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Cross section evaluation by spinor integration. II. The massive case in 4D

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In this paper, we continue our study of calculating the cross section by the spinor method, i.e., performing the phase space integration using the spinor method. We have focused on the case where the physical momenta are massive and in pure four dimensions. We established the framework of such a new method and presented several examples, including two real progresses: $Z^0 \to l^+ l^- H$ and $q\bar{q} \to f\bar{f}H^0$.

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I. INTRODUCTION

In the Tevatron collider and the LHC, multiple final states are observed frequently. In order to check the standard model and looking forward to finding new physics beyond the standard model [1–4], we need to explore the problem of how to calculate the cross section efficiently and conveniently. In the past, the cross section was evaluated in the three-dimensional momentum space [5,6] and people have developed quite mature numerical techniques. For the applications of programs MADGRAPH, PYTHIA, ALPGEN, and SHERPA, the reader can check the references, for example, [7].

On the other hand, enormous progress has been made in the evaluation of one-loop amplitudes [2]. One such progress is the unitarity cut method originally proposed in [8,9]. With the twistor program initiated by Witten [10], the double cut phase space integration has been reduced to algebraic manipulation through the holomorphic anomaly [11–14]. Inspired by this simplification, in our first paper [15], we have explored how to apply the spinor integration method to the evaluation of the cross section for the massless case. There are some obvious advantages compared with the momentum integration method. First, the three-dimensional momentum space integration can be reduced to just one-dimensional integration and furthermore for the massless case, the integration region is just [0, 1].[1] Secondly, in the calculation, every step is manifestly Lorentz-invariant, thus we obtain compact analytic expressions.

Continuing our study for the massless case, in this paper we focus on the massive case. We will see that if all the mass is set to zero, the massless case will be reproduced. Different from the massless case, the integration variable L is no longer a null momentum. So we cannot apply the spinor method directly. However this problem has been solved in the unitarity cut method [16,17]. More accurately, we can write

$$\int d^4L = \int dz \int d^4\ell \delta^+(\ell^2)(2\ell \cdot K); \qquad L = \ell + zK, \tag{1.1}$$

where K is a fixed vector and z is a real number. Through this decomposition, we establish the general framework for the massive case by the spinor method.

In our first paper, we have emphasized the advantages of using the spinor method [15]. In the massive case, the constrained three-dimensional momentum space integration still can be reduced to a one-dimensional integration, plus possible Feynman integrations. In every step, we get a scalar type of integration, which is Lorentz-invariant. Furthermore, the integration region can be written directly. Though it is not simply [0, 1] like the massless case, it is only the simple functions of mass and energy.

The outline of this paper is as follows. In Sec. II, we first briefly review the four-dimensional unitarity cut method. Then we take the Faddeev-Popov trick to establish the general framework.

In Sec. III, we apply our method to the pure phase space integration for two, three, and four outgoing particles as well as some simple examples to demonstrate the main idea and features. These are the basis for practical and more complicated applications.

In Sec. IV, we calculate two practical examples and summarize some experience of performing the integrations.

A summary of our results with some comments is given in Sec. V.

II. FRAMEWORK TO USE SPINOR METHOD

In this section, we will set up the spinor integration method for massive particles in four dimensions. Then we apply this method to the phase space integration where the integration region (i.e., the $\int dx$) of one dimensionless parameter is determined by the kinematical discussion. This region corresponds to the boundary of the whole phase space of outgoing momenta. One important difference, compared to the massless case, is that the integration region will be functions of masses of outgoing particles.

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[1]The result of unitarity integration maybe written as one Feynman parameter integration over rational functions.

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A. The spinor integration method for massive cuts

Here, we briefly review the spinor integration method for massive cuts (or sometimes called the "unitarity cut method") [16–18]. The Lorentz-invariant phase space (LIPS) of a massive double cut is defined by inserting two δ functions representing the cut conditions:

$$I = \int d^4\tilde{\ell}\delta^+(\tilde{\ell}^2 - m_1^2)\delta((K-\tilde{\ell})^2 - m_2^2), \tag{2.1}$$

where $\tilde{\ell}$ is the internal loop momentum and K the total momentum through the unitarity cut. Because $\tilde{\ell}$ is a massive momentum, to use the spinor integration method we need decompose it as

$$\tilde{\ell} = \ell + zK, \qquad \ell^2 = 0; \qquad \int d^4\tilde{\ell} = \int dz d^4\ell\delta^+(\ell^2)(2\ell\cdot K), \tag{2.2}$$

where ℓ is a null four-momentum, and can be expressed with spinor variables as

$$\ell = tP_{\lambda\tilde{\lambda}}, \qquad P_{\lambda\tilde{\lambda}} = \lambda\tilde{\lambda}; \qquad \int d^4\ell\delta^+(\ell^2) = \int\langle\lambda\,d\lambda\rangle[\tilde{\lambda}\,d\tilde{\lambda}]\int t dt. \tag{2.3}$$

Under this decomposition Eq. (2.1) becomes

$$\begin{aligned} I &= \int dz d^4\ell\delta^+(\ell^2)(2\ell\cdot K)\delta^+(z^2K^2 + 2zK\cdot\ell - m_1^2) \\ &\quad\times\delta^+((1-2z)K^2 - 2K\cdot\ell + m_1^2 - m_2^2) \\ &= \int dz((1-2z)K^2 + m_1^2 - m_2^2) \\ &\quad\times\delta^+(z(1-z)K^2 + z(m_1^2 - m_2^2) - m_1^2) \\ &\quad\times\int\langle\lambda\,d\lambda\rangle[\tilde{\lambda}\,d\tilde{\lambda}]\frac{(1-2z)K^2 + m_1^2 - m_2^2}{\langle\lambda|K|\tilde{\lambda}]^2}, \\ t &= \frac{(1-2z)K^2 + m_1^2 - m_2^2}{\langle\lambda|K|\tilde{\lambda}]}. \end{aligned} \tag{2.4}$$

The first line of the result it depends only on the variable z, so we can use the δ function to eliminate z as follows:

$$z_\pm = \frac{(K^2 + m_1^2 - m_2^2) \pm \sqrt{\Delta[K, m_1, m_2]}}{2K^2}, \tag{2.5}$$

where we have defined

$$\Delta[K, m_1, m_2] = (K^2 - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2. \tag{2.6}$$

Between the two solutions of z, only one should be taken. To see that, we make a kinematical analysis. Choose a center-of-mass frame such that

$$K = (E > 0, 0, 0, 0), \qquad \tilde{\ell} = (a, b, 0, 0), \qquad K - \tilde{\ell} = (E - a, -b, 0, 0).$$

The mass-shell conditions require $a^2 - b^2 = m_1^2$ and $(E - a)^2 - b^2 = m_2^2$, so $a = (E^2 + m_1^2 - m_2^2)/2E$. In the decomposition $\tilde{\ell} = \ell + zK$, because the positive light cone with $\delta^+(\ell)$ has been chosen, we can write

$$\ell = (|b|, b, 0, 0), \qquad \tilde{\ell} = (|b| + zE, b, 0, 0).$$

Then $|b| + zE = a$. This means that only z_- is retained.[2] In the remainder of this paper, we always refer to z as z_-, if it is not explicitly illustrated.

Then Eq. (2.4) becomes

$$I = \int\langle\lambda\,d\lambda\rangle[\tilde{\lambda}\,d\tilde{\lambda}]\frac{(1-2z)K^2 + m_1^2 - m_2^2}{\langle\lambda|K|\tilde{\lambda}]^2}, \qquad t = \frac{(1-2z)K^2 + m_1^2 - m_2^2}{\langle\lambda|K|\tilde{\lambda}]}. \tag{2.7}$$

Equation (2.7) is our final form for the spinor integration with massive double cuts. For convenience we define

$$z[K, m_1, m_2] = \frac{\alpha[K, m_1; m_2] - \beta[K; m_1, m_2]}{2}, \qquad t = \frac{\beta K^2}{\langle\lambda|K|\tilde{\lambda}]}, \tag{2.8}$$

where

$$\alpha[K, m_1; m_2] \equiv \frac{K^2 + m_1^2 - m_2^2}{K^2}, \qquad \beta[K; m_1, m_2] \equiv \frac{\sqrt{\Delta[K, m_1, m_2]}}{K^2}. \tag{2.9}$$

Notice that when $m_1 = m_2 = 0$ we have $\alpha = \beta = 1$, thus reproducing the massless case. Finally the original $\tilde{\ell}$ can be parametrized as

$$\begin{aligned} \tilde{\ell} &= tP_{\lambda\tilde{\lambda}} + zK \\ &= \frac{K^2}{\langle\lambda|K|\tilde{\lambda}]}\left[\beta\left(P_{\lambda\tilde{\lambda}} - \frac{K\cdot P_{\lambda\tilde{\lambda}}}{K^2}K\right) + \alpha\frac{K\cdot P_{\lambda\tilde{\lambda}}}{K^2}K\right]. \end{aligned} \tag{2.10}$$

B. Spinor integration method for the physical phase space integration

Now, we explore how to apply the spinor integration method for massive cuts to the phase space integration. Just like the massless case, when there are only two outgoing particles, the spinor integration method can be applied directly without any modification. To see this explicitly, just write the phase space of the cross section:

[2] If $E < 0$, we need z_+.

$$I_2 = \prod_{f=1,2} \int \frac{d^4 L_f}{(2\pi)^3} \delta^+(L_f^2 - m_i^2)(2\pi)^4 \delta^4\Big(K - \sum_{f=1,2} L_f\Big) \sim \int d^4 L_1 \delta^+(L_1^2 - m_1^2)\delta^+((K - L_1)^2 - m_2^2), \tag{2.11}$$

which is exactly the same (namely the showing up of two δ functions) as the spinor integration method given in Eq. (2.1).

Things will be different when $n = 3$, where the physical phase space is given by

$$\begin{aligned} I_3 &= \prod_{i=1}^{3} \int \frac{d^4 L_i}{(2\pi)^3} \delta^+(L_i^2 - m_i^2)(2\pi)^4 \delta^4\Big(K - \sum L_i\Big) \\ &\quad \times f(L_1, L_2, L_3) \\ &= \int \frac{d^4 L_3}{(2\pi)^3} \delta^+(L_3^2 - m_3^2) \int \frac{d^4 L_2}{(2\pi)^2} \delta^+(L_2^2 - m_2^2) \\ &\quad \times \delta^+((K - L_3 - L_2)^2 - m_1^2) f(L_2, L_3) \\ &= \int \frac{d^4 L_3}{(2\pi)^3} \delta^+(L_3^2 - m_3^2)\tilde{f}(L_3). \end{aligned} \tag{2.12}$$

The problem we meet here is just the same as the massless case. The integration over L_2 with two δ functions can be performed by the spinor integration method. However, there is only one δ function in the integration over L_3. In order to apply the spinor method recursively and continuously, we need insert one more δ function like the Faddeev-Popov method.

Similarly to the massless case [15], we consider the following integration:

$$\begin{aligned} I_x &\equiv \int dx \delta((xK - L_3)^2 - m_3^2) \\ &= \int dx \delta(x^2 K^2 - x(2K \cdot L_3) + L_3^2 - m_3^2), \end{aligned}$$

where the δ function has two solutions,

$$x_i = \frac{(2K \cdot L_3) \pm \sqrt{(2K \cdot L_3)^2 - 4K^2(L_3^2 - m_3^2)}}{2K^2}. \tag{2.13}$$

Using the on-shell condition $L_3^2 = m_3^2$, it reduces to

$$x_- = 0, \qquad x_+ = \frac{2K \cdot L_3}{K^2}. \tag{2.14}$$

We find that $x_- = 0$ is always a root. However, when $x = 0$, we have $\delta(L_3^2 - m_3^2)$ which does not give an independent δ function. So $x_- = 0$ should be excluded from our consideration. For another root x_+, from $(K - L_3)^2 = K^2 + m_3^2 - 2K \cdot L_3 \geq (m_1 + m_2)^2$, we have

$$2K \cdot L_3 \leq K^2 + m_3^2 - (m_1 + m_2)^2, \tag{2.15}$$

which gives the upper bound of x_+. For the lower bound, considering the center-of-mass frame where $K = (E, 0, 0, 0)$, $L_3 = (E_3, p, 0, 0)$ with $E_3^2 - p^2 = m_3^2$, we have $2K \cdot L_3 \geq 2Em_3$, i.e., $(2K \cdot L_3)^2 \geq 4K^2 m_3^2$.

Putting all consideration together we have

$$\begin{aligned} I_x &\equiv \int_{x_0}^{x_1} dx \delta((xK - L_3)^2 - m_3^2) \\ &= \int_{x_0}^{x_1} dx \frac{\delta(x - x_+)}{|2x_+ K^2 - (2K \cdot L_3)|} \\ &= \frac{1}{|\sqrt{(2K \cdot L_3)^2 - 4K^2(L_3^2 - m_3^2)}|}, \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} x_0[K, m_3] &\equiv \sqrt{\frac{4m_3^2}{K^2}}, \\ x_1[K, m_3; m_t] &\equiv \frac{K^2 + m_3^2 - m_t^2}{K^2} \\ &= \sqrt{x_0[K, m_3]^2 + \Lambda[K; m_3, m_t]^2}, \end{aligned} \tag{2.17}$$

where $\Lambda[K; m_3, m_t] = \sqrt{\Delta[K, m_3, m_t]}/K^2$ with $m_t = m_1 + m_2$. Using $K^2 \geq (m_1 + m_2 + m_3)^2$, it is easy to see that $\Lambda[K; m_3, m_t]^2 \geq 0$ and thus $x_1 \geq x_0$.

Now Eq. (2.12) can be written as

$$\begin{aligned} I_3 &= \frac{1}{(2\pi)^3} \int d^4 L_3 \delta^+(L_3^2 - m_3^2) \\ &\quad \times |\sqrt{(2K \cdot L_3)^2 - 4K^2(L_3^2 - m_3^2)}| \\ &\quad \times \int_{x_0}^{x_1} dx \delta((xK - L_3)^2 - m_3^2)\tilde{f}(L_3). \end{aligned}$$

Decomposing $L_3 = \ell + zK$ with $\ell^2 = 0$, then

$$\begin{aligned} I_3 &= \frac{1}{(2\pi)^3} \int dz d^4\ell \delta^+(\ell^2)(2\ell \cdot K) \\ &\quad \times \delta^+(z^2 K^2 + 2zK \cdot \ell - m_3^2)(2L_3 \cdot K) \\ &\quad \times \int_{x_0}^{x_1} dx \delta^+(x^2 K^2 - 2xK \cdot L_3)\tilde{f}(\ell) \\ &= \frac{1}{(2\pi)^3} \int_{x_0}^{x_1} dx x K^2 \int dz (x - 2z) \\ &\quad \times K^2 \delta^+(z(x - z)K^2 - m_3^2) \\ &\quad \times \int d^4\ell \delta^+(\ell^2)\delta^+(x(x - 2z)K^2 - 2xK \cdot \ell)\tilde{f}(\ell). \end{aligned} \tag{2.18}$$

One by-product of the above procedure is

$$2K \cdot L_3 = xK^2. \tag{2.19}$$

By solving the δ function $\delta^+(z(x - z)K^2 - m_3^2)$ and the similar kinematical discussion as in Sec. II A, we get

$$z = \frac{xK^2 - \sqrt{K^2(x^2K^2 - 4m_3^2)}}{2K^2} = \frac{x}{2} - \frac{\sqrt{x^2 - x_0^2}}{2},$$
$$x - 2z = \sqrt{x^2 - x_0^2}. \tag{2.20}$$

Continue the evaluation as

$$I_3 = \frac{c}{(2\pi)^3} \int_{x_0}^{x_1} dx x K^2 \int \langle\lambda|d\lambda\rangle[\tilde{\lambda}|d\tilde{\lambda}] \times \int t dt \delta^+(x(x-2z)K^2 - xt\langle\lambda|K|\tilde{\lambda}])\tilde{f}(\lambda, \tilde{\lambda}, t)$$
$$= \frac{c}{(2\pi)^3} \int_{x_0}^{x_1} dx x K^2 \int \langle\lambda|d\lambda\rangle[\tilde{\lambda}|d\tilde{\lambda}] \frac{(x-2z)K^2}{x\langle\lambda|K|\tilde{\lambda}]^2} \times \tilde{f}(\lambda, \tilde{\lambda}, t),$$
$$t = \frac{(x-2z)K^2}{\langle\lambda|K|\tilde{\lambda}]} = \frac{c}{(2\pi)^3} \int_{x_0}^{x_1} dx (K^2)^2 \sqrt{x^2 - x_0^2} \times \int \langle\lambda|d\lambda\rangle[\tilde{\lambda}|d\tilde{\lambda}] \frac{\tilde{f}(\lambda, \tilde{\lambda}, t)}{\langle\lambda|K|\tilde{\lambda}]^2},$$
$$t = \frac{K^2\sqrt{x^2 - x_0^2}}{\langle\lambda|K|\tilde{\lambda}]}, \tag{2.21}$$

where $c = \pi/2$ is related to the Jacobi of changing integration variables and the way we have taken the residues.

Finally we arrive

$$I_3 = \int \frac{d^4L_3}{(2\pi)^3} \delta^+(L_3^2 - m_3^2)\tilde{f}(L_3)$$
$$= \frac{\pi}{2(2\pi)^3} \int_{x_0}^{x_1} dx (K^2)^2 \sqrt{x^2 - x_0^2} \int \langle\lambda|d\lambda\rangle[\tilde{\lambda}|d\tilde{\lambda}] \times \frac{\tilde{f}(\lambda, \tilde{\lambda}, t)}{\langle\lambda|K|\tilde{\lambda}]^2},$$
$$t = \frac{K^2\sqrt{x^2 - x_0^2}}{\langle\lambda|K|\tilde{\lambda}]}. \tag{2.22}$$

This is our key setup in this paper. Notice that when $m_1 = m_2 = m_3 = 0$, $x_0 = 0$, so Eq. (2.22) reduces to

$$I_3 = \int \frac{d^4L_3}{(2\pi)^3} \delta^+(L_3^2)\tilde{f}(L_3)$$
$$= \frac{\pi}{2(2\pi)^3} \int_0^1 dx (K^2)^2 x \int \langle\lambda|d\lambda\rangle[\tilde{\lambda}|d\tilde{\lambda}] \frac{\tilde{f}(\lambda, \tilde{\lambda}, t)}{\langle\lambda|K|\tilde{\lambda}]^2},$$
$$t = \frac{K^2 x}{\langle\lambda|K|\tilde{\lambda}]}, \tag{2.23}$$

which is the familiar massless case presented in [15].

In the end, let us give a remark. The integration region of $x \in [x_0, x_1]$ depends on the dynamical momentum K as well as mass parameters m_3 and m_{total}. Because of this, the roles of m_3 and m_{total} are not obviously symmetric. If we define

$$x = \sqrt{x_0^2 + \Lambda^2 u^2}, \tag{2.24}$$

the integration region of u will be $u \in [0, 1]$ which does not depend on external momenta and masses anymore. Under this transformation we have

$$I_3 = \frac{\pi}{2(2\pi)^3} \int_0^1 du (K^2)^2 \frac{\Lambda^3 u^2}{\sqrt{x_0^2 + \Lambda^2 u^2}} \int \langle\lambda|d\lambda\rangle[\tilde{\lambda}|d\tilde{\lambda}] \times \frac{\tilde{f}(\lambda, \tilde{\lambda}, t)}{\langle\lambda|K|\tilde{\lambda}]^2},$$
$$t = \frac{K^2\Lambda u}{\langle\lambda|K|\tilde{\lambda}]}, \tag{2.25}$$

and

$$L_3 = \frac{K^2\Lambda u}{\langle\lambda|K|\tilde{\lambda}]} \lambda\tilde{\lambda} + \left(\frac{\sqrt{x_0^2 + \Lambda^2 u^2}}{2} - \frac{\Lambda u}{2}\right)K. \tag{2.26}$$

This transformation will become even simpler when $m_3 = 0$ where we get just a linear transformation $x = \Lambda u$. Although Eq. (2.25) may look simpler, for some calculations we find that Eq. (2.25) is, in general, not better than Eq. (2.22) and readers may use whichever one they like. In the later part of this paper, we will use the form of Eq. (2.22).

III. SIMPLE EXAMPLES

In this section, we present some very simple examples to demonstrate our method, especially the integration region of x. We denote the physical phase space integration of n outgoing particles as $I_n^{s \text{ or } m}(f; K)$, where s stands for the spinor method and m the momentum method. The K is the sum of momenta of these n particles and f is a general function.

A. The pure phase space integration with two outgoing particles

This integral can be performed directly by the spinor method as we have analyzed in last section.

Spinor integration method: The integration is given by

$$I_2^s(1; K) = \int \frac{d^4L_2}{(2\pi)^3} \frac{d^4L_1}{(2\pi)^3} \delta^+(L_2^2 - m_2^2) \times \delta^+(L_1^2 - m_1^2)(2\pi)^4\delta^4(K - L_2 - L_1) \tag{3.1}$$

$$= \frac{1}{(2\pi)^2} \int d^4L_1 \delta^+(L_1^2 - m_1^2)\delta^+((K - L_1)^2 - m_2^2). \tag{3.2}$$

According to Eq. (2.7), one gets

$$I_2^s(1;K) = \frac{\pi}{2(2\pi)^2}\int\langle\lambda\lambda\rangle[\tilde\lambda\,\tilde\lambda]\frac{(1-2z)K^2+m_1^2-m_2^2}{\langle\lambda|K|\tilde\lambda]^2} = \frac{1}{(2\pi)^2}\frac{\pi}{2}\beta = \frac{1}{2(2\pi)^2}\frac{\sqrt{(K^2-m_1^2-m_2^2)^2-4m_1^2m_2^2}}{K^2}, \quad (3.3)$$

which is obviously symmetric between m_1, m_2.

Momentum integration method: It is given by

$$I_2^m(1;K) = \int\frac{dL_1^3}{(2\pi)^3 2E_1}\frac{dL_2^3}{(2\pi)^3 2E_2}(2\pi)^4\delta^4(K-L_2-L_1). \quad (3.4)$$

Taking the center-of-mass frame, where $K=(E,0,0,0)$ and $L_1=(E_1,k_1,0,0)$, yields

$$I_2^m(1;K) = \frac{1}{(2\pi)^2}\int\frac{k_1^2dk_1}{2E_2}d\Omega\,\delta^+(E^2-2EE_1+m_1^2-m_2^2) = \frac{1}{(2\pi)^2}\frac{\pi}{2}\beta. \quad (3.5)$$

B. The pure phase space integration with three outgoing particles

Spinor integration method: From Eq. (2.12) with the result $I_2^s(1;K)$ in the previous subsection we have

$$I_3^s(1,K) = \int\frac{d^4L_3}{(2\pi)^3}\delta^+(L_3^2-m_3^2)I_2^s(1;K-L_3) \quad (3.6)$$

$$= \frac{\pi}{2(2\pi)^3}\int_{x_0}^{x_1}dx(K^2)^2\sqrt{x^2-x_0^2}\int\langle\lambda|d\lambda\rangle[\tilde\lambda|d\tilde\lambda] \times\frac{I_2^s(1;K-L_3)}{\langle\lambda|K|\tilde\lambda]^2}, \quad (3.7)$$

where $I_2^s(1;K-L_3)$ depends on $(K-L_3)^2$ only. But using $L_3^2-m_3^2=0$ and $(xK-L_3)^2-m_3^2=0$, we can find

$$(K-L_3)^2 = (1-x)K^2+m_3^2, \quad (3.8)$$

which does not depend on λ, $\tilde\lambda$ at all. Thus

$$I_3^s(1,K) = \frac{\pi}{2(2\pi)^3}\int_{x_0}^{x_1}dxK^2\sqrt{x^2-x_0^2}I_2^s(1;K-L_3), \quad (3.9)$$

where x, x_1 is given by (2.17). This expression is obviously symmetric between m_1, m_2, but not for m_3. However, it is easy to check numerically that the final result is indeed symmetric among all the mass parameters.

Momentum integration method: The integration is

$$I_3^m(1;K) = \int\frac{d^3L_1}{(2\pi)^3 2E_1}\frac{d^3L_2}{(2\pi)^3 2E_2}\frac{d^3L_3}{(2\pi)^3 2E_3}\times(2\pi)^4\delta^4(K-L_1-L_2-L_3) \quad (3.10)$$

$$= \int\frac{d^3L_3}{(2\pi)^3 2E_3}I_2^m(1;K-L_3). \quad (3.11)$$

In the center-of-mass frame, $K=(E,0,0,0)$, $L_3=(E_3,p,0,0)$ with $E_3^2-p^2=m_3^2$, thus $(E-E_3)^2-p^2\geq(m_1+m_2)^2$, i.e., $E_3\leq(E^2+m_3^2-(m_1+m_2)^2/2E)$. Namely, the integration region of E_3 is

$$m_3\leq E_3\leq\frac{E^2+m_3^2-(m_1+m_2)^2}{2E}. \quad (3.12)$$

Using this we have

$$I_3^m(1;K) = \frac{1}{(2\pi)^2}\int dE_3\sqrt{E_3^2-m_3^2}I_2^m(1;K-L_3). \quad (3.13)$$

In order to show this is identical to Eq. (3.9), we can make a transformation $2E_3/E\to x$. Then

$$m_3\leq E_3\leq\frac{E^2+m_3^2-(m_1+m_2)^2}{2E}\to x_0\leq x\leq x_1, \qquad 2E\sqrt{E_3^2-m_3^2}\to K^2\sqrt{x^2-x_0^2}. \quad (3.14)$$

It is obvious that $I_3^m(1;K)=I_3^s(1,K)$.

C. The pure phase space integration with four outgoing particles

Here we will only present the expression using the spinor method. The pure phase space is

$$I_4^s(1;K) = \int\frac{d^4L_4}{(2\pi)^3}\delta^+(L_4^2-m_4^2)I_3^s(1;K-L_4). \quad (3.15)$$

Using the recursive method we get

$$I_4^s(1;K) = \frac{1}{64(2\pi)^5}\int_{x_0^{(4)}}^{x_1^{(4)}}dx^{(4)}K^2\sqrt{(x^{(4)})^2-(x_0^{(4)})^2}\times\int_{x_0^{(3)}}^{x_1^{(3)}}dx^{(3)}(K-L_4)^2\sqrt{(x^{(3)})^2-(x_0^{(3)})^2}\times I_2^s(1;K-L_3-L_4), \quad (3.16)$$

where naively we have following boundary values:

$$x_0^{(3)} = \sqrt{\frac{4m_3^2}{(K-L_4)^2}}, \quad x_1^{(3)} = \frac{(K-L_4)^2 + m_3^2 - (m_1+m_2)^2}{(K-L_4)^2}. \tag{3.17}$$

However, similarly to Eq. (3.8), we can find that

$$\begin{aligned}(K-L_4)^2 &= (1-x^{(4)})K^2 + m_4^2\\ (K-L_4-L_3)^2 &= (1-x^{(3)})(K-L_4)^2 + m_3^2\\ &= (1-x^{(3)})(1-x^{(4)})K^2\\ &\quad + (1-x^{(3)})m_4^2 + m_3^2.\end{aligned} \tag{3.18}$$

Thus we have

$$\begin{aligned}x_0^{(4)} &= \sqrt{\frac{4m_4^2}{K^2}},\\ x_1^{(4)} &= \frac{K^2 + m_4^2 - (m_1+m_2+m_3)^2}{K^2}\\ x_0^{(3)} &= \sqrt{\frac{4m_3^2}{(1-x^{(4)})K^2 + m_4^2}},\\ x_1^{(3)} &= \frac{(1-x^{(4)})K^2 + m_4^2 + m_3^2 - (m_1+m_2)^2}{(1-x^{(4)})K^2 + m_4^2}.\end{aligned} \tag{3.19}$$

Putting (3.18) and (3.19) into (3.16), we get the analytic expression for the pure phase space of four arbitrary massive particles

$$\begin{aligned}I_4^s(1;K) &= \frac{1}{64(2\pi)^5}\int_{x_0^{(4)}}^{x_1^{(4)}} dx^{(4)}\sqrt{K^2((x^{(4)})^2K^2 - 4m_4^2)}\int_{x_0^{(3)}}^{x_1^{(3)}} dx^{(3)}\\ &\quad\times\frac{[((1-x^{(4)})K^2 + m_4^2)((x^{(3)})^2((1-x^{(4)})K^2 + m_4^2) - 4m_3^2)]^{1/2}}{(1-x^{(3)})(1-x^{(4)})K^2 + (1-x^{(3)})m_4^2 + m_3^2}\\ &\quad\times[((1-x^{(3)})(1-x^{(4)})K^2 + (1-x^{(3)})m_4^2 + m_3^2 - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2]^{1/2}.\end{aligned} \tag{3.20}$$

The expression is not obviously symmetric among (m_1, m_2, m_3, m_4) by our choice of the order of integrations. However, it is easy to check by the numerical method that the final result is indeed symmetric among all masses.

D. The phase space integration of three outgoing particles with $f = (2L_1 \cdot L_2)(2L_1 \cdot L_3)$

Here we calculate a relatively complicated example with $f = (2L_1 \cdot L_2)(2L_1 \cdot L_3)$.

Spinor integration method: The integration can be directly written as

$$I_3^s(f;K) = \int \frac{d^4L_3}{(2\pi)^3}\delta^+(L_3^2 - m_3^2)I_2^s(f;K'), \quad K' = K - L_3, \tag{3.21}$$

where using the momentum conservation, f can be written as

$$f = (2L_1\cdot K' - 2m_1^2)(2L_1\cdot(K-K')). \tag{3.22}$$

Using the Eq. (2.10) with K replaced by K', we can simplify f further as

$$f = (\alpha K'^2 - 2m_1^2)\times\left(\beta K'^2\frac{\langle\lambda|K|\tilde\lambda]}{\langle\lambda|K'|\tilde\lambda]} + (\alpha-\beta)K'\cdot K - \alpha K'^2\right), \tag{3.23}$$

where

$$\alpha = \frac{K'^2 + m_1^2 - m_2^2}{K'^2}, \qquad \beta = \frac{\sqrt{\Delta[K', m_1, m_2]}}{K'^2}. \tag{3.24}$$

Now we calculate $I_2^s(f;K')$ using the simplified version f. It is given by

$$I_2^s(f;K') = \frac{\pi}{2(2\pi)^2}\alpha\beta(K'^2 - m_1^2 - m_2^2)(K'\cdot K - K'^2). \tag{3.25}$$

When we put it back into $I_3^s(f;K)$, we need to know that

$$K'^2 = (1-x)K^2 + m_3^2$$
$$K'\cdot K = K^2 - zK^2 - \frac{1}{2}t\langle\lambda|K|\tilde\lambda] = K^2 - \frac{1}{2}xK^2,$$

where we have used the relation $t = \frac{(x-2z)K^2}{\langle\lambda|K|\tilde\lambda]}$. This means $I_2^s(f;K')$ does not contain explicitly λ and $\tilde\lambda$ for the spinor integration over L_3. So we get immediately

$$I_3^s(f;K) = \frac{1}{16(2\pi)^3}\int_{x_0}^{x_1} dx K^2\sqrt{x^2 - x_0^2}\times\alpha\beta((1-x)K^2 + m_3^2 - m_1^2 - m_2^2)\times\left(\frac{1}{2}xK^2 - m_3^2\right), \tag{3.26}$$

where x_0, x_1 and α, β are given by (2.17) and (3.24) respectively.

Momentum integration method: The integration is

$$I_3^m(f;K) = \int\frac{d^3L_3}{(2\pi)^3 2E_3}I_2^m(f;K'). \tag{3.27}$$

$I_2^m(f;K')$ can be calculated as follows:

$$I_2^m(f;K') = \frac{1}{(2\pi)^2}\int \frac{d^3L_1}{2E_1}\delta^+((K'-L_1)^2 - m_2^2) \times (2L_1\cdot K' - 2m_1^2)(2L_1\cdot(K-K')). \quad (3.28)$$

Choose a center-of-mass frame, such that $K' = (E', 0, 0, 0)$, $L_1 = (E_1, 0, 0, k_1)$ with $E_1^2 - k_1^2 = m_1^2$ and $K = (E, \vec{p})$. The angle between $\vec{p}$ and $\vec{L}_1$ is θ. Then

$$\begin{aligned} I_2^m(f;K') &= \frac{1}{(2\pi)^2}\int \frac{k_1^2 dk_1}{2E_1}\int_{-1}^{1} dy \int_0^{2\pi} d\varphi \delta^+(E'^2 - 2E'E_1 + m_1^2 - m_2^2) \times (2E_1E' - 2m_1^2)(2(EE_1 - ypk_1) - 2E_1E') \\ &= \frac{1}{2\pi}\int k_1 dE_1 \delta^+(E'^2 - 2E'E_1 + m_1^2 - m_2^2)(2E_1E' - 2m_1^2)(2EE_1 - 2E_1E') \\ &= \frac{1}{8\pi}\frac{E'^2 + m_1^2 - m_2^2}{E'}\frac{\sqrt{\Delta[E', m_1, m_2]}}{E'^2}(E'^2 - m_1^2 - m_2^2)(E - E'). \end{aligned} \quad (3.29)$$

This is identical to $I_2^s(f;K')$ in the center-of-mass frame. To calculate $I_3^m(f;K)$ simply, we need to choose the center-of-mass frame of K which is not the one we have used for $I_2^m(f;K')$, thus we need to write $I_2^m(f;K')$ as the Lorentz-invariant form, which is not so straightforward sometimes.

Here we use the Lorentz-invariant form $I_2^s(f;K')$ given by the spinor method to go further. Taking the center-of-mass frame where $K = (E, 0, 0, 0)$, then

$$I_3^m(f;K) = \frac{1}{4(2\pi)^3}\int dE_3\sqrt{E_3^2 - m_3^2}\frac{E^2 - 2EE_3 + m_3^2 + m_1^2 - m_2^2}{(E^2 - 2EE_3 + m_3^2)^2}\sqrt{\Delta[E', m_1, m_2]} \times (E^2 - 2EE_3 + m_3^2 - m_1^2 - m_2^2)(EE_3 - m_3^2). \quad (3.30)$$

It is equal to $I_3^s(f;K)$, which can be easily checked by making a transform $2E_3/E \to x$ as in Sec. III B.

IV. PRACTICAL APPLICATIONS

In our previous section we have done some simple examples. However, these examples do not involve the real amplitudes. In this section we will discuss the phase space integration of two simple real physics processes with three outgoing particles. These two examples are presented in the following two references: [19,20].

A. Z^0 decays into lepton pairs and spin-0 bosons

This example discusses the decay reaction

$$Z^0 \to l^+ l^- H, \quad (4.1)$$

where l stands for the electron or muon with $m_l = 0$ and H for the Higgs boson with $m_H \neq 0$. The invariant matrix element squared is given by Eq. 2.10a in [19]. According to the Glashow-Weinberg-Salam model (Eq. 3.1 and Eq. 3.2 in [19]), the matrix element squared can be written as

$$\bar{\sum_{\text{pol}}}|M|^2 = \frac{cM_Z^2}{(Q^2 - M_Z^2)^2}[Q^2M_Z^2 + 4(k\cdot l_+)(k\cdot l_-)],$$
$$c = \frac{2}{3}(a^2 + b^2)B_1^2, \quad (4.2)$$

where k is the total momentum of Z^0, M_Z the mass of Z^0, and

$$a = \frac{g}{\cos\theta_W}\left(\frac{1}{4} - \sin^2\theta_W\right), \qquad b = -\frac{g}{4\cos\theta_W},$$
$$B_1 = \frac{g}{M_Z\cos\theta_W}, \qquad Q = l_+ + l_-.$$

We evaluate this by first evaluating the phase space integration over l_+ and l_H. From Eq. (3.3), we can easily get

$$\begin{aligned} I_2^s\left(\bar{\sum_{\text{pol}}}; k - l_-\right) &= \frac{\pi}{2(2\pi)^2}\int \langle\lambda d\lambda\rangle[\tilde{\lambda}d\tilde{\lambda}]\frac{(k-l_-)^2 - M_H^2}{\langle\lambda|k - l_-|\tilde{\lambda}]^2} \\ &\quad \times \frac{cM_Z^2}{(t\langle\lambda|l_-|\tilde{\lambda}] - M_Z^2)^2} \\ &\quad \times (t\langle\lambda|l_-|\tilde{\lambda}]M_Z^2 + 2(k\cdot l_-)t\langle\lambda|k|\tilde{\lambda}]) \\ &= \frac{c\pi}{2(2\pi)^2}\int \langle\lambda d\lambda\rangle[\tilde{\lambda}d\tilde{\lambda}] \\ &\quad \times \frac{((k-l_-)^2 - M_H^2)^2}{\langle\lambda|P_1|\tilde{\lambda}]^2\langle\lambda|P_2|\tilde{\lambda}]}\langle\lambda|R|\tilde{\lambda}], \end{aligned} \quad (4.3)$$

where

$$P_1 = (k - l_-) - \frac{(k-l_-)^2 - M_H^2}{M_Z^2}l_-, \qquad P_2 = k - l_-$$
$$R = l_- + \frac{2(k\cdot l_-)}{M_Z^2}k.$$

Introducing one Feynman parameter we can continue to

$$I_2^s\Big(\bar{\sum_{\rm pol}};k-l_-\Big)=\frac{c\pi}{2(2\pi)^2}\int_0^1 dy\int\langle\lambda d\lambda\rangle[\tilde\lambda d\tilde\lambda]\times((k-l_-)^2-M_H^2)^2\frac{2y\langle\lambda|R|\tilde\lambda]}{\langle\lambda|P|\tilde\lambda]^3}=\frac{c\pi}{2(2\pi)^2}\int_0^1 dy((k-l_-)^2-M_H^2)^2\times\frac{y2P\cdot R}{(P^2)^2},\quad(4.4)$$

where

$$2P\cdot R=2k\cdot l_-+\frac{2(k\cdot l_-)}{M_Z^2}(sy+u+k^2)$$
$$P^2=sy+u,$$
$$s=\frac{1}{M_Z^2}((2k\cdot l_-)^2-(k^2-M_H^2)(2k\cdot l_-)),$$
$$u=k^2-2k\cdot l_-.$$

Now we put (4.4) into

$$I_3^s\Big(\bar{\sum_{\rm pol}};k\Big)=\int\frac{d^4l_-}{(2\pi)^3}\delta^+(l_-^2)I_2^s\Big(\bar{\sum_{\rm pol}};k-l_-\Big).\quad(4.5)$$

To continue, we exchange the integration order of $\int dy$ and $\int d^4l_-$. Using $2k\cdot l_-=t\langle\lambda|k|\tilde\lambda]=k^2x$ and performing the spinor integration, we finally arrive at

$$I_3\Big(\bar{\sum_{\rm pol}};k\Big)=\frac{c}{16(2\pi)^3}\int_0^{(k^2-M_H^2)/k^2}dx(k^2x)^2\times\int_0^1 dy(k^2-k^2x-M_H^2)^2\times\frac{y(1+\frac{1}{M_Z^2}(sy+u+k^2))}{(sy+u)^2},\quad(4.6)$$

where

$$s=-\frac{k^2x}{M_Z^2}(k^2(1-x)-M_H^2),\qquad u=k^2(1-x).$$

We can perform the integral over y to yield

$$I_3\Big(\bar{\sum_{\rm pol}};k\Big)=\frac{c}{16(2\pi)^3}\int_0^{(k^2-M_H^2)/k^2}dxM_Z^2\Big((k^2x+M_Z^2)\ln\Big(1-\frac{x(k^2(1-x)-M_H^2)}{(1-x)M_Z^2}\Big)-\frac{(1-x)M_Z^2(k^2+M_Z^2)}{(1-x)(k^2x-M_Z^2)-xM_H^2}+\frac{k^2x(k^2(x-1)+M_H^2)}{M_Z^2}-k^2-M_Z^2\Big),\quad(4.7)$$

which can be integrated further to get analytic expression if one wants. Notice that $k^2=M_Z^2$, Eq. (4.7) can be simplified further as

$$I_3\Big(\bar{\sum_{\rm pol}};k\Big)=\frac{2M_Z^4}{384(\pi)^3}(a^2+b^2)\frac{g^2}{M_Z^2\cos^2\theta_W}\int_0^{1-(M_H^2/M_Z^2)}dx\Big((x+1)\ln\frac{1+x^2+x(\frac{M_H^2}{M_Z^2}-2)}{1-x}+\frac{2(1-x)}{1+x^2+x(\frac{M_H^2}{M_Z^2}-2)}+x^2+x\Big(\frac{M_H^2}{M_Z^2}-1\Big)-2\Big).\quad(4.8)$$

Taking the normalization factor $(2M_Z)^{-1}$ in the calculation of the cross section into account, we find that the integrand of Eq. (4.8) is just Eq. 3.4 in [19] by verifying $x=x_-$ and $\delta=M_H/M_Z$ in the center-of-mass frame.

B. The production of Higgs bosons in $p\bar{p}$ collisions

For the second real example, we consider the quark-antiquark-annihilation mechanism $q\bar{q}\to f\bar{f}H^0$ in [20]. The corresponding cross section and the matrix element squared are, respectively, given by Eq. 2.2 and Eq. 2.3 in [20]. We write the cross section as

$$I_3^s(H^{\mu\nu}q^{\mu\nu};Q)=c\int d^4k\delta^+(k^2-m_f^2)\int d^4\bar{k}\delta^+(\bar{k}^2-m_f^2)\times\int d^4h\delta^+(h^2-m_H^2)\times\delta^4(Q-k-\bar{k}-h)H^{\mu\nu}q^{\mu\nu}=c\int d^4k\delta^+(k^2-m_f^2)I_2^s(H^{\mu\nu}q^{\mu\nu};Q'),$$
$$Q'=Q-k,\quad(4.9)$$

where $k(\bar{k})$ and h are, respectively, the momentum of the heavy quark (antiquark) $f(\bar{f})$ and the Higgs. Q is the total momentum. We have absorbed all the common constant factors including the π factor into c. $H^{\mu\nu}q^{\mu\nu}$ is given by

$$\begin{aligned} H^{\mu\nu}q^{\mu\nu} = {}& \frac{32}{(2h\cdot\bar{k}+m_H^2)(2h\cdot k+m_H^2)}\bigg\{Q^2(Q\cdot h)^2\bigg[1+\frac{(4m_f^2-m_H^2)Q^2}{(2h\cdot\bar{k}+m_H^2)(2h\cdot k+m_H^2)}\bigg] \\ &+\bigg[(Q^2+m_H^2-4m_f^2)+\frac{2Q\cdot h(4m_f^2-m_H^2)}{(2h\cdot\bar{k}+m_H^2)}\bigg]\bigg[\frac{Q^2}{2}m_f^2-2k\cdot qk\cdot\bar{q}\bigg] \\ &+\bigg[(Q^2+m_H^2-4m_f^2)+\frac{2Q\cdot h(4m_f^2-m_H^2)}{(2h\cdot k+m_H^2)}\bigg]\bigg[\frac{Q^2}{2}m_f^2-2\bar{k}\cdot q\bar{k}\cdot\bar{q}\bigg] \\ &-(Q^2+m_H^2-4m_f^2)[2k\cdot q\bar{k}\cdot\bar{q}+2k\cdot\bar{q}\,\bar{k}\cdot q-Q^2k\cdot\bar{k}]\bigg\}. \end{aligned} \tag{4.10}$$

Notice that

$$2h\cdot\bar{k}+m_H^2=Q^2-2Q\cdot\bar{k}\qquad 2h\cdot k+m_H^2=Q^2-2Q\cdot k\qquad 2Q\cdot h=(Q^2-2Q\cdot k)+(Q^2-2Q\cdot\bar{k})$$
$$2k\cdot\bar{k}=(Q^2+m_H^2-2m_f^2)-(Q^2-2Q\cdot k)-(Q^2-2Q\cdot\bar{k}).$$

To simplify the calculation, we rearrange $H^{\mu\nu}q^{\mu\nu}$ as

$$\begin{aligned} H^{\mu\nu}q^{\mu\nu} = {}& 32\bigg\{\frac{1}{4}Q^2\bigg[\frac{Q^2-2Q\cdot k}{Q^2-2Q\cdot\bar{k}}+2+\frac{Q^2-2Q\cdot\bar{k}}{Q^2-2Q\cdot k}\bigg] \\ &+\frac{1}{4}(4m_f^2-m_H^2)(Q^2)^2\bigg[\frac{1}{(Q^2-2Q\cdot\bar{k})^2}+\frac{2}{(Q^2-2Q\cdot\bar{k})(Q^2-2Q\cdot k)}+\frac{1}{(Q^2-2Q\cdot k)^2}\bigg] \\ &+\bigg[\frac{Q^2}{(Q^2-2Q\cdot\bar{k})(Q^2-2Q\cdot k)}+\frac{4m_f^2-m_H^2}{(Q^2-2Q\cdot\bar{k})^2}\bigg][Q^2m_f^2-2k\cdot q2k\cdot\bar{q}] \\ &-\frac{Q^2+m_H^2-4m_f^2}{(Q^2-2Q\cdot\bar{k})(Q^2-2Q\cdot k)}\bigg[2k\cdot q2\bar{k}\cdot\bar{q}-\frac{1}{2}Q^2((Q^2+m_H^2-2m_f^2)-(Q^2-2Q\cdot k))\bigg] \\ &-\frac{(Q^2+m_H^2-4m_f^2)Q^2}{2(Q^2-2Q\cdot k)}\bigg\}, \end{aligned} \tag{4.11}$$

where we have used the symmetry between k and $\bar{k}$.

Now we can start the calculation. First we will perform the phase space integration over $\bar{k}$ and h. Then we perform the left k integration.

1. The integration $I_2^s(H^{\mu\nu}q^{\mu\nu};Q')$

First we simplify the input according to Eq. (2.10), i.e.,

$$\bar{k}=\frac{Q'^2}{\langle\lambda|Q'|\tilde{\lambda}]}\bigg[\beta\bigg(P_{\lambda\tilde{\lambda}}-\frac{Q'\cdot P_{\lambda\tilde{\lambda}}}{Q'^2}Q'\bigg)+\alpha\frac{Q'\cdot P_{\lambda\tilde{\lambda}}}{Q'^2}Q'\bigg],\qquad Q^2-2Q\cdot\bar{k}=\frac{Q'^2}{\langle\lambda|Q'|\tilde{\lambda}]}\langle\lambda|P_1|\tilde{\lambda}], \tag{4.12}$$

with

$$P_1=-\beta\bigg(Q-\frac{Q'\cdot Q}{Q'^2}Q'\bigg)-\bigg(\alpha\frac{Q'\cdot Q}{Q'^2}-\frac{Q^2}{Q'^2}\bigg)Q'=-\beta Q_1+\alpha_1Q'.$$

By checking Eq. (4.11), we find that there are four nontrivial integrations that should be attacked. We do them one by one.

Type I: $f_1=1/(Q^2-2Q\cdot\bar{k})$.

The integration is

$$I_2^s(f_1;Q') = \int\langle\lambda d\lambda\rangle[\tilde\lambda d\tilde\lambda]\frac{\beta\alpha_1}{\langle\lambda|\alpha_1 Q'|\tilde\lambda]\langle\lambda|P_1|\tilde\lambda]} = \int_0^1 dy\int\langle\lambda d\lambda\rangle[\tilde\lambda d\tilde\lambda]\frac{\beta\alpha_1}{\langle\lambda|yP_1+(1-y)\alpha_1 Q'|\tilde\lambda]^2}$$
$$= \int_0^1 dy\frac{4\beta\alpha_1 Q'^2}{-y^2\beta^2\Sigma+4\alpha_1^2(Q'^2)^2}, \tag{4.13}$$

where

$$\Sigma = (2Q\cdot Q')^2 - 4Q^2Q'^2.$$

Type II: $f_2 = Q^2 - 2Q\cdot\bar k$.
The integral is

$$I_2^s(f_2;Q') = \int\langle\lambda d\lambda\rangle[\tilde\lambda d\tilde\lambda]\frac{\beta Q'^2}{\langle\lambda|Q'|\tilde\lambda]^2}\frac{Q'^2}{\langle\lambda|Q'|\tilde\lambda]}\langle\lambda|P_1|\tilde\lambda] = \beta Q'\cdot P_1 = \beta\alpha_1 Q'^2. \tag{4.14}$$

Type III: $f_3 = 1/(Q^2-2Q\cdot\bar k)^2$.

$$I_2^s(f_3;Q') = \int\langle\lambda d\lambda\rangle[\tilde\lambda d\tilde\lambda]\frac{\beta Q'^2}{\langle\lambda|Q'|\tilde\lambda]^2}\frac{1}{(Q^2-2Q\cdot\bar k)^2} \tag{4.15}$$

$$= \int\langle\lambda d\lambda\rangle[\tilde\lambda d\tilde\lambda]\frac{\beta/Q'^2}{\langle\lambda|P_1|\tilde\lambda]^2} \tag{4.16}$$

$$= \frac{4\beta}{-\beta^2\Sigma+4\alpha_1^2(Q'^2)^2}. \tag{4.17}$$

Type IV: $f_4 = 2\bar k\cdot\bar q/(Q^2-2Q\cdot\bar k)$.
Using

$$2\bar k\cdot\bar q = \frac{Q'^2}{\langle\lambda|Q'|\tilde\lambda]}\langle\lambda|P_2|\tilde\lambda], \qquad P_2 = \beta\left(\bar q - \frac{Q'\cdot\bar q}{Q'^2}Q'\right) + \alpha\frac{Q'\cdot\bar q}{Q'^2}Q', \tag{4.18}$$

we get

$$I_2^s(f_3;Q') = \int\langle\lambda d\lambda\rangle[\tilde\lambda d\tilde\lambda]\frac{\beta Q'^2}{\langle\lambda|Q'|\tilde\lambda]^2}\frac{\langle\lambda|P_2|\tilde\lambda]}{\langle\lambda|P_1|\tilde\lambda]} \tag{4.19}$$

$$= \int_0^1 dy\int\langle\lambda d\lambda\rangle[\tilde\lambda d\tilde\lambda]\frac{2\beta\alpha_1^2 Q'^2(1-y)\langle\lambda|P_2|\tilde\lambda]}{\langle\lambda|yP_1+(1-y)\alpha_1 Q'|\tilde\lambda]^3} \tag{4.20}$$

$$= \int_0^1 dy\frac{\beta\alpha_1^2 Q'^2(1-y)[(-y\beta^2(1-\frac{Q'\cdot Q}{Q'^2})+\alpha_1\alpha)2Q\cdot\bar q - (y\beta^2\frac{Q'\cdot Q}{Q'^2}+\alpha_1\alpha)2k\cdot\bar q]}{(-y^2\beta^2\frac{\Sigma}{4Q'^2}+\alpha_1^2 Q'^2)^2}. \tag{4.21}$$

Now substituting above four types of integrations into $I_2^s(H^{\mu\nu}q^{\mu\nu};Q')$ and with some algebraic manipulation, we can get

$$\frac{1}{32}I_2^s(H^{\mu\nu}q^{\mu\nu};Q') = \Big[\frac{1}{4}Q^2(Q^2-2Q\cdot k)+\frac{1}{2}(Q^2)^2\frac{6m_f^2-m_H^2}{Q^2-2Q\cdot k}-\frac{Q^2 2k\cdot q2k\cdot\bar{q}}{Q^2-2Q\cdot k}-\frac{1}{2}Q^2(Q^2+m_H^2-4m_f^2)$$
$$+\frac{1}{2}Q^2\frac{(Q^2+m_H^2-4m_f^2)(Q^2+m_H^2-2m_f^2)}{Q^2-2Q\cdot k}\Big]\int_0^1 dy\frac{4\beta\alpha_1 Q'^2}{-y^2\beta^2\Sigma+4\alpha_1^2(Q'^2)^2}$$
$$-\frac{(Q^2+m_H^2-4m_f^2)(2k\cdot q)}{Q^2-2Q\cdot k}\int_0^1 dy\beta\alpha_1^2Q'^2(1-y)$$
$$\times\frac{[(-y\beta^2(1-\frac{Q'\cdot Q}{Q'^2})+\alpha_1\alpha)(2Q\cdot\bar{q})-(y\beta^2\frac{Q'\cdot Q}{Q'^2}+\alpha_1\alpha)(2k\cdot\bar{q})]}{(-y^2\beta^2\frac{\Sigma}{4Q'^2}+\alpha_1^2Q'^2)^2}$$
$$+\frac{\beta(4m_f^2-m_H^2)}{-\beta^2\Sigma+4\alpha_1^2(Q'^2)^2}[(Q^2)^2+4(Q^2m_f^2-2k\cdot q2k\cdot\bar{q})]+\frac{(4m_f^2-m_H^2)\beta(Q^2)^2}{4(Q^2-2Q\cdot k)^2}$$
$$+\frac{Q^2\beta\alpha_1Q'^2-2Q^2\beta(Q^2+m_H^2-4m_f^2)}{4(Q^2-2Q\cdot k)}+\frac{1}{2}\beta Q^2. \tag{4.22}$$

2. The integration $I_3^s(f;Q')$

Now we do the left k integration using (2.22) with

$$x_0=\sqrt{\frac{4m_f^2}{Q^2}},\qquad x_1=\frac{Q^2+m_f^2-(m_f+m_H)^2}{Q^2} \tag{4.23}$$

and the following relations:

$$Q\cdot k=\frac{x}{2}Q^2,\qquad (Q-k)^2=(1-x)Q^2+m_f^2\qquad 2k\cdot q=\frac{(x-2z_1)\langle\lambda|q|\tilde{\lambda}]Q^2}{\langle\lambda|Q|\tilde{\lambda}]}+2z_1Q\cdot q$$
$$2k\cdot\bar{q}=\frac{(x-2z_1)\langle\lambda|\bar{q}|\tilde{\lambda}]Q^2}{\langle\lambda|Q|\tilde{\lambda}]}+2z_1Q\cdot\bar{q}.$$

From $I_2^s(H^{\mu\nu}q^{\mu\nu};Q')$, the terms containing $Q\cdot k$ and $(Q-k)^2$ do not depend on k, and thus can be done easily just as in the example of the pure phase space integration with three outgoing particles. Then we need only perform the following two types of nontrivial integrations:

$$I_3^s(2\bar{q}\cdot k;Q)=\int_{x_0}^{x_1}dx(Q^2)^2\sqrt{x^2-x_0^2}\int\langle\lambda|d\lambda\rangle[\tilde{\lambda}|d\tilde{\lambda}]\frac{1}{\langle\lambda|Q|\tilde{\lambda}]^2}\Big(\frac{(x-2z_1)\langle\lambda|\bar{q}|\tilde{\lambda}]}{\langle\lambda|Q|\tilde{\lambda}]}Q^2+2z_1Q\cdot\bar{q}\Big)$$
$$=\int_{x_0}^{x_1}dx(Q^2)^2\sqrt{x^2-x_0^2}\Big[(x-2z_1)\frac{Q\cdot\bar{q}}{Q^2}+\frac{2z_1Q\cdot\bar{q}}{Q^2}\Big]=\int_{x_0}^{x_1}dxQ^2\sqrt{x^2-x_0^2}xQ\cdot\bar{q}, \tag{4.24}$$

$$I_3^s(2k\cdot q2k\cdot\bar{q};Q)=\int_{x_0}^{x_1}dx(Q^2)^2\sqrt{x^2-x_0^2}\int\langle\lambda|d\lambda\rangle[\tilde{\lambda}|d\tilde{\lambda}]\frac{1}{\langle\lambda|Q|\tilde{\lambda}]^2}\times\Big(\frac{(x-2z_1)^2\langle\lambda|q|\tilde{\lambda}]\langle\lambda|\bar{q}|\tilde{\lambda}]}{\langle\lambda|Q|\tilde{\lambda}]^2}(Q^2)^2$$
$$+2z_1Q\cdot q\frac{(x-2z_1)\langle\lambda|\bar{q}|\tilde{\lambda}]}{\langle\lambda|Q|\tilde{\lambda}]}Q^2+2z_1Q\cdot\bar{q}\frac{(x-2z_1)\langle\lambda|q|\tilde{\lambda}]}{\langle\lambda|Q|\tilde{\lambda}]}Q^2+2z_1Q\cdot\bar{q}2z_1Q\cdot q\Big)$$
$$=\int_{x_0}^{x_1}dx(Q^2)^2\sqrt{x^2-x_0^2}\Big((x-2z_1)^2\frac{(2Q\cdot\bar{q})(2Q\cdot q)-Q^2(q\cdot\bar{q})}{3Q^2}+2z_1Q\cdot q(x-2z_1)\frac{Q\cdot\bar{q}}{Q^2}$$
$$+2z_1Q\cdot\bar{q}(x-2z_1)\frac{Q\cdot q}{Q^2}+\frac{2z_1Q\cdot\bar{q}2z_1Q\cdot q}{Q^2}\Big)$$
$$=\int_{x_0}^{x_1}dxQ^2\sqrt{x^2-x_0^2}\Big(\frac{(2Q\cdot\bar{q})(2Q\cdot q)}{3Q^2}(x^2Q^2-m_f^2)-\frac{q\cdot\bar{q}}{3}(x^2Q^2-4m_f^2)\Big). \tag{4.25}$$

3. The final result

Substituting $I_2^s(H^{\mu\nu}q^{\mu\nu};Q')$, $I_3^s(2k\cdot q2k\cdot\bar{q};Q)$, and $I_3^s(2\bar{q}\cdot k;Q)$ into Eq. (4.9) yields

$$\frac{1}{32c}I_3^s(H^{\mu\nu}q^{\mu\nu};Q)=\int_{x_0}^{x_1}dx\sqrt{x^2-x_0^2}\bigg\{\frac{1}{2(1-x)}\int_0^1 dy\frac{\beta(2-2\alpha+\alpha x)}{-y^2\beta^2(x^2-x_0^2)+(2-2\alpha+\alpha x)^2}T_1-\frac{(Q^2+m_H^2-4m_f^2)}{Q^2(1-x)}\times\int_0^1 dy\frac{4\beta(1-y)(2-2\alpha+\alpha x)^2}{(-y^2\beta^2(x^2-x_0^2)+(2-2\alpha+\alpha x)^2)^2}T_2+\frac{\beta(4m_f^2-m_H^2)/Q^2}{-\beta^2(x^2-x_0^2)+(2-2\alpha+\alpha x)^2}T_3+T_4\bigg\}, \tag{4.26}$$

where

$$\begin{aligned}T_1&=(Q^2x+m_H^2-4m_f^2)^2+(Q^2+m_H^2-4m_f^2)^2+4(Q^2+m_H^2-4m_f^2)m_f^2+2Q^2(6m_f^2-m_H^2)\\&\quad+\frac{4}{3}\bigg(q\cdot\bar{q}(x^2Q^2-4m_f^2)-(2Q\cdot\bar{q})(2Q\cdot q)\bigg(x^2-\frac{m_f^2}{Q^2}\bigg)\bigg),\\T_2&=\bigg(y\beta^2\bigg(x-\frac{2m_f^2}{Q^2}\bigg)+\alpha(2-2\alpha+\alpha x)\bigg)x(Q\cdot\bar{q})(Q\cdot q)+\frac{1}{6}(y\beta^2(2-x)+\alpha(2-2\alpha+\alpha x))\\&\quad\times\bigg(q\cdot\bar{q}(x^2Q^2-4m_f^2)-(2Q\cdot\bar{q})(2Q\cdot q)\bigg(x^2-\frac{m_f^2}{Q^2}\bigg)\bigg),\\T_3&=(Q^2)^2+4Q^2m_f^2+\frac{4}{3}\bigg(q\cdot\bar{q}(x^2Q^2-4m_f^2)-(2Q\cdot\bar{q})(2Q\cdot q)\bigg(x^2-\frac{m_f^2}{Q^2}\bigg)\bigg),\\T_4&=\frac{\beta(4m_f^2-m_H^2)Q^2}{4(1-x)^2}+\frac{\beta(2-2\alpha+\alpha x)(Q^2)^2-4(Q^2+m_H^2-4m_f^2)Q^2\beta}{8(1-x)}+\frac{1}{2}\beta(Q^2)^2.\end{aligned} \tag{4.27}$$

The corresponding parameters are

$$\alpha=\frac{(1-x)Q^2+2m_f^2-m_H^2}{(1-x)Q^2+m_f^2},\qquad \beta=\frac{[((1-x)Q^2-m_H^2)^2-4m_f^2m_H^2]^{1/2}}{(1-x)Q^2+m_f^2}. \tag{4.28}$$

In Fig. 1, we display $\frac{1}{32c}I_3^s$ versus the c.m. energy $\sqrt{s}$ of the $p\bar{p}$ by the numerical method. Notice that the displayed $\frac{1}{32c}I_3^s$ is not the real cross section since the dynamical factor c given in the original reference has not been included and the real cross section is I_3^s (so the decay behavior of I_3 at high energy cannot be observed from this figure). Here, we emphasize two points. First, by our spinor method, almost all calculations have been reduced to reading out the residues of poles and making some algebraic manipulations. Thus although the analytic expression looks long, the calculation is kindly trivial.

Second, we can take an appropriate integration order to simplify the process according to the structure of the integrand. Usually we should first perform the integrations over those variables, with respect to which the structure of the integrand is relatively simple. In this example, we have leave k as the last integration variable.[3] This is because the integrand Eq. (4.11) does not contain h explicitly.

Notice that different integration ordering, i.e., integrating over p_1 first and then p_2 or integrating over p_2 first and then p_1, will in general give different-looking expressions. For example, in the expression of (3.20), we have fixed arbitrarily the ordering m_1, m_2, m_3, m_4. Different ordering will end up with different integration regions although the final result should be the same. Furthermore, if we have left one particle unintegrated while others have been integrated, then we will get the corresponding differential cross section for this particle. Thus different integration ordering

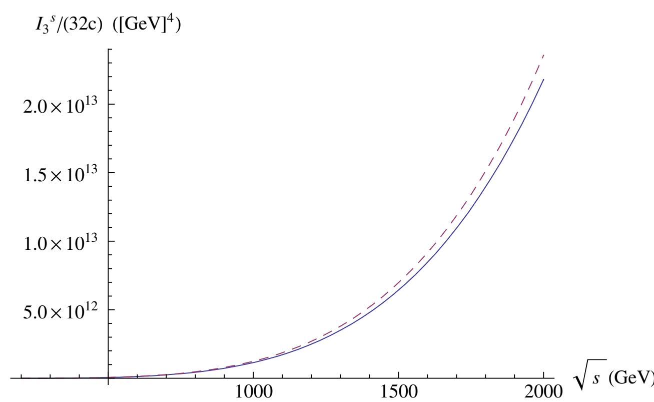

FIG. 1 (color online). $\frac{1}{32c}I_3^s$ as a function of $\sqrt{s}$ for the $p\bar{p}$ collision. The dashed and continuous curves, respectively, represent two sets of parameters: $m_H=10$ GeV, $m_f=4.5$ GeV and $m_H=30$ GeV, $m_f=35$ GeV.

[3]Because of the symmetry of k and $\bar{k}$, it is the same if we leave $\bar{k}$ as the last integration variable.

will give different differential cross sections for different particles.

V. CONCLUSION

Originating from the application of the spinor method to the massless case, in this paper, we have established the framework to process the massive case. From the examples presented in the paper, the advantages of our method is further manifested.

First, the manifestly Lorentz-invariant form of the result in each step is gotten naturally. This ensures that the recursive method can be applied conveniently especially when the number of outgoing particles is large. In this process, we do not need take any specified reference frame as when using the momentum integration method.

Second, the integration regions can be written straightforwardly according to Eq. (2.17), while with the momentum integration method, one has to pursue exhaustively the specifying of many variables (for example, angles and module variables). Note that in our method, for the massive case the region is not so simple as the massless case; it is only the functions of mass and energy.

Finally, the salient point is that the constrained three-dimensional momentum space integration is reduced to a one-dimensional integration, plus possible Feynman integrations. However, in this large simplification, we just pay a little extra price, namely, the integration over λ and $\tilde{\lambda}$ which can be obtained by reading out residues of corresponding poles.

In this paper, our new method has shown the value of practical calculations. As we have mentioned in the introduction, our method provides compact analytic expressions for the cross section. Thus we can investigate the analytic structure using these expressions. We think it is an interesting direction. Also, in this paper we have just touched the tree-level result. It is our goal to combine these analytic expressions with one-loop results to see if we can improve the current numerical next-to-leading order algorithm, especially the infrared singularity subtraction. A regularization scheme is mandatory and we need consider the general D-dimensional case. All these questions will be our future projects.

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