approximation (20). One finds a good qualitative agreement between the two. The maximum growth rate we find is $\tau^{-1} \equiv \omega_I = 1.7 \times 10^{-3} M^{-1}$, where τ is the e-folding time. We would like to emphasize that this growth rate is 4 orders of magnitude stronger than has been previously found.

What are the observable consequences of this instability? Let us consider, for example, the neutral spinless pion π^0 whose mass is $\mu \simeq 134.97\ \mathrm{MeV/c^2}$. The superradiant instability is most pronounced for $M\mu \simeq 0.469$, which corresponds to a primordial black hole of mass $M \simeq 9.3 \times 10^{11}$ Kg [15]. For the superradiant instability to be effective, the lifetime of the neutral pion, $\tau_{1/2} \simeq 8.2 \times 10^{-17}$ sec, should be significantly longer than the time scale associated with the instability, $\tau = (1.7 \times 10^{-3})^{-1} \mathrm{GM/c^3} \simeq 1.3 \times 10^{-21}$ sec. This condition is indeed satisfied by more than 4 orders of magnitude. One therefore concludes that the superradiant instability in the neutral pion channel may indeed manifest itself for primordial black holes.

In summary, we have studied analytically the instability of rotating black holes to perturbations of massive scalar fields. Former analytical estimations [12–14] of the time scale associated with the instability were restricted to the regimes $M\mu \gg 1$ and $M\mu \ll 1$. In these two limits the growth rate of the field was found to be extremely weak. However, subsequent numerical investigations [12,14,15] have indicated that the instability is actually greatest in the regime $M\mu = O(1)$, where unfortunately the previous analytical approximations are not suitable to describe the dynamics of this instability. Motivated by these numerical studies, we have provided here, for the first time, an analytic treatment of the superradiant instability (the black-hole bomb mechanism) in the physically most interesting regime $M\mu \sim \frac{1}{2}$, where the instability is most pronounced. We find an instability growth rate of $\tau^{-1} \equiv \omega_I = 1.7 \times 10^{-3} M^{-1}$ for the fastest growing mode—4 orders of magnitude *stronger* than has been previously estimated.

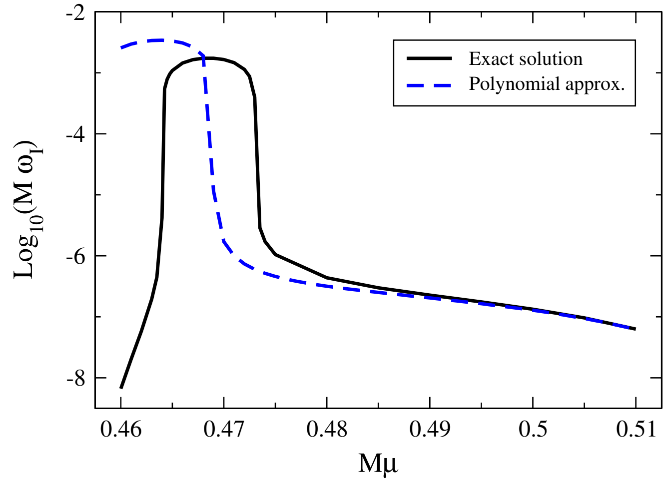

FIG. 1 (color online). Superradiant instability for maximally rotating black holes ($a \simeq M$). The results are for the $l = m = 1$ mode, the mode with the greatest instability. The growth rate $M\omega_I$ is shown as a function of the dimensionless product $M\mu$. We display results for the direct solutions of both the exact resonance condition (19) and the polynomial approximation (20). The maximum growth rate is $\tau^{-1} \equiv \omega_I = 1.7 \times 10^{-3} M^{-1}$, where τ is the e-folding time.

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