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Relativistic effects in the double S- and P-wave charmonium production in e^+e^- annihilation

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On the basis of perturbative QCD and the relativistic quark model we calculate relativistic and bound state corrections in the pair production of S-wave and P-wave charmonium states. Relativistic factors in the production amplitude connected with the relative motion of heavy quarks and the transformation law of the bound state wave function to the reference frame of the moving S- and P-wave mesons are taken into account. For the gluon and quark propagators entering the production vertex function we use a truncated expansion in the ratio of the relative quark momenta to the center-of-mass energy $\sqrt{s}$ up to the second order. The relativistic treatment of the wave functions makes all such second order terms convergent, thus allowing the reliable calculation of their contributions to the production cross section. Relativistic corrections to the quark bound state wave functions in the rest frame are considered by means of the QCD generalization of the standard Breit potential. It turns out that the examined effects change essentially the nonrelativistic results of the cross section for the reaction $e^+ + e^- \to J/\Psi(\eta_c) + \chi_{cJ}(h_c)$ at the center-of-mass energy $\sqrt{s} = 10.6$ GeV.

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I. INTRODUCTION

The large rate for the exclusive double charmonium production measured at the Belle and *BABAR* experiments [1,2] reveals definite problems in the theoretical description of these processes [3–5] on the basis of nonrelativistic QCD (NRQCD) at the leading order in α_s. Many theoretical efforts were made in order to improve the calculation of the production cross section $e^+ + e^- \to J/\Psi + \eta_c$. They included the analysis of other production mechanisms for the state $J/\Psi + \eta_c$ [6,7] and the calculation of relativistic corrections and corrections of next-to-leading order in α_s which could change essentially the initial nonrelativistic result [8–18]. New calculation made recently in [18] included a number of improvements connected with the resummation of a class of relativistic corrections, the inclusion of the contribution that arises from the interference between the relativistic corrections and the corrections of next-to-leading order in α_s and some others. As a result the discrepancy between the theoretical prediction for $\sigma(e^+ + e^- \to J/\Psi + \eta_c)$ and experimental measurements has been resolved in [18]. But despite the evident successes achieved in the solution of this problem on the basis of NRQCD and also in the light cone method and quark potential models, the double charmonium production in e^+e^- annihilation remains an interesting task. On the one hand, the reason is that there exist the production processes of the P- and D-wave charmonium states which should be investigated as the production of S-wave states. On the other hand, the variety of the used approaches and the model parameters in this problem raises the question about the comparison of the obtained results resulting in a better understanding of the quark-gluon dynamics. Two main sources of the enhancement of the nonrelativistic cross section for the double charmonium production are revealed to the present in all used approaches: the radiative corrections of order $O(\alpha_s)$ and the relative motion of c quarks forming the bound states.

In this work we continue the investigation of the exclusive double charmonium production in e^+e^- annihilation on the basis of a relativistic quark model [14,19–21] in the case of S- and P-wave charmonium states. The relativistic quark model provides the solution in many tasks of heavy quark physics. Several basic features of the quark model and NRQCD are very similar. The expansion of the production amplitude and heavy quark interaction operator over relative momenta of heavy quarks which is used in quark models is very close to the expansions in the framework of NRQCD. Basic nonperturbative NRQCD matrix elements can be expressed in terms of the quark bound state wave functions. The relativistic quark model gives the possibility to study the question about a broadening of the meson wave functions due to the account of special class of relativistic corrections which can lead to the increase of the double charmonium production cross sections. Thus, the aim of this study consists in the calculation of relativistic effects in the production processes $e^+ + e^- \to J/\Psi(\eta_c) + \chi_{cJ}(h_c)$ on the basis of a quasipotential approach [14,19] which allows to keep relativistic corrections both in the production amplitude and quark bound state wave functions.

II. GENERAL FORMALISM

We consider the following reactions $e^+ + e^- \to J/\Psi(\eta_c) + \chi_{cJ}(h_c)$, where the final state consists of the pair of S wave (J/Ψ or η_c) and P wave (χ_{c0}, χ_{c1}, χ_{c2} or h_c) charm mesons. The diagrams that give contributions to the amplitude of these processes in the leading order of the QCD coupling constant α_s are presented in Fig. 1. Two

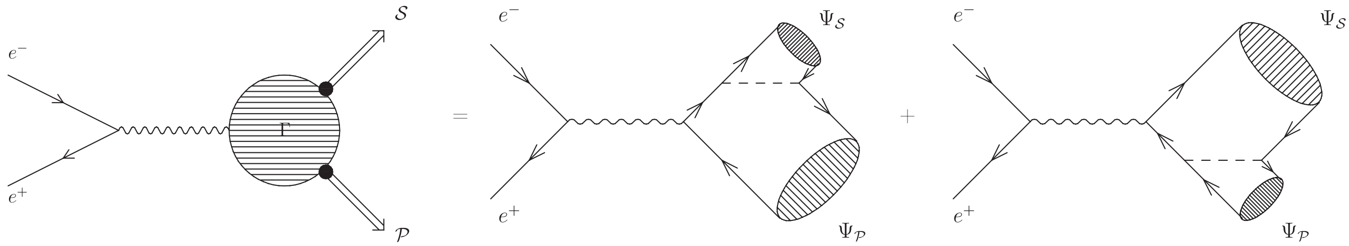

FIG. 1. The production amplitude of a pair of S- and P-wave charmonium states in e^+e^- annihilation. $\mathcal{S}$ denotes the S-state meson and $\mathcal{P}$ the P-wave meson. The wavy line shows the virtual photon and the dashed line corresponds to the gluon. Γ is the production vertex function.

other diagrams can be obtained by corresponding permutations. There are two stages of the production process. In the first stage, which is described by perturbative QCD, the virtual photon γ^* produces four heavy c quarks and $\bar{c}$ antiquarks with the following four-momenta:

$$p_{1,2} = \tfrac{1}{2}P \pm p, \qquad (p \cdot P) = 0; \qquad q_{1,2} = \tfrac{1}{2}Q \pm q, \qquad (q \cdot Q) = 0, \tag{1}$$

where $P(Q)$ are the total four-momenta, $p = L_P(0, \mathbf{p})$, $q = L_P(0, \mathbf{q})$ are the relative four-momenta obtained from the rest frame four-momenta $(0, \mathbf{p})$ and $(0, \mathbf{q})$ by the Lorentz transformation to the system moving with the momenta P, Q. In the second nonperturbative stage, quark-antiquark pairs form the final mesons.

Let examine the production amplitude of the S-wave vector state (J/Ψ) and P-wave states χ_{cJ} $(J = 0, 1, 2)$, which can be presented in the form [14,21]

$$\mathcal{M}(p_-, p_+, P, Q) = \frac{8\pi^2\alpha\alpha_s\mathcal{Q}_c}{3s}\bar{v}(p_+)\gamma^\beta u(p_-)\int\frac{d\mathbf{p}}{(2\pi)^3} \times \int\frac{d\mathbf{q}}{(2\pi)^3} \times Sp\{\Psi^{\mathcal{S}}(p, P) \times \Gamma_1^{\beta\nu}(p, q, P, Q)\Psi^{\mathcal{P}}(q, Q)\gamma_\nu + \Psi^{\mathcal{P}}(q, Q)\Gamma_2^{\beta\nu}(p, q, P, Q) \times \Psi^{\mathcal{S}}(p, P)\gamma_\nu\}, \tag{2}$$

where α is the fine structure constant and $\mathcal{Q}_c$ is the c-quark electric charge. The relativistic S- and P-wave functions of the bound quarks $\Psi^{\mathcal{S},\mathcal{P}}$ accounting for the transformation from the rest frame to the moving one with four-momenta P, Q, are

$$\Psi^{\mathcal{S}}(p, P) = \frac{\Psi_0^{\mathcal{S}}(\mathbf{p})}{\left[\frac{\epsilon(p)}{m}\frac{(\epsilon(p)+m)}{2m}\right]}\left[\frac{\hat{v}_1 - 1}{2} + \hat{v}_1\frac{\mathbf{p}^2}{2m(\epsilon(p)+m)} - \frac{\hat{p}}{2m}\right] \times \hat{\varepsilon}_{\mathcal{S}}^*(1 + \hat{v}_1)\left[\frac{\hat{v}_1 + 1}{2} + \hat{v}_1\frac{\mathbf{p}^2}{2m(\epsilon(p)+m)} + \frac{\hat{p}}{2m}\right], \tag{3}$$

$$\Psi^{\mathcal{P}}(q, Q) = \frac{\Psi_0^{\mathcal{P}}(\mathbf{q})}{\left[\frac{\epsilon(q)}{m}\frac{(\epsilon(q)+m)}{2m}\right]}\left[\frac{\hat{v}_2 - 1}{2} + \hat{v}_2\frac{\mathbf{q}^2}{2m(\epsilon(q)+m)} + \frac{\hat{q}}{2m}\right] \times \hat{\varepsilon}_{\mathcal{P}}^*(Q, S_z)(1 + \hat{v}_2)\left[\frac{\hat{v}_2 + 1}{2} + \hat{v}_2\frac{\mathbf{q}^2}{2m(\epsilon(q)+m)} - \frac{\hat{q}}{2m}\right], \tag{4}$$

where the hat is a notation for the contraction of the four vector with the Dirac matrices, $v_1 = P/M_{\mathcal{S}}$, $v_2 = Q/M_{\mathcal{P}}$; ε_S is the polarization vector of the vector charmonium J/Ψ; $\varepsilon_{\mathcal{P}}(Q, S_z)$ is the polarization vector of the spin-triplet state χ_{cJ}, $\epsilon(p) = \sqrt{p^2 + m^2}$ and m is the c-quark mass. The relativistic wave functions in Eqs. (3) and (4) are equal to the product of the wave functions in the rest frame $\Psi_0^{\mathcal{S},\mathcal{P}}$ and the spin-1 projection operators that are accurate at all orders in $|\mathbf{p}|/m$ [14,21]. The expression of the spin projector in a slightly different form has been derived primarily in [22] in the framework of NRQCD. Our derivation of relations (3) and (4) accounts for the transformation law of the bound state wave functions from the rest frame to the moving one with four-momenta P and Q. This transformation law was discussed in the Bethe-Salpeter approach in [23] and in the quasipotential method in [24]. At the leading order in α_s the vertex functions $\Gamma_{1,2}^{\beta\nu}(p, P; q, Q)$ can be written as

$$\Gamma_1^{\beta\nu}(p, P; q, Q) = \gamma_\mu\frac{(\hat{l} - \hat{q}_1 + m)}{(l - q_1)^2 - m^2 + i\epsilon}\gamma_\beta D^{\mu\nu}(k_1) + \gamma_\beta\frac{(\hat{p}_1 - \hat{l} + m)}{(l - p_1)^2 - m^2 + i\epsilon}\gamma_\mu D^{\mu\nu}(k_1), \tag{5}$$

$$\Gamma_2^{\beta\nu}(p, P; q, Q) = \gamma_\beta\frac{(\hat{q}_2 - \hat{l} + m)}{(l - q_2)^2 - m^2 + i\epsilon}\gamma_\mu D^{\mu\nu}(k_2) + \gamma_\mu\frac{(\hat{l} - \hat{p}_2 + m)}{(l - p_2)^2 - m^2 + i\epsilon}\gamma_\beta D^{\mu\nu}(k_2), \tag{6}$$

where the gluon momenta are $k_1 = p_1 + q_1$, $k_2 = p_2 + q_2$ and $l^2 = s = (P + Q)^2 = (p_- + p_+)^2$, p_-, p_+ are four-momenta of the electron and positron. The dependence on the relative momenta of c quarks is presented both in the gluon propagator $D_{\mu\nu}(k)$ and quark propagator as well as in the relativistic wave functions (3) and (4). Taking into account that the ratio of the relative quark momenta p and q to the energy $\sqrt{s}$ is small, we expand the inverse denominators of quark and gluon propagators in the form

$$\frac{1}{(l-q_{1,2})^2-m^2}=\frac{2}{s}\left[1-\frac{2M_{\mathcal{S}}^2-M_{\mathcal{P}}^2-4m^2}{2s}-\frac{2q^2}{s}\pm\frac{4(lq)}{s}+\frac{16(lq)^2}{s^2}+\cdots\right], \quad (7)$$

$$\frac{1}{(l-p_{1,2})^2-m^2}=\frac{2}{s}\left[1-\frac{2M_{\mathcal{P}}^2-M_{\mathcal{S}}^2-4m^2}{2s}-\frac{2p^2}{s}\pm\frac{4(lp)}{s}+\frac{16(lp)^2}{s^2}+\cdots\right], \quad (8)$$

$$\frac{1}{k_{2,1}^2}=\frac{4}{s}\left[1-\frac{4(p^2+q^2+2pq)}{s}\pm\frac{4(lp+lq)}{s}+\frac{16}{s^2}[(lp)^2+(lq)^2+2(lp)(lq)]+\cdots\right]. \quad (9)$$

In the expansions (7)–(9) we accounted for terms of second order in relative momenta p and third order in relative momenta q. Substituting (7)–(9), (3) and (4) in (2) we preserve relativistic factors involved in the denominators of the relativistic wave functions (3) and (4), but in the numerator of the amplitude (2) we take into account corrections of second order in $|\mathbf{p}|/m$ and up to fourth order in $|\mathbf{q}|/m$. This provides the convergence of the resulting momentum integrals. The expansion of the production amplitude in $|\mathbf{p}|/m$ and $|\mathbf{q}|/m$ is quite close to the approach used in the framework of NRQCD [25]. Then the angular integrals are calculated using the following relations:

$$\int\frac{\Psi_0^{\mathcal{S}}(\mathbf{p})}{\left[\frac{\epsilon(p)}{m}\frac{(\epsilon(p)+m)}{2m}\right]}\frac{d\mathbf{p}}{(2\pi)^3}=\frac{1}{\sqrt{2}\pi}\int_0^\infty\frac{p^2R_{\mathcal{S}}(p)}{\left[\frac{\epsilon(p)}{m}\frac{(\epsilon(p)+m)}{2m}\right]}dp, \quad (10)$$

$$\int q_\mu\frac{\Psi_0^{\mathcal{P}}(\mathbf{q})}{\left[\frac{\epsilon(q)}{m}\frac{(\epsilon(q)+m)}{2m}\right]}\frac{d\mathbf{q}}{(2\pi)^3}=-i\varepsilon_{\mathcal{P}\mu}(Q,L_z)\frac{1}{\pi\sqrt{6}}\times\int_0^\infty q^3\frac{R_{\mathcal{P}}(q)}{\left[\frac{\epsilon(q)}{m}\frac{(\epsilon(q)+m)}{2m}\right]}dq, \quad (11)$$

$$\int p_\mu p_\nu\frac{\Psi_0^{\mathcal{S}}(\mathbf{p})}{\left[\frac{\epsilon(p)}{m}\frac{(\epsilon(p)+m)}{2m}\right]}\frac{d\mathbf{p}}{(2\pi)^3}=-\frac{1}{6\sqrt{2}\pi}(g_{\mu\nu}-v_{1\mu}v_{1\nu})\times\int_0^\infty p^4\frac{R_{\mathcal{S}}(p)}{\left[\frac{\epsilon(p)}{m}\frac{(\epsilon(p)+m)}{2m}\right]}dp, \quad (12)$$

$$\int\frac{q_\alpha q_\beta q_\gamma\Psi_0^{\mathcal{P}}(\mathbf{q})}{\left[\frac{\epsilon(q)}{m}\frac{(\epsilon(q)+m)}{2m}\right]}\frac{d\mathbf{q}}{(2\pi)^3}=\frac{i}{5\pi\sqrt{6}}[\varepsilon_\gamma(Q,L_z)P_{\alpha\beta}+\varepsilon_\alpha(Q,L_z)P_{\gamma\beta}+\varepsilon_\beta(Q,L_z)P_{\alpha\gamma}]\times\int_0^\infty\frac{q^5R_{\mathcal{P}}(q)}{\left[\frac{\epsilon(q)}{m}\frac{(\epsilon(q)+m)}{2m}\right]}dq, \quad (13)$$

where $P_{\alpha\beta}=(g_{\alpha\beta}-v_{2\alpha}v_{2\beta})$, $R_{\mathcal{S}}(p)$, $R_{\mathcal{P}}(q)$ are the radial momentum wave functions of S- and P-wave charmonium states, $\varepsilon_\mu(Q,L_z)$ is the polarization vector in orbital space. Whereas Eqs. (10) and (11) are used for the leading order contribution, Eqs. (12) and (13) contain the relativistic corrections of the necessary order. For a specific P-wave state, summing over S_z and L_z in the amplitude (2) can be further simplified as [26]

$$\sum_{S_z,L_z}\langle 1,L_z;1,S_z|J,J_z\rangle\varepsilon^*_{\mathcal{P}\alpha}(Q,L_z)\varepsilon^*_{\mathcal{P}\beta}(Q,S_z)=\begin{cases}\frac{1}{\sqrt{3}}(g_{\alpha\beta}-v_{2\alpha}v_{2\beta}), & J=0,\\ \frac{i}{\sqrt{2}}\epsilon_{\alpha\beta\sigma\rho}v_2^\sigma\epsilon^{*\rho}(Q,J_z), & J=1,\\ \epsilon_{\alpha\beta}(Q,J_z), & J=2,\end{cases} \quad (14)$$

where $\langle 1,L_z;1,S_z|J,J_z\rangle$ are the Clebsch-Gordon coefficients. Calculating the trace in the amplitude (2) by means of expressions (3)–(6) and (14) and the system FORM [27], we find that the tensor parts of four amplitudes describing the production of S- and P-wave charmonium states in the used approximation have the following structure:

$$S_{1,\beta}(J/\Psi+\chi_{c0})=A_1\varepsilon^*_{S\beta}+A_2v_{1\beta}(v_2\varepsilon^*_S)+A_3v_{2\beta}(v_2\varepsilon^*_S), \quad (15)$$

$$S_{2,\beta}(J/\Psi+\chi_{c1})=B_1\varepsilon_{\alpha\lambda\gamma\beta}v_1^\alpha v_2^\lambda\varepsilon^{*\gamma}(Q,J_z)(v_2\varepsilon^*_S)+B_2\varepsilon_{\alpha\lambda\gamma\beta}v_2^\alpha\varepsilon_S^{*\lambda}\varepsilon^{*\gamma}(Q,J_z)+B_3v_{1\beta}\varepsilon_{\alpha\lambda\gamma\sigma}v_1^\alpha v_2^\lambda\varepsilon_S^{*\gamma}\varepsilon^{*\sigma}(Q,J_z)+B_4v_{2\beta}\varepsilon_{\alpha\lambda\gamma\sigma}v_1^\alpha v_2^\lambda\varepsilon_S^{*\gamma}\varepsilon^{*\sigma}(Q,J_z), \quad (16)$$

$$S_{3,\beta}(J/\Psi+\chi_{c2})=\varepsilon^*_{\alpha\sigma}(Q,J_z)[C_1\varepsilon_S^{*\alpha}g_{\sigma\beta}+C_2v_1^\alpha(v_2\varepsilon_S^*)g_{\beta\sigma}+C_3v_1^\alpha\varepsilon_S^{*\sigma}v_{1\beta}+C_4v_1^\alpha\varepsilon_S^{*\sigma}v_{2\beta}+C_5v_1^\alpha v_1^\sigma\varepsilon^*_{S\beta}+C_6v_1^\alpha v_1^\sigma v_1^\beta(_2\varepsilon_S^*)+C_7v_1^\alpha v_1^\sigma v_{2\beta}(v_2\varepsilon_S^*)], \quad (17)$$

$$S_{4,\beta}(\eta_c+h_c)=D_1v_{1\beta}(v_1\varepsilon^*(Q,L_z))+D_2v_{2\beta}(v_1\varepsilon^*(Q,L_z))+D_3\varepsilon^*_\beta(Q,L_z), \quad (18)$$

where the coefficients A_i, B_i, C_i, D_i can be presented as sums of terms containing the factors $u=M_\mathcal{P}/(M_\mathcal{P}+M_\mathcal{S})$, $\kappa=m/(M_\mathcal{P}+M_\mathcal{S})$ and $C_{ij}=c^i(p)c^j(q)=[(m-\varepsilon(p))/(m+\varepsilon(p))]^i[(m-\varepsilon(q))/(m+\varepsilon(q))]^j$, preserving terms with $i+j\leq 2$, and $r^2=(M_\mathcal{P}+M_\mathcal{S})^2/s$ up to terms of order $O(r^4)$. Exact analytical expressions for these coefficients are sufficiently lengthy (compare with the results written in Appendix A of our previous paper [19]), so, we present in Appendix A of this work only their approximate numerical form using the observed meson masses and the c-quark mass $m=1.55$ GeV.

Introducing the scattering angle θ between the electron momentum $\mathbf{p}_e$ and momentum $\mathbf{P}$ of the J/Ψ meson, we can calculate the differential cross section $d\sigma/d\cos\theta$ and then the total cross section σ as a function of r^2. We find it useful to present the double charmonium production cross sections in the form ($k=0$, 0, 1, 2, 3 corresponds to χ_{c0}, χ_{c1}, χ_{c2} and h_c):

$$\sigma(J/\Psi(\eta_c)+\chi_{cJ}(h_c))=\frac{\alpha^2\alpha_s^2Q_c^2\pi r^2\sqrt{1-r^2}\sqrt{1-r^2(2u-1))}}{6912\kappa^2u^9(1-u)^9}\times\frac{|\tilde R_\mathcal{S}(0)|^2|\tilde R'_\mathcal{P}(0)|^2}{s(M_\mathcal{P}+M_\mathcal{S})^8}\sum_{i=0}^7F_i^{(k)}(r^2)\omega_i, \quad (19)$$

where the functions $F_i^{(k)}$ ($k=0$, 1, 2, 3) are written explicitly in Appendix B,

$$\tilde R_\mathcal{S}(0)=\frac{1}{2\pi^2}\int_0^\infty p^2R_S(p)\frac{(\epsilon(p)+m)}{2\epsilon(p)}dp, \quad (20)$$

$$\tilde R'_\mathcal{P}(0)=\frac{1}{3}\sqrt{\frac{2}{\pi}}\int_0^\infty q^3R_P(q)\frac{(\epsilon(q)+m)}{2\epsilon(q)}dq. \quad (21)$$

The parameters ω_i can be expressed in terms of momentum integrals I_n, J_n as follows:

$$I_n=\int_0^\infty p^2R_\mathcal{S}(p)\frac{(\epsilon(p)+m)}{2\epsilon(p)}\left(\frac{m-\epsilon(p)}{m+\epsilon(p)}\right)^n dp,\quad J_n=\int_0^\infty q^3R_\mathcal{P}(q)\frac{(\epsilon(q)+m)}{2\epsilon(q)}\left(\frac{m-\epsilon(q)}{m+\epsilon(q)}\right)^n dq, \quad (22)$$

$$\omega_0=1,\quad \omega_1=\frac{I_1}{I_0},\quad \omega_2=\frac{I_2}{I_0},\quad \omega_3=\omega_1^2,\quad \omega_4=\frac{J_1}{J_0},\quad \omega_5=\frac{J_2}{J_0},\quad \omega_6=\omega_4^2,\quad \omega_7=\omega_1\omega_4. \quad (23)$$

On the one part, in the potential quark model the relativistic corrections, connected with the relative motion of heavy c quarks, are involved in the production amplitude (2) and the cross section (19) through the different relativistic factors. They are determined in the final expression (19) by the specific parameters ω_i The momentum integrals (22) for the parameters ω_i are convergent and we calculate them numerically, using the wave functions obtained by the numerical solution of the Schrödinger equation. The exact form of the wave functions $\Psi_0^\mathcal{S}(\mathbf{p})$ and $\Psi_0^\mathcal{P}(\mathbf{q})$ is important for improving the accuracy of the calculation of relativistic contributions to the cross section (19). It is sufficient to note that the double charmonium production cross section $\sigma(s)$ contains the factor $|R_\mathcal{S}(0)|^2|R'_\mathcal{P}(0)|^2$ in the nonrelativistic approximation. Small changes of the numerical values of the bound state wave functions at the origin lead to substantial changes of the final results. In the approach based on nonrelativistic QCD this problem is closely related to the determination of the color-singlet matrix elements for the charmonium [25]. Thus, on the other part, there are relativistic corrections to the bound state wave functions $\Psi_0^\mathcal{S}(\mathbf{p})$, $\Psi_0^\mathcal{P}(\mathbf{q})$. In order to take them into account, we suppose that the dynamics of a $c\bar c$ pair is determined by the QCD generalization of the standard Breit Hamiltonian [28–30]:

$$H=H_0+\Delta U_1+\Delta U_2,\quad H_0=2\sqrt{\mathbf{p}^2+m^2}-2m-\frac{C_F\alpha_s}{r}+Ar+B, \quad (24)$$

$$\Delta U_1(r)=-\frac{C_F\alpha_s^2}{4\pi r}[2\beta_0\ln(\mu r)+a_1+2\gamma_E\beta_0],\quad a_1=\frac{31}{3}-\frac{10}{9}n_f,\quad \beta_0=11-\frac{2}{3}n_f, \quad (25)$$

$$\Delta U_2(r)=-\frac{C_F\alpha_s}{2m^2r}\left[\mathbf{p}^2+\frac{\mathbf{r}(\mathbf{r}\mathbf{p})\mathbf{p}}{r^2}\right]+\frac{\pi C_F\alpha_s}{m^2}\delta(\mathbf{r})+\frac{3C_F\alpha_s}{2m^2r^3}(\mathbf{S}\mathbf{L})-\frac{C_F\alpha_s}{2m^2}\left[\frac{\mathbf{S}^2}{r^3}-3\frac{(\mathbf{S}\mathbf{r})^2}{r^5}-\frac{4\pi}{3}(2\mathbf{S}^2-3)\delta(\mathbf{r})\right]-\frac{C_AC_F\alpha_s^2}{2mr^2}, \quad (26)$$

where n_f is the number of flavors, $C_A=3$ and $C_F=4/3$ are the color factors of the SU(3) color group. $\mathbf{S}$ is the spin of the $(c\bar c)$ state and $\mathbf{L}$ is the orbital angular momentum. For the dependence of the QCD coupling constant $\alpha_s(\mu^2)$ on the renormalization point μ^2 we use the leading order result

TABLE I. Numerical values of the relativistic parameters (20), (21), and (23) in the double charmonium production cross section (19).

Meson ($c\bar{c}$)	$n^{2S+1}L_J$	J^{PC}	$\tilde{R}_{\mathcal{S}}(0)$, GeV$^{3/2}$	$\tilde{R}'_{\mathcal{P}}(0)$, GeV$^{5/2}$	$\omega_1(\mathcal{S})$ or $\omega_4(\mathcal{P})$	$\omega_2(\mathcal{S})$ or $\omega_5(\mathcal{P})$
J/Ψ	1^3S_1	1^{--}	0.81	-	-0.20	0.0078
η_c	1^1S_0	0^{-+}	0.92	-	-0.20	0.0087
χ_{c0}	1^3P_0	0^{++}	-	0.19	-0.15	0.0065
χ_{c1}	1^3P_1	1^{++}	-	0.18	-0.14	0.0065
χ_{c2}	1^3P_2	2^{++}	-	0.18	-0.15	0.0065
h_c	1^1P_1	1^{+-}	-	0.18	-0.14	0.0065

$$\alpha_s(\mu^2) = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda^2)}. \qquad (27)$$

The typical momentum transfer scale in a quarkonium is of order of the quark mass, so we set the renormalization scale $\mu = m$ and $\Lambda = 0.168$ GeV, which gives $\alpha_s = 0.314$ for the charmonium states. The parameters of the linear potential $A = 0.18$ GeV^2 and $B = -0.16$ GeV have usual values of quark models. Starting with the Hamiltonian (24) we construct the effective potential model based on the Schrödinger equation and find its numerical solutions in the case of S- and P-wave charmonium [31]. The details of the used model are presented in Appendix C. Then we calculate the matrix elements entering in the expressions for the parameters ω_i and obtain the value of the production cross sections at $\sqrt{s} = 10.6$ GeV. So, we take into account relativistic corrections of order $O(v^2)$ in Eqs. (10)–(13) and (20)–(22) by means of the calculation of the bound state wave functions in the effective quark model. Basic parameters which determine our numerical results are written in Table I. The comparison of the obtained results with the previous calculations [3,4,32,33] and experimental data [1,2] is presented in Table II.

III. NUMERICAL RESULTS AND DISCUSSION

In this paper we have investigated the role of relativistic effects in the production processes of S- and P-wave ($c\bar{c}$) mesons in the quark model. In the present study of the production amplitude (2) we kept relativistic corrections of two types. The first type is determined by several functions depending on the relative quark momenta $\mathbf{p}$ and $\mathbf{q}$ and arising from the gluon propagator, the quark propagator and the relativistic meson wave functions (3) and (4). The corrections of second type originate from the perturbative treatment of the quark-antiquark interaction operator which leads to the different wave functions $\Psi_0^{\mathcal{S}}(\mathbf{p})$ and $\Psi_0^{\mathcal{P}}(\mathbf{q})$ for the S-wave and P-wave charmonium states, respectively. In addition, we systematically accounted for the bound state corrections working with the observed masses of S-wave mesons (J/Ψ, η_c) and P-wave mesons (χ_{cJ}, h_c). The calculated masses of S-wave and P-wave charmonium states agree well with experimental values [34] (see Table III). Note that the basic parameters of the model are kept fixed from the previous calculations of the meson mass spectra and decay widths [21,35,36]. The strong coupling constant entering in the production amplitude (2) is taken to be $\alpha_s = 0.24$ in accordance with the relation (27) at $\mu = 2m$.

Numerical results and their comparison with several previous calculations and experimental data are presented in Table II. Theoretically, there were two studies of the production $J/\Psi(\eta_c) + \chi_{cJ}(h_c)$ in e^+e^- annihilation in NRQCD [3,4]. They give 2.4 fb and 6.7 fb for the production of $J/\Psi + \chi_{c0}$. Such spread in results is explained by the different numerical values of the used parameters, i.e. the matrix elements, the mass of c quark m and the strong coupling constant α_s. The third investigation of the production of $J/\Psi + \chi_{c0}$ was done in the light cone formalism in [32] where the result 14.4 fb was obtained. The essential growth of the cross section in [32] at $\sqrt{s} = 10.6$ GeV is connected with the use of specific light cone wave functions describing the relative motion of heavy c quarks. The fourth study of the reaction $e^+ + e^- \to J/\Psi + \chi_{c0}$ was devoted to the next-to-leading order QCD corrections [33]. Here, it was shown that a sharp increase of the production cross section ($\sigma = 17.9$ fb) can be derived with the account of NLO in α_s contributions. On our opinion, all used approaches in this problem have

TABLE II. Comparison of the obtained results with previous theoretical predictions and experimental data.

State H_1H_2	$\sigma_{BABAR}\times$ $\mathrm{Br}_{H_2\to\mathrm{charged}\geq 2}$ (fb) [2]	$\sigma_{\mathrm{Belle}}\times$ $\mathrm{Br}_{H_2\to\mathrm{charged}\geq 2}$ (fb) [1]	σ_{NRQCD} (fb) [3]	σ (fb) [4]	σ (fb) [32]	σ (fb) [33]	Our result (fb)
$J/\Psi + \chi_{c0}$	$10.3 \pm 2.5^{+1.4}_{-1.8}$	$6.4 \pm 1.7 \pm 1.0$	2.40 ± 1.02	6.7	14.4	17.9(6.35)	4.79 ± 0.80
$J/\Psi + \chi_{c1}$			0.38 ± 0.12	1.1			1.07 ± 0.23
$J/\Psi + \chi_{c2}$			0.69 ± 0.13	1.6			1.10 ± 0.13
$\eta_c + h_c$			0.308 ± 0.017				0.24 ± 0.02

TABLE III. The parameters of the effective relativistic Hamiltonian.

Meson ($c\bar{c}$)	$n^{2S+1}L_J$	$\mathbf{p}^2_{\rm eff}$, GeV^2	$\tilde{m}$, GeV	E, GeV	β, GeV	b, GeV	$M^{\rm th}$, GeV	$M^{\rm exp}$, GeV, [34]
J/Ψ	1^3S_1	0.5	0.85	0.087	0.75	1.5	3.044	3.097
η_c	1^1S_0	0.5	0.85	0.087	0.75	1.5	2.989	2.980
χ_{c0}	1^3P_0	0.6	0.87	0.479	0.55	-	3.437	3.415
χ_{c1}	1^3P_1	0.6	0.87	0.479	0.55	-	3.479	3.511
χ_{c2}	1^3P_2	0.6	0.87	0.479	0.55	-	3.520	3.556
h_c	1^1P_1	0.6	0.87	0.479	0.55	-	3.486	3.526

both the advantages and drawbacks. The evident advantage of NRQCD and light cone approach consists in the use of the rigorous expansions of basic matrix elements in the heavy quark velocity v or light cone wave function moments. But as a result, we observe the appearance in NRQCD or light cone model numerous parameters connected with the nonperturbative matrix elements or higher twist wave functions. These additional parameters should be fixed with the reliable accuracy only after solution of many tasks in heavy quark physics and on the basis of experimental data. So, further successes of NRQCD or light cone approach are related with the determination of corresponding nonperturbative matrix elements. In the relativistic quark model we are also working with the nonperturbative parameters which are expressed through the bound state wave function. If we have the reliable approach for the construction of heavy quark interaction operator, we can obtain numerical predictions for different production cross sections. Unfortunately, the theoretical accuracy of the calculation of heavy quark potential is limited. As a result, the construction of the bound state wave function in the range of the relativistic momenta is not accurate as we expected. So, the problem of the theoretical accuracy remains also in the relativistic quark model. Nevertheless, we can hope that combining different methods for the investigation of heavy quark production processes we can obtain the independent predictions for numerous nonperturbative parameters in the theory and find the physical observables with essentially higher accuracy.

The exclusive double charmonium production cross section presented in the form (19) is convenient for a comparison with the results of NRQCD. Indeed, in the nonrelativistic limit, when $u = 1/2$, $\kappa = 1/4$, $\omega_i = 0$ ($i \geq 1$), $r^2 = 16$ m^2/s, the cross section (19) coincides with the calculation in [3]. In this limit the functions $F_0^{(k)}(r^2)$ transform into corresponding functions F_k from [3]. When we take into account bound state corrections working with observed meson masses, we get $u = M_{\mathcal{P}}/(M_{\mathcal{P}} + M_{\mathcal{S}}) \neq 1/2$, $\kappa = m/(M_{\mathcal{P}} + M_{\mathcal{S}}) \neq 1/4$. This leads to the modification of the general factor in (19) in comparison with the nonrelativistic theory and the form of the functions $F_0^{(k)}$ (see [3]). It follows from the numerical values of the parameters ω_i, presented in Table I, that the relativistic corrections amount to 15–20% in the production amplitude. Moreover, the relativistic effects decrease the values of the parameters $R_{\mathcal{S}}(0)$, $R'_{\mathcal{P}}(0)$, which transform into $\tilde{R}_{\mathcal{S}}(0)$, $\tilde{R}'_{\mathcal{P}}(0)$. In all considered reactions $e^+ + e^- \to J/\Psi(\eta_c) + \chi_{cJ}(h_c)$ the relativistic effects increase the nonrelativistic cross section, but in the case of the production $\eta_c + h_c$ the sum of bound state and relativistic corrections decreases the nonrelativistic cross section. It is necessary to point out once again that the essential effect on the value of the production cross sections $J/\Psi(\eta_c) + \chi_{cJ}(h_c)$ belongs to the parameters $\tilde{R}_{\mathcal{S}}(0)$, $\tilde{R}'_{\mathcal{P}}(0)$, α_s, m. Small changes in their values can lead to significant changes for the production cross sections. Comparing the values of the parameters $R_{\mathcal{S}}(0)$ ($|R_{J/\Psi,\eta_c}(0)|^2 = 0.9$; 1.2 GeV$^{3/2}$), $R'_{\mathcal{P}}(0)$ ($|R'_{\mathcal{P}}(0)|^2 = 0.043$ GeV$^{5/2}$) obtained in this study on the basis of quark model and in [33] we see that the values of the radial wave functions at the origin are very close, but our value for the derivative of the radial wave function at the origin is slightly smaller. The calculation of radiative corrections $O(\alpha_s)$ to the nonrelativistic cross section of the production $J/\Psi + \chi_{c0}$ was done recently in [33]. It evidently shows that one-loop corrections are considerable (factor $K = 2.8$ to nonrelativistic result). As a result, the total value of the cross section $J/\Psi + \chi_{c0}$ significantly increases. It should be noted that the difference between the theory and both *BABAR* and Belle experiments became threatening.

We presented a systematic treatment of relativistic effects in the S- and P-wave double charmonium production in e^+e^- annihilation. We explicitly separated two different types of relativistic contributions to the production amplitudes. The first type includes the relativistic v/c corrections to the wave functions and their relativistic transformations. The second type includes the relativistic $p/\sqrt{s}$ corrections emerging from the expansion of the quark and gluon propagators. The latter corrections were taken into account up to the second order. It is important to note that the expansion parameter $p/\sqrt{s}$ is very small. In our analysis of the production amplitudes we correctly take into account relativistic contributions of order $O(v^2/c^2)$ for the S-wave meson and corrections of orders $O(v^2/c^2)$ and $O(v^4/c^4)$ for the P-wave mesons. We cannot keep corrections of order $O(v^4/c^4)$ for the P-wave part of the amplitude (2) because they become divergent if we use expansions (7)–(9). Therefore, the basic theoretical uncertainty of our calculation is connected with the omitted terms of order $O(\mathbf{p}^4/m^4)$. Taking into account that the average value of the heavy quark velocity squared in the charmonium is $\langle v^2\rangle = 0.3$, we expect that they should not exceed 30% of the obtained relativistic contribution. These theoretical errors in the calculated production cross section at $\sqrt{s} = 10.6$ GeV are shown directly in Table II. In the cross section (19) we have neglected the terms containing the product of I_n and J_n with summary index ≥ 2 because their contribution has been found negligibly small. There are no another comparable uncertainties related to the choice of m or any other parameters of the model, since their values were fixed from our previous consideration of meson and baryon properties [21,35].

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APPENDIX A: THE COEFFICIENTS A_i, B_i, C_i, D_i ENTERING IN THE PRODUCTION AMPLITUDES (15)–(18)

These coefficients are the sums of the terms containing the parameters $u = M_{\mathcal{P}}/(M_{\mathcal{P}} + M_{\mathcal{S}})$ and $\kappa = m/(M_{\mathcal{P}} + M_{\mathcal{S}})$. We present A_i, B_i, C_i, D_i in numerical form using the observed meson masses and the mass of c quark $m = 1.55$ GeV.

$e^+ + e^- \to J/\Psi + \chi_{c0}$

$$A_1 = -7.05 + \frac{18.55}{r^2} + 0.013r^2 + C_{20}\Big(7.05 - \frac{18.55}{r^2} + 0.013r^2\Big) + C_{02}\Big(5.42 - \frac{5.41}{r^2} - 0.013r^2\Big) + C_{10}\Big(68.16 - \frac{63.92}{r^2} - 12.15r^2 + 0.089r^4\Big) + C_{01}\Big(23.29 - \frac{13.71}{r^2} - 8.11r^2 + 0.066r^4\Big) + C_{11}\Big(-76.83 + \frac{41.48}{r^2} + 62.06r^2 - 12.66r^4\Big), \tag{A1}$$

$$A_2 = -8.35 + 2.34r^2 + C_{20}(8.35 - 2.34r^2) + C_{02}(1.80 - 2.34r^2) + C_{10}(30.38 - 26.93r^2 + 4.00r^4) + C_{01}(5.96 - 9.62r^2 + 2.69r^4) + C_{11}(-19.22 + 36.18r^2 - 23.80r^4), \tag{A2}$$

$$A_3 = -0.99 + 0.99r^2 + C_{20}(0.99 - 0.99r^2) + C_{02}(1.98 - 0.99r^2) + C_{10}(1.65 - 9.72r^2 + 1.50r^4) + C_{01}(0.97 - 2.21r^2 + 0.94r^4) + C_{11}(-1.62 + 4.02r^2 - 5.24r^4). \tag{A3}$$

$e^+ + e^- \to J/\Psi + \chi_{c1}$

$$B_1 = -0.002 + 2.66r^2 + C_{20}(0.002 - 2.66r^2) - 2.66r^2C_{02} + C_{10}(-6.17 - 14.23r^2 + 4.59r^4) + C_{01}(-4.62 - 7.14r^2 + 3.13r^4) + C_{11}(11.56 + 28.13r^2 - 27.18r^4), \tag{A4}$$

$$B_2 = -2.10 - \frac{0.004}{r^2} + C_{20}\Big(2.10 + \frac{0.004}{r^2}\Big) + C_{10}\Big(14.99 - \frac{12.39}{r^2} - 3.34r^2 + 0.016r^4\Big) + C_{01}\Big(7.29 - \frac{9.28}{r^2} - 0.83r^2 + 0.01r^4\Big) + C_{11}\Big(-24.44 + \frac{23.21}{r^2} + 10.13r^2 - 0.98r^4\Big), \tag{A5}$$

$$B_3 = -1.63r^2 + 1.63r^2C_{20} + 1.63r^2C_{02} + C_{10}(10.64r^2 - 2.87r^4) + C_{01}(5.43r^2 - 1.20r^4) + C_{11}(-16.41r^2 + 13.44r^4), \tag{A6}$$

$$B_4 = -0.002 + 0.82r^2 + C_{20}(0.002 - 0.82r^2) - 0.82r^2C_{02} + C_{10}(-6.17 - 1.23r^2 + 1.77r^4) + C_{11}(11.56 + 1.35r^2 - 11.46r^4). \tag{A7}$$

$e^+ + e^- \to J/\Psi + \chi_{c2}$

$$C_1 = -1.84 + 1.84C_{20} + C_{10}(7.50 - 1.87r^2 + 0.01r^4) + C_{01}(3.72 - 0.97r^2 + 0.02r^4) + C_{11}(-23.15 + 11.46r^2 - 0.95r^4), \tag{A8}$$

$$C_2 = 2.63r^2 - 2.63r^2C_{20} - 2.63r^2C_{02} + C_{10}(-13.28r^2 + 4.55r^4) + C_{01}(-7.72r^2 + 2.96r^4) + C_{11}(44.33r^2 - 30.95r^4), \tag{A9}$$

$$C_3 = 1.59r^2 - 1.59r^2C_{20} - 1.59r^2C_{02} + C_{10}(-5.58r^2 + 2.85r^4) + C_{01}(-3.91r^2 + 1.10r^4) + C_{11}(12.99r^2 - 14.29r^4), \tag{A10}$$

$$C_4 = -0.80r^2 - 0.80r^2C_{20} + 0.80r^2C_{02} + C_{10}(1.41r^2 - 1.75r^4) + C_{01}(4.02r^2 - 1.51r^4) + C_{11}(-4.43r^2 + 12.19r^4), \quad (A11)$$

$$C_5 = -2.39r^2 - 2.39r^2C_{20} + 2.39r^2C_{02} + C_{10}(6.91r^2 - 4.33r^4) + C_{01}(10.29r^2 - 2.98r^4) + C_{11}(-32.09r^2 + 32.60r^4), \quad (A12)$$

$$C_6 = -2.37r^4C_{10} + C_{11}r^2(-12.59r^2 + 1.65r^4),$$
$$C_7 = 3.29r^4C_{10} + C_{11}r^2(-12.59r^2 + 1.65r^4). \quad (A13)$$

$e^+ + e^- \to \eta_c + h_c$

$$D_1 = -1.58 + 1.29r^2 + C_{20}(1.58 - 1.29r^2) + C_{02}(1.58 - 1.29r^2) + C_{10}(0.70 - 9.33r^2 + 2.97r^4) + C_{01}(1.35 - 5.29r^2 + 1.01r^4) + C_{11}(-0.60 + 19.42r^2 - 16.51r^4), \quad (A14)$$

$$D_2 = 0.14 - 1.14r^2 + C_{20}(-0.14 + 1.14r^2) + C_{02}(-3.73 + 1.14r^2) + C_{10}(7.11 + 2.77r^2 + 2.37r^4) + C_{01}(-0.12 + 4.72r^2 - 1.54r^4) + C_{11}(-6.08 - 3.69r^2 - 12.92r^4), \quad (A15)$$

$$D_3 = -1.49 + \frac{3.47}{r^2} + C_{20}\left(1.49 - \frac{3.47}{r^2}\right) + C_{10}\left(12.21 - \frac{15.99}{r^2} - 2.35r^2 + 0.02r^4\right) + C_{01}\left(6.76 - \frac{2.97}{r^2} - 0.80r^2 + 0.02r^4\right) + C_{11}\left(-32.42 + \frac{13.67}{r^2} + 13.22r^2 - 1.35r^4\right). \quad (A16)$$

APPENDIX B: THE FUNCTIONS $F_i^{(k)}(r^2)$ ($k = 0, 1, 2, 3$) ENTERING IN THE PRODUCTION CROSS SECTION (19)

$e^+ + e^- \to J/\Psi + \chi_{c0}$

$$F_0^{(0)} = 12.94r^2 + 690.48r^4 - 391.86r^6 + 39.82r^8 + 6.36r^{10}, \quad (B1)$$

$$F_1^{(0)} = -43.18r^2 - 4554.04r^4 + 5941.64r^6 - 1942.48r^8 + 89.61r^{10}, \quad (B2)$$

$$F_2^{(0)} = -25.87r^2 - 1380.96r^4 + 783.71r^6 - 79.64r^8 - 12.72r^{10}, \quad (B3)$$

$$F_3^{(0)} = 36.04r^2 + 7634.68r^4 - 15\,478.8r^6 + 10\,011.2r^8 - 2095.47r^{10}, \quad (B4)$$

$$F_4^{(0)} = -25.29r^2 - 996.50r^4 + 1961.67r^6 - 957.35r^8 + 97.10r^{10}, \quad (B5)$$

$$F_5^{(1)} = -38.81r^2 - 447.77r^4 + 601.11r^6 - 101.74r^8 - 12.87r^{10}, \quad (B6)$$

$$F_6^{(1)} = 12.36r^2 + 349.41r^4 - 1216.3r^6 + 1370.5r^8 - 579.05r^{10}, \quad (B7)$$

$$F_7^{(1)} = 84.44r^2 + 6382.47r^4 - 15\,011.8r^6 + 13\,175.8r^8 - 4056.79r^{10}. \quad (B8)$$

$e^+ + e^- \to J/\Psi + \chi_{c1}$

$$F_0^{(1)} = 165.06r^4 - 248.34r^6 + 75.88r^8 + 17.33r^{10}, \quad (B9)$$

$$F_1^{(1)} = -1655.33r^4 + 3347.61r^6 - 1829.98r^8 + 80.46r^{10}, \quad (B10)$$

$$F_2^{(1)} = -330.12r^4 + 496.69r^6 - 151.76r^8 - 34.67r^{10}, \quad (B11)$$

$$F_3^{(1)} = 4263.79r^4 - 10\,572.2r^6 + 8306.62r^8 - 1953.72r^{10}, \quad (B12)$$

$$F_4^{(1)} = -752.14r^4 + 1660.5r^6 - 979.66r^8 + 60.00r^{10}, \quad (B13)$$

$$F_5^{(1)} = -237.01r^4 + 390.21r^6 - 117.85r^8 - 35.64r^{10}, \quad (B14)$$

$$F_6^{(1)} = 920.55rr^4 - 2520.4r^6 + 2230.78r^8 - 612.50r^{10}, \quad (B15)$$

$$F_7^{(1)} = 7012.32r^4 - 18\,599.5r^6 + 15\,704.7r^8 - 4122.93r^{10}. \quad (B16)$$

$e^+ + e^- \to J/\Psi + \chi_{c2}$

$$F_0^{(2)} = 23.41r^2 + 99.63r^4 - 247.93r^6 + 110.30r^8 + 27.28r^{10}, \quad (B17)$$

$$F_1^{(2)} = -78.15r^2 - 931.82r^4 + 2499.9r^6 - 1804.84r^8 + 137.34r^{10}, \quad \text{(B18)}$$

$$F_2^{(2)} = -46.82r^2 - 199.27r^4 + 495.86r^6 - 220.60r^8 - 54.56r^{10}, \quad \text{(B19)}$$

$$F_3^{(2)} = 65.23r^2 + 1952.35r^4 - 5759.03r^6 + 5433.52r^8 - 1541.06r^{10}, \quad \text{(B20)}$$

$$F_4^{(2)} = -94.75r^2 - 1014.54r^4 + 2475.82r^6 - 1509.41r^8 + 50.08r^{10}, \quad \text{(B21)}$$

$$F_5^{(2)} = -70.23r^2 - 213.25r^4 + 608.04r^6 - 269.46r^8 - 55.66r^{10}, \quad \text{(B22)}$$

$$F_6^{(2)} = 95.89r^2 + 2191.11r^4 - 5632.06r^6 + 4350.3r^8 - 954.85rr^{10}, \quad \text{(B23)}$$

$$F_7^{(2)} = 316.34r^2 + 7972.42r^4 - 22\,933.8r^6 + 20\,774.7r^8 - 5658.04r^{10}. \quad \text{(B24)}$$

$e^+ + e^- \to \eta_c + h_c$

$$F_0^{(3)} = 11.69r^2 - 27.29r^4 + 35.00r^6 - 22.48r^8 + 6.02r^{10}, \quad \text{(B25)}$$

$$F_1^{(3)} = -10.35r^2 + 41.40r^4 - 185.00r^6 + 250.24r^8 - 141.63r^{10}, \quad \text{(B26)}$$

$$F_2^{(3)} = -23.375r^2 + 54.57r^4 - 70.00r^6 + 44.97r^8 - 12.04r^{10}, \quad \text{(B27)}$$

$$F_3^{(3)} = 2.29r^2 + 88.53r^4 + 5.76r^6 - 348.17r^8 + 451.27r^{10}, \quad \text{(B28)}$$

$$F_4^{(3)} = -19.99r^2 + 118.20r^4 - 215.81r^6 + 199.84r^8 - 86.92r^{10}, \quad \text{(B29)}$$

$$F_5^{(3)} = -35.06r^2 + 101.73r^4 - 115.65r^6 + 61.33r^8 - 12.52r^{10}, \quad \text{(B30)}$$

$$F_6^{(3)} = 8.55r^2 - 81.12r^4 + 268.40r^6 - 363.38r^8 + 234.48r^{10}, \quad \text{(B31)}$$

$$F_7^{(3)} = 17.70r^2 - 263.94r^4 + 959.68r^6 - 1671.7r^8 + 1317.3r^{10}. \quad \text{(B32)}$$

APPENDIX C: EFFECTIVE RELATIVISTIC HAMILTONIAN

There are different quark models for the description of the mass spectrum of the heavy quark bound states and their decay rates. Sometimes, the used quark interaction operators have a very lengthy form. So, we find useful for our calculation of the relativistic corrections in the bound state wave functions $\Psi_0^{\mathcal{S}}$, $\Psi_0^{\mathcal{P}}$ to take the known Breit potential (24) obtained in QCD. It contains a number of terms which should be transformed in order to use the program of numerical solution of the Schrödinger equation [31]. The rationalization of the kinetic energy operator can be presented in the following form [37]:

$$T = 2\sqrt{\mathbf{p}^2 + m^2} = 2\frac{\mathbf{p}^2 + m^2}{\sqrt{\mathbf{p}^2 + m^2}} \approx \frac{\mathbf{p}^2}{\tilde{m}} + \frac{2m^2}{\tilde{E}}, \quad \text{(C1)}$$

where $\tilde{m}$ is the effective mass of heavy quarks,

$$\tilde{m} = \frac{\tilde{E}}{2} = \sqrt{\mathbf{p}_{\rm eff}^2 + m^2}. \quad \text{(C2)}$$

$\mathbf{p}_{\rm eff}^2$ should be considered as a new parameter which effectively accounts for relativistic corrections in Eq. (C1) which has a nonrelativistic form. Numerical values of $\mathbf{p}_{\rm eff}^2$ for S- and P-wave charmonium states discussed in [13,18,20] are presented in Table III. In the case of S-wave states it is necessary to transform the δ-like terms of the potential. For this aim, we use the known smeared δ function of the Gaussian form [38]:

$$\tilde{\delta}(\mathbf{r}) = \frac{b^3}{\pi^{3/2}} e^{-b^2 r^2} \quad \text{(C3)}$$

with the additional parameter b which defines the hyperfine splitting in the $(c\bar{c})$ system. Since the numerical results are practically independent on b in the range of commonly used values $1.5 \div 2.2$, we take $b = 1.5$ GeV. The second term in the Breit potential (24), which also has to be transformed, takes the form

$$\Delta\tilde{U} = -\frac{2\alpha_s}{3m^2 r}\left[\mathbf{p}^2 - \frac{d^2}{dr^2}\right]. \quad \text{(C4)}$$

In order to replace it by the effective term containing the powerlike potentials, we use the approximate charmonium wave functions which can be written for S- and P-wave states as

$$\Psi_0^{\mathcal{S}}(r) = \frac{\beta^{3/2}}{\pi^{3/4}} e^{-(1/2)\beta^2 r^2},$$
$$\Psi_0^{\mathcal{P}}(r) = \sqrt{\frac{8}{3}} \frac{\beta^{3/2}}{\pi^{3/4}} \beta r e^{-(1/2)\beta^2 r^2} Y_{1m}(\theta, \phi). \quad (C5)$$

The wave functions (C5) give a good approximation of the true quark bound state wave functions in the region of nonrelativistic momenta. Using (C5), we transform (C4) as follows:

$$\Delta\tilde{U} \rightarrow \Delta\tilde{U}^{\rm eff} = -\frac{2\alpha_s}{3m^2 r}(mE - mB + \beta^2) - \frac{8\alpha_s^2}{9mr^2} + \frac{2\alpha_s A}{3m} + \frac{2\alpha_s \beta^4}{3m^2} r, \quad (C6)$$

where E is the quark-antiquark bound state energy which can be obtained from the Schrödinger equation with the Hamiltonian H_0. In order to derive (C6) we changed the operator $\mathbf{p}^2$ by its nonrelativistic expression: $\mathbf{p}^2\Psi_0 = m[E + \frac{4\alpha_s}{3r} - Ar - B]\Psi_0$. As a result of such transformations the potential ΔU_2 takes the following form in the case of S states:

J/Ψ-meson:

$$\Delta U_2(r) = \frac{20\pi\alpha_s}{9m^2} \frac{b^3}{\pi^{3/2}} e^{-b^2 r^2} - \frac{2\alpha_s}{3m^2 r}(mE - mB + \beta^2) - \frac{26\alpha_s^2}{9mr^2} + \frac{2\alpha_s \beta^4}{3m^2} r + \frac{2\alpha_s A}{3m}, \quad (C7)$$

η_c-meson:

$$\Delta U_2(r) = -\frac{4\pi\alpha_s}{3m^2} \frac{b^3}{\pi^{3/2}} e^{-b^2 r^2} - \frac{2\alpha_s}{3m^2 r}(mE - mB + \beta^2) - \frac{26\alpha_s^2}{9mr^2} + \frac{2\alpha_s \beta^4}{3m^2} r + \frac{2\alpha_s A}{3m}. \quad (C8)$$

Final expressions (C7) and (C8) contain the relativistic corrections with the same accuracy $O(|\mathbf{p}|^2/m^2)$ as the production amplitudes. A similar transformation of the Breit Hamiltonian can be done for the P-wave states. In Table III we present the results of the calculation of the charmonium mass spectrum and a comparison with the existing experimental data. The obtained masses agree with the experimental ones within an accuracy of 1–2%. So, we can suppose that the obtained effective Hamiltonian allows to account for relativistic corrections in the bound state wave functions with sufficiently good accuracy.

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