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Evolution of fluctuations near QCD critical point

M. A. Stephanov

Department of Physics, University of Illinois, Chicago, Illinois 60607, USA

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We propose to describe the time evolution of quasistationary fluctuations near QCD critical point by a system of stochastic Boltzmann-Langevin-Vlasov-type equations. We derive the equations and study the system analytically in the linearized regime. Known results for equilibrium stationary fluctuations as well as the critical scaling of diffusion coefficient are reproduced. We apply the approach to the long-standing question of the fate of the critical point fluctuations during the hadronic rescattering stage of the heavy-ion collision after chemical freeze-out. We find that if conserved particle number fluctuations survive the rescattering, so do, under a certain additional condition, the fluctuations of nonconserved quantities, such as mean transverse momentum. We derive a simple analytical formula for the magnitude of this memory effect.

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I. INTRODUCTION

Mapping the QCD phase diagram as a function of temperature T and baryochemical potential μ_B is one of the fundamental goals of heavy-ion collision experiments. QCD critical point is a distinct singular feature of the phase diagram. It is a ubiquitous property of QCD models based on the chiral symmetry breaking dynamics [1,2] (see [3] for review and further references). Locating the point using first-principle lattice calculations is a formidable challenge [4–8]. Recent progress and results are encouraging, but much work needs to be done to understand and constrain systematic errors (see, e.g., Refs. [9–11] and reviews [12,13] for further references and discussion).

If the critical point is located in the region accessible to heavy-ion collision experiments it can be discovered experimentally. The search for the critical point is planned at the Relativistic Heavy Ion Collider (RHIC) at BNL, the Super Proton Synchrotron (SPS) at CERN, the future Facility for Antiproton and Ion Research (FAIR) at GSI, and Nuclotron-based Ion Collider Facility (NICA) in Dubna [14–17].

The characteristic feature of a critical point is the increase and divergence of fluctuations. The nonmonotonous behavior of event-by-event fluctuations, measured in heavy-ion collisions, as a function of the initial collision energy is a signature of the QCD critical point [18,19]. The estimates of the magnitude of the fluctuations in [19] were based on the assumption of thermodynamic equilibrium, which is a reasonable first approximation at freeze-out. For such stationary fluctuations the probability of a given value of a fluctuating variable is proportional to the exponential of the entropy, i.e., to the number of microscopic states with that value of the variable [20,21].

In a dynamic environment of a heavy-ion collision, the system continuously evolves with time. As long as the evolution is slow enough compared to the typical reequilibration time, one can consider fluctuations as simply tracking the evolving equilibrium conditions. However, some fluctuating modes can be slower. In fact, it is precisely these slow modes which are of primary interest to us. These include fluctuations of conserved quantities and, most importantly, the critical fluctuations of the order parameter field σ at the critical point. Fluctuations must keep readjusting to the continuously drifting equilibrium value. Can this quasistationary dynamics of fluctuations be described quantitatively? The purpose of this paper is to achieve this.

We derive stochastic equations for the particle distribution functions as well as the critical mode using fluctuation-dissipation relation in Sec. III. We determine the corresponding equation for the correlators of the fluctuations in Sec. V and study its solution in Secs. VII, VIII, and IX. Finally, in Sec. X, as an example of the application, we answer analytically, in an idealized regime, the long-standing question of the fate of fluctuations during the hadronic rescattering phase. We discover a memory effect, which protects not only fluctuations of *conserved* quantities. Notations introduced throughout the paper are indexed in Appendix A.

II. COMPARISON WITH RELATED WORK

The time evolution of fluctuations has been considered previously using different methods and/or in different contexts. Below we review some of this work in order to point out the new ingredients as well as the results of our approach.

Quasistationary dynamics of fluctuations motivated Ref. [22]. The relaxation of the correlation length, as a proxy to the magnitude of fluctuations, was studied using a model equation. Here we shall address evolution of fluctuations on a microscopic level and directly in terms of observable quantities.

The quasistationary dynamics plays an essential role in the anomalous suppression of charge fluctuations, which has been proposed as a signature of the quark-gluon plasma formation at early times in Refs. [23,24], and more quantitatively analyzed in [25] (see also review Ref. [26]). The evolution follows a diffusion-type equation, which means that fluctuations of larger spatial extent relax slower. Therefore, as the size of the acceptance window is increased, the memory of the fluctuations goes further back in time, allowing to probe earlier stages of the fireball evolution.

The most interesting and not easily anticipated result of the approach we introduce here is the following. Although the memory effect is due to the slowness of the *conserved* charge fluctuations, the fluctuations of other quantities are also affected. For example, we show that the fluctuations of observables such as, e.g., mean transverse momentum p_T in the event also ''remember'' their earlier value. More precisely, if the chemical freeze-out (the freeze-out of inelastic reactions) has occurred near the critical point, the elastic collisions during the subsequent evolution of the fireball do not completely ''wash out'' the critical point contribution to the mean p_T fluctuations even on the time scales longer than typical collisional relaxation time. We can determine the magnitude of that effect by studying the microscopic nature and evolution of the slowest mode of fluctuations.

The evolution of fluctuations in the vicinity of the critical point has been studied numerically in Ref. [27]. The fluctuations were introduced by randomization of initial conditions, while the subsequent evolution was deterministic. The essential ingredient of the approach we introduce here is the full treatment of fluctuations. I.e., fluctuations are driven by a random external source, acting at all times. The strength of the source is determined by fluctuation-dissipation relation.

Stochastic Langevin-type equations were used to study hydrodynamic fluctuations near the critical point in [28]. However, experimental observables such as, e.g., mean p_T fluctuations, are not directly related to hydrodynamic variables. The new ingredient in the present approach is the use of kinetic Boltzmann equation. The degrees of freedom here are particle distribution functions, which directly translate into observable fluctuation measures. Strictly speaking, our approach lacks rigorous consistency of low-energy hydrodynamic description, and should be considered as only a model of the late hadronic stage of the heavy-ion collision. However, this relatively minor compromise allows us to address directly experimental fluctuation measures and make quantitative predictions, rather than limiting the study to density fluctuations. As a test of the new approach we shall derive some results of Ref. [28] in Sec. VIII B.

In a different context, the stochastic Boltzmann-Vlasov type equations have been used to estimate the rate for hot electroweak baryon number violation [29–31].

III. THE FORMALISM

To model the fireball evolving through the phase diagram near the critical point, we consider a relativistic system of particles interacting with a scalar field σ, coupled to a thermal bath at temperature T. We are considering late hadronic phase of the fireball expansion. The particle density is assumed to be already sufficiently small, so that their motion can be considered classically, using Boltzmann equation. The mass m of the scalar field, on the other hand, is considered to be sufficiently small compared to $2\pi T$, so that the field can be treated classically, using field equations. This condition is fulfilled sufficiently close to the critical point.

A. Equations of motion

Since the field σ is a Lorentz scalar, we assume that the coupling of it to the particles affects their mass (as opposed to, e.g., chemical potential, which is a Lorentz vector). As an example, one can consider coupling of nucleons $\sigma\bar{N}N$ or pions $\sigma\pi\pi$ to the σ field in the chiral sigma model. Thus we are led to consider classical motion of particles with variable mass $M(\sigma)$ [32], which depends on the local value of the scalar field σ. The action of the system is given by:

$$\mathcal{S} = \int d^3\boldsymbol{x}\frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma - U(\sigma)) - \int dsM(\sigma), \quad (1)$$

where the last integral is taken over the worldline of a particle with variable mass. The corresponding equations of motion are given by:

$$\partial^2\sigma + dU/d\sigma + \int ds(dM/d\sigma) = 0; \quad (2)$$

$$dp^\mu/d\tau = \partial^\mu M(\sigma), \quad \text{with} \quad p^\mu = Mdx^\mu/d\tau, \quad (3)$$

where $M = M(\sigma)$ is the local value of the variable particle mass. One can check that the motion governed by Eqs. (3) preserves $p^\mu p_\mu - M(\sigma)^2 = 0$ along the particle trajectory.

The Boltzmann equation for the distribution function $f(x, p)$ of such particles in the external field σ reads [33]:

$$\frac{p^\mu}{M}\frac{\partial f}{\partial x^\mu} + \partial^\mu M\frac{\partial f}{\partial p^\mu} + \mathcal{C}[f] = 0, \quad (4)$$

or in a more physically transparent, noncovariant, form:

$$\dot{f} + \boldsymbol{v}\cdot\boldsymbol{\nabla}f - (\boldsymbol{\nabla}M/\gamma)\cdot(\partial f/\partial\boldsymbol{p}) + \mathcal{C}[f]/\gamma = 0, \quad (5)$$

where

$$\boldsymbol{v} \equiv \boldsymbol{p}/(\gamma M) \quad \text{and} \quad \gamma \equiv (1 - \boldsymbol{v}^2)^{-1/2} \quad (6)$$

is the particle velocity and relativistic gamma-factor, respectively.

The collision integral $\mathcal{C}[f]$ in Eq. (4) gives the collision frequency for all particles with momentum $\boldsymbol{p}$ (near space-time point x) in the rest frame of those particles, while C/γ

is that frequency in the lab frame. Using equations of motion (3) one can show that the Boltzmann Eq. (4) (or (5)), implies continuity equation for the particle number current

$$\partial_\mu j^\mu + \int_p \mathcal{C}[f]/\gamma = 0, \quad \text{where } j^\mu \equiv \int_p f p^\mu/(M\gamma), \tag{7}$$

with

$$\int_p \equiv \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}. \tag{8}$$

I.e., particle number can only be changed (if at all) by collisions.

The equation of motion for the scalar field σ in the presence of particles with distribution $f(x, p)$, following Eq. (2), is given by

$$\partial^2\sigma + dU/d\sigma + (dM/d\sigma)\int_p f/\gamma = 0. \tag{9}$$

Coupled Eqs. (4) and (9) describe evolution of the particle distribution f and the scalar field σ. These equations are conceptually similar to Vlasov equations in electrodynamics. The difference is that the classical field σ is a Lorentz scalar. There is also certain limited similarity with the nuclear mean-field approach [34].

Our goal is to extend the above formalism to the description of fluctuations in the system. In application to linearized Boltzmann equation this has been done by Fox and Uhlenbeck and others [35–38]. Here we shall extend this formalism to linearized Boltzmann-Vlasov type coupled Eqs. (4) and (9).

B. Linearized equations

For a given constant field σ, the Boltzmann equation has a stationary solution, which is also constant in space, $f_\sigma(\boldsymbol{p})$, satisfying $\mathcal{C}[f_\sigma] = 0$. This is Boltzmann distribution for particles of mass $M(\sigma)$ at arbitrary values of temperature T and chemical potential μ:

$$f_\sigma(\boldsymbol{p}) = e^{\mu/T}e^{-\gamma(\boldsymbol{p})M/T}. \tag{10}$$

The values of T and μ depend on the total particle number (if it is conserved by collisions) and on total energy, if the system is closed, or by conditions of equilibrium with the thermal bath if it is open.

The equilibrium value of σ is determined by

$$dU/d\sigma + (dM/d\sigma)\int_p f_\sigma/\gamma = 0. \tag{11}$$

where the second term can be viewed diagrammatically as the contribution of a thermal tadpole.

We linearize the equations for σ and f by expanding around their equilibrium value. The deviation of f from its equilibrium value f_σ will be parametrized, as usual, by function h:

$$f = f_\sigma(1 + h). \tag{12}$$

The linearized Boltzmann equation then reads

$$\dot{h} - \dot{\sigma}g/(\gamma T) + \boldsymbol{v}\cdot\boldsymbol{\nabla}h + \mathcal{I}[h] = 0, \tag{13}$$

where

$$g \equiv dM/d\sigma \tag{14}$$

and $\mathcal{I}[h]$ is the linearized collision integral:

$$\mathcal{C}[f] = \gamma f_\sigma \mathcal{I}[h] + \mathcal{O}(h^2). \tag{15}$$

Note, that both $\mathcal{C}$ and $\mathcal{I}$ depend on the local value of the field σ (through the dependence of the particle mass M), and we used the property of the equilibrium distribution $\mathcal{C}[f_\sigma] = 0$.

Shifting the notation for σ so that $\sigma = 0$ is the equilibrium value (solution of Eq. (11)) we can write the linearized Eq. (9) as

$$\ddot{\sigma} - \boldsymbol{\nabla}^2\sigma + m^2\sigma + g\int_p f_0 h/\gamma = 0. \tag{16}$$

where we defined the "in-medium" mass m of the σ field quanta as

$$m^2 = m_0^2 + \frac{d}{d\sigma}\left(g\int_p f_\sigma/\gamma\right)_{\sigma=0}. \tag{17}$$

with $m_0^2 \equiv d^2U(0)/d\sigma^2$. The last term in Eq. (17) can be recognized as the one-loop thermal contribution to the vacuum mass m_0.

In the system we considered so far the dissipation (entropy increase) is entirely due to the collision term $\mathcal{C}[f]$. In a more general, and more realistic, case when the field σ interacts with other particles in a heat bath, one can describe the additional dissipation effects adding a term $\Gamma_0\dot{\sigma}$ into the left-hand side (lhs) of Eq. (16).

C. Noise and its correlators

Eqs. (5) and (9) describe evolution of functions f and σ averaged over the time scale of many particle collisions. Fluctuations of f and σ can be characterized by correlation functions (also averaged over many collisions).

In order to describe these fluctuations we follow the approach of Ref. [36] and introduce random noise terms. We shall determine the correlation functions of these noise terms following Refs. [35,36,39,40] in the linearized regime. For that purpose we shall cast equations in the following first-order form:

$$\dot{h} - \pi g/(\gamma T) + \boldsymbol{v}\cdot\boldsymbol{\nabla}h + \mathcal{I}[h] = \xi; \tag{18a}$$

$$\dot{\pi} + \Gamma_0\pi - \boldsymbol{\nabla}^2\sigma + m^2\sigma + g\int_p f_0 h/\gamma = \eta; \tag{18b}$$

$$\dot{\sigma} - \pi = 0. \tag{18c}$$

where we introduced noises ξ and η and, for generality, the additional dissipation term $\Gamma_0\pi$. It is worth pointing out that the field σ is stochastic with our without the noise η because of the coupling to the particles [last term on the lhs of Eq. (18b)].

To determine the correlators of the noises ξ and η, we use fluctuation-dissipation relation. The probability distribution of the fluctuating degrees of freedom [20,21]

$$\mathcal{P}[f,\sigma] \sim \exp S[f,\sigma] \tag{19}$$

is determined by the entropy function

$$S[f,\sigma] = \int d^3\boldsymbol{x}\Big[-\int_p f(\log f - 1) - \frac{1}{T}\Big(\frac{\pi^2}{2} + \frac{(\boldsymbol{\nabla}\sigma)^2}{2} + U(\sigma) + \int_p (M(\sigma)\gamma - \mu)f\Big)\Big], \tag{20}$$

where T is the temperature of the external heat bath. The first term is the well-known Boltzmann entropy (H-function), while the second term is simply $-(E-\mu N)/T$, where E is the energy of the system and μ is the chemical potential. This term is the contribution of external reservoir to the (fluctuations of) entropy.

In the linear approximation, we can consider noise to be Gaussian, and all nontrivial information about it to be in the correlators such as $\langle\xi(x_1,p_1)\xi(x_2,p_2)\rangle$, $\langle\eta(x_1)\eta(x_2)\rangle$ and $\langle\xi(x_1,p)\eta(x_2)\rangle$. To determine these correlators we expand the entropy to quadratic order (note cancellation of terms linear in h due to $\log f_\sigma = (\mu - M(\sigma)\gamma)/T$)

$$S^{(2)} = -\frac{1}{2}\int d^3\boldsymbol{x}\Big[\int_p f_0 h^2 + \frac{1}{T}(\pi^2 + (\boldsymbol{\nabla}\sigma)^2 + m^2\sigma^2)\Big], \tag{21}$$

where

$$m^2 = m_0^2 - T\int_p \frac{d^2 f_\sigma}{d\sigma^2} \tag{22}$$

is the same "in-medium" mass of the σ field quanta already defined in Eq. (17), as can be verified by using Eq. (10).

We follow Refs. [35,36,38] to define the "entropy matrix" (or, more precisely, operator) $\mathbb{E}$:

$$\mathbb{E}\begin{pmatrix} h \\ \pi \\ \sigma \end{pmatrix} = \begin{pmatrix} f_0 h \\ \pi/T \\ (-\boldsymbol{\nabla}^2\sigma + m^2\sigma)/T \end{pmatrix}, \tag{23}$$

so that Eq. (21) can be written as

$$S^{(2)} = -\frac{1}{2}\boldsymbol{h}\cdot\mathbb{E}\boldsymbol{h}, \tag{24}$$

where $\boldsymbol{h} \equiv (h,\pi,\sigma)$ denotes the (infinitely dimensional) "vector" whose components are the degrees of freedom of the system, with the scalar product defined as

$$(h,\pi,\sigma)\cdot(h',\pi',\sigma') \equiv \int d^3\boldsymbol{x}\Big[\int_p hh' + \pi\pi' + \sigma\sigma'\Big]. \tag{25}$$

Similarly, Eqs. (18) can be also cast in matrix (operator) form

$$\dot{\boldsymbol{h}} + \mathbb{G}\boldsymbol{h} = \boldsymbol{\xi}, \tag{26}$$

where

$$\mathbb{G}\begin{pmatrix} h \\ \pi \\ \sigma \end{pmatrix} = \begin{pmatrix} -\pi g/(\gamma T) + \boldsymbol{v}\cdot\boldsymbol{\nabla}h + \mathcal{I}[h] \\ \Gamma_0\pi - \boldsymbol{\nabla}^2\sigma + m^2\sigma + g\int_p f_0 h/\gamma \\ -\pi \end{pmatrix}. \tag{27}$$

Then the correlator of the noises, combined into a vector $\boldsymbol{\xi} = (\xi,\eta,0)$, can be expressed in terms of the matrix/operator $\mathbb{Q}$ defined as

$$\langle\boldsymbol{\xi}(t_1)\otimes\boldsymbol{\xi}(t_2)\rangle = 2\mathbb{Q}\delta(t_1 - t_2), \tag{28}$$

and given by the usual fluctuation-dissipation relation (see Refs. [35,36,39])

$$2\mathbb{Q} = \mathbb{G}\mathbb{E}^{-1} + \mathbb{E}^{-1}\mathbb{G}^\dagger. \tag{29}$$

Using Eqs. (23), (27), and (29), one can now find $\mathbb{Q}$:

$$2\mathbb{Q}\begin{pmatrix} h \\ \pi \\ \sigma \end{pmatrix} = \begin{pmatrix} \mathcal{I}[h/f_0] + \mathcal{I}^\dagger[h]/f_0 \\ 2\Gamma_0 T\pi \\ 0 \end{pmatrix} = \begin{pmatrix} (\mathcal{K} + \mathcal{K}^\dagger)[h] \\ 2\Gamma_0 T\pi \\ 0 \end{pmatrix}, \tag{30}$$

where we defined operator $\mathcal{K}$ as

$$\mathcal{I}[h] \equiv \mathcal{K}[f_0 h]. \tag{31}$$

One can show [40] that the operator $\mathcal{K}$ is self-adjoint for elastic collisions, but we leave equations in a more general form. Equation (30) together with Eq. (28) translates into the following explicit expression for the correlators:

$$\langle\xi(x_1,p_1)\xi(x_2,p_2)\rangle = (\mathcal{K} + \mathcal{K}^\dagger)(2\pi)^3\delta^3(\boldsymbol{p}_1 - \boldsymbol{p}_2) \times \delta^4(x_1 - x_2); \tag{32a}$$

$$\langle\eta(x_1)\eta(x_2)\rangle = 2\Gamma_0 T\delta^4(x_1 - x_2); \tag{32b}$$

$$\langle\xi(x_1,p_1)\eta(x_2)\rangle = 0. \tag{32c}$$

It is easy to recognize in Eq. (32a) the generalization of the result of Ref. [36]. One can also observe that the interaction between the particles and the field σ does not manifest itself in any modification of the corresponding noises. That should be expected given the physical origin of the noise: collisions and the interaction with the external reservoir. The correlations are local in coordinate space, which also correctly reflects their origin.

IV. STATIONARY, EQUILIBRIUM FLUCTUATIONS

The quantity directly accessible by experimental measurement is a two-particle correlator $\langle \delta f_{(1)} \delta f_{(2)} \rangle$. Before we begin studying time evolution of fluctuations let us derive the stationary, equilibrium value of the two-particle correlator and compare with existing results.

To linear order in fluctuations,

$$f = f_\sigma(1+h) = f_0(1 + h - g\sigma/(\gamma T)) + \mathcal{O}(\sigma^2). \quad (33)$$

The equal-time correlators of h and σ are contained in the matrix elements of the correlator of $\boldsymbol{h}$ which *in equilibrium* are given by

$$\langle \boldsymbol{h} \otimes \boldsymbol{h} \rangle = \mathbb{E}^{-1} \quad (34)$$

according to (24). Using the explicit expression (23) for the components of the entropy matrix we thus find

$$\begin{aligned} \langle \delta f_{(1)} \delta f_{(2)} \rangle &= \langle f_{(1)} f_{(2)} \rangle - f_{0(1)} f_{0(2)} \\ &= f_{0(1)} f_{0(2)} \left\langle \left(h - \frac{g\sigma}{\gamma T} \right)_{(1)} \left(h - \frac{g\sigma}{\gamma T} \right)_{(2)} \right\rangle \\ &= f_{0(1)} \delta_{(1,2)} + \frac{g^2}{T} \frac{f_{0(1)} f_{0(2)}}{\gamma_{(1)} \gamma_{(2)}} D_{(1,2)}, \end{aligned} \quad (35)$$

where subscripts (1) and (2) refer to the points in the phase space $(\boldsymbol{x}_1, \boldsymbol{p}_1)$ and $(\boldsymbol{x}_2, \boldsymbol{p}_2)$ where the scripted quantities are to be evaluated, $\delta_{(1,2)} = \delta^3(\boldsymbol{x}_1 - \boldsymbol{x}_2)(2\pi)^3 \delta^3(\boldsymbol{p}_1 - \boldsymbol{p}_2)$ and $D_{(1,2)} = (-\boldsymbol{\nabla}^2 + m^2)^{-1} \delta^3(\boldsymbol{x}_1 - \boldsymbol{x}_2)$. Integrating over $\boldsymbol{x}_{1,2}$ one obtains the known result for the equilibrium fluctuations of particles coupled to classical scalar field σ [19,41]:

$$V^{-1} \langle \delta \nu_{\boldsymbol{p}} \delta \nu_{\boldsymbol{p}'} \rangle = f_0 \delta_{\boldsymbol{p}\boldsymbol{p}'} + \frac{g^2}{m^2 T} \frac{f_0}{\gamma} \frac{f_0'}{\gamma'}, \quad (36)$$

where $V = \int d^3\boldsymbol{x}$ is the volume, $\delta_{\boldsymbol{p}\boldsymbol{p}'} \equiv (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{p}')$, and we denoted the momentum space distribution as

$$\nu_{\boldsymbol{p}} = \int d^3\boldsymbol{x} f(\boldsymbol{x}, \boldsymbol{p}), \quad \text{such that} \quad \int_{\boldsymbol{p}} \nu_{\boldsymbol{p}} = N \quad (37)$$

is the total number of the particles.

V. TIME EVOLUTION OF FLUCTUATIONS

Now we want to consider the time evolution of fluctuations as the parameters of the system, most importantly m, change.

If we take the initial probability distribution for fluctuating variables to be Gaussian, in a linear system the fluctuations will remain Gaussian at all times. This can be verified directly, by converting the generalized Langevin Eqs. (26) into corresponding Fokker-Plank equation for the probability distribution $P[\boldsymbol{h}; t]$:

$$\dot{P} = \frac{\partial}{\partial \boldsymbol{h}} \cdot \left(\mathbb{G} \boldsymbol{h} P + \mathbb{Q} \frac{\partial P}{\partial \boldsymbol{h}} \right). \quad (38)$$

Parametrizing the probability using (time-dependent) operator $\boldsymbol{\Sigma}$:

$$P = (\det \boldsymbol{\Sigma})^{-1/2} \exp\left[-\frac{1}{2} \boldsymbol{h} \cdot \boldsymbol{\Sigma}^{-1} \boldsymbol{h} \right], \quad (39)$$

and substituting into Eq. (38), one finds equation for $\boldsymbol{\Sigma}$:

$$\dot{\boldsymbol{\Sigma}} = -\mathbb{G}\boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbb{G}^\dagger + 2\mathbb{Q}. \quad (40)$$

Note that $\boldsymbol{\Sigma} = \mathbb{E}^{-1}$ is a stationary solution of this equation, as expected from Eq. (34) and the fact that

$$\langle \boldsymbol{h} \otimes \boldsymbol{h} \rangle = \boldsymbol{\Sigma}. \quad (41)$$

This verifies the fluctuation-dissipation relation (29).

The Eq. (40) for $(d/dt)\langle \boldsymbol{h} \otimes \boldsymbol{h} \rangle$ can be also derived directly, by applying Eqs. (26) and (28), without assuming Gaussianity (39).

Equation (40) can be formally integrated from initial time, taken to be 0, to arbitrary time t:

$$\boldsymbol{\Sigma}(t) = \mathbb{V}(t,0) \boldsymbol{\Sigma}(0) \mathbb{V}^\dagger(t,0) + 2 \int_0^t dt' \mathbb{V}(t,t') \mathbb{Q} \mathbb{V}^\dagger(t,t'), \quad (42)$$

where evolution operator $\mathbb{V}(t,t')$ satisfies

$$\dot{\mathbb{V}} = -\mathbb{G}(t)\mathbb{V}, \quad (43)$$

i.e.,

$$\mathbb{V}(t,t') = \mathcal{T} \exp\left(-\int_{t'}^t dt'' \mathbb{G}(t'') \right). \quad (44)$$

Equation (40) can be now used to study the time evolution of the fluctuations, provided, of course, they remain small, so that linear approximation is valid at all relevant times.

A more useful equivalent form of Eq. (40) is

$$\frac{d\tilde{\boldsymbol{\Sigma}}}{dt} = -\mathbb{G}\tilde{\boldsymbol{\Sigma}} - \tilde{\boldsymbol{\Sigma}}\mathbb{G}^\dagger - \frac{d(\mathbb{E}^{-1})}{dt}, \quad \text{where } \tilde{\boldsymbol{\Sigma}} \equiv \boldsymbol{\Sigma} - \mathbb{E}^{-1}. \quad (45)$$

In this form, and for $\mathbb{E} = \text{const}$, it describes relaxation of $\boldsymbol{\Sigma}$ to its equilibrium value $\mathbb{E}^{-1}$. The solution is given by

$$\tilde{\boldsymbol{\Sigma}}(t) = \mathbb{V}(t,0) \tilde{\boldsymbol{\Sigma}}(0) \mathbb{V}^\dagger(t,0) - \int_0^t dt' \mathbb{V}(t,t') \times \frac{d(\mathbb{E}^{-1})}{dt} \mathbb{V}^\dagger(t,t'), \quad (46)$$

which is equivalent to Eq. (42).

Determining the time-dependence more explicitly in the general case is a complicated task, and should perhaps be part of numerical modeling of a more realistic system. As an illustration of the use of Eq. (46) we shall address the important question of the fate of the critical point fluctuations after chemical freeze-out. To prepare for this, we shall briefly discuss the way conservation of particle number is reflected in the equations (Sec. VI), and then analyze

the evolution of fluctuations for time-independent $\mathbb{G}$ and $\mathbb{E}$ (Secs. VII and VIII).

VI. CONSERVATION OF PARTICLE NUMBER AND FLUCTUATIONS

Chemical freeze-out is a moment in the history of a heavy-ion collision fireball, when inelastic reactions become too infrequent to modify the chemical composition of the system. In other words, the number of particles of a given species is conserved during subsequent evolution.

To model the evolution past the chemical freeze-out, we shall require that the collision integral $\mathcal{C}[f]$ conserves the particle number. According to Eq. (7) this requires

$$\int_p \mathcal{C}[f]/\gamma = 0 \tag{47}$$

to be valid for all f. The linearized collision operator $\mathcal{I}$ in Eq. (15) therefore obeys, for all h,

$$\mathcal{M}[\mathcal{I}[h]] = 0 \tag{48}$$

For future convenience, we have introduced notation

$$\mathcal{M}[h] \equiv \frac{\int_p f_0 h}{\int_p f_0} \tag{49}$$

for the average over equilibrium distribution f_0. Imposing the condition that operator $\mathcal{K}$ defined in Eq. (31) is self-adjoint [40], one can see that Eq. (48) implies

$$\mathcal{I}[\text{const}] = 0. \tag{50}$$

In other words, operator $\mathcal{I}$ has a zero mode. This is also evident from the fact that constant h corresponds to changing the value of μ in the equilibrium distribution (10), and that $\mathcal{C}[f_\sigma] = 0$ for arbitrary μ.

VII. SOLVING EVOLUTION EQUATIONS WITH CONSTANT COEFFICIENTS

The evolution operator $\mathbb{V}$ in Eq. (44) for the system with time-independent $\mathbb{G}$ can be written in the form:

$$\mathbb{V}(t, t') = \sum_\lambda e^{-\lambda(t-t')} \boldsymbol{h}_\lambda \otimes \boldsymbol{\theta}_\lambda, \tag{51}$$

where the sum goes over all solutions of the following eigenvalue system

$$\lambda \boldsymbol{h}_\lambda = \mathbb{G}\boldsymbol{h}_\lambda, \tag{52}$$

and vectors $\boldsymbol{\theta}_\lambda$ form the dual (adjoint) basis with respect to the one formed by vectors $\boldsymbol{h}_\lambda$, i.e.,

$$\boldsymbol{\theta}_{\lambda_i} \cdot \boldsymbol{h}_{\lambda_j} = \delta_{ij}. \tag{53}$$

If the system is also spatially homogeneous, it is convenient to apply Fourier transformation with respect to the space coordinate $\boldsymbol{x}$ to Eqs. (52). Given the definition of operator $\mathbb{G}$ in Eq. (27), we find:

$$-\lambda h - \pi g/(\gamma T) + i\boldsymbol{v}\cdot\boldsymbol{q}h + \mathcal{I}[h] = 0; \tag{54a}$$

$$-\lambda\pi + \Gamma_0\pi + \boldsymbol{q}^2\sigma + m^2\sigma + g n_0 \mathcal{M}[h/\gamma] = 0; \tag{54b}$$

$$-\lambda\sigma - \pi = 0; \tag{54c}$$

where we defined

$$n_0 \equiv \int_p f_0 \tag{55}$$

-the equilibrium density of the particles.

VIII. THE SLOWEST MODE

The slowest mode corresponding to the lowest eigenvalue of the eigensystem (54) is of primary interest to us. In this section we shall determine it.

A. Zero mode

We begin by considering the simpler case $\boldsymbol{q} = 0$. Since operator $\mathcal{I}$ has a zero mode $h = \text{const}$ (50), let us separate it by writing

$$h = \tilde{h} + \bar{h}, \quad \text{where } \bar{h} \equiv \mathcal{M}[h]. \tag{56}$$

The zero eigenvalue $\lambda_0 = 0$ of Eqs. (54) corresponds to the solution such that $\tilde{h} = \pi = 0$ (so that Eq. (54a) is trivial) and

$$\boldsymbol{h}_{\lambda_0}: g n_0 \mathcal{M}[1/\gamma]\bar{h} = -m^2\sigma \qquad (\lambda_0 = 0). \tag{57}$$

Note that this mode is predominantly σ near the critical point (i.e, $\bar{h} \to 0$ as $m \to 0$). This may appear surprising, since the mode λ_0 is due to the particle number conservation, while σ is not a density of a conserved quantity. However, near the critical point the fluctuations of particle number density are dominated by their mixing with σ [3,42], whose fluctuations diverge. Equation (57) also shows that in the limit $g \to 0$ at fixed m, the λ_0 mode is predominantly $\bar{h}$, as it should be if σ is decoupled.

In order to find the dual vector $\boldsymbol{\theta}_{\lambda_0}$ we need some information about all other modes $\boldsymbol{h}_\lambda$, since $\boldsymbol{\theta}_{\lambda_0}$ must be orthogonal to them (53). Applying operator $\mathcal{M}$ to Eq. (54a) and using (48), we find, at $\boldsymbol{q} = 0$:

$$\boldsymbol{h}_\lambda: \bar{h} = \frac{g}{T}\mathcal{M}[1/\gamma]\sigma \qquad (\lambda \neq 0). \tag{58}$$

This equation contains all the information about the nonzero modes that we need to determine $\boldsymbol{\theta}_{\lambda_0}$.

In order to simplify subsequent linear algebra manipulations, we shall define two convenient basis vectors:

$$\hat{\boldsymbol{h}}: \tilde{h} = \pi = \sigma = 0 \quad \text{and} \quad \bar{h} = g\mathcal{M}[1/\gamma]/T; \tag{59}$$

$$\hat{\boldsymbol{\sigma}}: \tilde{h} = \pi = \bar{h} = 0 \quad \text{and} \quad \sigma = 1. \tag{60}$$

In terms of these vectors, Eq. (58) means that, for any nonzero eigenvalue $\lambda \neq 0$, eigenmodes are given by

$$\boldsymbol{h}_\lambda = \hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{h}} + (\text{terms with } \bar{h} = \sigma = 0), \tag{61}$$

while Eq. (57) for the zero mode can be written as

$$\boldsymbol{h}_{\lambda_0} = \Delta^2 \hat{\boldsymbol{\sigma}} - m^2 \hat{\boldsymbol{h}}, \tag{62}$$

where we defined

$$\Delta^2 \equiv \frac{g^2 n_0}{T} \mathcal{M}[1/\gamma]^2. \tag{63}$$

Equations (61) and (62) together determine the orientation and the length of the dual vector

$$\boldsymbol{\theta}_{\lambda_0} = \frac{1}{V} \frac{\hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\theta}}}{\Delta^2 + m^2}, \tag{64}$$

which satisfies the defining orthonormality conditions (53). We defined another convenient vector, related to (59):

$$\hat{\boldsymbol{\theta}} = \frac{T}{\Delta^2} f_0 \hat{\boldsymbol{h}}, \tag{65}$$

such that $\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{h}} = 1$. One can also check that $\boldsymbol{\theta}_{\lambda_0}$ is the $\lambda = 0$ eigenvector of $\lambda \boldsymbol{\theta}_\lambda = \mathbb{G}^\dagger \boldsymbol{\theta}_\lambda$, as it should be.

B. Hydrodynamic mode and diffusion coefficient

The mode λ_0, corresponding to conservation of the particle number, is hydrodynamic in the sense that, for small $\boldsymbol{q}$, $\lambda_0 = \mathcal{O}(\boldsymbol{q}^2)$. The ratio $\lambda_0/\boldsymbol{q}^2 = D$ defines the corresponding diffusion coefficient D, which we can extract from Eqs. (54).

As we did deriving Eq. (58) for nonzero modes at $\boldsymbol{q} = 0$, let us apply operator $\mathcal{M}$ to Eq. (54a). Now, at $\boldsymbol{q} \neq 0$, we find

$$\lambda \bar{h} - i \mathcal{M}[\boldsymbol{v} \cdot \boldsymbol{q} \tilde{h}] = \lambda \frac{g}{T} \mathcal{M}[1/\gamma] \sigma. \tag{66}$$

We need now to express $\tilde{h}$ in terms of $\bar{h}$ and σ using Eq. (54a). Since $\tilde{h} \to 0$ as $\boldsymbol{q} \to 0$, one can see that $\tilde{h}$ must begin at order $\boldsymbol{q}$. Keeping in Eq. (54a) only terms of $\mathcal{O}(\boldsymbol{q})$ we obtain:

$$i \boldsymbol{v} \cdot \boldsymbol{q} \bar{h} + \mathcal{I}[\tilde{h}] = 0. \tag{67}$$

We would need to invert operator $\mathcal{I}$ to express $\tilde{h}$ in terms of $\bar{h}$. For generic operator $\mathcal{I}$ we shall define function $\psi(\boldsymbol{v}^2)$, which solves the equation

$$\mathcal{I}[\boldsymbol{v}\psi] = \boldsymbol{v}. \tag{68}$$

The fact that solution can be found in this form follows from isotropy of the collision operator and equilibrium distribution function. In terms of ψ, the solution to Eq. (67) is given by

$$\tilde{h} = -i \boldsymbol{q} \cdot \boldsymbol{v} \psi \bar{h}. \tag{69}$$

Substituting this into Eq. (66) we find, instead of Eq. (58),

$$\bar{h}(\lambda_0 - D_0 \boldsymbol{q}^2) = \lambda_0 \frac{g}{T} \mathcal{M}[1/\gamma] \sigma, \tag{70}$$

where we denoted by D_0

$$D_0 \equiv \frac{1}{3} \mathcal{M}[\boldsymbol{v}^2 \psi] \tag{71}$$

the diffusion coefficient for the particle gas with *fixed* mass (the limit $g \to 0$). Putting together Eq. (70) and (57), which remains valid to the order in $\boldsymbol{q}^2$ we need, we find

$$\lambda_0 = D \boldsymbol{q}^2 + \mathcal{O}(\boldsymbol{q}^4), \quad \text{where } D = \frac{m^2}{\Delta^2 + m^2} D_0. \tag{72}$$

The fact that $D \to 0$ as $m^2 \to 0$ is to be expected on general grounds from the hydrodynamic relation $D = \bar{\sigma}/\chi$ [43], where $\bar{\sigma}$ is the conductivity, and χ is the susceptibility of the particle number, and the fact that $\chi \sim 1/m^2$ [28,43]. Within our microscopic approach:

$$\chi = \frac{\langle \delta N^2 \rangle}{VT} = \frac{n_0}{T} \frac{\Delta^2 + m^2}{m^2} \tag{73}$$

according to Eq. (93) which we encounter later.

IX. FASTER MODES

This section is a slight detour from the main thread of the paper. We have already accumulated all information about the zero and even nonzero modes, Eqs. (57) and (58), that we need to study the memory effect in fluctuations (Sec. X). However, it might still be interesting to look at the structure of the *nonzero* modes in more detail, to understand better the properties of the system of equations we are solving.

So far we have not used any information about the linear collision operator $\mathcal{I}$ beyond the conservation of the particle number and isotropy. For the sake of analytic transparency, and within this section only, we shall assume here that all eigenvalues, but one, of the operator $\mathcal{I}$ are equal to the same value τ^{-1}, which has the meaning of an average relaxation rate. The exception is the zero eigenvalue, corresponding to the condition (48). This approximation is well-known and is due to Refs. [44,45] (see also Ref. [46]).

Operator $\mathcal{I}$ should also respect the condition that operator $\mathcal{K}$, defined by (31) is self-adjoint. All the above conditions are satisfied by the operator

$$\mathcal{I}[h] = \tau^{-1}(h - \mathcal{M}[h]). \tag{74}$$

Another simplification we shall adopt in this section only corresponds to assuming $\tau^{-1} \gg \Gamma_0, m$, i.e., that τ^{-1} is much faster than any other rate in the problem.

We emphasize that, although these approximations are physically sensible, they are only used here to make a transparent analytic treatment possible, illustrating the properties of the system we study.

Substituting Eq. (58) back into (54a) and solving for $\tilde{h}$ we find, at $\boldsymbol{q} \to 0$,

$$\tilde{h} = \frac{\lambda \tau}{1 - \lambda \tau} \frac{g}{T} (\mathcal{M}[1/\gamma] - 1/\gamma) \sigma. \tag{75}$$

Now substituting h given by Eqs. (56), (58), and (75) into Eq. (54b), we find the equation determining the eigenvalues λ:

$$\lambda^2 - \lambda\Gamma_0 + m^2 + \frac{\Delta^2 - \lambda\tau\Delta_\Gamma^2}{1 - \lambda\tau} = 0. \tag{76}$$

where we used (63) and defined also

$$\Delta_\Gamma^2 \equiv \frac{g^2 n_0}{T}\mathcal{M}[1/\gamma^2]. \tag{77}$$

Equation (76) has three roots. For the scale hierarchy we consider, $\tau^{-1} \gg \Gamma_0, m$, there are two roots of order Γ_0 or m and one root of order τ^{-1}. The two smaller roots, to leading order in τ, satisfy the quadratic equation

$$\lambda^2 - \lambda\Gamma + \tilde{m}^2 = 0, \tag{78}$$

where

$$\tilde{m}^2 \equiv m^2 + \Delta^2; \tag{79}$$

$$\Gamma \equiv \Gamma_0 + \Delta_\Gamma^2\tau; \tag{80}$$

and thus

$$\lambda_{1,2} = -\Gamma/2 \pm i\sqrt{\tilde{m}^2 - (\Gamma/2)^2}. \tag{81}$$

At this point one can see that $\tilde{m}$ is the rest mass (pole mass) of the quasiparticle σ. It is different from the (static) screening mass m, and does not vanish at the critical point [28,47,48], where $m \to 0$.

Also, Eq. (80) shows that the full dissipation rate Γ contains contribution $\Delta_\Gamma^2\tau$ from the interaction of σ with the particles. In principle, one could start with $\Gamma_0 = 0$ (closed system) and consider the particle collisions to be the only source of the dissipation.

The third eigenvalue is given, to the leading nontrivial order in τ, by

$$\lambda_3 = \tau^{-1} + (\Delta^2 - \Delta_\Gamma^2)\tau. \tag{82}$$

(Applying Cauchy-Bunyakovsky-Schwarz inequality to Eqs. (63) and (77), one can see that $\Delta < \Delta_\Gamma$.)

Finally, any function $h_\perp$ satisfying

$$\mathcal{M}[h_\perp] = 0 \quad \text{and} \quad \mathcal{M}[h_\perp/\gamma] = 0 \tag{83}$$

solves the eigensystem with $\sigma = \pi = 0$ and $\lambda = \tau^{-1}$. The linear (eigen)space defined by Eqs. (83) is infinitely dimensional and, correspondingly, $\lambda = \tau^{-1}$ is an infinitely degenerate eigenvalue. This degeneracy is not lifted because modes $h_\perp$ do not mix with the modes corresponding to eigenvalues λ_i, $i = 0, 1, 2, 3$. This is a convenient feature of the Anderson-Witting approximation (74).

Returning to the λ_0 mode, with operator $\mathcal{I}$ in the form given by Eq. (74), Eq. (68) can be solved: $\psi = \tau$, and $D_0 = \mathcal{M}[\boldsymbol{v}^2]\tau/3$.

X. FROM CHEMICAL TO KINETIC FREEZE-OUT

A. Preliminaries

As an application of the formalism, let us consider the following long-standing problem. Assuming that the chemical freeze-out occurred near the critical point, how much of the fluctuation signal survives until kinetic freeze-out? A more precise and detailed answer to this problem will likely require a numerical simulation. Here we want to illustrate the mechanism, and make a simple estimate of the effect. To that end we shall make several simplifying assumptions, in order to maintain analytical control. In essence, we shall assume that separation of different relaxation-time scales is sufficiently large for us to be able to focus on only the most relevant modes.

Chemical freeze-out is characterized by "freezing" of *inelastic* reactions. This means that the number of each individual particles is conserved (particles in the same isospin multiplet could be considered as different internal states of the same particle, to allow for quasielastic collisions). This, in turn, means that any measure of fluctuations of a conserved number of particles should not change. More precisely, it can only change by diffusion, which we shall assume here to be the slowest scale in the problem (i.e., we work in the $\boldsymbol{q} \to 0$ limit).

On the other hand, fluctuations such as those of mean p_T, which is not a conserved quantity, must evolve between chemical and kinetic freeze-out, at which point they are "frozen" and eventually observable. The form and the amount of this evolution we shall now discuss.

During the interval between chemical and kinetic freeze-out the typical time scale, τ_e, of the evolution of the system is much slower than the inverse elastic collision rate τ and the scales Γ^{-1} associated with the relaxation of the σ field. In a realistic heavy-ion collision $\tau_e = \mathcal{O}(10 - 20)$ fm (order of fireball size), while Γ^{-1}, $\tau = \mathcal{O}(0.5 - 2)$ fm (typical hadronic scales). Thus we shall assume $\tau_e \gg \tau, \Gamma^{-1}$.

In order to be able to obtain analytically tangible solution we shall take into account the effect of the change of only one parameter: m-the screening mass of σ. Since the fluctuations are singular as $1/m^2$ near the critical point, the effect of change of m could be assumed to be dominant, compared with the change of, e.g., equilibrium distribution functions f_0 (e.g., via change of T), which we shall consider fixed, for simplicity. As a concrete example, one could consider evolution of m determined by the model in Ref. [22]. As we shall see, the actual time dependence of m will not matter, as long as it is slow, which is helped by critical slowing down [22].

The physically reasonable assumptions spelled out above are needed to make the analytic results attainable and usefully transparent. These assumptions can be relaxed, e.g., via a numerical simulation, at the expense of analytic control. Our main purpose here is to illustrate the mechanism in the most transparent way possible.

B. Evolution of fluctuations

We begin by determining the evolution operator $\mathbb{V}(t, t')$. If we choose the interval, $t - t'$, so small that we could neglect the change of $\mathbb{G}$ (i.e., change of m) and consider it constant, then we could integrate Eq. (44) and obtain $\mathbb{V}$ given by Eq. (51).

If the interval $t - t'$ is also large compared to relaxation scales λ_i^{-1}, for all $i \neq 0$, only the term corresponding to the zero mode $\lambda = \lambda_0 = \mathcal{O}(\boldsymbol{q}^2)$ will survive in Eq. (51):

$$\mathbb{V}(t, t') = e^{-\lambda_0(t-t')}\boldsymbol{h}_{\lambda_0} \otimes \boldsymbol{\theta}_{\lambda_0} + (\text{exp. small terms}). \quad (84)$$

In order to extend this result to longer time intervals over which the change of $\mathbb{G}$ cannot be neglected, we use the property

$$\mathbb{V}(t, t') = \mathbb{V}(t, t_n) \ldots \mathbb{V}(t_2, t_1)\mathbb{V}(t_1, t') \quad (85)$$

and subdivide $t - t'$ into smaller intervals satisfying $\lambda_i^{-1} \ll t_n - t_{n-1} \ll \tau_e$. Our assumption of scale hierarchy is needed to make such a choice possible. Using Eq. (84) we then find

$$V(t, t') = e^{-\int_{t'}^{t} \lambda_0 dt}\boldsymbol{h}_{\lambda_0}(t) \otimes \boldsymbol{\theta}_{\lambda_0}(t') \times (\boldsymbol{\theta}_{\lambda_0}(t) \cdot \boldsymbol{h}_{\lambda_0}(t_n)) \ldots (\boldsymbol{\theta}_{\lambda_0}(t_1) \cdot \boldsymbol{h}_{\lambda_0}(t')). \quad (86)$$

In order to evaluate dot products in Eq. (86), we use explicit form of eigenvectors $\boldsymbol{h}_{\lambda_0}$ and $\boldsymbol{\theta}_{\lambda_0}$, given by Eqs. (62) and (64), and find, e.g.,

$$\boldsymbol{\theta}_{\lambda_0}(t_1) \cdot \boldsymbol{h}_{\lambda_0}(t') = \frac{\Delta^2 + m^2(t')}{\Delta^2 + m^2(t_1)}. \quad (87)$$

There is a string of such factors in Eq. (86) and, multiplying them successively, one finds that all but the first and the last factor $\Delta^2 + m^2(t)$ cancel, leaving

$$\mathbb{V}(t, t') = e^{-\int_{t'}^{t} \lambda_0 dt}\boldsymbol{h}_{\lambda_0}(t) \otimes \boldsymbol{\theta}_{\lambda_0}(t')\frac{\Delta^2 + m^2(t')}{\Delta^2 + m^2(t)} = e^{-\int_{t'}^{t} \lambda_0 dt}\boldsymbol{h}_{\lambda_0}(t) \otimes \boldsymbol{\theta}_{\lambda_0}(t). \quad (88)$$

The only dependence on the initial time t' remains in the exponentially decaying prefactor. Since $\lambda_0 = D\boldsymbol{q}^2$, this prefactor is close to unity near the limit we have been working in: $\boldsymbol{q} \to 0$ (the fact that $D \sim m^2 \to 0$ near the critical point also helps). In general, the importance of the prefactor depends on the size of the region over which the fluctuations are measured. Since $1/\lambda_0$ is an estimate of the time, τ_D, it takes for a fluctuation to diffuse over this region, the factor can be estimated roughly as $\exp[-(t - t')/\tau_D]$. Below we shall consider the case when the region is large enough, so that $\tau_D \gg \tau_e$.

We are now ready to apply Eq. (46). We shall take the initial time $t = 0$ to be the time of chemical freeze-out, and the final time $t = t_{\rm k}$ the time of kinetic freeze-out. At chemical freeze-out, $t = 0$, the fluctuations are equilibrated and $\tilde{\Sigma} = 0$. Thus at kinetic freeze-out, $t = t_{\rm k}$, Eq. (46) gives, upon integration,

$$\tilde{\boldsymbol{\Sigma}}(t_{\rm k}) = \boldsymbol{h}_{\lambda_0}(t_{\rm k}) \otimes \boldsymbol{\theta}_{\lambda_0}(t_{\rm k}) \times (\mathbb{E}^{-1}(t_c) - \mathbb{E}^{-1}(t_{\rm k}))\boldsymbol{\theta}_{\lambda_0}(t_{\rm k}) \otimes \boldsymbol{h}_{\lambda_0}(t_{\rm k}). \quad (89)$$

A shorter way to derive Eq. (89) is to observe that the actual time-dependence of $m(t)$ is not important, as long as it is faster than the diffusion: $\tau_D \gg \tau_e$ (but $\tau_e \gg \lambda_i^{-1}$). Choosing $m(t)$ to have an (almost) instantaneous step from $m_{\rm c}$ to $m_{\rm k}$, and constant at all other times one can then find solution (89) using Eq. (46) with initial condition $\tilde{\Sigma}(0) = \mathbb{E}^{-1}(t_{\rm c}) - \mathbb{E}^{-1}(t_{\rm k})$.

Taking into account Eq. (23), which for convenience we write, using notations $\mathbb{1}_h$ for the unit operator $\mathbb{1}_h[h] = h$ and $\hat{\boldsymbol{\pi}}$ for the basis vector ($h = 0$, $\pi = 1$, $\sigma = 0$),

$$\mathbb{E}^{-1}(t) = f_0^{-1}\mathbb{1}_h + T\hat{\boldsymbol{\pi}} \otimes \hat{\boldsymbol{\pi}} + \frac{T}{m^2(t)}\hat{\boldsymbol{\sigma}} \otimes \hat{\boldsymbol{\sigma}}, \quad (90)$$

together with Eqs. (62) and (64), we find for $\boldsymbol{\Sigma} = \mathbb{E}^{-1} + \tilde{\boldsymbol{\Sigma}}$ at kinetic freeze-out time

$$\boldsymbol{\Sigma}(t_{\rm k}) = \mathbb{E}^{-1}(t_{\rm k}) + \left(\frac{T}{m_{\rm c}^2} - \frac{T}{m_{\rm k}^2}\right)\frac{\Delta^2\hat{\boldsymbol{\sigma}} - m_{\rm k}^2\hat{\boldsymbol{h}}}{\Delta^2 + m_{\rm k}^2} \otimes \frac{\Delta^2\hat{\boldsymbol{\sigma}} - m_{\rm k}^2\hat{\boldsymbol{h}}}{\Delta^2 + m_{\rm k}^2}, \quad (91)$$

where $m_{\rm c,k} = m(t_{\rm c,k})$ is the value of the σ screening mass at chemical/kinetic freeze-out. The last term, containing $1/m_{\rm c}^2$, is the memory effect, due to the freezing out of conserved particle number fluctuations.

C. Two-particle correlator and memory

In order to translate Eq. (91) into observed fluctuations, we should recall that $\boldsymbol{\Sigma} = \langle \boldsymbol{h} \otimes \boldsymbol{h} \rangle$ and apply Eqs. (35) (all but the last equality) to calculate the 2-particle correlator. The fluctuations of the momentum space distribution of particles (37) at kinetic freeze-out are thus given by

$$V^{-1}\langle \delta\nu_{\boldsymbol{p}}\delta\nu_{\boldsymbol{p}'}\rangle = f_0\delta_{\boldsymbol{p}\boldsymbol{p}'} + \frac{g^2}{m_{\rm k}^2 T}\frac{f_0}{\gamma}\frac{f_0'}{\gamma'} + \frac{g^2}{T}\left(\frac{1}{m_{\rm c}^2} - \frac{1}{m_{\rm k}^2}\right) \times f_0 f_0' \times \frac{\Delta^2/\gamma + m_{\rm k}^2\mathcal{M}[1/\gamma]}{\Delta^2 + m_{\rm k}^2} \cdot \frac{\Delta^2/\gamma' + m_{\rm k}^2\mathcal{M}[1/\gamma]}{\Delta^2 + m_{\rm k}^2}, \quad (92)$$

which should be compared to Eq. (36) with $m = m_{\rm k}$. To check that the effect of the additional term is to preserve the particle number (multiplicity) fluctuation at the value it attained at *chemical* freeze-out, let us calculate that fluctuation by integrating Eq. (92) over momenta $\boldsymbol{p}$ and $\boldsymbol{p}'$ and using Eq. (37). Normalizing by the total number $\langle N \rangle = n_0 V$ for convenience, we find

$$\frac{\langle(\delta N)^2\rangle}{\langle N\rangle} = 1 + \frac{\Delta^2}{m_{\rm k}^2} + \left(\frac{1}{m_{\rm c}^2} - \frac{1}{m_{\rm k}^2}\right)\Delta^2 = 1 + \frac{\Delta^2}{m_{\rm c}^2}, \quad (93)$$

where we used definition (63). We see that, as expected, the effect of the memory term is to keep multiplicity fluctuations from changing after chemical freeze-out.

The effect which is less obvious is that the memory term also contributes to fluctuations of quantities which are *not* conserved. We shall keep discussion as general as possible, but to be less abstract, we shall consider fluctuations of mean transverse momentum p_T per event, which is one of the most common "intensive" measures of fluctuations. This fluctuation can be also expressed via the correlator (92) (see, e.g., Ref. [19]):

$$\langle(\delta p_T)^2\rangle = \frac{1}{\langle N\rangle^2}\int_{\boldsymbol{p}}\int_{\boldsymbol{p}'}(p_T - \bar{p}_T)(p'_T - \bar{p}_T)\langle\delta\nu_{\boldsymbol{p}}\delta\nu_{\boldsymbol{p}'}\rangle, \quad (94)$$

where we defined

$$\bar{p}_T \equiv \mathcal{M}[p_T]. \quad (95)$$

Normalizing by $\langle N\rangle$ to remove trivial system-size scaling, we find

$$\langle N\rangle\langle(\delta p_T)^2\rangle = \mathcal{M}[(p_T - \bar{p}_T)^2] + \frac{g^2 n_0}{T}\mathcal{M}[(p_T - \bar{p}_T)/\gamma]^2 \times\left(\frac{1 - r_m}{m_{\rm k}^2} + \frac{r_m}{m_{\rm c}^2}\right), \quad (96)$$

where we introduced

$$r_m = \left(\frac{\Delta^2}{\Delta^2 + m_{\rm k}^2}\right)^2. \quad (97)$$

Equation (96) shows that the critical contribution $\mathcal{O}(1/m_{\rm c}^2)$ can, under certain conditions, survive through the hadronic rescattering stage until kinetic freeze-out. Compared to the value at chemical freeze-out, the $\mathcal{O}(1/m_{\rm c}^2)$ term is attenuated by the factor r_m (97) which, if the σ screening mass at kinetic freeze-out, $m_{\rm k}$, is of order Δ or smaller, is a non-negligible fraction of unity.

D. Estimating the memory factor

Let us now estimate the memory factor (97). The value of r_m depends quite strongly on the ratio of $m_{\rm k}$ to Δ. For fluctuations to survive, $m_{\rm k}/\Delta$ cannot be large.

The estimate for Δ can be made using Eq. (63). In order to do this correctly we need to generalize our analysis to include more than one species of particles: nucleons (2 spin and 2 isospin states), pions, etc. We then find that the expression for r_m in Eq. (97) still holds, with Δ^2 receiving contributions from all species:

$$\Delta^2 = \Delta^2_{\rm nucleons} + \Delta^2_{\rm pions} + \ldots. \quad (98)$$

Choosing, for example, top SPS energy freeze-out conditions $T = 168$ MeV and $\mu_B = 266$ MeV [49], we find for the contribution of nucleons $\Delta_{\rm nucleons} \approx 430.(g_p/10.)$ MeV. We take $g_p \approx m_p/f_\pi \sim 10$ as an estimate of the coupling of σ to protons.

The estimate for the contribution of pions is $\Delta_{\rm pions} \approx 110.(g_\pi/2.)$ MeV, where $g_\pi \approx G/m_\pi \sim 2.$, using the estimate for G from Ref. [19]. The estimates for the contribution of antinucleons and kaons are similarly small, compared to $\Delta_{\rm nucleons}$. Summing in quadratures, increases the estimate for Δ by less than 10% over $\Delta_{\rm nucleons}$: $\Delta \approx 460$ MeV.

Thus, at top SPS energy, the critical p_T fluctuations survive at least half as well as the particle multiplicity fluctuations ($r_m > 1/2$) until kinetic freeze-out, if the σ screening mass at the freeze-out does not exceed $m_{\rm k} < \sqrt{\sqrt{2}-1}\Delta \approx 300$ MeV.

XI. SUMMARY, DISCUSSION AND OUTLOOK

In summary, we introduced an approach to studying time-dependent quasistationary fluctuations near QCD critical point by combining *stochastic* Boltzmann equation with an equation of motion for a scalar field, describing the "soft" critical mode. We obtained the general solution of the linearized system and studied its relaxation modes. We focused on the slowest (diffusion) mode and analyzed its effect on the evolution of fluctuations after chemical freeze-out.

One of the consequences of our analysis is the following prediction. Under the conditions that particle number fluctuations are frozen after chemical freeze-out, the fluctuations of *nonconserved* quantities, such as, e.g., mean p_T, are also preserved over time scales longer than collisional relaxation-time τ. The strength of this effect crucially depends on the ratio of the σ screening mass $m_{\rm k}$ at kinetic freeze-out to Δ [see Eq. (97) and Sec. X D].

In other words, while for the multiplicity fluctuations to be preserved after chemical freeze-out the kinematic window of acceptance must be large enough [25,26], for the p_T fluctuations to be preserved, additional condition is necessary: $m_k < \Delta$. We find that, e.g., at top SPS energies, p_T fluctuations can survive the hadronic rescattering at least half as well as the particle multiplicity fluctuations for $m_{\rm k} < 300$ MeV.

The origin of this effect is the mixing between the critical mode σ and the conserved particle number density [see discussion after Eq. (57)]. E.g., when $\Delta \gg m$, the mode $\boldsymbol{h}_{\lambda_0}$, which is kept from relaxing by the particle number conservation, is almost the same as σ, Eq. (62). The fluctuations of σ involved in $\boldsymbol{h}_{\lambda_0}$ must keep the magnitude they reached at chemical freeze-out, contributing the term $\sim 1/m_{\rm c}^2$ into Eqs. (93) and (96). While multiplicity fluctuations in Eqs. (93) are frozen, the p_T fluctuations evolve, with contribution of the mode $\boldsymbol{h}_{\lambda_0}$ decreasing with increasing m as the factor r_m.

At the same time, the fluctuations of σ *alone*, with particle number fixed [i.e., obeying Eq. (58)], equilibrate

on a short time scale, Γ^{-1}, tracking the evolution of m. This equilibrated mode of fluctuations contributes $1/m_{\rm k}^2$ term into Eq. (96).

In this paper we focused on fluctuations of one particle species, treating the rest of the hadron gas as a heat bath. This simplification allowed us to follow the evolution of fluctuations analytically and expose the mechanism behind the memory effect in the most transparent way. This analysis could be generalized to the case of multiple particle species, carrying (different values of) the same conserved charge, as well as the case of multiple conserved quantities (baryon number, isospin, etc.). Taking into account fluctuations of conserved energy and momentum would be necessary, for example, to obtain correct $m \to 0$ scaling of the diffusion coefficient [28,43]. We leave this to future work.

We also neglected the effects of quantum statistics for simplicity. Although these are relatively small under realistic conditions (few percent, as estimated by mean occupation numbers $\mathcal{M}[f_0]$), this approximation could be removed. For the most part this would require replacing the equilibrium distribution in Eq. (10) with Bose-Einstein or Fermi-Dirac distribution and factors f_0 in equations such as Eq. (92) with $f_0(1 \pm f_0)$. This would also imply that the collision integral $\mathcal{C}[f]$ has Uehling-Uhlenbeck form [50]. The influence functional method [51,52] could be used to derive the corresponding equations.

We would like to stress that, although we did use relaxation-time approximation to obtain more explicit formulas for *nonzero* modes λ in Sec. IX, the results pertaining to the memory effect, which rely on the properties of the zero mode λ_0 studied in Sec. VIII, are valid beyond relaxation-time approximation.

A numerical simulation of the stochastic Eqs. (18) should allow to take into account more detailed properties of the heavy-ion collision evolution, such as inhomogeneity, anisotropy and flow. The evolution of the σ mass m can be described self-consistently, using Eq. (22), conceptually reminiscent of nuclear mean-field approach [34], or disoriented chiral condensate studies [53,54].

We also deliberately limited our analysis to linearized regime and focused on quadratic moments of fluctuations. The stronger singular behavior of higher moments of fluctuations makes them more attractive signatures of the QCD critical point [55]. A study of the higher-order moments would require generalization of the analysis to nonlinear equations such as (5) and (9).

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APPENDIX A: NOTATIONS

$\mathcal{C}[f]$-collision integral (4);
D-diffusion constant (72);
D_0-same, at $g = 0$, (71);
$\mathbb{E}$-"entropy matrix" (24) and (23);
f-short for $f(\boldsymbol{x}, \boldsymbol{p}; t)$, nonequilibrium distribution function (4);
f_σ-equilibrium distribution function for given background σ (10);
f_0 or f_0'-short for $f_0(\boldsymbol{p})$ or $f_0(\boldsymbol{p}')$, as above, for $\sigma = 0$;
$\mathbb{G}$-operator (27), acting on $\boldsymbol{h}$ gives "drift" terms in stochastic Eqs. (18) and (26);
g-coupling of particles to σ (14);
h-short for $h(\boldsymbol{x}, \boldsymbol{p}; t)$ relative deviation of f from f_σ (12);
$\bar{h}$-mean value of h (56);
$\tilde{h}$-deviation of h from $\bar{h}$ (56);
$\boldsymbol{h}$-generalized vector (h, π, σ) (24);
$\boldsymbol{h}_\lambda$-eigenmode of $\mathbb{G}$ with eigenvalue λ (52);
$\hat{\boldsymbol{h}}$-convenient basis vector (59);
$\mathcal{I}$-linearized collision integral (15);
$\mathcal{K}$-linear operator related to $\mathcal{I}$ by Eq. (31);
M or $M(\sigma)$-particle mass for given σ (1);
$\mathcal{M}[h]$-mean value of h (49);
m_0-vacuum mass of the field σ (16), $d^2U(0)/d\sigma^2$;
m-thermal screening mass of σ (17) and (22);
$\tilde{m}$-thermal pole mass (rest energy) of σ quasiparticles (79) and (81);
$m_{\rm c}$ or $m_{\rm k}$-screening masses of σ at chemical or kinetic freeze-out;
N-total number of particles;
n_0-equilibrium density of particles (55);
$\boldsymbol{p}$ or $\boldsymbol{p}'$-particle momentum variable in f;
p_T-the magnitude of the component of $\boldsymbol{p}$ transverse to beam axis;
$\boldsymbol{q}$-Fourier conjugate to $\boldsymbol{x}$ in $h(\boldsymbol{x}, \boldsymbol{p}; t)$;
$\mathbb{Q}$-matrix of noise correlators (28);
r_m-memory factor (97);
$S^{(2)}$-quadratic terms in the entropy (21);
T-temperature of the external bath (20);
U-$U(\sigma)$ potential for σ, (1);
V-$\int d^3\boldsymbol{x}$, 3-volume;
$\mathbb{V}$-evolution operator (43);
$\boldsymbol{v}$-short for $\boldsymbol{v}(\boldsymbol{p})$, particle velocity (6);
Γ_0-relaxation rate of σ due to interaction with the external thermal bath only (18b);
Γ-the full relaxation rate of σ (78) and (80);
γ or γ'-$\gamma(\boldsymbol{p})$ or $\gamma(\boldsymbol{p}')$, relativistic factor (6);
Δ^2-Eq. (63) and also $\tilde{m}^2 - m^2$ (79);
$\delta_{\boldsymbol{p}\boldsymbol{p}'}$-$(2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{p}')$, (36);
η-$\eta(x)$, Langevin noise in Eq. (18b);
$\boldsymbol{\theta}_\lambda$-dual vector to $\boldsymbol{h}_\lambda$ (53);
$\hat{\boldsymbol{\theta}}$-see Eq. (65);
λ_0-smallest eigenvalue of (52);
$\nu_{\boldsymbol{p}}$-momentum space distribution, (37);
ξ-$\xi(\boldsymbol{x}, \boldsymbol{p}; t)$, noise in Boltzmann Eq. (18a);

$\boldsymbol{\xi}$-noise vector $(\xi, \eta, 0)$ (26) and (28);
π-canonical momentum for σ, (18c);
$\boldsymbol{\Sigma}$-matrix of correlators $\langle \boldsymbol{h} \otimes \boldsymbol{h} \rangle$, (41);
$\tilde{\boldsymbol{\Sigma}}$-deviation of $\boldsymbol{\Sigma}$ from equilibrium (45);
σ-$\sigma(x)$, scalar field, critical mode;
$\hat{\boldsymbol{\sigma}}$-basis vector (60);
τ-collisional relaxation time (74);
τ_e-fireball evolution time scale, Sec. X A;
ψ-$\psi(\boldsymbol{v}^2)$, solution to Eq. (68);
$\int_p$ -see Eq. (8);
$\cdot$-the scalar product is defined in Eq. (25).

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