Anisotropic hydrodynamics, bulk viscosities, and *r*-modes of strange quark stars with strong magnetic fields

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In strong magnetic fields the transport coefficients of strange quark matter become anisotropic. We determine the general form of the complete set of transport coefficients in the presence of a strong magnetic field. By using a local linear response method, we calculate explicitly the bulk viscosities $\zeta_\perp$ and $\zeta_\parallel$ transverse and parallel to the B field, respectively, which arise due to the nonleptonic weak processes $u + s \leftrightarrow u + d$. We find that for magnetic fields $B < 10^{17}$ G, the dependence of $\zeta_\perp$ and $\zeta_\parallel$ on the field is weak, and they can be approximated by the bulk viscosity for the zero magnetic field. For fields $B > 10^{18}$ G, the dependence of both $\zeta_\perp$ and $\zeta_\parallel$ on the field is strong, and they exhibit de Haas–van Alphen–type oscillations. With increasing magnetic field, the amplitude of these oscillations increases, which eventually leads to negative $\zeta_\perp$ in some regions of parameter space. We show that the change of sign of $\zeta_\perp$ signals a hydrodynamic instability. As an application, we discuss the effects of the new bulk viscosities on the *r*-mode instability in rotating strange quark stars. We find that the instability region in strange quark stars is affected when the magnetic fields exceed the value $B = 10^{17}$ G. For fields which are larger by an order of magnitude, the instability region is significantly enlarged, making magnetized strange stars more susceptible to *r*-mode instability than their unmagnetized counterparts.

I. INTRODUCTION

Neutron stars provide a natural laboratory to study extremely dense matter. In the interiors of such stars, the density can reach up to several times the nuclear saturation density, $n_0 \simeq 0.16$ fm^{-3}. At such high densities quarks could be squeezed out of nucleons to form quark matter [1–3]. The true ground state of dense quark matter at high densities and low temperatures remains an open problem due to the difficulty of solving nonperturbative quantum chromodynamics (QCD). It has been suggested that strange quark matter that consists of comparable numbers of *u*, *d*, and *s* quarks may be the stable ground state of normal quark matter [4]. This has led to the conjecture that the family of compact stars may have members consisting entirely of quark matter (so-called strange stars) and/or members featuring quark cores surrounded by a hadronic shell (hybrid stars) [5].

Observationally, it is very challenging to distinguish the various types of compact objects, such as the strange stars, hybrid stars, and ordinary neutron stars. Their early cooling behavior is dominated by neutrino emission which is a useful probe of the internal composition of compact stars. Thus, cooling simulations provide an effective test of the nature of compact stars [6–16]. However, many theoretical uncertainties and the current amount of data on the surface temperatures of neutron stars leave sufficient room for speculations [17–19]. Another useful avenue for testing the internal structure and composition of compact stars is astroseismology, i.e., the study of the phenomena related to stellar vibrations [20–25]. In particular, there are a number of instabilities which are associated with the oscillations of rotating stars. Here we will be concerned with the so-called *r*-mode instability (see Refs. [22,23] for reviews). This instability is known to limit the angular velocity of rapidly rotating compact stars. The *r*-mode and related instabilities in rotating neutron stars are damped by the shear and bulk viscosities of matter; therefore these are important ingredients of theoretically modeling rapidly rotating stars. Such models and their microscopic input can then be constrained via the observations of rapidly rotating pulsars, such as the Crab pulsar and the millisecond pulsars.

For quark matter in chemical equilibrium, the shear viscosity is dominated by strong interactions between quarks. The bulk viscosity, however, is dominated by flavor-changing weak processes, whereas strong interactions play a secondary role. For normal (nonsuperconducting) strange quark matter, the bulk viscosity is dominated by the nonleptonic process [26–30]

$$u + s \rightarrow u + d, \tag{1a}$$
$$u + d \rightarrow u + s, \tag{1b}$$

since the contributions of the leptonic processes $u + e \leftrightarrow d + \nu$ and $u + e \leftrightarrow s + \nu$ are suppressed due to much smaller phase spaces. The bulk viscosity of various phases of quark matter has been studied extensively; see Refs. [24,26–41].

Compact stars are strongly magnetized. Neutron star observations indicate that the magnetic field is of the order of $B \sim 10^{12}$–10^{13} G at the surface of ordinary pulsars. Magnetars—strongly magnetized neutron stars—may feature even stronger magnetic fields of the order of 10^{15}–10^{16} G [42–48]. An upper limit on the magnetic field can be set through the virial theorem. Gravitational equilibrium of stars is compatible with magnetic fields of the order of 10^{18}–10^{20} G [49–51]. In such a strong magnetic field, not only the thermodynamical but also the hydrodynamical properties of stellar matter will be significantly affected. In particular, due to the large magnetization of strange quark matter the fluid will be strongly anisotropic in a strong magnetic field (we note here that the magnetization of ordinary neutron matter is small [52]). Therefore, there is a need to develop an anisotropic hydrodynamic theory to describe strongly magnetized matter in compact stars. As we show below, the matter is completely described in terms of eight viscosity coefficients, which include six shear viscosities and two bulk viscosities.

In this paper, we will carry out a theoretical study of the anisotropic hydrodynamics of magnetized strange quark matter and will calculate the two bulk viscosities. We will also discuss the implications of the anisotropic bulk viscosities on the r-mode instability in rotating quark stars.

The paper is organized as follows. The formalism of anisotropic hydrodynamics for magnetized strange quark matter is developed in Sec. II. In Sec. III we apply the local linear response method to derive explicit expressions for bulk viscosities. The stability of the fluid under a strong magnetic field is analyzed in Sec. IV. Section V contains our numerical results for the bulk viscosities. The damping of the r-mode instability in rotating quark stars by the bulk viscosity is studied in Sec. VI. Section VII contains our summary. We use natural units $\hbar = k_B = c = 1$. The metric tensor is $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. We will use the SI system of units in our equations involving electromagnetism; however, we will quote the strength of the magnetic field in centimeter-gram-second units (Gauss), as is common in the literature on compact stars.

II. ANISOTROPIC HYDRODYNAMICS

A. Ideal hydrodynamics

Hydrodynamics arises as an effective theory valid in the long-wavelength, low-frequency limit where the energy-momentum tensor $T^{\mu\nu}$, the conserved baryon current n_B^{μ}, the conserved electric current n_e^{μ}, the entropy density flux s^{μ}, etc., are expanded in terms of gradients of the four-velocity u^{μ} and the thermodynamic parameters of the system, such as the temperature T, baryon chemical potential μ_B, etc. The hydrodynamic equations can be expressed as conservation laws for the total energy-momentum tensor $T^{\mu\nu}$, as well as baryon and electric currents, n_B^{μ} and n_e^{μ}. The zeroth-order terms in the expansion correspond to an ideal fluid and we shall use the index

0 to label them. In the presence of an electromagnetic field, the zeroth-order terms can be generally written as [53,54]

$$T_0^{\mu\nu} = T_{F0}^{\mu\nu} + T_{EM}^{\mu\nu},$$
$$T_{F0}^{\mu\nu} = \varepsilon u^{\mu} u^{\nu} - P\Delta^{\mu\nu} - \tfrac{1}{2}(M^{\mu\lambda}F_{\lambda}{}^{\nu} + M^{\nu\lambda}F_{\lambda}{}^{\mu}),$$
$$n_{B0}^{\mu} = n_B u^{\mu}, \qquad n_{e0}^{\mu} = n_e u^{\mu}, \qquad s_0^{\mu} = s u^{\mu}, \qquad (2)$$

where ε, P, n_B, n_e, and s are the local energy density, thermodynamic pressure, baryon number density, electric charge density, and entropy density, respectively, measured in the rest frame of the fluid. $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^{\mu}u^{\nu}$ is the projector on the directions orthogonal to u^{μ}.

Here $T_{EM}^{\mu\nu} = -F^{\mu\lambda}F^{\nu}{}_{\lambda} + g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}/4$ is the energy-momentum tensor of the electromagnetic field. $F^{\mu\nu}$ is the field-strength tensor which can be decomposed into components parallel and perpendicular to u^{μ} as

$$F^{\mu\nu} = F^{\mu\lambda}u_{\lambda}u^{\nu} - F^{\nu\lambda}u_{\lambda}u^{\mu} + \Delta^{\mu}{}_{\alpha}F^{\alpha\beta}\Delta_{\beta}{}^{\nu}$$
$$\equiv E^{\mu}u^{\nu} - E^{\nu}u^{\mu} + \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}(u_{\alpha}B_{\beta} - u_{\beta}B_{\alpha}), \quad (3)$$

where in the second line we have introduced the four-vectors $E^{\mu} \equiv F^{\mu\nu}u_{\nu}$ and $B^{\mu} \equiv \epsilon^{\mu\nu\alpha\beta}F_{\nu\alpha}u_{\beta}/2$ with $\epsilon^{\mu\nu\alpha\beta}$ being the totally antisymmetric Levi-Civita tensor. In the rest frame of the fluid, $u^{\mu} = (1, \mathbf{0})$, we have $E^0 = B^0 = 0$, $E^i = F^{i0}$, and $B^i = -\epsilon^{ijk}F_{jk}/2$, which are precisely the electric and magnetic fields in this frame. Therefore, E^{μ} and B^{μ} are nothing but the electric and magnetic fields measured in the frame where the fluid moves with a velocity u^{μ}.

The antisymmetric tensor $M^{\mu\nu}$ is the polarization tensor which describes the response to the applied field strength $F^{\mu\nu}$. For example, if Ω is the thermodynamic potential of the system, $M^{\mu\nu} \equiv -\partial\Omega/\partial F_{\mu\nu}$. For later use, we also define the in-medium field-strength tensor $H^{\mu\nu} \equiv F^{\mu\nu} - M^{\mu\nu}$. In analogy to $F^{\mu\nu}$ we can decompose $M^{\mu\nu}$ and $H^{\mu\nu}$ as

$$M^{\mu\nu} = (P^{\nu}u^{\mu} - P^{\mu}u^{\nu}) + \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}(M_{\beta}u_{\alpha} - M_{\alpha}u_{\beta}),$$
$$H^{\mu\nu} = (D^{\mu}u^{\nu} - D^{\nu}u^{\mu}) + \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}(H_{\beta}u_{\alpha} - H_{\alpha}u_{\beta}),$$
$$(4)$$

with $P^{\mu} \equiv -M^{\mu\nu}u_{\nu}$, $M^{\mu} \equiv \epsilon^{\mu\nu\alpha\beta}M_{\nu\alpha}u_{\beta}/2$, $D^{\mu} \equiv H^{\mu\nu}u_{\nu}$, and $H^{\mu} \equiv \epsilon^{\mu\nu\alpha\beta}H_{\nu\alpha}u_{\beta}/2$.

In the rest frame of the fluid, the nontrivial components of these tensors are $(F^{10}, F^{20}, F^{30}) = \mathbf{E}$, $(F^{32}, F^{13}, F^{21}) = \mathbf{B}$, $(M^{10}, M^{20}, M^{30}) = -\mathbf{P}$, $(M^{32}, M^{13}, M^{21}) = \mathbf{M}$, $(H^{10}, H^{20}, H^{30}) = \mathbf{D}$, and $(H^{32}, H^{13}, H^{21}) = \mathbf{H}$. Here $\mathbf{P}$ and $\mathbf{M}$ are the electric polarization vector and magnetization vector, respectively. In the linear approximation they are related to the fields $\mathbf{E}$ and $\mathbf{B}$ by $\mathbf{P} = \chi_e\mathbf{E}$ and $\mathbf{M} = \chi_m\mathbf{B}$, with χ_e and χ_m being the electric and magnetic susceptibilities. The four-vectors E^{μ}, B^{μ}, $\cdots$ are all spacelike, $E^{\mu}u_{\mu} = 0, B^{\mu}u_{\mu} = 0, \cdots$, and normalized as $E^{\mu}E_{\mu} = -E^2, B^{\mu}B_{\mu} = -B^2, \cdots$, where $E \equiv |\mathbf{E}|$ and $B \equiv |\mathbf{B}|$.

Since the electric field is much weaker than the magnetic field in the interior of a neutron star, we will neglect it in most of the following discussion. Upon introducing the four-vector $b^\mu \equiv B^\mu/B$, which is parallel to B^μ and is normalized by the condition $b^\mu b_\mu = -1$, and the antisymmetric tensor $b^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta} b_\alpha u_\beta$, we can write

$$F^{\mu\nu} = -Bb^{\mu\nu}, \qquad M^{\mu\nu} = -Mb^{\mu\nu},$$
$$H^{\mu\nu} = -Hb^{\mu\nu}, \tag{5}$$

with $M \equiv |\mathbf{M}|$ and $H \equiv |\mathbf{H}|$.

The Maxwell equation $\epsilon^{\mu\nu\alpha\beta} \partial_\beta F_{\nu\alpha} = 0$ takes the form

$$\partial_\nu (B^\mu u^\nu - B^\nu u^\mu) = 0. \tag{6}$$

Its nonrelativistic form, which is known as the induction equation, is given by

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \qquad \nabla \cdot \mathbf{B} = 0, \tag{7}$$

where $\mathbf{v}$ is the three-velocity of the fluid. Contracting Eq. (6) with b_μ gives

$$\theta + D\ln B - u^\nu b^\mu \partial_\mu b_\nu = 0, \tag{8}$$

where $\theta \equiv \partial_\mu u^\mu$ and $D \equiv u^\mu \partial_\mu$. The second Maxwell equation can be written as

$$\partial_\mu H^{\mu\nu} = n_e^\nu, \tag{9}$$

whose nonrelativistic form is

$$\nabla \cdot \mathbf{D} - \mathbf{H} \cdot \nabla \times \mathbf{v} = n_e^0,$$
$$\nabla \times (\mathbf{H} - \mathbf{D} \times \mathbf{v}) - \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \times \partial_t \mathbf{v} = \mathbf{n}_e, \tag{10}$$

where n_e^0 is the electric charge density and $\mathbf{n}_e$ is the corresponding current. It is useful to rewrite the energy-momentum tensor in the following form [55–57],

$$T_{F0}^{\mu\nu} = \varepsilon u^\mu u^\nu - P_\perp \Xi^{\mu\nu} + P_\parallel b^\mu b^\nu,$$
$$T_{EM}^{\mu\nu} = \tfrac{1}{2} B^2 (u^\mu u^\nu - \Xi^{\mu\nu} - b^\mu b^\nu), \tag{11}$$
$$\Xi^{\mu\nu} \equiv \Delta^{\mu\nu} + b^\mu b^\nu,$$

where $\Xi^{\mu\nu}$ is the projection tensor on the direction perpendicular to both u^μ and b^μ. We have defined the transverse and longitudinal pressures $P_\perp = P - MB$ and $P_\parallel = P$ relative to b^μ; here P is the thermodynamic pressure. In the absence of a magnetic field, the fluid is isotropic and $P_\perp = P_\parallel = P$. In the local rest frame of fluid, we have $b^\mu = (0, 0, 0, 1)$ (without loss of generality, we choose the z axis along the direction of the magnetic field); hence the electromagnetic tensor takes the usual form, while $T_{F0}^{\mu\nu} = \text{diag}(\varepsilon, P_\perp, P_\perp, P_\parallel)$.

Next we would like to check the consistency of the terms that appear in $T_{F0}^{\mu\nu}$ with the formulas of standard thermodynamics involving electromagnetic fields. By using the thermodynamic relation

$$\varepsilon = Ts + \mu_B n_B + \mu_e n_e - P, \tag{12}$$

and the conservation equations for n_{B0}^μ, n_{e0}^μ, and s_0^μ in ideal hydrodynamics, one can show that the hydrodynamic equation $u_\nu \partial_\mu T_0^{\mu\nu} = 0$ together with the Maxwell equation (8) implies

$$D\varepsilon = TDs + \mu_B Dn_B + \mu_e Dn_e - MDB, \tag{13}$$

which is consistent with the standard thermodynamic relation

$$d\varepsilon = Tds + \mu_B dn_B + \mu_e dn_e - MdB. \tag{14}$$

One should note that the potential energy $-MB$ has already been included in our definition of ε. Otherwise, new terms $-MB$, $-D(MB)$, and $-d(MB)$ should be added to the left-hand sides of Eqs. (12)–(14), respectively. Thus, we conclude that our hydrodynamical equations are consistent with well-known thermodynamic relations.

B. Navier-Stokes-Fourier-Ohm theory

By keeping the first-order terms of the derivative expansion of conserved quantities, one obtains the Navier-Stokes-Fourier-Ohm theory. In this theory, $T^{\mu\nu}$, n_B^μ, n_e^μ, and s^μ can be generally expressed as

$$T^{\mu\nu} = T_0^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu + \tau^{\mu\nu},$$
$$n_B^\mu = n_B u^\mu + j_B^\mu,$$
$$n_e^\mu = n_e u^\mu + j_e^\mu, \tag{15}$$
$$s^\mu = su^\mu + j_s^\mu,$$

where h^μ, $\tau^{\mu\nu}$, j_B^μ, j_e^μ, and j_s^μ are the dissipative fluxes. They all are orthogonal to u^μ; this reflects the fact that the dissipation in the fluid should be spatial. We shall assume that j_s^μ can be expressed as a linear combination of h^μ, j_B^μ, and j_e^μ [58,59]. This allows us to incorporate the fact that the entropy flux is determined by the energy-momentum and baryon number diffusion fluxes. Thus,

$$j_s^\mu = \gamma h^\mu - \alpha_B j_B^\mu - \alpha_e j_e^\mu, \tag{16}$$

with the coefficients γ, α_e, and α_B being functions of thermodynamic variables.

Next, the hydrodynamic equations are specified by utilizing the conservation laws of the total energy-momentum $T^{\mu\nu}$, the baryon number density flow n_B^μ, electric current n_e^μ, and the second law of thermodynamics,

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu n_B^\mu = 0,$$
$$\partial_\mu n_e^\mu = 0, \qquad T\partial_\mu s^\mu \geq 0. \tag{17}$$

To discuss the dissipative parts, let us first define the four-velocity u^μ, since it is not unique when energy exchange by thermal conduction is allowed for. We will use the Landau-Lifshitz frame in which u^μ is chosen to be parallel to the energy density flow, so that $h^\mu = 0$. Upon projecting the first equation of Eq. (17) on u^ν and after

some straightforward manipulations, we find

$$(\varepsilon + P)\theta + D\varepsilon - \tau^{\mu\nu}\partial_\mu u_\nu + MDB = j_e^\lambda u^\nu F_{\nu\lambda}. \quad (18)$$

Combining Eq. (18) and (12), and the second equation in Eq. (17), we arrive at

$$T\partial_\mu s^\mu = \tau^{\mu\nu}w_{\mu\nu} + (\mu_B - T\alpha_B)\partial_\mu j_B^\mu - Tj_B^\mu\nabla_\mu\alpha_B$$
$$+ (\mu_e - T\alpha_e)\partial_\mu j_e^\mu - j_e^\mu(T\nabla_\mu\alpha_e + E_\mu), \quad (19)$$

where $\nabla_\mu \equiv \Delta_{\mu\nu}\partial^\nu$ and $w^{\mu\nu} \equiv \frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu)$. For a thermodynamically and hydrodynamically stable system, Eq. (19) should be non-negative. This implies

$$\alpha_B = \beta\mu_B, \qquad \alpha_e = \beta\mu_e, \qquad \tau^{\mu\nu} = \eta^{\mu\nu\alpha\beta}w_{\alpha\beta},$$
$$j_B^\mu = -\kappa^{\mu\nu}T\nabla_\nu\alpha_B, \qquad j_e^\mu = -\sigma^{\mu\nu}(T\nabla_\nu\alpha_e + E_\nu), \quad (20)$$

where $\beta \equiv 1/T$, $\eta^{\mu\nu\alpha\beta}$ is the rank-four tensor of viscosity coefficients, and $\kappa^{\mu\nu}$ and $\sigma^{\mu\nu}$ are thermal and electrical conductivity tensors with respect to the diffusion fluxes of baryon number density and electric charge density. By definition, $\eta^{\mu\nu\alpha\beta}$ is symmetric in the pairs of indices α, β and μ, ν. It necessarily satisfies the condition $\eta^{\mu\nu\alpha\beta}(B^\sigma) = \eta^{\alpha\beta\mu\nu}(-B^\sigma)$, which is Onsager's symmetry principle for transport coefficients. Similarly, the tensors $\kappa^{\mu\nu}$ and $\sigma^{\mu\nu}$ should satisfy the conditions $\kappa^{\mu\nu}(B^\lambda) = \kappa^{\nu\mu}(-B^\lambda)$ and $\sigma^{\mu\nu}(B^\lambda) = \sigma^{\nu\mu}(-B^\lambda)$. Furthermore, all the tensors of transport coefficients $\eta^{\mu\nu\alpha\beta}$, $\kappa^{\mu\nu}$, and $\sigma^{\mu\nu}$ must be orthogonal to u^μ by definition.

As we have seen, the appearance of the magnetic field makes the system anisotropic. Such anisotropy is specified by the vector b^μ, so that the tensors $\eta^{\mu\nu\alpha\beta}$, $\sigma^{\mu\nu}$, and $\kappa^{\mu\nu}$ should be in general expressed in terms of u^μ, b^μ, $g^{\mu\nu}$, and $b^{\mu\nu}$. All independent irreducible tensor combinations having the symmetry of $\eta^{\mu\nu\alpha\beta}$ and which are orthogonal to u^μ are [60](see also the Appendix)

(i) $\Delta^{\mu\nu}\Delta^{\alpha\beta}$,

(ii) $\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}$,

(iii) $\Delta^{\mu\nu}b^\alpha b^\beta + \Delta^{\alpha\beta}b^\mu b^\nu$,

(iv) $b^\mu b^\nu b^\alpha b^\beta$,

(v) $\Delta^{\mu\alpha}b^\nu b^\beta + \Delta^{\nu\beta}b^\mu b^\alpha + \Delta^{\mu\beta}b^\nu b^\alpha + \Delta^{\nu\alpha}b^\mu b^\beta$,

(vi) $\Delta^{\mu\alpha}b^{\nu\beta} + \Delta^{\nu\beta}b^{\mu\alpha} + \Delta^{\mu\beta}b^{\nu\alpha} + \Delta^{\nu\alpha}b^{\mu\beta}$,

(vii) $b^{\mu\alpha}b^\nu b^\beta + b^{\nu\beta}b^\mu b^\alpha + b^{\mu\beta}b^\nu b^\alpha + b^{\nu\alpha}b^\mu b^\beta$,

(viii) $b^{\mu\alpha}b^{\nu\beta} + b^{\mu\beta}b^{\nu\alpha}$. (21)

All independent irreducible tensor combinations having the symmetry of $\kappa^{\mu\nu}$ and $\sigma^{\mu\nu}$ and which are orthogonal to u^μ are

$$\begin{aligned}&\text{(i)} \ \Delta^{\mu\nu},\\ &\text{(ii)} \ b^\mu b^\nu,\\ &\text{(iii)} \ b^{\mu\nu}.\end{aligned} \quad (22)$$

In accordance with the number of tensors (21) and (22), a fluid in a magnetic field in general has eight independent viscosity coefficients, three independent thermal conduction coefficients, and three independent electrical conductivities. They may be defined as the coefficients in the following decompositions for the viscous stress tensor, heat flux, and electric charge flux:

$$\tau^{\mu\nu} = 2\eta_0(w^{\mu\nu} - \Delta^{\mu\nu}\theta/3) + \eta_1(\Delta^{\mu\nu} - \tfrac{3}{2}\Xi^{\mu\nu})(\theta - \tfrac{3}{2}\phi)$$
$$- 2\eta_2(b^\mu\Xi^{\nu\alpha}b^\beta + b^\nu\Xi^{\mu\alpha}b^\beta)w_{\alpha\beta}$$
$$- \eta_3(2b^{\mu\alpha}b^{\nu\beta}w_{\alpha\beta} - \Xi^{\mu\alpha}w_\alpha{}^\nu - \Xi^{\nu\alpha}w_\alpha{}^\mu)$$
$$- 2\eta_4(\Xi^{\mu\alpha}b^{\nu\beta} + \Xi^{\nu\alpha}b^{\mu\beta})w_{\alpha\beta}$$
$$+ 2\eta_5(b^{\mu\alpha}b^\nu b^\beta + b^{\nu\alpha}b^\mu b^\beta)w_{\alpha\beta}$$
$$+ \tfrac{3}{2}\zeta_\perp\Xi^{\mu\nu}\phi + 3\zeta_\parallel b^\mu b^\nu\varphi, \quad (23)$$

$$j_B^\mu = \kappa T\nabla^\mu\alpha_B - \kappa_1 b^\mu b^\nu T\nabla_\nu\alpha_B - \kappa_2 b^{\mu\nu}T\nabla_\nu\alpha_B, \quad (24)$$

$$j_e^\mu = \sigma(\nabla^\mu\alpha_e + E^\mu) - \sigma_1 b^\mu b^\nu(\nabla^\mu\alpha_e + E^\mu)$$
$$- \sigma_2 b^{\mu\nu}(\nabla^\mu\alpha_e + E^\mu), \quad (25)$$

where $\phi \equiv \Xi^{\mu\nu}w_{\mu\nu}$, $\varphi \equiv b^\mu b^\nu w_{\mu\nu}$, and $\tau^{\mu\nu}$ is constructed so that the η's are the coefficients of its traceless parts; i.e., they can be regarded as shear viscosities. ζ's are the coefficients of the parts with nonzero trace and can be considered as bulk viscosities. The κ's and σ's are thermal and electrical conductivities, respectively.

Now the divergence of entropy density flux (19) can be explicitly written as

$$T\partial_\mu s^\mu = 2\eta_0(w^{\mu\nu} - \tfrac{1}{3}\Delta^{\mu\nu}\theta)(w_{\mu\nu} - \tfrac{1}{3}\Delta_{\mu\nu}\theta) + \eta_1(\theta - \tfrac{3}{2}\phi)^2 + 2\eta_2(b^\mu b_\rho w^{\rho\nu} - b^\nu b_\rho w^{\rho\mu})(b_\mu b^\rho w_{\rho\nu} - b_\nu b^\rho w_{\rho\mu})$$
$$+ \eta_3(b^{\mu\rho}w_\rho{}^\nu - b^{\nu\rho}w_\rho{}^\mu)(b_{\mu\rho}w^\rho{}_\nu - b_{\nu\rho}w^\rho{}_\mu) + \tfrac{3}{2}\zeta_\perp\phi^2 + 3\zeta_\parallel\varphi^2 - \kappa T^2\nabla^\mu\alpha_B\nabla_\mu\alpha_B + \kappa_1 T^2(b^\mu\nabla_\mu\alpha_B)^2$$
$$- \sigma(T\nabla^\mu\alpha_e + E^\mu)(T\nabla_\mu\alpha_e + E_\mu) + \sigma_1(Tb^\mu\nabla_\mu\alpha_e + E^\mu b_\mu)^2. \quad (26)$$

One should note that the terms corresponding to the transport coefficients η_4, η_5, κ_2, and σ_2 in Eqs. (23)–(25) do not contribute to the divergence of the entropy density flux. For stable systems, all the other transport coefficients must be positive definite according to the second law of thermodynamics. In Sec. IV we will demonstrate explicitly that negative bulk viscosities $\zeta_\perp$ or/and $\zeta_\parallel$ indeed cause an instability in the hydrodynamic evolution of strange stars.

To conclude this section, we compare our definition of the viscosity coefficients in Eq. (23) with the definition given in Ref. [60] for nonrelativistic fluid, which reads

$$
\begin{aligned}
\tau_{ij} &= 2\tilde{\eta}(w_{ij} - \delta_{ij}\theta/3) + \tilde{\zeta}\delta_{ij}\theta + \tilde{\eta}_1(2w_{ij} - \delta_{ij}\theta + \delta_{ij}w_{kl}b_k b_l - 2w_{ik}b_k b_j - 2w_{jk}b_k b_i + b_i b_j\theta + b_i b_j w_{kl}b_k b_l) \\
&\quad + 2\tilde{\eta}_2(w_{ik}b_k b_j + w_{jk}b_k b_i - 2b_i b_j w_{kl}b_k b_l) + \tilde{\eta}_3(w_{ik}b_{jk} + w_{jk}b_{ik} - w_{kl}b_{ik}b_j b_l - w_{kl}b_{jk}b_i b_l) \\
&\quad + 2\tilde{\eta}_4(w_{kl}b_{il}b_j b_k + w_{kl}b_{jl}b_i b_k) + \tilde{\zeta}_1(\delta_{ij}w_{kl}b_k b_l + b_i b_j\theta),
\end{aligned} \tag{27}
$$

where $b_{ij} \equiv \epsilon_{ijk}b_k$ and the remaining notations are self-explanatory.

Our viscosity coefficients in Eq. (23) are related to the coefficients in Eq. (27) by

$$
\begin{aligned}
\eta_0 &= \tilde{\eta} + \tilde{\eta}_1, & \eta_1 &= \tfrac{3}{4}(\tilde{\eta}_1 + \tfrac{1}{2}\tilde{\zeta}_1 - \tfrac{3}{2}\tilde{\zeta}), \\
\eta_2 &= \tilde{\eta}_2 - \tilde{\eta}_1, & \eta_4 &= \tfrac{1}{2}\tilde{\eta}_3, & \eta_5 &= \tilde{\eta}_4, \\
\zeta_\perp &= \tilde{\zeta} + \tfrac{1}{3}\tilde{\zeta}_1, & \zeta_\parallel &= \tilde{\zeta} + \tfrac{4}{3}\tilde{\zeta}_1.
\end{aligned} \tag{28}
$$

In Ref. [60] there is no term that corresponds to our η_3. The reason is that Ref. [60] considers the combination of vectors (viii) in Eq. (21) as dependent on the others; in the Appendix we will show that (at least for relativistic fluids) all the combinations (i)–(viii) are linearly independent. Note that the transport coefficients in Eq. (26) appear as prefactors of quadratic forms; therefore the second law of thermodynamics requires that these coefficients must be positive definite for stable ensembles. This is not manifest in Eq. (27).

III. BULK VISCOSITIES

The typical oscillation frequency of neutron stars is of the order of magnitude of the rotation frequency, $1 \text{ s}^{-1} \lesssim \omega \lesssim 10^3 \text{ s}^{-1}$. The most important microscopic processes which dissipate energy on the corresponding time scales are the weak processes.

The compression and expansion of strange quark matter with nonzero strange quark mass will drive the system out of equilibrium. The processes (1a) and (1b) are the most efficient microscopic processes that restore local chemical equilibrium. Therefore, the bulk viscosities are determined mainly by the processes (1a) and (1b). In this section we will derive analytical expressions for the bulk viscosities $\zeta_\perp$ and $\zeta_\parallel$ [61].

Let us imagine an isotropic flow $\mathbf{v}(t) \sim e^{i\omega t}$ which characterizes the stellar oscillation. If there are no dissipative processes, such an oscillation will drive the system from one instantaneous equilibrium state to another instantaneous equilibrium state. The appearance of dissipation changes the picture: during the oscillations the thermodynamic quantities will differ from their equilibrium values.

Let us explore how the thermodynamic quantities evolve during the flow oscillation.

In general, we can write the change of baryon density $n_B \equiv (n_u + n_d + n_s)/3$ induced by the oscillation of the fluid as

$$
n_B(t) = n_{B0} + \delta n_B(t), \qquad \delta n_B = \delta n_B^{\text{eq}} + \delta n_B', \tag{29}
$$

where n_{B0} is the static (time-independent) equilibrium value, δn_B^{eq} denotes the equilibrium value shift from n_{B0} due to the volume change, and $\delta n_B'$ denotes the instantaneous departure from the equilibrium value. Because processes (1a) and (1b) conserve baryon number, $\delta n_B'(t)$ can be set to zero, if we neglect other microscopic processes. Then δn_B can be determined through the continuity equation of ideal hydrodynamics,

$$
\delta n_B(t) = -\frac{n_{B0}}{i\omega}\theta. \tag{30}
$$

Since processes (1a) and (1b) also conserve the sum $n_d + n_s$, a similar argument leads to the relation

$$
\delta n_d + \delta n_s = -\frac{n_{d0} + n_{s0}}{i\omega}\theta. \tag{31}
$$

When the system is driven out of chemical equilibrium, the chemical potential of the s quark will be slightly different from that of the d quark. Let us denote this difference by $\delta\mu = \mu_s - \mu_d = \delta\mu_s - \delta\mu_d$, with $\delta\mu_f$ being the deviation of μ_f from its static equilibrium value. Up to linear order in the deviation we find

$$
\delta\mu(t) \simeq \left(\frac{\partial\mu_s}{\partial n_s}\right)_0 \delta n_s - \left(\frac{\partial\mu_d}{\partial n_d}\right)_0 \delta n_d, \tag{32}
$$

where n_f denotes the number density of quarks of flavor f and the subscript 0 indicates that the quantity in the bracket is computed in static equilibrium state. δn_s and δn_d are the deviations of s-quark and d-quark densities from their static equilibrium value. In the final expressions they should be functions of θ.

The instantaneous departure from equilibrium is restored by the weak processes (1a) and (1b). Adopting the linear approximation, this can be described by

$$\Gamma_d - \Gamma_s = \lambda \delta \mu, \qquad \lambda > 0, \tag{33}$$

where Γ_d and Γ_s are the rates of processes (1a) and (1b), respectively. If the weak processes are turned off, one should have

$$\delta \dot{n}_f^{\text{eq}} = -n_{f0} \theta = n_{f0} \frac{\dot{\delta n_B}}{n_{B0}}, \tag{34}$$

where the dot denotes the time derivative. After turning on the weak processes, we have

$$\dot{\delta n}_u = \dot{\delta n}_B, \qquad \dot{\delta n}_d = n_{d0} \frac{\dot{\delta n_B}}{n_{B0}} + \lambda \delta \mu(t),$$
$$\dot{\delta n}_s = n_{s0} \frac{\dot{\delta n_B}}{n_{B0}} - \lambda \delta \mu(t). \tag{35}$$

This system of coupled linear first-order equations is closed by substituting Eqs. (30)–(32). It is then easy to obtain the solution,

$$\delta n_u = -n_{u0} \frac{\theta}{i\omega},$$
$$\delta n_d = -\frac{i\omega n_{d0} + \lambda (\partial \mu_s / \partial n_s)_0 (n_{d0} + n_{s0})}{i\omega + \lambda A} \frac{\theta}{i\omega}, \tag{36}$$
$$\delta n_s = -\frac{i\omega n_{s0} + \lambda (\partial \mu_d / \partial n_d)_0 (n_{d0} + n_{s0})}{i\omega + \lambda A} \frac{\theta}{i\omega},$$

where the coefficient A is defined by

$$A = \left(\frac{\partial \mu_s}{\partial n_s} \right)_0 + \left(\frac{\partial \mu_d}{\partial n_d} \right)_0. \tag{37}$$

The parallel and transverse components of the pressure $P_\parallel$ and $P_\perp$ can be written as

$$P_\parallel = P_\parallel^{\text{eq}} + \delta P_\parallel', \qquad P_\perp = P_\perp^{\text{eq}} + \delta P_\perp', \tag{38}$$

and

$$P_{\parallel/\perp}^{\text{eq}} \simeq P_{\parallel/\perp}^0 + \sum_f \left(\frac{\partial P_{\parallel/\perp}}{\partial n_f} \right)_0 \delta n_f^{\text{eq}} + \left(\frac{\partial P_{\parallel/\perp}}{\partial B} \right)_0 \delta B,$$
$$\delta P_{\parallel/\perp}' \simeq \sum_f \left(\frac{\partial P_{\parallel/\perp}}{\partial n_f} \right)_0 \delta n_f', \tag{39}$$

where

$$\delta n_f' \equiv \delta n_f - \delta n_f^{\text{eq}}. \tag{40}$$

The small departure of the magnetic field δB can be calculated by the variation of Eq. (8). One finds

$$\delta B = -\frac{2}{3} \frac{B}{i\omega} \theta. \tag{41}$$

A direct calculation then gives

$$\delta n_u' = 0, \qquad \delta n_d' = \frac{\lambda C_\parallel}{i\omega + \lambda A} \frac{\theta}{i\omega},$$
$$\delta n_s' = -\frac{\lambda C_\parallel}{i\omega + \lambda A} \frac{\theta}{i\omega}, \tag{42}$$

where we introduced the coefficient $C_\parallel$ as

$$C_\parallel \simeq n_{d0} \left(\frac{\partial \mu_d}{\partial n_d} \right)_0 - n_{s0} \left(\frac{\partial \mu_s}{\partial n_s} \right)_0. \tag{43}$$

Now we obtain

$$\delta P_{\parallel/\perp}' = -\frac{\lambda C_\parallel C_{\parallel/\perp}}{i\omega + \lambda A} \frac{\theta}{i\omega}, \tag{44}$$

with $C_\perp$ defined as

$$C_\perp \simeq C_\parallel - XB, \qquad X = \left(\frac{\partial \mathcal{M}}{\partial n_d} \right)_0 - \left(\frac{\partial \mathcal{M}}{\partial n_s} \right)_0. \tag{45}$$

Then the deviation of $T^{\mu\nu}$ from its equilibrium value can be written as

$$\delta T^{\mu\nu} = -\text{Re} \delta P_\perp' \Xi^{\mu\nu} + \text{Re} \delta P_\parallel' b^\mu b^\nu. \tag{46}$$

For isotropic flows, we have

$$\tau^{\mu\nu} = \zeta_\perp \Xi^{\mu\nu} \theta - \zeta_\parallel b^\mu b^\nu \theta. \tag{47}$$

By comparing the above two expressions, we obtain

$$\zeta_\parallel = \frac{\lambda C_\parallel^2}{\omega^2 + \lambda^2 A^2}, \tag{48}$$

and

$$\zeta_\perp = \frac{\lambda C_\perp C_\parallel}{\omega^2 + \lambda^2 A^2}. \tag{49}$$

Expressions (48) and (49) show that the bulk viscosities $\zeta_\perp$ and $\zeta_\parallel$ are functions of the perturbation frequency ω, the weak rate λ, and the thermodynamic quantities $C_\parallel$, $C_\perp$, A. From the derivation above we can convince ourselves that these expressions should be valid also in the case of color-superconducting matter. For the zero magnetic field, Eq. (49) reduces to Eq. (48), which, with parameters $C_\parallel$, A, and λ taken in the absence of magnetic field, gives the expression for the usual bulk viscosity ζ_0 defined in isotropic hydrodynamics.

Both $\zeta_\perp$ and $\zeta_\parallel$ attain their maxima in the limit of zero frequency, $\zeta_\parallel^{\text{max}} = C_\parallel^2 / (\lambda A^2)$, $\zeta_\perp^{\text{max}} = C_\parallel C_\perp / (\lambda A^2)$; and the maxima are inversely proportional to the weak inter-action rate. At high frequency, $\omega \gg \lambda A$, $\zeta_\perp$ and $\zeta_\parallel$ fall off as $1/\omega^2$. For practical applications to cold strange stars, where the chemical potential is much larger than the temperature, the quantities $C_\parallel$, $C_\perp$, and A can be evaluated in the zero-temperature limit. Their dependence on temperature is weak. Contrary to this, the coefficient λ depends strongly on temperature: for normal quark matter, λ has a power-law dependence on T (see Sec. V); for a fully paired color-superconducting phase, the weak rate is ex-

ponentially suppressed by a Boltzmann factor $e^{-\Delta/T}$ with Δ being the superconducting gap. Consequently, $\zeta_\perp$ and $\zeta_\parallel$ depend exponentially on T [24,32–37,40,41].

Before we find the numerical values for the bulk viscosities, we first need to analyze the stability of magnetized strange quark matter. We observe that according to Eq. (49) negative values of $\zeta_\perp$ are *a priori* not excluded. In the following we analyze the consequences and implications of negative $\zeta_\perp$ on the stability of the system.

IV. STABILITY ANALYSIS

A. Mechanical stability

Stable equilibrium in a self-gravitating fluid, such as in strange quark stars, is attained through the balance of gravity and pressure. The gravitational equilibrium requires that both components of the pressure $P_\parallel$ and $P_\perp$ should be positive (otherwise the star will undergo a gravitational collapse). At zero temperature, the one-loop thermodynamic pressure $P = T \ln \mathcal{Z}/V$ of noninteracting strange quark matter, where $\mathcal{Z}$ is the grand partition function, is given by

$$
P = \sum_{f=u,d,s} \frac{N_c q_f B}{4\pi^2} \sum_{n=0}^{n_{max}^f} \nu_n \Bigg[\mu_f \sqrt{\mu_f^2 - m_f^2 - 2nq_f B}
$$
$$
- (m_f^2 + 2q_f Bn) \ln \frac{\mu_f + \sqrt{\mu_f^2 - m_f^2 - 2q_f Bn}}{\sqrt{m_f^2 + 2q_f Bn}} \Bigg], \quad (50)
$$

where q_f is the absolute value of electric charge, μ_f and m_f are the chemical potential and the mass of quark of flavor f, n labels the Landau levels, $\nu_n = 2 - \delta_{0n}$ is the degree of degeneracy of each Landau level, and $n_{max}^f = \text{Int}[(\mu_f^2 - m_f^2)/(2q_f B)]$ is the highest Landau level for quarks of flavor f. By differentiating Eq. (50) with respect to B one can easily get the magnetization as

$$
M = \sum_f \sum_{n=0}^{n_{max}^f} \nu_n \frac{N_c q_f}{4\pi^2} \Bigg[\mu_f \sqrt{\mu_f^2 - m_f^2 - 2nq_f B}
$$
$$
- (m_f^2 + 4nq_f B) \ln \frac{\mu_f + \sqrt{\mu_f^2 - m_f^2 - 2nq_f B}}{\sqrt{m_f^2 + 2nq_f B}} \Bigg]. \quad (51)
$$

In Fig. 1 we illustrate the magnetization as a function of magnetic field at zero temperature. The parameters are chosen as

$$
\mu_u = \mu_d = \mu_s = 400 \text{ MeV}, \qquad m_s = 150 \text{ MeV},
$$
$$
m_u = m_d = 5 \text{ MeV}. \quad (52)
$$

On average, the magnetization increases when B grows and eventually becomes constant when $B > B_c$, where

$$
B_c \equiv \text{Max}_f \{(\mu_f^2 - m_f^2)/(2q_f)\}. \quad (53)
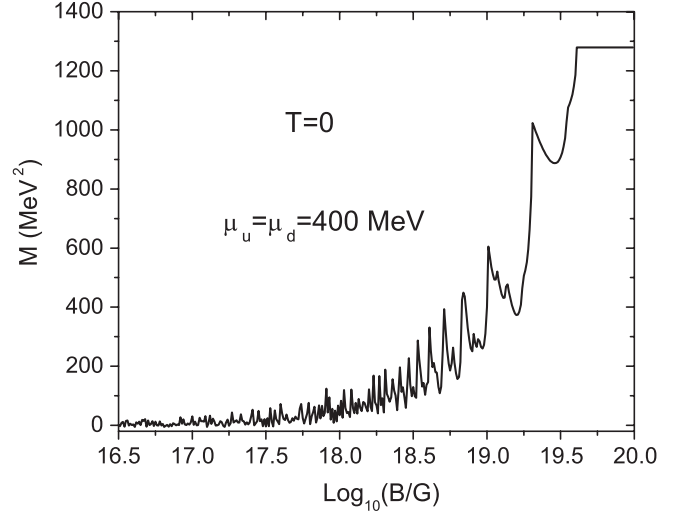$$

FIG. 1. The magnetization of strange quark matter as function of the magnetic field B.

However, the detailed structure of the magnetization exhibits strong de Haas–van Alphen oscillations [63]. This oscillatory behavior is of the same origin as the de Haas–van Alphen oscillations of the magnetization in metals and originates from the quantization of the energy levels associated with the orbital motion of charged particles in a magnetic field. The irregularity of this oscillation shown in Fig. 1 is due to the unequal masses and charges of u, d, and s quarks.

When $B > B_c$, all quarks are confined to their lowest Landau level and their transverse motions are frozen. In this case, the longitudinal pressure $P_\parallel \propto B$, so the magne-
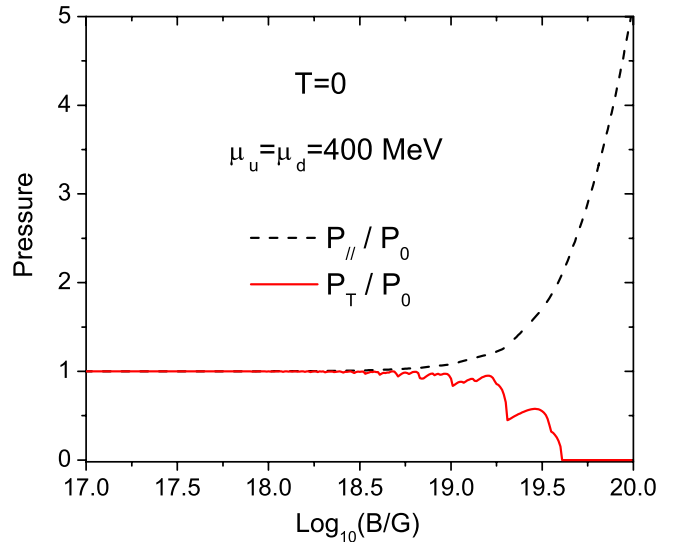

FIG. 2 (color online). The parallel $P_\parallel$ (dashed black curve) and transverse $P_\perp$ (solid red curve) pressures of strange quark matter as functions of the magnetic field B in units of the pressure P_0 for zero magnetic field.

tization $M \equiv \partial P / \partial B = P/B$ is independent of B, and the transverse pressure $P_\perp = P - MB$ of the system vanishes. This behavior is evident in Fig. 2.

Thus, we conclude that when all the quarks are confined to their lowest Landau level, the transverse pressure vanishes. The system therefore becomes mechanically unstable and would collapse due to the gravity [64]. This phenomenon establishes an upper limit on the magnetic field sustained by a quark star. Given $\mu_d \sim 400$ MeV, this upper limit is roughly $B_c \sim 10^{20}$ G as shown in Fig. 2.

B. Thermodynamic stability

The thermodynamic stability requires that the local entropy density should reach its maximum in the equilibrium state [65]. The total energy density of the fluid and the magnetic field is

$$\varepsilon_{\text{total}} = Ts + \mu_f n_f - P + \frac{B^2}{2}, \qquad (54)$$

and the corresponding first law of thermodynamics, in variational form, is

$$\delta \varepsilon_{\text{total}} = T\delta s + \mu_f \delta n_f + H\delta B, \qquad (55)$$

where H is the strength of the magnetic field and δ stands for a small departure of a given quantity from its equilibrium value. Varying Eq. (55) on both sides and taking into account that $\varepsilon_{\text{total}}$, n_f, and B are independent variational variables, one obtains

$$\delta^2 s = -\frac{1}{T} \delta\mu_f \delta n_f - \frac{1}{T} \delta H \delta B = -\frac{1}{T} \delta x^T \chi \delta x, \quad (56)$$

where $\delta x = (\delta n_u, \delta n_d, \delta n_s, \delta B)$ and

$$\chi = \begin{pmatrix} \frac{\partial \mu_u}{\partial n_u} & 0 & 0 & \frac{\partial \mu_u}{\partial B} \\ 0 & \frac{\partial \mu_d}{\partial n_d} & 0 & \frac{\partial \mu_d}{\partial B} \\ 0 & 0 & \frac{\partial \mu_s}{\partial n_s} & \frac{\partial \mu_s}{\partial B} \\ \frac{\partial H}{\partial n_u} & \frac{\partial H}{\partial n_d} & \frac{\partial H}{\partial n_s} & \frac{\partial H}{\partial B} \end{pmatrix}_0 . \qquad (57)$$

The thermodynamical stability criteria require that $\delta s = 0$, $\delta^2 s \leq 0$, or, equivalently, χ is positive definite. Taking into account the relation $(\partial H / \partial n_f)_0 = -(\partial \mu_f / \partial B)_0$, it is easy to show that these criteria are equivalent to the requirement

$$\left(\frac{\partial n_f}{\partial \mu_f}\right)_0 \geq 0, \qquad \left(\frac{\partial M}{\partial B}\right)_0 \leq 1. \qquad (58)$$

From Eq. (50) we obtain

$$n_{f0} = \frac{N_c q_f B}{2\pi^2} \sum_{n=0}^{n_{\max}^f} \nu_n \sqrt{\mu_f^2 - m_f^2 - 2nq_f B}, \qquad (59)$$

$$\frac{\partial n_{f0}}{\partial \mu_f} = \frac{N_c q_f B}{2\pi^2} \sum_{n=0}^{n_{\max}^f} \nu_n \frac{\mu_f}{\sqrt{\mu_f^2 - m_f^2 - 2nq_f B}};$$

then it is evident that the condition $(\partial n_f / \partial \mu_f)_0 \geq 0$ is
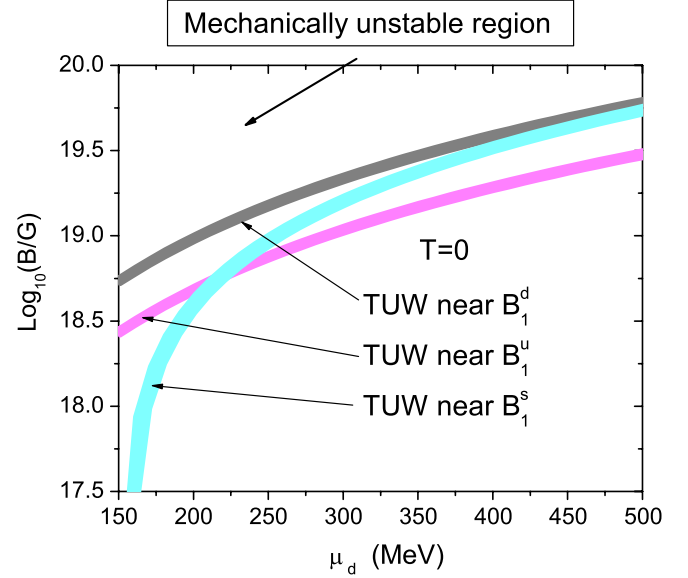

FIG. 3 (color online). The first three thermodynamically unstable windows for strange quark matter in the $\log B - \mu_d$ plane for $T = 0$ and $\mu_u = \mu_d$.

always satisfied. However, $(\partial M / \partial B)_0$ is divergent when B approaches a $n \neq 0$ Landau level for each flavor quark from below,

$$\left(\frac{\partial M}{\partial B}\right)_0 \to \frac{N_c q_f B}{\pi^2} \frac{(nq_f)^2}{\mu_f \sqrt{\mu_f^2 - m_f^2 - 2nq_f B}},$$

when $B \to B_n^f \equiv \frac{\mu_f^2 - m_f^2}{2nq_f} - 0^+$. \qquad (60)

This shows that strange quark matter will be thermodynamically unstable just below each Landau level B_n^f ($n \neq 0$) for every flavor f. The first three thermodynamically unstable windows (TUWs) associated, respectively, with B_1^d, B_1^u, and B_1^s are illustrated in the $\log B - \mu$ plane in Fig. 3 for our parameters (52). The TUW is actually very narrow. One may conjecture that such an instability may lead to formation of magnetic domains [63]. The presence of possible magnetic domains in neutron star crusts was discussed in Ref. [66]; furthermore, such a possibility for color-flavor-locked quark matter was pointed out in Ref. [67]. We will not pursue here the study of domain structure and related physics, since among other things, this will require us to specify the geometry of the system.

C. Hydrodynamic stability

In this subsection we address the problem of hydrodynamic stability within the theory presented in Sec. II. The fluid is said to be stable if it returns to its initial state after a transient perturbation. Otherwise, i.e., when the perturbation grows and takes the fluid into another state, the fluid is unstable. Our particular goal here is to determine whether a

small, plane-wave perturbation around a homogeneous equilibrium state grows for nonzero $\zeta_\perp$ and $\zeta_\parallel$ [58]. All the other transport coefficients are set to zero. For this purpose, it is sufficient to solve the hydrodynamic and Maxwell equations that are linearized around the homogeneous equilibrium state

$$\partial_\mu \delta T^{\mu\nu} = 0, \qquad \partial_\mu \delta n_B^\mu = 0, \qquad \partial_\mu \delta n_e^\mu = 0,$$
$$\partial_\mu \delta H^{\mu\nu} = 0, \qquad \epsilon^{\mu\nu\alpha\beta} \partial_\nu \delta F_{\alpha\beta} = 0, \qquad (61)$$

where

$$\delta T^{\mu\nu} = \delta T_{F0}^{\mu\nu} + \delta T_{EM}^{\mu\nu} + \delta\tau^{\mu\nu},$$
$$\delta T_{F0}^{\mu\nu} = \delta\varepsilon u^\mu u^\nu - \delta P_\perp \Xi^{\mu\nu} + \delta P_\parallel b^\mu b^\nu$$
$$+ (\varepsilon + P_\perp)(\delta u^\mu u^\nu + u^\mu \delta u^\nu)$$
$$+ (P_\parallel - P_\perp)(\delta b^\mu b^\nu + b^\mu \delta b^\nu),$$
$$\delta T_{EM}^{\mu\nu} = B\delta B(u^\mu u^\nu - \Xi^{\mu\nu} - b^\mu b^\nu)$$
$$+ B^2(\delta u^\mu u^\nu + u^\mu \delta u^\nu - \delta b^\mu b^\nu - b^\mu \delta b^\nu),$$
$$\delta\tau^{\mu\nu} = \zeta_\perp \Xi^{\mu\nu}\delta\phi + \zeta_\parallel b^\mu b^\nu \delta\varphi + \zeta_\perp \delta\Xi^{\mu\nu}\phi$$
$$+ \zeta_\parallel \delta(b^\mu b^\nu)\varphi,$$
$$\delta n_B^\mu = \delta n_B u^\mu + n_B \delta u^\mu,$$
$$\delta n_e^\mu = \delta n_e u^\mu + n_e \delta u^\mu,$$
$$\delta H^{\mu\nu} = \delta H b^{\mu\nu} + H\delta b^{\mu\nu},$$
$$\delta F^{\mu\nu} = \delta B b^{\mu\nu} + B\delta b^{\mu\nu},$$
$$\delta\Xi_{\mu\nu} = \delta b^\mu b^\nu + b^\mu \delta b^\nu - \delta u^\mu u^\nu - u^\mu \delta u^\nu,$$
$$\delta\phi = \tfrac{1}{2}\Xi^{\mu\nu}(\partial_\mu \delta u_\nu + \partial_\nu \delta u_\mu),$$
$$\delta\varphi = \tfrac{1}{2}b^\mu b^\nu(\partial_\mu \delta u_\nu + \partial_\nu \delta u_\mu). \qquad (62)$$

Upon linearizing the normalization conditions $u^\mu u_\mu = 1$, $b^\mu b_\mu = -1$, $u^\mu b_\mu = 0$, one finds that the perturbed variables need to satisfy the constraints

$$\delta u^\mu u_\mu = \delta b^\mu b_\mu = u^\mu \delta b_\mu + \delta u^\mu b_\mu = 0. \qquad (63)$$

In the equations above the perturbations are assumed to have the form $\delta Q = \delta Q_0 \exp(ikx)$, where δQ_0 is constant, and the unperturbed quantities are independent of space and time. The hydrodynamic and Maxwell equations need to be supplemented by an equation of state in order to close the system. The linearized equation of state is given by

$$\delta P = c_s^2 \delta\varepsilon + M\delta B, \qquad (64)$$

where $c_s^2 \equiv (\partial P/\partial\varepsilon)_B$ is the speed of sound.

In the most general case Eqs. (61)–(64) constitute 15 independent equations, but the equations associated with the conservation of n_B and n_e are decoupled from the others if the equation of state is taken in the form (64). Therefore, we are left with 13 equations. We work in the rest frame of the equilibrium fluid, $u^\mu = (1, 0, 0, 1)$ and $b^\mu = (0, 0, 0, 1)$ and choose as independent variables

$$\delta Y_i = \{\delta u_1, \delta u_2, \delta u_3, \delta b_0, \delta b_1, \delta b_2, \delta\varepsilon, \delta P_\parallel, \delta P_\perp,$$
$$\delta B, \delta M, \delta\phi, \delta\varphi\}. \qquad (65)$$

(N.B. One can choose other independent variables, but the results do not change). The 13 linear equations can be collected into the following matrix form:

$$G_{ij}\delta Y_j = 0. \qquad (66)$$

The matrix G has the following form:

$$G = \begin{pmatrix}
-k_2 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & k_2 & -k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-Hk_0 & -Hk_0 & 0 & 0 & Hk_3 & Hk_3 & 0 & 0 & 0 & k_1+k_2 & -k_1-k_2 & 0 & 0 \\
k_3 & k_3 & 0 & 0 & -k_0 & -k_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Bk_1 & Bk_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_0 & 0 & 0 & 0 \\
-hk_1 & -hk_2 & -hk_3 & -HBk_3 & 0 & 0 & k_0 & 0 & 0 & Bk_0 & 0 & 0 & 0 \\
-hk_0 & -hk_0 & 0 & 0 & HBk_3 & HBk_3 & 0 & 0 & k_1+k_2 & B(k_1+k_2) & 0 & -\zeta_\perp(k_1+k_2) & 0 \\
0 & 0 & -hk_0 & -HBk_0 & HBk_1 & HBk_2 & 0 & k_3 & 0 & -Bk_3 & 0 & 0 & \zeta_\parallel k_3 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & M & B & 0 & 0 \\
ik_1 & ik_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -ik_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -c_s^2 & 1 & 0 & -M & 0 & 0 & 0
\end{pmatrix}, \qquad (67)$$

where $h = \varepsilon + P + HB$ is the total enthalpy. The exponential plane-wave solutions for frequencies k_0 and wave vectors $\mathbf{k}$ satisfy the dispersion relations given by

$$\det G = 0. \qquad (68)$$

For the modes propagating parallel (longitudinal modes) or

perpendicular (transverse modes) to the magnetic field, Eq. (68) has simple solutions.

1. *Transverse modes,* $k_3 = 0$. There are two types of transverse modes. One is solely determined by the Maxwell equations and has the following dispersion relation,

$$k_0 = \pm k_\perp, \tag{69}$$

where $k_\perp^2 = k_1^2 + k_2^2$, and describes simply an electromagnetic wave. Another solution has the dispersion relation

$$
\begin{aligned}
k_0 &= \frac{i\zeta_\perp k_\perp^2 \pm \sqrt{4(\varepsilon + P_\parallel)(\varepsilon + P_\perp)c_s^2 k_\perp^2 - k_\perp^4 \zeta_\perp^2}}{2(\varepsilon + P_\parallel)} \\
&\approx \pm \sqrt{\frac{\varepsilon + P_\perp}{\varepsilon + P_\parallel}} c_s k_\perp + \frac{i\zeta_\perp k_\perp^2}{2(\varepsilon + P_\parallel)},
\end{aligned} \tag{70}
$$

where the second approximate relation is valid in the long-wavelength limit. This solution represents a sound wave propagating perpendicular to the magnetic field. The speed of this sonic wave is $\sqrt{(\varepsilon + P_\perp)/(\varepsilon + P_\parallel)}c_s$ and is smaller than the speed c_s of a ordinary sound wave. It is seen that positive $\zeta_\perp$ implies dissipation of the sonic wave, i.e., a decay of the initial disturbance. We conclude that the fluid flow is stable in the case. However, we see that for negative $\zeta_\perp$, the initial disturbance grows and the fluid is unstable. Thus, we conclude that negative transverse bulk viscosity implies hydrodynamic instability via growth of transverse sound waves.

2. *Longitudinal modes,* $k_1 = k_2 = 0$. We find three types of longitudinal modes. The first one is again the electromagnetic wave with the dispersion relation

$$k_0 = \pm k_3. \tag{71}$$

The second one is a transverse wave oscillating perpendicularly to the magnetic field, but traveling along the magnetic field lines. It has the dispersion relation

$$k_0 = \pm \sqrt{\frac{BH}{\varepsilon + P_\parallel + BH}} k_3. \tag{72}$$

This mode is the Alfven wave whose speed is equal to $\sqrt{BH/(\varepsilon + P_\parallel + BH)}$. The third longitudinal mode has the following dispersion relation:

$$
\begin{aligned}
k_0 &= \frac{i\zeta_\parallel k_3^2 \pm \sqrt{4(\varepsilon + P_\parallel)^2 c_s^2 k_3^2 - k_3^4 \zeta_\parallel^2}}{2(\varepsilon + P_\parallel)} \\
&\approx \pm c_s k_3 + \frac{i\zeta_\parallel k_3^2}{2(\varepsilon + P_\parallel)}.
\end{aligned} \tag{73}
$$

This mode represents an ordinary sound wave with dis-

sipation due to the longitudinal bulk viscosity $\zeta_\parallel$. It is obvious that if $\zeta_\parallel < 0$ this mode will not decay, but rather grow, thus leading to hydrodynamic instability.

In the next section, we will show that for certain values of the parameters, the transverse bulk viscosity $\zeta_\perp$ could be indeed negative. We emphasize here that this does not imply a violation of the second law of thermodynamics, but rather this manifests a hydrodynamic instability of the ground state; i.e., small perturbations will take the system via this hydrodynamic instability to a new state. A candidate state is the one which has inhomogeneous (domain) structure. Both the structure of the new state and the transition from the homogeneous to the inhomogeneous state are interesting problems which are beyond the scope of this study. However, we would like to point out a number analogous cases where a negative transport coefficient indicates instability towards formation of a new state with domain structure. One such case is the negative resistivity (also known as the Gunn effect) in certain semiconducting materials [68,69]. Another case is the negative (effective) shear viscosity, which is extensively studied in the literature [70–73]. Finally, negative bulk viscosity has been investigated in different contexts in [74,75].

V. RESULTS FOR THE BULK VISCOSITIES

In order to calculate the bulk viscosities, we need to determine the coefficients A, $C_\parallel$, $C_\perp$, and λ. From Eqs. (50), (51), and (59) we obtain in a straightforward manner

$$
\begin{aligned}
A &= \left(\frac{N_c q_s B}{2\pi^2} \sum_{n=0}^{n_{max}^s} \nu_n \frac{\mu_s}{\sqrt{\mu_s^2 - m_s^2 - 2nq_s B}}\right)^{-1} \\
&+ \left(\frac{N_c q_d B}{2\pi^2} \sum_{n=0}^{n_{max}^d} \nu_n \frac{\mu_d}{\sqrt{\mu_d^2 - m_d^2 - 2nq_d B}}\right)^{-1},
\end{aligned}
$$

$$
\begin{aligned}
C_\parallel &= n_{d0}\left(\frac{N_c q_d B}{2\pi^2} \sum_{n=0}^{n_{max}^d} \nu_n \frac{\mu_d}{\sqrt{\mu_d^2 - m_d^2 - 2nq_d B}}\right)^{-1} \\
&- n_{s0}\left(\frac{N_c q_s B}{2\pi^2} \sum_{n=0}^{n_{max}^s} \nu_n \frac{\mu_s}{\sqrt{\mu_s^2 - m_s^2 - 2nq_s B}}\right)^{-1},
\end{aligned}
$$

$$
C_\perp = C_\parallel - \left(\frac{\partial M}{\partial \mu_d}\bigg/\frac{\partial n_{d0}}{\partial \mu_d} - \frac{\partial M}{\partial \mu_s}\bigg/\frac{\partial n_{s0}}{\partial \mu_s}\right)B, \tag{74}
$$

where

$$
\frac{\partial M}{\partial \mu_f} = \frac{N_c q_f}{2\pi^2} \sum_{n=0}^{n_{max}^f} \nu_n \frac{\mu_f^2 - m_f^2 - 3nq_f B}{\sqrt{\mu_f^2 - m_f^2 - 2nq_f B}}. \tag{75}
$$

The rate λ of the weak processes (1a) and (1b) should also be affected by a strong magnetic field. The major effect of a magnetic field on λ is to modify the phase space

of weak processes (1a) and (1b) [76,77]. Taking this into account, one obtains

$$\lambda = \frac{64\pi^5}{5} \tilde{G}^2 \mu_d T^2 \left(\frac{q_u B}{2\pi^2} \sum_{n=0}^{n_{max}^u} \nu_n \frac{1}{\sqrt{\mu_u^2 - m_u^2 - 2nq_u B}} \right)^2$$

$$\times \left(\frac{q_d B}{2\pi^2} \sum_{n=0}^{n_{max}^d} \nu_n \frac{1}{\sqrt{\mu_d^2 - m_d^2 - 2nq_d B}} \right)$$

$$\times \left(\frac{q_s B}{2\pi^2} \sum_{n=0}^{n_{max}^s} \nu_n \frac{1}{\sqrt{\mu_s^2 - m_s^2 - 2nq_s B}} \right), \qquad (76)$$

where $\tilde{G}^2 \equiv G_F^2 \sin^2\theta_C \cos^2\theta_C = 6.46 \times 10^{-24}$ MeV^{-4} is the Fermi constant.

(1) When the magnetic field is much smaller than the typical chemical potential, say, $q_d B \ll \mu_d^2$, its effect on the bulk viscosities is negligible. For typical parameters (52), this condition holds up to $B \sim 10^{17}$ G. In this case, the system is practically isotropic; $\zeta_\parallel$ and $\zeta_\perp$ are effectively degenerate with the isotropic ζ_0, the bulk viscosity of unmagnetized matter. The zero magnetic field limit results can be obtained easily by replacing $\sum_{n=0}^{n_{max}^f} q_f B \rightarrow 2k_{Ff}^2$, and $B \rightarrow 0$. The bulk viscosity for zero magnetic field ζ_0 as function of oscillation frequency ω for various temperatures is shown in Fig. 4. The "shoulder" structure and the temperature dependence of ζ_0 are easily understood from Eq. (48) and have been widely discussed in the literature [24,28–37,40,41].

(2) When the magnetic field is extremely large, say, $B \gg B_c \sim 10^{20}$ G, for our choice of parameters (52), all the quarks are confined in their lowest Landau level. In this case we obtain
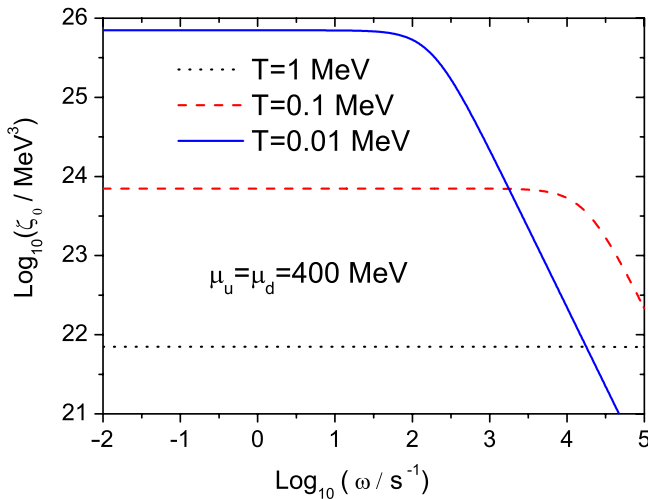
$$A = \frac{2\pi^2}{N_c q_d \mu_d B} (k_{Fd} + k_{Fs}), \qquad C_\parallel = \frac{m_s^2 - m_d^2}{\mu_d},$$

$$C_\perp = 0, \qquad \lambda = \frac{4G^2 q_u^2 q_d^2 B^4 T^2}{5\pi^3 \mu_u^2 k_{Fs}}, \qquad (77)$$

and therefore

$$\zeta_\perp = 0, \qquad \zeta_\parallel \approx \frac{45 m_s^4 \mu_u^2 k_{Fs}}{16\pi \tilde{G}^2 q_u^2 B^2 T^2 (k_{Fs} + k_{Fd})^2}. \qquad (78)$$

We used the parameters (52) and assumed physically interesting frequencies $\omega < 10^4$ s^{-1}. The bulk viscosity $\zeta_\perp$ vanishes as a consequence of vanishing $P_\perp$ when $B > B_c$. Since $\zeta_\parallel$ is now inversely proportional to B^2 it approaches zero for large B. Therefore, both $\zeta_\perp$ and $\zeta_\parallel$ are suppressed for large B. In constrast, $P_\parallel$ is enhanced by the extremely large magnetic field (see Fig. 2).

(3) When the magnetic field is strong, but not strong enough to confine all the quarks to their lowest Landau level, the situation becomes complicated. For our chosen parameters (52), this situation roughly corresponding to the interval 10^{17} G $< B < 10^{20}$ G. In this case, a finite number of Landau levels is occupied, and the essential observation is that $C_\parallel$ and $C_\perp$ can be negative. The behaviors of $C_\parallel$ and $C_\perp$ are shown in Fig. 5 as functions of B. Let us concentrate on the few levels just above the value 10^{19} G. When B grows passing over B_n^d or B_n^s for each n, both $C_\parallel$ and $C_\perp$ change their sign. More importantly, they have always opposite signs. Therefore, in this region, $\zeta_\perp$ is negative, which leads to hydrodynamic instability (see the analysis in Sec. IV C).

FIG. 4 (color online). The isotropic bulk viscosity ζ_0 at zero magnetic field as a function of the oscillation frequency ω for $\mu_u = \mu_d = 400$ MeV at $T = 0.01$ (solid blue curve) 0.1 (dashed red curve), and 1 (dotted black curve) MeV.
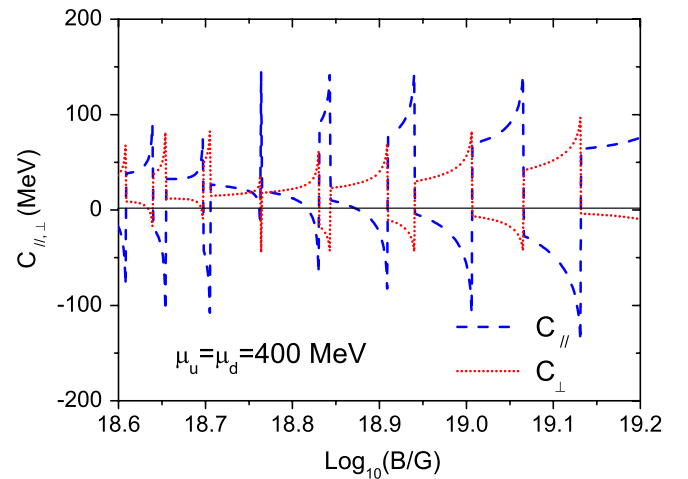

FIG. 5 (color online). Coefficients $C_\parallel$ (dashed blue curve) and $C_\perp$ (dotted red curve) as functions of B at zero temperature.

The numerical values of the bulk viscosities $\zeta_\parallel$ and $\zeta_\perp$ are shown in Fig. 6 as functions of B. The parameters are those given in Eq. (52). We also fix the temperature $T = 0.1$ MeV and oscillation frequency $\omega = 2\pi \times 10^3$ s^{-1}. Both $\zeta_\parallel$ and $\zeta_\perp$ have "quasiperiodic" oscillatory dependence on the magnetic field. The two boundaries of each "period" correspond to a pair of neighboring B_n^f, $f = u, d,$ s, and $n = 0, 1, 2 \cdots$, and hence the period is roughly $\Delta B \sim 2q_f B^2/k_{Ff}^2$ for large B. Therefore, on average, the period increases as B grows. The amplitude of these oscillations also grows with increasing magnetic field until $B \simeq B_c$. Thereafter all the quarks are confined to their lowest Landau levels and $\zeta_\perp$ vanishes. From Fig. 6 we see that the magnitudes of $\zeta_\perp$ and $\zeta_\parallel$ can be 100 to 200 times larger than their zero-field value ζ_0. Because of the unequal

masses and charges of u, d, and s quarks, $\zeta_\parallel$ and $\zeta_\perp$ behave very irregularly. We illustrate the zoomed-in curves around $B = 10^{17}$ and $B = 10^{18}$ G in the subpanels, which look more regular. The quasiperiodic structures are more evident in these subpanels.

The most unusual feature seen in Fig. 6 is that for a wide range of field values, the transverse bulk viscosity $\zeta_\perp$ is negative. Therefore, strange quark matter in this region is hydrodynamically unstable. Besides this hydrodynamical instability, near each B_n^f, there is a narrow window in which thermodynamical instability arises. We depict $\zeta_\parallel$ and $\zeta_\perp$ in these unstable regions by dashed red curves. The solid blue curves correspond to the stable regime.

The magnetic field in a compact star need not be homogeneous and may have a complicated structure with poloidal and toroidal components. Furthermore, the fields will be functions of position in the star because of the density dependence of the parameters of the theory. Furthermore, the instabilities, described above, may lead to fragmentation of matter and formation of domain structures, where the regions with magnetic fields are separated from those without magnetic field by domain walls. Accordingly, only the averaged viscosities over some range of magnetic fields have practical sense for assessing the large-scale behavior of matter. Averaging over many oscillation periods in the stable region, we find that the averaged values of $\zeta_\parallel$ and $\zeta_\perp$ are much more regular, with their magnitudes restricted from 0 to several ζ_0 (see Fig. 7). In obtaining the curves in Fig. 7, we have eliminated the viscosities lying in the unstable regime. The solid black curves are obtained by averaging over a short period $\Delta \log_{10}(B/G) = 0.05$. The period was chosen such that the most rapid fluctuations are smeared out, but the oscillating structures over a larger scale are intact. The dashed red curves correspond to averaging over an even longer period, $\Delta \log_{10}(B/G) = 0.5$. The result of long-period averaging is that $\zeta_\parallel$ first increases slowly and then drops down quickly once $B > 10^{18.5}$ G; similarly, $\zeta_\perp$ first slowly decreases and then drops down very fast for $B \sim 10^{18.5}$ G. Such a dropping behavior reflects the fact that a large number of quarks are beginning to occupy the lowest Landau level.

We note that the appearances of thermodynamic, mechanical, and hydrodynamical instabilities are all induced by the Landau quantization of the quark levels, i.e., are quantum mechanical in nature. More precisely, they are all due to the interplay between the Landau levels and the Fermi momentum (reflected in the quantity B_n^f). Additionally, the hydrodynamical instability requires that the quark matter is paramagnetized. Although we did our analysis by using the free quark gas approximation, Eq. (50), it should be valid as long as there are sharp Fermi surfaces (low temperature), quantized Landau levels (high magnetic field), and paramagnetization. The appearances of these instabilities are expected to be a robust feature for such systems.
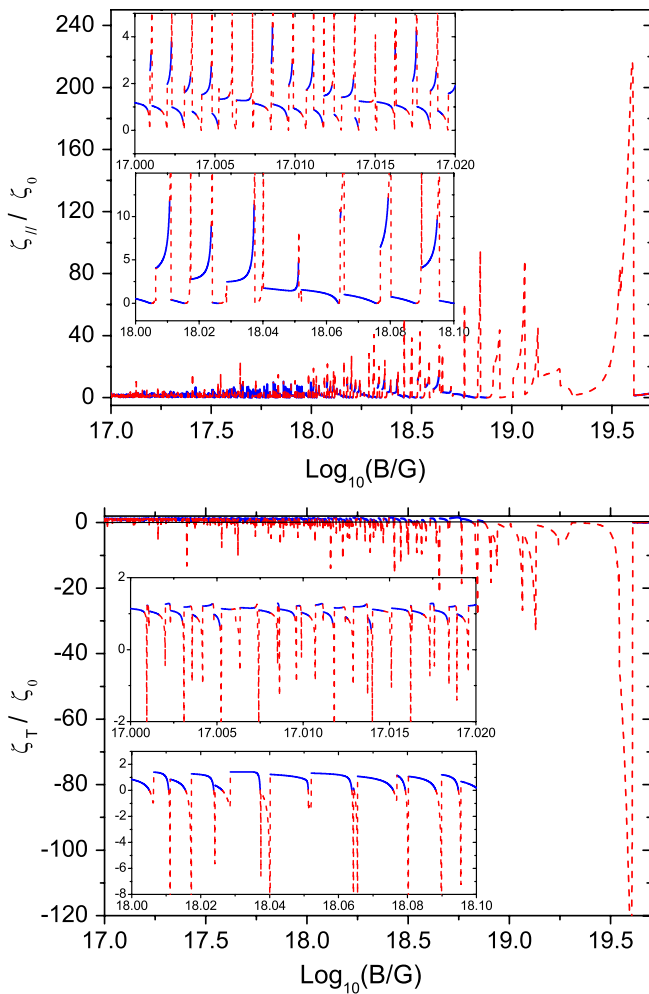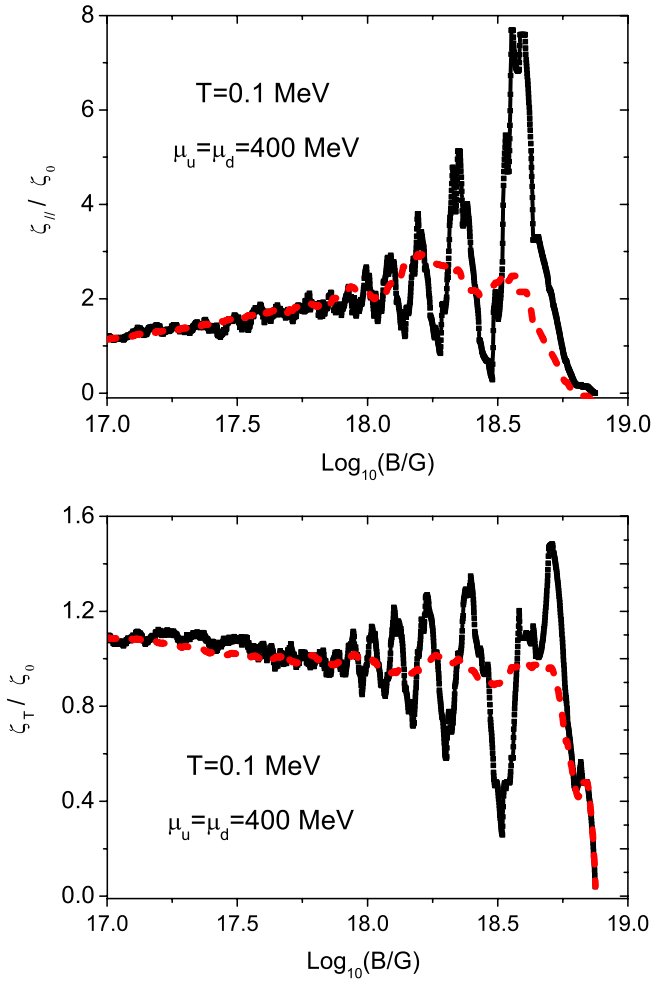

FIG. 6 (color online).   Bulk viscosities $\zeta_\parallel$ and $\zeta_\perp$ scaled by the isotropic bulk viscosity ζ_0 as functions of the magnetic field B at fixed frequency $\omega = 2\pi \times 10^3$ s^{-1} and temperature $T = 0.1$ MeV. The dashed red curves correspond to viscosities lying in the unstable regions and would be not physically reachable. Subpanels show the amplifications around 10^{17} and 10^{18} G. Our parameters are given in Eq. (52).

FIG. 7 (color online). The averaged bulk viscosities $\zeta_\parallel$ and $\zeta_\perp$ scaled by the isotropic bulk viscosity ζ_0 as functions of the magnetic field B at fixed frequency $\omega = 2\pi \times 10^3$ s^{-1} and temperature $T = 0.1$ MeV. The solid black curve corresponds to averaging over a short period [$\Delta \log_{10}(B/G) = 0.05$], while the dashed red curve corresponds to averaging over a long period [$\Delta \log_{10}(B/G) = 0.5$]. The viscosities lying in the unstable regions have been eliminated in the averaging.

We also checked that if one imposes the neutrality condition, i.e., the condition $2n_u = n_d + n_s$, there is only minor quantitative change, while the qualitative conclusions are almost unchanged. Our choice of chemical potential $\mu_u = \mu_d = 400$ MeV roughly corresponds to the choice of $n_B \sim 4 - 5n_0$ for neutral strange quark matter.

VI. r-MODE INSTABILITY WINDOW

The purpose of this section is to discuss the damping of the r-modes of Newtonian models of strange stars by dissipation driven by the bulk viscosities $\zeta_\perp$ and $\zeta_\parallel$. As is well-known, rotating equilibrium configurations of self-gravitating fluids are susceptible to instabilities at high rotation rates. Starting from the mass-shedding limit and going down with the rotation rate, the first instability point

corresponds to the dynamical instability of the $l = 2$ and $m = 2$ mode. This bar-mode instability is independent of the dissipative processes inside the star and occurs at values of the kinetic to potential energy ratio $T/W \sim 0.27$ [78]. For smaller rotation rates two secular instabilities with $l = 2$ arise, each corresponding to a sign of $m = \pm 2$. For incompressible fluids at constant density the T/W values for the onset of secular instabilities coincide. One instability is driven by the viscosity; the other instability is driven by the gravitational radiation. For realistic stars the T/W values for the onset of these instabilities do not coincide; relativity and other factors shift the viscosity-driven instability to higher values of T/W. At the same time the gravitational radiation instability is shifted to lower values of T/W. The gravitational radiation instability arises for the modes which are retrograde in the corotating frame, while prograde in the (distant) laboratory frame. The underlying mechanism is the well established Chandrasekhar-Friedman-Schutz (CFS) mechanism [79,80]. The bulk and shear viscosities can prevent the development of the CFS instabilities, except in a certain window in the rotation and temperature plane.

In the following we shall concentrate on axial modes of Newtonian stars, the so-called r-modes, which are known to undergo a CFS-type instability. Our main goal will be to assess the role of strong magnetic fields and bulk viscosity on the stability of these objects. We shall adopt the formalism of Refs. [22,81,82] for our study of the damping of the r-modes by bulk viscosity. For the sake of simplicity we shall describe both fluid mechanics and gravity in the Newtonian approximation.

The equations that describe the dynamical evolution of the star are

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{79a}$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla(h - \Phi) \equiv -\nabla U, \tag{79b}$$

$$\nabla^2 \Phi = -4\pi G \rho, \tag{79c}$$

where h is defined by the integral

$$h(P) \equiv \int_0^P \frac{dP'}{\rho(P')}. \tag{80}$$

The quantity ρ is the mass density of the fluid which is assumed to satisfy a barotropic equation of state, $\rho = \rho(P)$. Φ is the gravitational potential and G is the gravitational constant. The potential U is used to determine the velocity field $\mathbf{v}$.

The oscillation modes of a uniformly rotating star can be completely described in terms of two perturbation potentials $\delta U \equiv U - U_0$ and $\delta \Phi \equiv \Phi - \Phi_0$, where U_0 and Φ_0 are the potentials that correspond to the equilibrium configuration of the star. We assume that the time and azimuthal angular dependence of any perturbed quantity is described by $\propto e^{i\tilde\omega t + im\varphi}$, where m is an integer and $\tilde\omega$ is the frequency of the mode in the laboratory frame. Let Ω

denote the rotation frequency of the star and ω denote the frequency of the perturbed quantity measured in the corotating frame [which corresponds to the ω in Eqs. (48) and (49) because we will work in the corotating frame]. For small Ω there is a simple relation between Ω, $\tilde{\omega}$, and ω [81,82],

$$\omega = \tilde{\omega} + m\Omega. \tag{81}$$

By linearizing the Euler equation (79b) around the equilibrium configuration, the velocity perturbation δv^a is determined by [81,82]

$$\delta v^a = iQ^{ab}\nabla_b\delta U. \tag{82}$$

The tensor Q^{ab} is a function of $\tilde{\omega}$ and the rotation frequency Ω of the star,

$$Q^{ab} = \frac{1}{(\tilde{\omega} + m\Omega)^2 - 4\Omega^2}\Bigg[(\tilde{\omega} + m\Omega)\delta^{ab}$$
$$- \frac{4\Omega^2}{\tilde{\omega} + m\Omega}z^a z^b - 2i\nabla^a v_0^b\Bigg], \tag{83}$$

where **z** is a unit vector pointing along the rotation axis of the equilibrium star, which we assume to be parallel to the magnetic field, i.e., $z_i = b_i$ in Cartesian coordinate system. Here $v_0 = r\Omega \sin\theta\hat{\varphi}$ is the fluid velocity of the equilibrium star.

Having the linearized Euler equation, one proceeds to the linearization of the mass continuity equation (79a) and the equation for the gravitational potential (79c); one finds

$$\nabla_a(\rho Q^{ab}\nabla_b\delta U) = -(\tilde{\omega} + m\Omega)(\delta U + \delta\Phi)d\rho/dh,$$
$$\nabla^2\delta\Phi = -4\pi G(\delta U + \delta\Phi)d\rho/dh. \tag{84}$$

These equations, together with the appropriate boundary conditions at the surface of the star for δU and at infinity for $\delta\Phi$, determine the potentials δU and $\delta\Phi$.

For slowly rotating stars, Eq. (84) can be solved order by order in Ω,

$$\delta U = R^2\Omega^2\Bigg[\delta U_0 + \delta U_2\frac{\Omega^2}{\pi G\rho} + O(\Omega^4)\Bigg],$$
$$\delta\Phi = R^2\Omega^2\Bigg[\delta\Phi_0 + \delta\Phi_2\frac{\Omega^2}{\pi G\rho} + O(\Omega^4)\Bigg], \tag{85}$$

where R is the radius of a nonrotating star. Since we need only the perturbed velocity, we will focus on δU in the following discussion. The zeroth-order contribution to the r-mode is generated by the following form of the potential δU_0,

$$\delta U_0 = \alpha\left(\frac{r}{R}\right)^{m+1}P_{m+1}^m(\cos\theta)e^{i\tilde{\omega}t+im\varphi}, \tag{86}$$

$$\tilde{\omega} = -\frac{(m-1)(m+2)}{m+1}\Omega, \tag{87}$$

where α is an arbitrary dimensionless constant and $P_l^m(x)$

are the associated Legendre polynomials. It has been shown that the most unstable mode is the one with $m = 2$ [22,83]; therefore we shall consider only this case in the following discussion. Substituting δU_0 into Eq. (82) one obtains the first-order perturbed velocity,

$$\delta \mathbf{v}_0 = \alpha' R\Omega\left(\frac{r}{R}\right)^m\mathbf{Y}_{mm}^B(\theta, \varphi)e^{i\omega t}, \tag{88}$$

where $\alpha' = \alpha\sqrt{\pi(m + 1)^3(2m + 1)!/m}$ and $\mathbf{Y}_{lm}^B(\theta, \varphi)$ is the magnetic-type spherical harmonic function,

$$\mathbf{Y}_{lm}^B(\theta, \varphi) = \frac{\mathbf{r} \times \nabla Y_{lm}}{\sqrt{l(l+1)}}. \tag{89}$$

It is straightforward to check that the first-order perturbed velocity satisfies

$$\frac{\partial\delta v_{0z}}{\partial z} = 0, \qquad \nabla \cdot \delta\mathbf{v}_0 = 0; \tag{90}$$

therefore it does not contribute to the dissipation due to the bulk viscosities $\zeta_\perp$ and $\zeta_\parallel$. In order to see how the bulk viscosities damp the r-mode instability one must consider next-to-first order, i.e., the third-order perturbed velocity which is generated by the potential δU_2. One cannot determine analytically δU_2 from Eqs. (84) and (85), but the angular structure of δU_2 can be well represented by the following spherical harmonics expansion [82],

$$\delta U_2 = \alpha f_1(r)P_{m+1}^1(\cos\theta)e^{i\tilde{\omega}t+im\varphi}$$
$$+ \alpha f_2(r)P_{m+3}^m(\cos\theta)e^{i\tilde{\omega}t+im\varphi}. \tag{91}$$

The functions $f_1(r)$ and $f_2(r)$ have been determined numerically in Ref. [82]. A useful approximation is provided by the following simple expressions,

$$f_1(r) = -0.1294\left(\frac{r}{R}\right)^3 - 0.0044\left(\frac{r}{R}\right)^4 + 0.1985\left(\frac{r}{R}\right)^5$$
$$- 0.0388\left(\frac{r}{R}\right)^6, \tag{92}$$

$$f_2(r) = -0.0092\left(\frac{r}{R}\right)^3 + 0.0136\left(\frac{r}{R}\right)^4 - 0.0273\left(\frac{r}{R}\right)^5$$
$$- 0.0024\left(\frac{r}{R}\right)^6, \tag{93}$$

which excellently fit the numerical result. We will use Eqs. (92) and (93) in the following numerical calculation.

The energy of r-modes comes both from the velocity perturbation and the perturbation of the gravitational potential. For slowly rotating stars, the main contribution comes from the velocity perturbation [22,23,83,84]. Then, the energy of the r-mode measured in the corotating frame is

$$\tilde{E} = \frac{1}{2}\int\rho\delta\mathbf{v}^*\cdot\delta\mathbf{v}d^3\mathbf{r}. \tag{94}$$

Assuming spherical symmetry, we have

$$\tilde{E} = \frac{1}{2}\alpha'^2\Omega^2 R^{-2m+2}\int_0^R \rho r^{2m+2}dr. \qquad (95)$$

This energy will be dissipated both by gravitational radiation and by the thermodynamic transport in the fluid [22,23],

$$\frac{d\tilde{E}}{dt} = \left(\frac{d\tilde{E}}{dt}\right)_G + \left(\frac{d\tilde{E}}{dt}\right)_T. \qquad (96)$$

The dissipation rate due to gravitational radiation is given by [22,23,85]

$$\left(\frac{d\tilde{E}}{dt}\right)_G = -\tilde{\omega}(\tilde{\omega} + m\Omega)\sum_{l\geq 2}N_l\omega^{2l}[|\delta D_{lm}|^2 + |\delta J_{lm}|^2], \qquad (97)$$

where

$$N_l = \frac{4\pi G(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}. \qquad (98)$$

δD_{lm} and δJ_{lm} are the mass and current multipole moments of the perturbation,

$$\delta D_{lm} = \int \delta\rho r^l Y_{lm}^* d^3\mathbf{r},$$
$$\delta J_{lm} = 2\sqrt{\frac{l}{l+1}}\int r^l(\rho\delta\mathbf{v} + \delta\rho\mathbf{v})\cdot\mathbf{Y}_{lm}^{B*}d^3\mathbf{r}. \qquad (99)$$

Taking into account Eq. (87) one obtains

$$\tilde{\omega}(\tilde{\omega} + m\Omega) = -\frac{2(m-1)(m+2)}{(m+1)^2}\Omega^2 < 0, \qquad (100)$$

which implies that the total sign of $(d\tilde{E}/dt)_G$ is positive: gravitational radiation always increases the energy of the r-modes.

In order to compare the relative strengths of different dissipative processes, it is convenient to introduce the dissipative time scales defined by

$$\tau_i \equiv -\frac{2\tilde{E}}{(d\tilde{E}/dt)_i}, \qquad (101)$$

where the index i labels the dissipative process.

The lowest-order contribution to $(d\tilde{E}/dt)_G$ comes from the current multipole moment δJ_{ll}. For the most important case $l = m = 2$, this leads to the following time scale (derived for a simple polytropic equation of state $P \propto \rho^2$) [82,83]:

$$\frac{1}{\tau_G} = -\frac{1}{3.26}\left(\frac{\Omega^2}{\pi G\rho}\right)^3 \text{ s}^{-1}. \qquad (102)$$

The bulk viscosities $\zeta_\perp$ and $\zeta_\parallel$ dissipate the energy of the r-mode according to

$$\left(\frac{d\tilde{E}}{dt}\right)_{\zeta_\perp} = -\frac{3}{2}\int \zeta_\perp\left|\frac{\partial\delta v_x}{\partial x} + \frac{\partial\delta v_y}{\partial y}\right|^2 d^3\mathbf{r},$$
$$\left(\frac{d\tilde{E}}{dt}\right)_{\zeta_\parallel} = -3\int \zeta_\parallel\left|\frac{\partial\delta v_z}{\partial z}\right|^2 d^3\mathbf{r}. \qquad (103)$$

Accordingly, the time scales $\tau_{\zeta_\perp}$ and $\tau_{\zeta_\parallel}$ are given by

$$\tau_{\zeta_\perp,\zeta_\parallel} = -\frac{2\tilde{E}}{(d\tilde{E}/dt)_{\zeta_\perp,\zeta_\parallel}}. \qquad (104)$$

Currently, the shear viscosities $\eta_1-\eta_5$ of strange quark matter are not known. In order to determine the damping of the r-mode by shear viscosity, we take as a crude estimate the value of η_0 in the absence of a magnetic field [62]

$$\eta_0 \simeq \eta = 5.5\times 10^{-3}\alpha_s^{-5/3}\mu_d^{14/3}\times T^{-5/3}, \qquad (105)$$

where α_s is the coupling constant of strong interaction. We will choose the value $\alpha_s = 0.1$ and apply Eq. (105) to highly degenerate 3-flavor quark matter with equal chemical potentials of all flavors ($\mu_u \simeq \mu_d \simeq \mu_s$). The contribution to the energy dissipation rate $\tilde{E}$ due to shear viscosity η now becomes

$$\left(\frac{d\tilde{E}}{dt}\right)_\eta = -\int \eta|w_{ij} - \delta_{ij}\theta/3|^2 d^3\mathbf{r}. \qquad (106)$$

Assuming a uniform mass density star, the time scale τ_η can be simply expressed as [85]

$$\frac{1}{\tau_\eta} = \frac{7\eta}{\rho R^2}. \qquad (107)$$

The total time scale $\tau(\Omega, T)$ is given by the following sum,

$$\frac{1}{\tau} \equiv \frac{1}{\tau_G} + \frac{1}{\tau_{\zeta_\perp}} + \frac{1}{\tau_{\zeta_\parallel}} + \frac{1}{\tau_\eta}, \qquad (108)$$

which characterizes how fast the r-mode decays. Most importantly, if the sign of τ is negative the amplitude of the r-mode will not decay; rather it will increase with time. Thus, it is important to determine the critical angular velocity Ω_c for the onset of instability

$$\frac{1}{\tau(\Omega_c, T)} = 0. \qquad (109)$$

At a given temperature, stars with $\Omega > \Omega_c$ will be unstable due to gravitational radiation.

Figure. 8 shows the critical angular velocity Ω_c of a strange quark star with mass $M = 1.4M_\odot$ and radius $R = 10$ km as a function of the magnetic field B near 10^{18} G. The temperature is fixed as $T = 0.001$ MeV and other parameters are taken according to Eq. (52). In obtaining Fig. 8, we have taken into account the thermodynamical and hydrodynamical stability conditions. The solid blue curves correspond to the thermodynamically and hydrodynamically stable region, while the dashed red curves
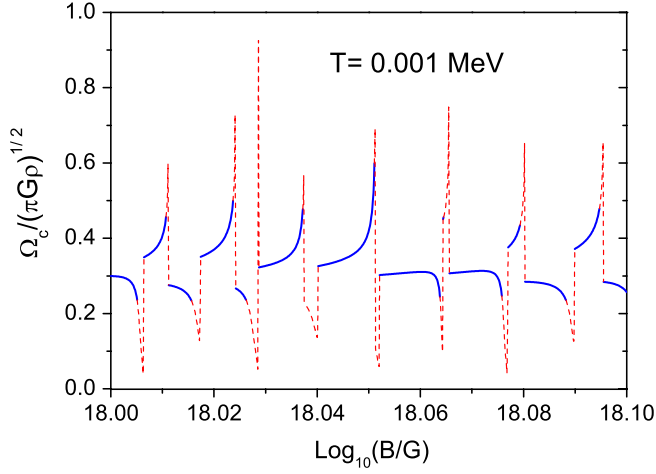
FIG. 8 (color online). The critical angular velocity Ω_c of a strange quark star as a function of magnetic field B at temperature $T = 0.001$ MeV. The dashed red lines correspond to the unstable, while the solid blue lines correspond to the stable regime.

correspond to unstable regions. The critical angular velocity is strongly oscillating with increasing B. This behavior is due to the oscillating nature of the bulk viscosities $\zeta_\parallel$ and $\zeta_\perp$ as shown in Fig. 6. Thus, this macroscopic behavior originates from a purely quantum mechanical effect, namely, the Landau quantization of the energy levels of quarks. As discussed for the bulk viscosities $\zeta_\parallel$ and $\zeta_\perp$ shown in Fig. 7, averaging is needed to obtain physically relevant quantities. In Fig. 9 we show the averaged critical angular velocity at various temperatures. The solid black curves are obtained by averaging over a short period $\Delta \log(B/G) = 0.05$, whereas the dashed red curves correspond to averaging over a long period $\Delta \log(B/G) = 0.5$. It is seen that after short-period averaging, the critical angular velocity (solid black curves) shows regular oscillation, the amplitude of which is growing as the B field increases. The critical angular velocity Ω_c displays a sharp drop for fields $B \lesssim 10^{18.5}$ G (dashed red curves), which is the consequence of the sharp drop of $\zeta_\parallel$ and $\zeta_\perp$ shown in Fig. 7. Thus, we conclude that for extremely large magnetic fields, the critical angular velocity at which the r-mode instability sets in could be significantly lower than in the absence of magnetic field.

Figure. 10 shows the window of the r-mode instability in the $\Omega - \log_{10} T$ plane for a strange quark star of mass $M = 1.4 M_\odot$ and radius $R = 10$ km. The regions above the respective curves correspond to the parameter space where the r-mode oscillations are unstable; i.e., a star in this region will rapidly spin down by emission of gravitational waves. The dashed green curve corresponds to vanishing bulk viscosities $\zeta_\parallel = \zeta_\perp = 0$. The solid black curve represents the (in)stability window of an unmagnetized strange quark star. The curves with symbols show the typical instability window for magnetic fields around
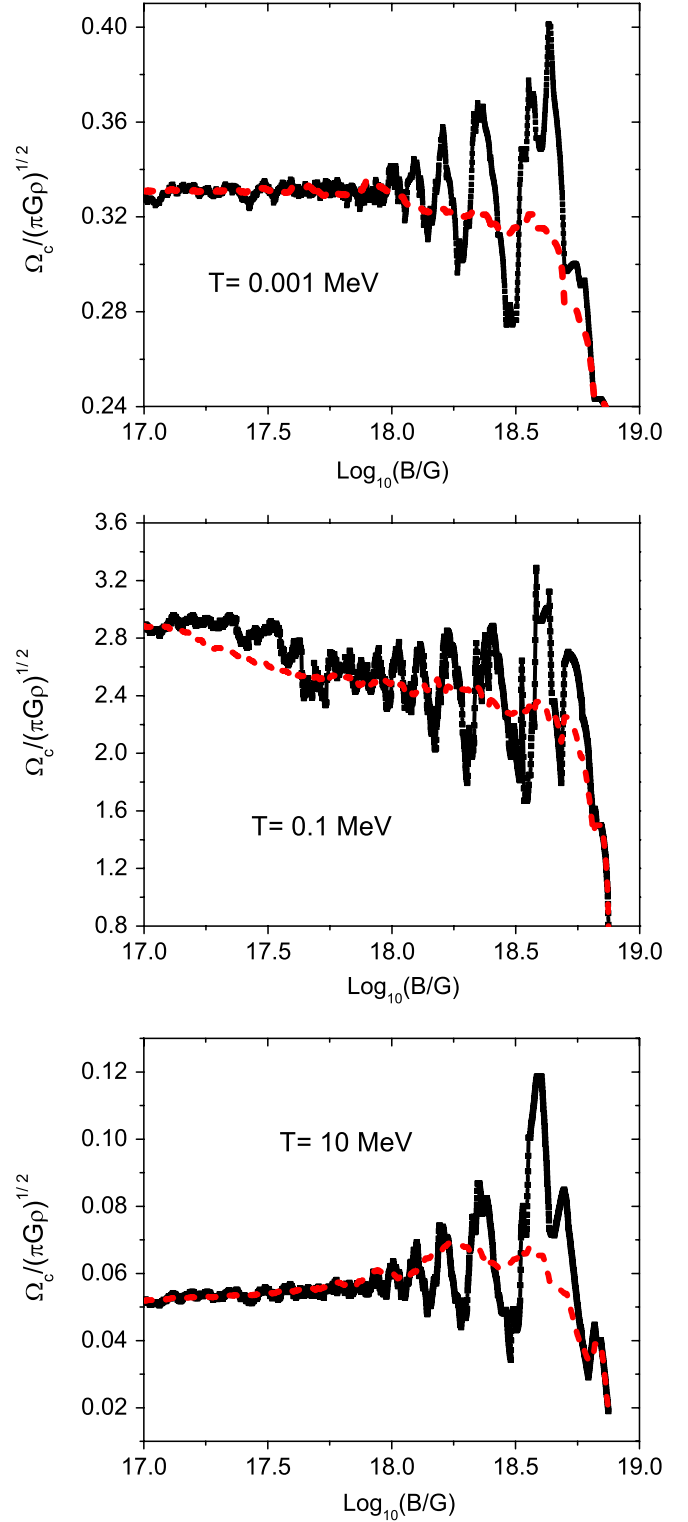

FIG. 9 (color online). The averaged critical angular velocity Ω_c of a strange quark star as a function of magnetic field B at various temperatures $T = 0.001$, 0.1, and 10 MeV. The solid black curve corresponds to averaging over a short period $[\Delta \log(B/G) = 0.05]$, while the dashed red curve corresponds to averaging over a long period $[\Delta \log(B/G) = 0.5]$.
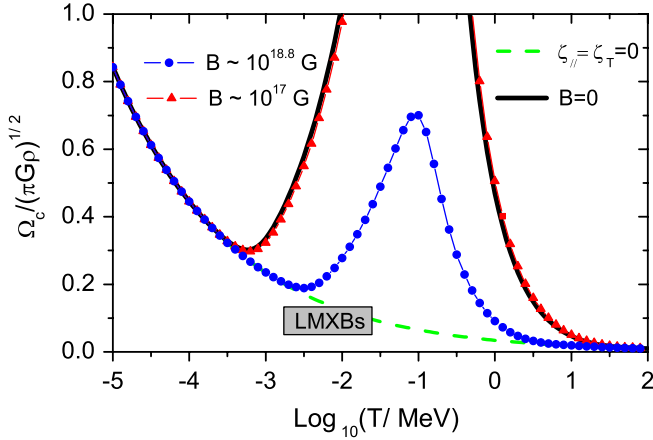
FIG. 10 (color online). The r-mode instability window for a strange quark star. The star is stable below the respective curves. The dashed green curve corresponds to vanishing bulk viscosities $\zeta_\parallel = \zeta_\perp = 0$. The solid black curve represents the window of unmagnetized strange quark matter. The curves with symbols show the typical behavior of the instability window when the magnetic fields are around 10^{17} G (red curve) and $10^{18.8}$ G (blue curve). The shadowed box represents typical temperatures (2×10^7–3×10^8 K) and rotation frequencies (300–700 Hz) of the majority of observed LMXBs [86].

10^{17} G (the red curve marked by triangles) and $10^{18.8}$ G (the blue curve, marked by circles). The symboled curves are obtained by using the bulk viscosities averaged over the period $\Delta \log_{10}(B/G) = 0.5$. For low temperatures, $T < 0.3$ keV, the r-mode instability is suppressed mainly by the shear viscosity; at these low temperatures the bulk viscosities are an insignificant source of damping, independent of how large the magnetic field is. However, for larger temperatures the bulk viscosities dominate the damping of r-mode oscillations. For magnetic fields below $B \sim 10^{17}$ G, the critical rotation frequency is almost independent of the B field. The r-mode instability window increases as the magnetic field grows. For fields $B > 10^{18}$ G it is a very sensitive function of the field, as a consequence of the rapid variation of the bulk viscosities with the field. Asymptotically, the instability window can become significantly larger than the window at the zero magnetic field (see also Fig. 9). For completeness, Fig. 10 also shows the observed distribution of low mass x-ray binaries (LMXBs) by the shadowed box, which corresponds to the typical temperatures (2×10^7–3×10^8 K) and rotation frequencies (300–700 Hz) of the majority of observed LMXBs [86]. It is seen that even in the case of extremely large magnetic fields, our instability window is consistent with the current LMXB data.

VII. SUMMARY

In this paper we have studied anisotropic hydrodynamics of strongly magnetized matter in compact stars. We find that there are in general eight viscosity coefficients: six of them are identified as shear viscosities, and the other two, $\zeta_\perp$ and $\zeta_\parallel$, are bulk viscosities [see Eq. (23)]. We applied our formalism to magnetized strange quark matter and gave explicit expressions for the bulk viscosities [Eqs. (48) and (49)] due to the nonleptonic weak reactions (1a) and (1b). Because of the Landau quantization of the energy levels of charged particles in a strong magnetic field, the magnetic field dependence of $\zeta_\parallel$ and $\zeta_\perp$ is very complicated and exhibits "quasiperiodic" de Haas–van Alphen–type oscillations (see Fig. 6). For a magnetic field $B \leq 10^{17}$ G the effect of the magnetic field on the transport coefficients is small and the bulk viscosities can be well approximated by their zero-field values. For large fields $10^{17} \leq B \leq 10^{20}$ G the viscosities are substantially modified; $\zeta_\perp$ may even become negative for some values of the B field. We showed that negative $\zeta_\perp$ render the fluid hydrodynamically unstable.

For a number of reasons (density dependence of parameters along the star profile, formation of domains, intrinsic multicomponent nature of the magnetic field) the dependence of the transport coefficients on the magnetic fields is needed at different resolutions (i.e., they require some suitable averaging over a range of magnetic fields). We have provided such averages over an increasingly larger scale. We find that if the averaging period is small the bulk viscosities show regular oscillations, the amplitudes of which increase with magnetic field (see the solid black curves in Fig. 7). These oscillations are smoothed out if we further increase the averaging scale. At this larger scale the most interesting feature is the rapid drop in the bulk viscosity of the matter due to the confinement of quarks to the lowest Landau level; this occurs for magnetic fields in excess of $B > 10^{18.5}$ G (see the dashed red curves in Fig. 7).

As an application, we utilized our computed anisotropic bulk viscosities to study the problem of damping of r-mode oscillations in rotating Newtonian stars. We find that the instability window increases as the magnetic field is increased above the value $B > 10^{17}$ G. By increasing the field one covers the entire range of parameter space which lies between the two extremes: the case when bulk viscosity vanishes (dashed green curve in Fig. 10, which corresponds to extremely large magnetic fields $B \gtrsim 10^{19}$ G for which the bulk viscosity drops to zero), and the case when the magnetic field is absent (solid black curve in Fig. 10). The found novel dependence of the r-mode instability window on the magnetic field may help to distinguish quark stars from ordinary neutron stars with strong magnetic fields, since the latter are much more difficult to magnetize. It would be interesting to see whether the objects that lie in between these extremes, e.g., hybrid configurations featuring quark cores and hadronic envelopes (see Ref. [87] and references therein), may interpolate smoothly between the physics of ordinary and strange compact objects.

APPENDIX: LINEAR INDEPENDENCE OF COMPONENTS IN EQ. (21)

The purpose of this appendix is to show explicitly that the eight different decompositions in Eq. (21) are linearly independent. To this end, let us write down a general linear combination of the eight different decompositions,

$$a_1(\mathrm{i}) + a_2(\mathrm{ii}) + \cdots + a_8(\mathrm{viii}) = 0. \tag{A1}$$

If (i)–(viii) are linearly independent, the coefficients a_1–a_8 should all vanish for any values of the vectors u^μ and b^μ. First, it is easy to see that the components (vi) and (vii) are independent of the other components, because they have odd parity under reflection $b^\mu \rightarrow -b^\mu$, whereas the other six components have even parity under this transformation.

Besides that, it is obvious that (vi) and (vii) are independent of each other. Therefore, we only need to treat the linear equation

$$a_1(\mathrm{i}) + \cdots + a_5(\mathrm{v}) + a_8(\mathrm{viii}) = 0. \tag{A2}$$

By contracting the indices μ and ν, we obtain the following three conditions:

$$\begin{aligned} 3a_1 + 2a_2 - a_3 + 2a_8 &= 0, \\ 3a_3 - a_4 + 4a_5 + 2a_8 &= 0, \\ 3a_1 + 2a_2 - 12a_3 + a_4 - 4a_5 &= 0. \end{aligned} \tag{A3}$$

Contracting Eq. (A1) with b^μ and b^ν we find the following two additional conditions:

$$a_1 + a_3 = 0, \qquad 2a_2 - a_3 + a_4 - 4a_5 = 0. \tag{A4}$$

Contracting the indices ν and α in Eq. (61), we obtain one further condition:

$$a_1 + 4a_2 - 2a_3 + a_4 - 6a_5 = 0. \tag{A5}$$

The nontrivial solution of the set of Eqs. (A2)–(A5) is

$$a_1 = a_3 = 0, \qquad a_2 = a_4/2 = a_5 = -a_8. \tag{A6}$$

Then, we have the following condition,

$$a_2(b^{\mu\alpha}b^{\nu\beta} + b^{\nu\alpha}b^{\mu\beta} - \Xi^{\mu\alpha}\Xi^{\nu\beta} - \Xi^{\nu\alpha}\Xi^{\mu\beta}) = 0. \tag{A7}$$

The only possible solution is $a_2 = 0$, which thus proves the independence of (i)–(viii).

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