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Perturbative quantization of two-dimensional space-time noncommutative QED

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Using the method of perturbative quantization in the first order approximation, we quantize a nonlocal QED-like theory including fermions and bosons whose interactions are described by terms containing higher order space-time derivatives. As an example, the two-dimensional space-time noncommutative QED (NC-QED) is quantized perturbatively up to $\mathcal{O}(e^2, \theta^3)$, where e is the NC-QED coupling constant and θ is the noncommutativity parameter. The resulting modified Lagrangian density is shown to include terms consisting of first order time-derivative and higher order space-derivatives of the modified field variables that satisfy the ordinary equal-time commutation relations up to $\mathcal{O}(e^2, \theta^3)$. Using these commutation relations, the canonical current algebra of the modified theory is also derived.

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I. INTRODUCTION

Theories including higher order time derivatives appear in different areas of physics. In general relativity, for instance, the quantum corrections added to the original lower-derivative theory contain higher derivatives of the metric [1]. This occurs also in the case of cosmic strings [2], and in Dirac's relativistic model of classical radiating electron [3,4]. In contrast to the naive expectation, the presence of an unconstrained higher-derivative term, no matter how small it may appear, makes the new theory dramatically different from its original lower-derivative counterpart. As it is shown in [4], unconstrained higher-derivative theories have more degrees of freedom than lower-derivative theories, and they lack a lower-energy bound. Classically, there is a no-go theorem by Ostrogradski [5], who essentially showed why no more than two time derivatives of the fundamental dynamical variables appear in the laws of physics. Ostrogradski's result is that there is a linear instability in the Hamiltonian associated with Lagrangians which depend upon more than one time derivative in such a way that the dependence cannot be eliminated by partial integration [5,6] (for a modern review of higher-derivative theories see [6]).

There is also a large class of theories containing higher derivatives that do not suffer the above problems. Nonlocal theories, where the nonlocality is regulated by a natural small parameter contain, in general, implicit constraints which keep the number of degrees of freedom constant and maintain a lower-energy bound [4]. Nonlocality naturally appears in effective theories in a low-energy limit that are derived from a larger theory with some degrees of freedom frozen out [4]. A good example is Wheeler-Feynman electrodynamics [7], in which the degrees of freedom of the electromagnetic field are frozen out [4]. Space-time noncommutative field theories that arise from open string in a background *electric* field,[1] are another example of nonlocal low-energy effective field theories consisting of an infinite number of temporal and spatial derivatives in the interaction part of the corresponding Lagrangian densities. The embedding of noncommutative field theories into string theory is maybe relevant to understanding the inevitable breakdown of our familiar notions of space and time at short distances in quantum gravity [8,9]. Whereas space-space noncommutative theories suffer from a mixing of ultraviolet and infrared singularities in their perturbative dynamics [10], the space-time noncommutative theories seem to be seriously acausal and inconsistent with conventional Hamiltonian evolution [11]. Besides they do not have a unitary S matrix [12]. Indeed the breakdown of unitarity in a theory consisting of higher order time derivative and the above mentioned Ostrogradskian instability are closely related [6]. However, as it is shown in [13], the unitarity of the space-time noncommutative theories can be restored and the path integral quantization can be performed. This progress suggests that space-time noncommutative theories can be incorporated in the framework of canonical quantization [14]. Different canonical approaches are suggested in [14,15]. In [15], first a general Hamiltonian formalism is developed for nonlocal field theories in d space-time dimensions by considering auxiliary $d+1$ dimensional field theories which are local with respect to the evolution in time. The case of noncommutative φ^3 theory is then considered as an example. In [14], a modification of the Poisson bracket (PB) suitable for a canonical analysis of space-time noncommutative field theories is constructed.

Another possibility to quantize the space-time noncommutative gauge theories is to use the perturbative quantization introduced in [16–18]. In [16,17], the method of

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[1]Note that space-space noncommutative field theory describes the low-energy limit of string theory in a background *magnetic* field.

perturbative quantization is used to define the Poisson structure and Hamiltonian of generic higher-derivative classical and quantum field theories. This method is independently developed in [18]. In [16], the perturbative quantization of noncommutative gauge theories is discussed qualitatively, as an example, but no explicit calculation is performed. In [17], the same method is used to quantize the Lagrange function involving higher order time derivatives for both bosons and fermions in $0 + 1$ dimensions. As an example the supersymmetric noncommutative Wess-Zumino model is considered. In all these examples higher order time derivatives appear in the interaction part of the Lagrangian. Recently, this method is also used in a series of paper by Reyes *et al.* [19,20] to quantize specific models, where the nonlocal higher time-derivative terms do not appear in the interaction part of the Lagrangian density.

The aim of the present paper is to quantize a two-dimensional noncommutative QED perturbatively in the first order approximation using the method described in [16–18]. The resulting effective Lagrangian density of the theory will be also presented in terms of modified field variables that satisfy the ordinary equal-time commutation relations order by order in perturbation theory. The organization of the paper is as follows. In Sec. II, we will develop the general framework of perturbative quantization for a $D + 1$ dimensional QED-like theory including bosons and fermions whose interactions are described by terms containing higher order space-time derivatives. In Sec. III, after introducing the noncommutative Moyal product, involving an infinite number of space-time derivatives, we will quantize $1 + 1$ dimensional space-time noncommutative theory perturbatively up to $\mathcal{O}(e^2, \theta^3)$, where e is the NC-QED coupling constant and θ is the noncommutativity parameter. The effective Lagrangian density of the theory including the first order time derivative and higher order space-derivatives is presented in Sec. IV. It includes a bosonic and a fermionic part. Whereas the fermionic part is modified in this order of expansion, the bosonic part remains unchanged. In Sec. V, using the Dirac brackets of the modified fields, we will determine, as a by product, the canonical algebra of the global NC-$U_V(1)$ vector currents of the original noncommutative theory up to $\mathcal{O}(e^2, \theta^3)$. Section VI is devoted to discussions.

II. CANONICAL QUANTIZATION OF MODIFIED QED INCLUDING HIGHER ORDER TIME DERIVATIVES

Let us consider the Lagrangian density of modified QED including higher order space-time derivatives of fermionic and bosonic degree of freedom ψ, $\bar{\psi}$, and A_μ,

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}}(\psi, \partial_\mu \psi, \partial_\mu \partial_\nu \psi, \ldots; \bar{\psi}, \partial_\mu \bar{\psi}, \partial_\mu \partial_\nu \bar{\psi}, \ldots; A_\rho, \partial_\mu A_\rho, \partial_\mu \partial_\nu A_\rho, \ldots). \tag{2.1}$$

Here, we have assumed that higher order space-time derivatives appear only in the interaction part, $\mathcal{L}_{\text{int}}$. The kinetic term, $\mathcal{L}_{\text{kin}}$, is therefore the ordinary kinetic Lagrangian density of free QED in $D + 1$ dimensions

$$\mathcal{L}_{\text{kin}} = \bar{\psi} i \gamma^\mu \partial_\mu \psi - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2, \tag{2.2}$$

where $\mathcal{F}_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor of ordinary QED and ξ is the gauge fixing parameter. In this section, we will perturbatively quantize the theory described by (2.1) up to order $\mathcal{O}(e^2)$, where e is the coupling constant of bosons and fermions. To this purpose, we will first introduce the corresponding fermionic and bosonic symplectic two-forms, from which the nontrivial Poisson algebra of these fields can be derived. After replacing higher order time derivatives with the corresponding space-derivatives using the Euler-Lagrange equation of motion arising from (2.1) up to $\mathcal{O}(e)$, the field variables $\bar{\psi}$, ψ, and A_μ will be appropriately redefined so that the modified field variables satisfy the ordinary fundamental Poisson brackets. The resulting Poisson algebra will be then quantized using the well-known Dirac quantization prescription.

A. General structure of symplectic two-forms and Poisson brackets

Let us start by varying the action $S = \int d^D x dt \mathcal{L}$, with $\mathcal{L}$ from (2.1),

$$\begin{aligned}\delta S = &\int d^D x dt \partial_{\nu_1}\bigg(\sum_{m,k=0}^{\infty}(-1)^k \partial_{\mu_1}\ldots\partial_{\mu_k}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_k}\partial_{\nu_1}\partial_{\nu_2}\ldots\partial_{\nu_{m+1}}\psi)}\bigg)\delta(\partial_{\nu_2}\ldots\partial_{\nu_{m+1}}\psi)\\ &+\int d^D x dt \delta(\partial_{\nu_2}\ldots\partial_{\nu_{m+1}}\bar{\psi})\partial_{\nu_1}\bigg(\sum_{m,k=0}^{\infty}(-1)^k \partial_{\mu_1}\ldots\partial_{\mu_k}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_k}\partial_{\nu_1}\partial_{\nu_2}\ldots\partial_{\nu_{m+1}}\bar{\psi})}\bigg)\\ &+\int d^D x dt \partial_{\nu_1}\bigg(\sum_{m,k=0}^{\infty}(-1)^k \partial_{\mu_1}\ldots\partial_{\mu_k}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_k}\partial_{\nu_1}\partial_{\nu_2}\ldots\partial_{\nu_{m+1}}A_\sigma)}\bigg)\delta(\partial_{\nu_2}\ldots\partial_{\nu_{m+1}}A_\sigma) + \text{EoM},\end{aligned} \tag{2.3}$$

where EoM is the space-time integral of the Euler-Lagrange equation of motions

$$\sum_{k=0}^{\infty}(-1)^k\partial_{\mu_1}\ldots\partial_{\mu_k}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_k}\psi)}=0,$$
$$\sum_{k=0}^{\infty}(-1)^k\partial_{\mu_1}\ldots\partial_{\mu_k}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_k}\bar{\psi})}=0, \quad (2.4)$$
$$\sum_{k=0}^{\infty}(-1)^k\partial_{\mu_1}\ldots\partial_{\mu_k}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_k}A_\sigma)}=0,$$

multiplied by $\delta\psi$ from right, $\delta\bar{\psi}$ from left, and δA_μ, respectively. Neglecting the surface terms with respect to spatial coordinates, the remaining terms in (2.3) can be written as

$$\delta S=\int d^Dxdt\sum_{m=0}^{\infty}\partial_0(\Pi_{\psi^{(m)}}\delta\psi^{(m)}-\delta\bar{\psi}^{(m)}\Pi_{\bar{\psi}^{(m)}}+\Pi^{\sigma}_{A^{(m)}}\delta A^{(m)}_{\sigma}). \quad (2.5)$$

Here, the superscripts (m) denote the m-th order time derivative of the corresponding fields, and the canonical momenta corresponding to fermions $\Pi_{\psi^{(m)}}$, $\Pi_{\bar{\psi}^{(m)}}$, and to bosons $\Pi^{\sigma}_{A^{(m)}}$ are given by

$$\Pi_{\psi^{(m)}}=\sum_{k=0}^{\infty}(-1)^k\partial_{\mu_1}\ldots\partial_{\mu_k}\frac{\partial^R\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_k}\psi^{(m+1)})},$$
$$\Pi_{\bar{\psi}^{(m)}}=\sum_{k=0}^{\infty}(-1)^k\partial_{\mu_1}\ldots\partial_{\mu_k}\frac{\partial^L\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_k}\bar{\psi}^{(m+1)})}, \quad (2.6)$$
$$\Pi^{\sigma}_{A^{(m)}}=\sum_{k=0}^{\infty}(-1)^k\partial_{\mu_1}\ldots\partial_{\mu_k}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_k}A^{(m+1)}_{\sigma})}.$$

Here, ∂^R and ∂^L are the right and left derivatives, respectively. Using (2.5) the symplectic two-form $\Omega(t)$ is defined as[2]

$$\Omega(t)=\sum_{m=0}^{\infty}\int d^Dx(X^{(m)}(t;\mathbf{x})-Y^{(m)}(t;\mathbf{x})+Z^{(m)}(t;\mathbf{x})), \quad (2.7)$$

where

$$X^{(m)}(t;\mathbf{x})=d\Pi^{\beta}_{\psi^{(m)}}(t;\mathbf{x})\wedge d\psi^{(m)}_{\beta}(t;\mathbf{x}),$$
$$Y^{(m)}(t;\mathbf{x})=d\bar{\psi}^{(m)}_{\alpha}(t;\mathbf{x})\wedge d\Pi^{\alpha}_{\bar{\psi}^{(m)}}(t;\mathbf{x}), \quad (2.8)$$
$$Z^{(m)}(t;\mathbf{x})=d\Pi^{\mu}_{A^{(m)}}(t;\mathbf{x})\wedge dA^{(m)}_{\mu}(t;\mathbf{x}).$$

To determine the Poisson brackets of fermionic and bosonic degrees of freedom, we will introduce the following equivalent representation of the symplectic two-form $\Omega(t)$ from (2.7):

$$\Omega(t)=\Omega_g(t)+\Omega_f(t), \quad (2.9)$$

where $\Omega_g(t)$, $\Omega_f(t)$ are the gauge and the fermionic parts of $\Omega(t)$, respectively. The gauge part is given by

$$\Omega_g(t)\equiv\frac{1}{2}\int d^Dxd^Dx'W_{ab}(t;\mathbf{x},\mathbf{x}')dz^a(t;\mathbf{x})\wedge dz^b(t;\mathbf{x}'), \quad (2.10)$$

where the phase space variables are a $(D+1)\times(m+1)$-dimensional vector

$$z^a\equiv(A^0,\ldots A^D,\dot{A}^0,\ldots,\dot{A}^D,\ldots A^{0(m)},\ldots,A^{D(m)},\ldots),$$

with $m\in\{0,\ldots,\infty\}$ determining the order of time derivatives acting on A_μ, with $\mu=0,1,\ldots D$. In (2.10), the operator W_{ab} is determined using $Z^{(m)}(\mathbf{x})$ from (2.8). Using $\Omega_g(t)$ from (2.10), the Poisson bracket between z^a is defined as

$$\{z^a(t;\mathbf{x}),z^b(t;\mathbf{x}')\}_{\rm PB}=W^{ab}(t;\mathbf{x},\mathbf{x}'), \quad (2.11)$$

where $W^{ab}(t;\mathbf{x},\mathbf{x}')$ is the inverse of the operator $W_{ab}(t;\mathbf{x},\mathbf{x}')$ appearing in (2.10) and satisfies

$$W^{ab}(t;\mathbf{x},\mathbf{x}')=-W^{ba}(t;\mathbf{x}',\mathbf{x}). \quad (2.12)$$

It is defined by the orthogonality relation

$$\int d^Dx'W_{ab}(t;\mathbf{x},\mathbf{x}')W^{bc}(t;\mathbf{x}',\mathbf{x}'')=\delta^D(\mathbf{x}-\mathbf{x}'')\delta_a{}^c. \quad (2.13)$$

As for the fermionic part of $\Omega(t)$, it is defined similarly by

$$\Omega_f(t)\equiv\frac{1}{2}\int d^Dxd^Dx'(W^{\alpha\beta}_{\bar{\psi}\psi}(t;\mathbf{x},\mathbf{x}')\times d\bar{\psi}_\alpha(t;\mathbf{x})\wedge d\psi_\beta(t;\mathbf{x}')+W^{\alpha\beta}_{\psi\bar{\psi}}(t;\mathbf{x},\mathbf{x}')\times d\psi_\beta(t;\mathbf{x})\wedge d\bar{\psi}_\alpha(t;\mathbf{x}')). \quad (2.14)$$

Here, $W_{\bar{\psi}\psi}$ and $W_{\psi\bar{\psi}}$ can be derived using $X^{(m)}(t;\mathbf{x})$, and $Y^{(m)}(t;\mathbf{x})$ appearing in (2.8). They include derivatives acting on $\bar{\psi}$ and ψ, respectively. Using further

$$W^{\alpha\beta}_{\psi\bar{\psi}}(t;\mathbf{x},\mathbf{x}')=W^{\alpha\beta}_{\bar{\psi}\psi}(t;\mathbf{x}',\mathbf{x}), \quad (2.15)$$

and $d\psi_\beta(t;\mathbf{x}')\wedge d\bar{\psi}_\alpha(t;\mathbf{x})=d\bar{\psi}_\alpha(t;\mathbf{x})\wedge d\psi_\beta(t;\mathbf{x}')$, the two-form (2.14) can be brought in a simpler form

$$\Omega_f(t)=\int d^Dxd^Dx'W^{\alpha\beta}_{\bar{\psi}\psi}(t;\mathbf{x},\mathbf{x}')d\bar{\psi}_\alpha(t;\mathbf{x})\wedge d\psi_\beta(t;\mathbf{x}'). \quad (2.16)$$

The operators $W_{\bar{\psi}\psi}$ and $W_{\psi\bar{\psi}}$ can be determined using the definition $X^{(m)}$ and $Y^{(m)}$ from (2.8). Their inverse operators are denoted by $W^{\psi\bar{\psi}}$ and $W^{\bar{\psi}\psi}$, respectively. They can be determined using the orthogonality relations

[2] The relative minus sign between $X^{(m)}$ and $Y^{(m)}$ appears also in [21].

$$\int d^D x' W^{\alpha\beta}_{\bar{\psi}\psi}(t;\mathbf{x},\mathbf{x}') W^{\psi\bar{\psi}}_{\beta\rho}(t;\mathbf{x}',\mathbf{x}'') = \delta^D(\mathbf{x}-\mathbf{x}'')\delta^{\alpha}{}_{\rho},$$
$$\int d^D x' W^{\alpha\beta}_{\psi\bar{\psi}}(t;\mathbf{x},\mathbf{x}') W^{\bar{\psi}\psi}_{\beta\rho}(t;\mathbf{x}',\mathbf{x}'') = \delta^D(\mathbf{x}-\mathbf{x}'')\delta^{\alpha}{}_{\rho}. \tag{2.17}$$

Similar to bosonic Poisson brackets, the Poisson brackets between the fermionic fields ψ and $\bar{\psi}$ are defined as

$$\{\psi(t;\mathbf{x}), \bar{\psi}(t;\mathbf{x}')\}_{\rm PB} = W^{\psi\bar{\psi}}(t;\mathbf{x},\mathbf{x}'),$$
$$\{\bar{\psi}(t;\mathbf{x}), \psi(t;\mathbf{x}')\}_{\rm PB} = W^{\bar{\psi}\psi}(t;\mathbf{x},\mathbf{x}'). \tag{2.18}$$

The inverse operators $W^{\psi\bar{\psi}}$ and $W^{\bar{\psi}\psi}$ have, similar to their inverse operators $W_{\bar{\psi}\psi}$ and $W_{\psi\bar{\psi}}$, the property

$$W^{\psi\bar{\psi}}_{\alpha\beta}(t;\mathbf{x},\mathbf{x}') = W^{\bar{\psi}\psi}_{\alpha\beta}(t;\mathbf{x}',\mathbf{x}). \tag{2.19}$$

B. Modified Poisson brackets

As we have mentioned at the beginning of this section, in the perturbative quantization introduced in [16–18], the field variables ψ, $\bar{\psi}$, and A_μ are to be redefined so that the corresponding Poisson brackets are the same as in the ordinary QED. To do this, we will consider first the equation of motion of free fermionic and bosonic fields arising from (2.2),[3]

$$\gamma^\mu \partial_\mu \psi = 0, \qquad \partial_\mu \bar{\psi}\gamma^\mu = 0, \qquad \Box A_\mu = 0. \tag{2.20}$$

Then, using (2.20), we will replace the time derivatives of fermionic and bosonic fields by the corresponding space derivatives as

$$\partial_0^n \psi_\alpha = \begin{cases} n=2p, & \partial_0^{2p}\psi_\alpha = \partial_i^{2p}\psi_\alpha, \\ n=2p+1, & \partial_0^{2p+1}\psi_\alpha = (-\gamma^0\gamma^i)_{\alpha\beta}\partial_i^{2p+1}\psi^\beta, \end{cases}$$
$$\partial_0^n \bar{\psi}_\alpha = \begin{cases} n=2p, & \partial_0^{2p}\bar{\psi}_\alpha = \partial_i^{2p}\bar{\psi}_\alpha, \\ n=2p+1, & \partial_0^{2p+1}\bar{\psi}_\alpha = \partial_i^{2p+1}\bar{\psi}^\beta(\gamma^0\gamma^i)_{\beta\alpha}, \end{cases}$$
$$\partial_0^n A^\nu = \begin{cases} n=2p, & \partial_0^{2p}A^\mu = \partial_i^{2p}A^\mu, \\ n=2p+1, & \partial_0^{2p+1}A^\mu = \partial_i^{2p}\dot{A}^\mu, \end{cases} \tag{2.21}$$

where a summation over $i = 1, \ldots, D$ is to be performed. The above replacement leads to a modification of the Poisson brackets defined in the previous section and eventually to a perturbative quantization of the theory up to second order in the coupling constant e.

To modify the Poisson brackets of the fermionic and bosonic fields, let us now consider δS from (2.5) and replace the time derivatives with the corresponding space derivatives using (2.21). We arrive after a lengthy but straightforward computation at

$$\delta S = \sum_{p=0}^{\infty}\int d^D x\{(\partial_i^{2p}\hat{\Pi}_{\psi^{(2p)}} + \partial_i^{2p+1}\hat{\Pi}_{\psi^{(2p+1)}}\gamma^0\gamma^i)\delta\psi$$
$$- \delta\bar{\psi}(\partial_i^{2p}\hat{\Pi}^{\alpha}_{\bar{\psi}^{(2p)}} - \gamma^0\gamma^i\partial_i^{2p+1}\hat{\Pi}_{\bar{\psi}^{(2p+1)}})$$
$$+ \partial_i^{2p}\hat{\Pi}^{\mu}_{A^{(2p)}}\delta A_\mu + \partial_i^{2p}\hat{\Pi}^{\mu}_{A^{(2p+1)}}\delta\dot{A}_\mu\}, \tag{2.22}$$

where the superscripts (p) denote the p-th order space derivative of the corresponding fields. Using (2.22) the modified symplectic two-form is given by

$$\hat{\Omega}(t) = \int d^D x(dP^{\alpha}_{\psi}\wedge d\psi_\alpha - d\bar{\psi}_\alpha \wedge dP^{\alpha}_{\bar{\psi}} + dP^{\mu}_{A}\wedge dA_\mu + dP^{\mu}_{\dot{A}}\wedge d\dot{A}_\mu), \tag{2.23}$$

where comparing to (2.22), the modified momenta P_ψ, $P_{\bar{\psi}}$, P^μ_A, and $P^\mu_{\dot{A}}$ read

$$P_\psi = \sum_{p=0}^{\infty}(\partial_i^{2p}\hat{\Pi}_{\psi^{(2p)}} + \partial_i^{2p+1}\hat{\Pi}_{\psi^{(2p+1)}}\gamma^0\gamma^i),$$
$$P_{\bar{\psi}} = \sum_{p=0}^{\infty}(\partial_i^{2p}\hat{\Pi}_{\bar{\psi}^{(2p)}} - \gamma^0\gamma^i\partial_i^{2p+1}\hat{\Pi}_{\bar{\psi}^{(2p+1)}}),$$
$$P^\mu_A = \sum_{p=0}^{\infty}\partial_i^{2p}\hat{\Pi}^{\mu}_{A^{(2p)}}, \qquad P^\mu_{\dot{A}} = \sum_{p=0}^{\infty}\partial_i^{2p}\hat{\Pi}^{\mu}_{A^{(2p+1)}}. \tag{2.24}$$

Here, $\hat{\Pi}_\psi$, $\hat{\Pi}_{\bar{\psi}}$, $\hat{\Pi}^\mu_A$, and $\hat{\Pi}^\mu_{\dot{A}}$ are defined in (2.6). In a perturbative quantization up to $\mathcal{O}(e^2)$, the modified momenta from (2.24) can be separated in a free and an interaction part proportional to the coupling constant e,

$$P_\psi = i\bar{\psi}\gamma^0 + ie\bar{\psi}\overleftarrow{\mathcal{J}}_\psi\gamma^0, \qquad P_{\bar{\psi}} = -ie\gamma^0\overrightarrow{\mathcal{J}}_{\bar{\psi}}\psi,$$
$$P^\mu_A = -\dot{A}^\mu - e\xi^\mu_A, \qquad P^\mu_{\dot{A}} = -e\xi^\mu_{\dot{A}}, \tag{2.25}$$

where $\mathcal{J}_\psi$, $\mathcal{J}_{\bar{\psi}}$, ξ^σ_A, and $\xi^\sigma_{\dot{A}}$ can be determined from (2.24).[4] In (2.25), the operators $\mathcal{J}_\psi$ and $\mathcal{J}_{\bar{\psi}}$ include derivatives that act left on $\bar{\psi}$ and right on ψ, respectively. This is denoted by left and right arrows. Using (2.25), the modified symplectic two-form $\hat{\Omega}(t)$ of modified QED described by the Lagrangian (2.1) and (2.2) is then given by

[3]Here, the gauge fixing parameter is chosen to be $\xi = 1$.

[4]The notation in (2.25) is particularly suitable for a Lagrangian of the form (2.1) and (2.2).

$$\hat{\Omega}(t) = \int d^D x\bigg\{d\bar{\psi} \wedge [i\gamma^0 + ie(\overleftarrow{\mathcal{J}}_{\psi}\gamma^0 + \gamma^0\overrightarrow{\mathcal{J}}_{\bar{\psi}})]d\psi + ie\bar{\psi}\bigg(\frac{\delta\overleftarrow{\mathcal{J}}_{\psi}}{\delta A^{\rho}}\gamma^0\bigg)dA^{\rho} \wedge d\psi + ie\bar{\psi}\bigg(\frac{\delta\overleftarrow{\mathcal{J}}_{\psi}}{\delta \dot{A}^{\rho}}\gamma^0\bigg)d\dot{A}^{\rho} \wedge d\psi$$
$$+ ied\bar{\psi} \wedge dA^{\rho}\bigg(\gamma^0\frac{\delta\overrightarrow{\mathcal{J}}_{\bar{\psi}}}{\delta A^{\rho}}\bigg)\psi + ied\bar{\psi} \wedge d\dot{A}^{\rho}\bigg(\gamma^0\frac{\delta\overrightarrow{\mathcal{J}}_{\bar{\psi}}}{\delta \dot{A}^{\rho}}\bigg)\psi + e\bigg(\frac{\delta\xi^A_{\mu}}{\delta A^{\nu}}\bigg)dA^{\mu} \wedge dA^{\nu} + e\bigg(\frac{\delta\xi^{\dot{A}}_{\mu}}{\delta \dot{A}^{\nu}}\bigg)d\dot{A}^{\mu} \wedge d\dot{A}^{\nu}$$
$$+ \bigg[g_{\mu\nu} + e\bigg(\frac{\delta\xi^A_{\mu}}{\delta \dot{A}^{\nu}} - \frac{\delta\xi^{\dot{A}}_{\nu}}{\delta A^{\mu}}\bigg)\bigg]dA^{\mu} \wedge d\dot{A}^{\nu}\bigg\} + \mathcal{O}(e^2). \tag{2.26}$$

To determine the symplectic two-form (2.26) we have used the fact that for a theory described by (2.1) and (2.2), ξ_A and $\xi_{\dot{A}}$ are functions of A_μ and its derivatives. To determine the corresponding Poisson algebra, let us consider the modified symplectic two-form (2.26), which can be formally given as

$$\hat{\Omega}(t) = \int d^D x(\hat{W}^{\bar{\psi}\psi}_{\alpha\beta}d\bar{\psi}^{\alpha} \wedge d\psi^{\beta} + \hat{W}^{A\psi}_{\mu\alpha}dA^{\mu} \wedge d\psi^{\alpha}$$
$$+ \hat{W}^{\dot{A}\psi}_{\mu\alpha}d\dot{A}^{\mu} \wedge d\psi^{\alpha} + \hat{W}^{\bar{\psi}A}_{\alpha\mu}d\bar{\psi}^{\alpha} \wedge dA^{\mu}$$
$$+ \hat{W}^{\bar{\psi}\dot{A}}_{\alpha\mu}d\bar{\psi}^{\alpha} \wedge d\dot{A}^{\mu} + \hat{W}^{\dot{A}A}_{\mu\nu}d\dot{A}^{\mu} \wedge dA^{\nu}$$
$$+ \hat{W}^{AA}_{\mu\nu}dA^{\mu} \wedge dA^{\nu} + \hat{W}^{\dot{A}\dot{A}}_{\mu\nu}d\dot{A}^{\mu} \wedge d\dot{A}^{\nu}) + \mathcal{O}(e^2). \tag{2.27}$$

Here, the coefficients $\hat{W}_{ab}$ with $a, b \in \{\bar{\psi}, \psi, A, \dot{A}\}$, can be read from (2.26). Note that in (2.27), $\mu, \nu = 0, \ldots, D$ are the space-time indices and $\alpha, \beta = 1, \ldots, 2^{((D+1)/2)}$ are the spinor indices. Comparing with the original symplectic two-form $\Omega(t)$ from (2.9), whose bosonic and fermionic parts are given in (2.10) and (2.14), new bases appear in the (perturbatively) modified phase space of the theory. Note that among the terms in $\hat{\Omega}(t)$ from (2.26) and (2.27), the terms proportional to $dA^\mu \wedge dA^\nu$ and $d\dot{A}^\mu \wedge d\dot{A}^\nu$ survive only if their coefficients are antisymmetric in μ and ν. According to our definitions from previous section, the modified Poisson brackets of the fermionic and bosonic degrees of freedom can be determined by inverting $\hat{W}_{ab}$ using the orthonormality relations similar to (2.13) and (2.17). Up to $\mathcal{O}(e^2)$, the modified Poisson brackets read

$$\hat{W}^{\dot{A}A}_{\mu\nu}(t; \mathbf{x}, \mathbf{x}') = \{\dot{A}_{\mu}(t; \mathbf{x}), A_{\nu}(t; \mathbf{x}')\}_{\rm PB} = \bigg[g_{\mu\nu} + e\bigg(\frac{\delta\xi^{\dot{A}}_{\nu}}{\delta A^{\mu}} - \frac{\delta\xi^A_{\mu}}{\delta \dot{A}^{\nu}}\bigg)\bigg]\delta^D(\mathbf{x} - \mathbf{x}'),$$
$$\hat{W}^{A\dot{A}}_{\mu\nu}(t; \mathbf{x}, \mathbf{x}') = \{A_{\mu}(t; \mathbf{x}), \dot{A}_{\nu}(t; \mathbf{x}')\}_{\rm PB} = \bigg[-g_{\mu\nu} + e\bigg(\frac{\delta\xi^A_{\nu}}{\delta \dot{A}^{\mu}} - \frac{\delta\xi^{\dot{A}}_{\mu}}{\delta A^{\nu}}\bigg)\bigg]\delta^D(\mathbf{x} - \mathbf{x}'),$$
$$\hat{W}^{\dot{A}\dot{A}}_{\mu\nu}(t; \mathbf{x}, \mathbf{x}') = \{\dot{A}_{\mu}(t; \mathbf{x}), \dot{A}_{\nu}(t; \mathbf{x}')\}_{\rm PB} = e\bigg(\frac{\delta\xi^A_{\mu}}{\delta A^{\nu}} - \frac{\delta\xi^A_{\nu}}{\delta A^{\mu}}\bigg)\delta^D(\mathbf{x} - \mathbf{x}'),$$
$$\hat{W}^{AA}_{\mu\nu}(t; \mathbf{x}, \mathbf{x}') = \{A_{\mu}(t; \mathbf{x}), A_{\nu}(t; \mathbf{x}')\}_{\rm PB} = e\bigg(\frac{\delta\xi^{\dot{A}}_{\mu}}{\delta \dot{A}^{\nu}} - \frac{\delta\xi^{\dot{A}}_{\nu}}{\delta \dot{A}^{\mu}}\bigg)\delta^D(\mathbf{x} - \mathbf{x}'),$$
$$\hat{W}^{A\bar{\psi}}_{\mu\alpha}(t; \mathbf{x}, \mathbf{x}') = \{A_{\mu}(t; \mathbf{x}), \bar{\psi}_{\alpha}(t; \mathbf{x}')\}_{\rm PB} = e\bar{\psi}^{\beta}\bigg(\frac{\delta\overleftarrow{\mathcal{J}}_{\psi}}{\delta \dot{A}^{\mu}}\bigg)_{\beta\alpha}\delta^D(\mathbf{x} - \mathbf{x}'),$$
$$\hat{W}^{A\psi}_{\mu\alpha}(t; \mathbf{x}, \mathbf{x}') = \{A_{\mu}(t; \mathbf{x}), \psi_{\alpha}(t; \mathbf{x}')\}_{\rm PB} = e\bigg(\frac{\delta\overrightarrow{\mathcal{J}}_{\bar{\psi}}}{\delta \dot{A}^{\mu}}\bigg)_{\alpha\beta}\psi^{\beta}\delta^D(\mathbf{x} - \mathbf{x}'), \tag{2.28}$$
$$\hat{W}^{\dot{A}\bar{\psi}}_{\mu\alpha}(t; \mathbf{x}, \mathbf{x}') = \{\dot{A}_{\mu}(t; \mathbf{x}), \bar{\psi}_{\alpha}(t; \mathbf{x}')\}_{\rm PB} = -e\bar{\psi}^{\beta}\bigg(\frac{\delta\overleftarrow{\mathcal{J}}_{\psi}}{\delta A^{\mu}}\bigg)_{\beta\alpha}\delta^D(\mathbf{x} - \mathbf{x}'),$$
$$\hat{W}^{\dot{A}\psi}_{\mu\alpha}(t; \mathbf{x}, \mathbf{x}') = \{\dot{A}_{\mu}(t; \mathbf{x}), \psi_{\alpha}(t; \mathbf{x}')\}_{\rm PB} = -e\bigg(\frac{\delta\overrightarrow{\mathcal{J}}_{\bar{\psi}}}{\delta A^{\mu}}\bigg)_{\alpha\beta}\psi^{\beta}\delta^D(\mathbf{x} - \mathbf{x}'),$$
$$\hat{W}^{\psi\bar{\psi}}_{\alpha\beta}(t; \mathbf{x}, \mathbf{x}') = \{\psi_{\alpha}(t; \mathbf{x}), \bar{\psi}_{\beta}(t; \mathbf{x}')\}_{\rm PB} = [-i\gamma^0 + ie(\overrightarrow{\mathcal{J}}_{\bar{\psi}}\gamma^0 + \gamma^0\overleftarrow{\mathcal{J}}_{\psi})]_{\alpha\beta}\delta^D(\mathbf{x} - \mathbf{x}').$$

C. Field redefinition and Dirac quantization

In this section, we will redefine the fermionic and bosonic field variables so that their corresponding Poisson brackets are the same as in ordinary QED. To do this let us first define the modified gauge field $\tilde{A}_\mu$ and its corresponding modified canonical momentum $\tilde{\Pi}_\mu$ as

$$\tilde{A}_\mu \equiv A_\mu - e\xi_\mu^{\dot{A}}, \qquad \tilde{\Pi}_\mu \equiv -\dot{A}_\mu - e\xi_\mu^{A}, \tag{2.29}$$

and determine the bosonic Poisson bracket

$$\{\tilde{A}_\mu(t;\mathbf{x}), \tilde{\Pi}_\nu(t;\mathbf{x}')\}_{\text{PB}},$$

using the Poisson brackets (2.28). In (2.29), ξ_μ^A and $\xi_\mu^{\dot{A}}$ are the same as introduced in (2.25). Replacing $\tilde{\Pi}_\nu$ from (2.29) in the above Poisson bracket, we arrive first at

$$\begin{aligned}\{\tilde{A}_\mu(t;\mathbf{x}), \tilde{\Pi}_\nu(t;\mathbf{x}')\} = & -\{A_\mu(t;\mathbf{x}), \dot{A}_\nu(t;\mathbf{x}')\} \\ & - e\{A_\mu(t;\mathbf{x}), \xi_\nu^A(t;\mathbf{x}')\} \\ & + e\{\xi_\mu^{\dot{A}}(t;\mathbf{x}), \dot{A}_\nu(t;\mathbf{x}')\} + \mathcal{O}(e^2),\end{aligned} \tag{2.30}$$

where we have skipped the subscript PB. To evaluate the first two terms on the right-hand side of the above relations, we use the Poisson brackets (2.28). The remaining three terms in (2.30) can be determined using the standard definition of the Poisson bracket of two functionals $\mathcal{F}$, $\mathcal{G}$ that depend on the dynamical variables (η, π_η) including the fermionic and bosonic field variables and their derivatives [20],

$$\begin{aligned}\{\mathcal{F}(t;\mathbf{x}), \mathcal{G}(t;\mathbf{x}')\}_{\text{PB}} \equiv & \int d^Dz d^Dz' \bigg(\frac{\delta\mathcal{F}(t;\mathbf{x})}{\delta\eta(t;\mathbf{z})}\frac{\delta\mathcal{G}(t;\mathbf{x}')}{\delta\pi_\eta(t;\mathbf{z}')} \\ & \mp \frac{\delta\mathcal{F}(t;\mathbf{x})}{\delta\pi_\eta(t;\mathbf{z}')}\frac{\delta\mathcal{G}(t;\mathbf{x}')}{\delta\eta(t;\mathbf{z})}\bigg) \\ & \times\{\eta(t;\mathbf{z}), \pi_\eta(t;\mathbf{z}')\}_{\text{PB}}.\end{aligned} \tag{2.31}$$

Here, the minus (plus) sign corresponds to bosonic (fermionic) fields η. Separating the space and time components of the indices μ and ν in (2.30) and after a lengthy but straightforward calculation, the canonical equal-time Poisson bracket of $\tilde{A}_\mu$ and its conjugate momentum $\tilde{\Pi}_\nu$ reads

$$\{\tilde{A}_\mu(t;\mathbf{x}), \tilde{\Pi}_\nu(t;\mathbf{x}')\}_{\text{PB}} = g_{\mu\nu}\delta^D(\mathbf{x}-\mathbf{x}') + \mathcal{O}(e^2). \tag{2.32}$$

This is, up to $\mathcal{O}(e^2)$, the standard Poisson bracket of ordinary QED which can be quantized in the standard Dirac quantization procedure, i.e. replacing

$$\begin{aligned}&\{\cdot,\cdot\}_{\text{PB}} \to -i[\cdot,\cdot]_{\text{DB}}, \quad \text{for bosons,} \\ &\{\cdot,\cdot\}_{\text{PB}} \to -i\{\cdot,\cdot\}_{\text{DB}}, \quad \text{for fermions,}\end{aligned} \tag{2.33}$$

we arrive at

$$[\tilde{A}_\mu(t;\mathbf{x}), \tilde{\Pi}_\nu(t;\mathbf{x}')]_{\text{DB}} = ig_{\mu\nu}\delta^D(\mathbf{x}-\mathbf{x}') + \mathcal{O}(e^2). \tag{2.34}$$

Here, the subscript DB is the Dirac bracket. Similarly, one finds

$$[\tilde{A}_\mu(t;\mathbf{x}), \tilde{A}_\nu(t;\mathbf{x}')]_{\text{DB}} = [\tilde{\Pi}_\mu(t;\mathbf{x}), \tilde{\Pi}_\nu(t;\mathbf{x}')]_{\text{DB}} = \mathcal{O}(e^2). \tag{2.35}$$

The redefined fermionic fields ψ and $\bar{\psi}$ are given by

$$\tilde{\psi} = \psi + e\overrightarrow{\mathcal{J}}_{\bar{\psi}}\psi, \qquad \tilde{\bar{\psi}} = \bar{\psi} + e\bar{\psi}\overleftarrow{\mathcal{J}}_\psi, \tag{2.36}$$

where $\mathcal{J}_{\bar{\psi}}$ and $\mathcal{J}_\psi$ are the same as introduced in (2.25). Replacing (2.36) in the corresponding Poisson bracket

$$\{\tilde{\psi}(t;\mathbf{x}), \tilde{\bar{\psi}}(t;\mathbf{x}')\}_{\text{PB}},$$

and using the modified algebra from (2.28) and the definition of Poisson bracket from (2.31), we arrive at

$$\begin{aligned}\{\tilde{\psi}(t;\mathbf{x}), \tilde{\bar{\psi}}(t;\mathbf{x}')\}_{\text{PB}} = & \{\psi(t;\mathbf{x}), \bar{\psi}(t;\mathbf{x}')\} \\ & + e\{\psi(t;\mathbf{x}), \bar{\psi}(t;\mathbf{x}')\}\overleftarrow{\mathcal{J}}_\psi \\ & + e\overrightarrow{\mathcal{J}}_{\bar{\psi}}\{\psi(t;\mathbf{x}), \bar{\psi}(t;\mathbf{x}')\} + \mathcal{O}(e^2) \\ = & -i\gamma_0\delta^D(\mathbf{x}-\mathbf{x}') + \mathcal{O}(e^2).\end{aligned} \tag{2.37}$$

This leads, after replacing the Poisson bracket by the Dirac bracket using (2.33), to

$$\{\tilde{\bar{\psi}}(t;\mathbf{x}), \tilde{\psi}(t;\mathbf{x}')\}_{\text{DB}} = \gamma^0\delta^D(\mathbf{x}-\mathbf{x}') + \mathcal{O}(e^2), \tag{2.38}$$

which is up to $\mathcal{O}(e^2)$ the ordinary canonical equal-time anticommutation relation of ordinary QED. Apart from (2.37), it can easily be checked that the Poisson brackets of the modified gauge fields $\tilde{A}_\mu$ from (2.29) with the modified fermions $\tilde{\psi}$, $\tilde{\bar{\psi}}$ from (2.36), i.e. $\{\tilde{A}_\mu(t;\mathbf{x}), \tilde{\psi}(t;\mathbf{x}')\}$ and $\{\tilde{A}_\mu(t;\mathbf{x}), \tilde{\bar{\psi}}(t;\mathbf{x}')\}$, vanish, as is expected also from ordinary QED.

In the next section, we will first introduce the Lagrangian density of two-dimensional space-time noncommutative QED as an example of a modified QED including higher order time derivatives. The modified Poisson brackets of fermionic and bosonic degrees of freedom will be then determined perturbatively up to order two in the coupling constant e and order three in the noncommutativity parameter θ.

III. MODIFIED POISSON BRACKETS OF TWO-DIMENSIONAL SPACE-TIME NONCOMMUTATIVE QED

The noncommutative gauge theory is characterized by the replacement of the familiar product of functions with the $\star$ product defined by

$$f(x)\star g(x) \equiv \exp\bigg(\frac{i\theta_{\mu\nu}}{2}\frac{\partial}{\partial\xi_\mu}\frac{\partial}{\partial\zeta_\nu}\bigg)f(x+\xi)g(x+\zeta)|_{\xi=\zeta=0}. \tag{3.1}$$

In two space-time dimensions, $\theta_{\mu\nu}$, being an antisymmetric matrix and reflecting the noncommutativity of space and time coordinates, reduces to $\theta_{\mu\nu} = \theta\epsilon_{\mu\nu}$, where $\epsilon_{\mu\nu}$ is the two-dimensional Levi-Cività symbol. The $\star$ product satisfies the identity

$$\int_{-\infty}^{+\infty} d^d x f(x) \star g(x) = \int_{-\infty}^{+\infty} d^d x g(x) \star f(x) = \int_{-\infty}^{+\infty} d^d x f(x) g(x), \tag{3.2}$$

and is associative

$$\int_{-\infty}^{+\infty} d^d x (f \star g \star h)(x) = \int_{-\infty}^{+\infty} d^d x (h \star f \star g)(x) = \int_{-\infty}^{+\infty} d^d x (g \star h \star f)(x). \tag{3.3}$$

Here, d is the dimension of space-time coordinates. Let us consider the Lagrangian density of two-dimensional space-time noncommutative QED including the fermionic and the bosonic fields

$$\mathcal{L} = i\bar{\psi} \star \gamma^\mu \partial_\mu \psi - e\bar{\psi} \star \gamma^\mu A_\mu \star \psi - \frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu) \star (\partial_\nu A^\nu), \tag{3.4}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]_\star$, and $[A_\mu, A_\nu]_\star = A_\mu \star A_\nu - A_\nu \star A_\mu$. The corresponding Euler-Lagrangian equation of motion for the fermionic and bosonic fields are given by

$$\begin{aligned} &\gamma^\mu \partial_\mu \psi + ie\gamma^\mu A_\mu \star \psi = 0, \\ &\partial_\mu \bar{\psi} \gamma^\mu - ie\bar{\psi}\gamma^\mu \star A_\mu = 0, \\ &D_\mu F^{\mu\nu} + \frac{1}{\xi}\partial^\nu \partial_\mu A^\mu = eJ^\nu, \end{aligned} \tag{3.5}$$

with $D_\mu = \partial_\mu + ie[A_\mu, \cdot]_\star$. The covariant $U_V(1)$ vector current J^ν in (3.5) is defined by

$$J^\mu(x) \equiv -\psi_\beta(x) \star \bar{\psi}_\alpha(x)(\gamma^\mu)^{\alpha\beta}. \tag{3.6}$$

In two space-time dimensions, the above theory can be regarded as a higher order time-derivative theory, where the higher order time derivatives appear only in the interaction part. To show this, let us separate the Lagrangian (3.4) in a free and an interaction part $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$. Using the relations (3.2) and (3.3), the $\star$ product in the free part of the Lagrangian (3.4) can be removed, leaving us with the ordinary free part of commutative QED Lagrangian

$$\mathcal{L}_0 = \bar{\psi} i\gamma^\mu \partial_\mu \psi - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2, \tag{3.7}$$

where $\mathcal{F}_{\mu\nu}$ is the commutative field strength tensor [see below (2.2) for its definition]. The interaction part of (3.4), after removing one of the $\star$ products, can be separated into two parts, $\mathcal{L}_{\text{int}} \equiv \mathcal{L}_{\text{int}}^{(1)} + \mathcal{L}_{\text{int}}^{(2)}$. The first part includes the interaction of fermions ψ and $\bar{\psi}$ with the gauge field A_μ. It is given by

$$\begin{aligned} \mathcal{L}_{\text{int}}^{(1)} &= +e\psi_\beta \star \bar{\psi}_\alpha (\gamma^\lambda)_{\alpha\beta} A_\lambda, \\ &= +e\sum_{n=0}^{\infty}\left(\frac{i\theta}{2}\right)^n \frac{1}{n!}\epsilon^{\mu_1\nu_1}\ldots\epsilon^{\mu_n\nu_n}(\partial_{\mu_1}\ldots\partial_{\mu_n}\psi_\beta) \\ &\quad \times (\partial_{\nu_1}\ldots\partial_{\nu_n}\bar{\psi}_\alpha)(\gamma^\lambda)^{\alpha\beta}A_\lambda. \end{aligned} \tag{3.8}$$

The second part consists of bosons self-interaction from the gauge kinetic term in (3.4). It is given by

$$\begin{aligned} \mathcal{L}_{\text{int}}^{(2)} &= -\frac{ie}{2}\mathcal{F}_{\mu\nu}[A^\mu, A^\nu]_\star + \frac{e^2}{4}[A_\mu, A_\nu]_\star [A^\mu, A^\nu]_\star, \\ &= -ie\mathcal{F}_{\mu\nu}\sum_{p=0}^{\infty}\left(\frac{i\theta}{2}\right)^{2p+1}\frac{1}{(2p+1)!}\epsilon^{\alpha_1\beta_1}\cdots\epsilon^{\alpha_{2p+1}\beta_{2p+1}}(\partial_{\alpha_1}\cdots\partial_{\alpha_{2p+1}}A^\mu)(\partial_{\beta_1}\cdots\partial_{\beta_{2p+1}}A^\nu) \\ &\quad + e^2\sum_{p,s=0}^{\infty}\left(\frac{i\theta}{2}\right)^{2p+2s+2}\frac{1}{(2p+1)!(2s+1)!}\epsilon^{\rho_1\sigma_1}\cdots\epsilon^{\rho_{2p+1}\sigma_{2p+1}}\epsilon^{\alpha_1\beta_1}\cdots\epsilon^{\alpha_{2s+1}\beta_{2s+1}}(\partial_{\rho_1}\cdots\partial_{\rho_{2p+1}}A_\mu) \\ &\quad \times (\partial_{\sigma_1}\cdots\partial_{\sigma_{2p+1}}A_\nu)(\partial_{\alpha_1}\cdots\partial_{\alpha_{2s+1}}A^\mu)(\partial_{\beta_1}\cdots\partial_{\beta_{2s+1}}A^\nu). \end{aligned} \tag{3.9}$$

In (3.8) and (3.9), the definition of the $\star$ product from (3.1) is used. Using (2.6), the canonical conjugate momenta corresponding to ψ, $\bar{\psi}$ and A_μ are given by

$$\Pi^{\sigma}_{A^{(m)}} = -\dot{A}^{\sigma}\delta^{m0} - ie\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{r=0}^{\ell}\sum_{s=0}^{k-\ell}\left(\frac{i\theta}{2}\right)^{k+m+1}\frac{[(-1)^{\ell+1}+(-1)^{k+m+\ell+1}]}{(k+m+1)!}\binom{\ell}{r}\binom{k-\ell}{s}(\partial_0^{r+k-\ell}\partial_x^{s+\ell+m+1}A_\mu)$$
$$\times\,\partial_0^{\ell-r}\partial_x^{k-\ell-s}\mathcal{F}^{\mu\sigma} - ie\sum_{p=0}^{\infty}\left(\frac{i\theta}{2}\right)^{2p+1}\frac{\epsilon^{\alpha_1\beta_1}\cdots\epsilon^{\alpha_{2p+1}\beta_{2p+1}}}{(2p+1)!}(\partial_{\alpha_1}\cdots\partial_{\alpha_{2p+1}}A^0)(\partial_{\beta_1}\cdots\partial_{\beta_{2p+1}}A^{\sigma})\delta^{m0} + \mathcal{O}(e^2),$$
$$\Pi^{\beta}_{\psi^{(m)}} = -e\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{r=0}^{\ell}\sum_{s=0}^{k-\ell}\left(\frac{i\theta}{2}\right)^{m+k+1}\frac{(-1)^{\ell}}{(m+k+1)!}\binom{\ell}{r}\binom{k-\ell}{s}(\partial_x^{s+m+\ell+1}\partial_0^{r+k-\ell}\bar{\psi}_\alpha)(\partial_0^{\ell-r}\partial_x^{k-\ell-s}A_\lambda)(\gamma^\lambda)^{\alpha\beta}$$
$$+\, i\delta^{m0}\bar{\psi}_\alpha(\gamma_0)^{\alpha\beta},$$
$$\Pi^{\alpha}_{\bar{\psi}^{(m)}} = -e\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{r=0}^{\ell}\sum_{s=0}^{k-\ell}\left(\frac{i\theta}{2}\right)^{m+k+1}\frac{(-1)^{k+m+\ell+1}}{(m+k+1)!}\binom{\ell}{r}\binom{k-\ell}{s}(\partial_x^{s+m+\ell+1}\partial_0^{r+k-\ell}\psi_\beta)(\partial_0^{\ell-r}\partial_x^{k-\ell-s}A_\lambda)(\gamma^\lambda)^{\alpha\beta}, \quad (3.10)$$

where in $\Pi^{\sigma}_{A^{(m)}}$ the gauge fixing parameter is chosen to be $\xi = 1$, and the terms of order e^2 are neglected. In what follows, we will use the notations introduced in Sec. II to determine separately the corresponding Poisson brackets to the gauge field A_μ and the fermionic fields $\bar{\psi}$, ψ up to order $\mathcal{O}(e^2, \theta^3)$.

A. Poisson brackets of gauge fields

Using (3.9), the Lagrangian density of the gauge fields is, up to order $\mathcal{O}(e^2, \theta^3)$, given by

$$\mathcal{L}_g = -\tfrac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \tfrac{1}{2}(\partial_\mu A^\mu)^2 + \frac{e\theta}{2}\epsilon^{\alpha\beta}\partial_\alpha A^\mu\partial_\beta A^\nu\mathcal{F}_{\mu\nu} + \mathcal{O}(e^2,\theta^3),$$
$$= -\tfrac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + e\theta((\partial_x A_0)\dot{A}_1 - (\partial_x A_1)\dot{A}_0)\mathcal{F}_{01} + \mathcal{O}(e^2,\theta^3), \quad (3.11)$$

where an integration by part is performed in the θ-independent part of the Lagrangian. In this order of perturbative expansion, the above Lagrangian includes only first order time derivatives of A_μ. It is therefore not necessary to modify the theory in the sense of replacing the higher order time derivatives by the corresponding space derivatives using the corresponding equation of motion $\Box A_\mu = 0$ from (2.20) and the resulting relations from (2.21).[5] In what follows, we will nevertheless determine the symplectic two-forms and the corresponding Poisson brackets using the formulation introduced in Sec. II. As it turns out the Poisson brackets receive corrections of order $e\theta$, that vanish by taking the commutative limit $\theta \to 0$. To start, let us consider (2.10) and choose $z^a = (A^0, A^1, \dot{A}^0, \dot{A}^1)$ as the phase space variables. The bosonic symplectic two-form is given by

$$\Omega_g(t) = \frac{1}{2}\int dx dx' W_{ab}(t; x, x') dz^a(t; x) \wedge dz^b(t; x'), \quad (3.12)$$

or equivalently by

$$\Omega_g(t) = \int dx (Z^{(0)} + Z^{(1)}),$$

where, according to (2.8), $Z^{(m)}$ is given as

$$Z^{(m)} = d\Pi^0_{A^{(m)}} \wedge dA_0^{(m)} + d\Pi^1_{A^{(m)}} \wedge dA_1^{(m)}.$$

Using the general definition of $\Pi^{\sigma}_{A^{(m)}}$ from (3.10), the momenta $\Pi_{\sigma}^{A^{(m)}}$ in the order $\mathcal{O}(e^2, \theta^3)$ read

$$P_0^A = \Pi_0^A = -\dot{A}_0 - e\theta\mathcal{F}_{01}(\partial_x A_1) + \mathcal{O}(e^2,\theta^3),$$
$$P_1^A = \Pi_1^A = -\dot{A}_1 - e\theta\{2\dot{A}_1\partial_x A_0 - (\partial_x A_0)^2 - \dot{A}_0\partial_x A_1\} + \mathcal{O}(e^2,\theta^3),$$
$$P_0^{\dot{A}} = \Pi_0^{\dot{A}} = \mathcal{O}(e,\theta^3),$$
$$P_1^{\dot{A}} = \Pi_1^{\dot{A}} = \mathcal{O}(e,\theta^3). \quad (3.13)$$

Here, $\partial_x \equiv \frac{\partial}{\partial x^1}$ and $A_\mu^{(m=0)}$ and $A_\mu^{(m=1)}$ are denoted by A_μ and $\dot{A}_\mu$, respectively. Combining the above expressions we arrive at

$$Z^{(0)} = d(-\dot{A}_0 + e\theta(\mathcal{F}_{10}\partial_x A_1)) \wedge dA_0 + \{d\dot{A}_1 + e\theta(2d(\dot{A}_1\partial_x A_0) - 2(\partial_x A_0)d(\partial_x A_0) - d(\dot{A}_0\partial_x A_1))\} \wedge dA_1 + \mathcal{O}(e^2,\theta^3),$$
$$Z^{(1)} = \mathcal{O}(e,\theta^3). \quad (3.14)$$

Performing appropriate integrations by part and using the antisymmetry of the bosonic wedge product, the expressions in (3.14) can be simplified and we arrive up to $\mathcal{O}(e^2, \theta^3)$ at

[5] According to our notations from the previous section, in this case the "hatted" and "unhatted" quantities are equal up to $\mathcal{O}(e^2, \theta^3)$.

$$W_{ab}(t;x,x') \simeq \begin{pmatrix} 0 & -e\theta(\mathcal{F}_{01}\partial_x - \partial_x\mathcal{F}_{01}) & 1 & -e\theta(\partial_x A_1) \\ -e\theta(\mathcal{F}_{01}\partial_x + 2\partial_x\mathcal{F}_{01}) & 0 & -e\theta(\partial_x A_1) & -(1+2e\theta(\partial_x A_0)) \\ -1 & e\theta(\partial_x A_1) & 0 & 0 \\ e\theta(\partial_x A_1) & 1+2e\theta(\partial_x A_0) & 0 & 0 \end{pmatrix}\delta(x-x'). \tag{3.15}$$

To determine the bosonic Poisson brackets the above matrix is to be inverted. After a lengthy but straightforward computation we arrive at the inverse matrix $W^{-1}_{ab}(t;x,x')$ up to $\mathcal{O}(e^2,\theta^3)$,

$$W^{-1}_{ab}(t;x,x') \simeq \begin{pmatrix} 0 & 0 & -1 & e\theta(\partial_x A_1) \\ 0 & 0 & e\theta(\partial_x A_1) & 1-2e\theta(\partial_x A_0) \\ 1 & -e\theta(\partial_x A_1) & 0 & e\theta(\mathcal{F}_{01}\partial_x - \partial_x\mathcal{F}_{01}) \\ -e\theta(\partial_x A_1) & -(1-2e\theta(\partial_x A_0)) & e\theta(\mathcal{F}_{01}\partial_x + 2\partial_x\mathcal{F}_{01}) & 0 \end{pmatrix}\delta(x-x'), \tag{3.16}$$

that leads to the Poisson brackets of the bosonic fields

$$\{z^a(t;x), z^b(t;x')\} = W^{ab}(t;x,x') + \mathcal{O}(e^2,\theta^3), \tag{3.17}$$

with z^a, $z^b \in (A^0, A^1, \dot{A}^0, \dot{A}^1)$, and $W^{ab}(t;x,x')$ given in (3.16).

B. Modified Poisson brackets of fermionic fields

To determine the Poisson brackets corresponding to the fermionic fields ψ and $\bar{\psi}$, let us consider first the fermionic Lagrangian density up to $\mathcal{O}(e^2,\theta^3)$, which is given by

$$\begin{aligned}\mathcal{L}_f = {}& i\bar{\psi}\gamma^\mu\partial_\mu\psi - e\bar{\psi}\gamma^\lambda\psi A_\lambda + \frac{ie\theta}{2}\dot{\bar{\psi}}\gamma^\lambda\partial_x\psi A_\lambda \\ & - \frac{ie\theta}{2}\partial_x\bar{\psi}\gamma^\lambda\dot{\psi}A_\lambda + \frac{e\theta^2}{8}\ddot{\bar{\psi}}\gamma^\lambda\partial_x^2\psi A_\lambda \\ & + \frac{e\theta^2}{8}\partial_x^2\bar{\psi}\gamma^\lambda\ddot{\psi}A_\lambda - \frac{e\theta^2}{4}\partial_x\dot{\bar{\psi}}\gamma^\lambda\partial_x\dot{\psi}A_\lambda \\ & + \mathcal{O}(e^2,\theta^3).\end{aligned} \tag{3.18}$$

In (3.18) higher order time derivatives, denoted by dots, appear only in the interaction part of the Lagrangian density. In what follows, we will determine the modified fermionic Poisson brackets using the general definition of the fermionic part of the symplectic two-form $\Omega(t)$ from (2.7), i.e.

$$\Omega_f(t) = \sum_{m=0}^{1}\int dx(X^{(m)}(t;x) - Y^{(m)}(t;x)),$$

with $X^{(m)}$ and $Y^{(m)}$ given in (2.8). Equivalently, in 1 + 1 dimensions,

$$\begin{aligned}\Omega_f(t) = \frac{1}{2}\int & dxdx'(W_{\bar{\psi}\psi}(t;x,x')d\bar{\psi}(t;x)\wedge d\psi(t;x') \\ & + W_{\psi\bar{\psi}}(t;x,x')d\psi(t;x)\wedge d\bar{\psi}(t;x')),\end{aligned}$$

from (2.14) can be used. The modification will be performed by replacing the higher order time derivatives by the corresponding space derivatives. To do this, we will first use the equation of motion from (3.5) up to order $\mathcal{O}(e)$,

$$\begin{aligned}&\gamma^\mu\partial_\mu\psi \approx 0, \quad \text{leading to} \quad \dot{\psi} = -\gamma^5\partial_x\psi, \\ &\partial_\mu\bar{\psi}\gamma^\mu \approx 0, \quad \text{leading to} \quad \dot{\bar{\psi}} = \partial_x\bar{\psi}\gamma^5,\end{aligned} \tag{3.19}$$

where in two dimensions $\gamma^5 = \gamma^0\gamma^1$. Then, using (3.19), higher order time derivatives acting on ψ and $\bar{\psi}$ can be replaced by the corresponding higher order space derivatives as[6]

$$\partial_0^n\psi_\alpha = \begin{cases} n = 2p, & \partial_0^{2p}\psi_\alpha = \partial_x^{2p}\psi_\alpha, \\ n = 2p+1, & \partial_0^{2p+1}\psi_\alpha = (-\gamma^5)_{\alpha\beta}\partial_x^{2p+1}\psi^\beta, \end{cases} \tag{3.20}$$

$$\partial_0^n\bar{\psi}_\alpha = \begin{cases} n = 2p, & \partial_0^{2p}\bar{\psi}_\alpha = \partial_x^{2p}\bar{\psi}_\alpha, \\ n = 2p+1, & \partial_0^{2p+1}\bar{\psi}_\alpha = \partial_x^{2p+1}\bar{\psi}^\beta(\gamma^5)_{\beta\alpha}. \end{cases} \tag{3.21}$$

Using the general expressions of the fermionic momenta $\Pi_{\psi^{(m)}}$ and $\Pi_{\bar{\psi}^{(m)}}$ from (3.10) and replacing higher order time derivatives with the corresponding higher order space derivatives using (3.20) and (3.21), we arrive at the modified momenta,

$$\begin{aligned}\hat{\Pi}^\beta_\psi = {}& i(\gamma^0)^{\alpha\beta}\bar{\psi}_\alpha - \frac{ie\theta}{2}\partial_x\bar{\psi}_\alpha(\gamma^\lambda)^{\alpha\beta}A_\lambda \\ & - \frac{e\theta^2}{8}\partial_x^2\bar{\psi}_\alpha((\gamma^\lambda\gamma^5)^{\alpha\beta}\partial_x A_\lambda + (\gamma^\lambda)^{\alpha\beta}\partial_0 A_\lambda) \\ & + \mathcal{O}(e^2,\theta^3), \\ \hat{\Pi}^\alpha_{\bar{\psi}} = {}& \frac{ie\theta}{2}(\gamma^\lambda)^{\alpha\beta}\partial_x\psi_\beta A_\lambda - \frac{e\theta^2}{8}((\gamma^\lambda\gamma^5)^{\alpha\beta}\partial_x A_\lambda \\ & + (\gamma^\lambda)^{\alpha\beta}\partial_0 A_\lambda)\partial_x^2\psi_\beta + \mathcal{O}(e^2,\theta^3), \\ \hat{\Pi}^\beta_{\dot{\psi}} = {}& \frac{e\theta^2}{8}\partial_x^2\bar{\psi}_\alpha(\gamma^\lambda)^{\alpha\beta}A_\lambda + \mathcal{O}(e^2,\theta^3), \\ \hat{\Pi}^\alpha_{\dot{\bar{\psi}}} = {}& \frac{e\theta^2}{8}A_\lambda(\gamma^\lambda)^{\alpha\beta}\partial_x^2\psi_\beta + \mathcal{O}(e^2,\theta^3).\end{aligned} \tag{3.22}$$

[6]See also (2.21) for a general $D + 1$ dimensional case.

Using at this stage (3.22), the modified $X^{(m)}$, $Y^{(m)}$ with $m = 0, 1$ from (2.8), are explicitly given by

$$\begin{aligned}
\hat{X}^{(0)} &= d\bar{\psi}\bigg(i\gamma^0 - \frac{ie\theta}{2}\overleftarrow{\partial}_x(\gamma^\lambda A_\lambda) \\
&\quad - \frac{e\theta^2}{8}\overleftarrow{\partial}_x^2[\gamma^\lambda\gamma^5\partial_x A_\lambda + \gamma^\lambda\partial_0 A_\lambda]\bigg)\wedge d\psi + \mathcal{O}(e^2,\theta^3), \\
\hat{Y}^{(0)} &= d\bar{\psi}\wedge\bigg(\frac{ie\theta}{2}(\gamma^\lambda A_\lambda)\vec{\partial}_x \\
&\quad - \frac{e\theta^2}{8}[\gamma^\lambda\gamma^5\partial_x A_\lambda + \gamma^\lambda\partial_0 A_\lambda]\vec{\partial}_x^2\bigg)d\psi + \mathcal{O}(e^2,\theta^3), \\
\hat{X}^{(1)} &= d\bar{\psi}\bigg(\frac{e\theta^2}{8}\overleftarrow{\partial}_x^2(\gamma^\lambda A_\lambda)\bigg)\wedge d\dot{\psi} \\
&= d\bar{\psi}\bigg(-\frac{e\theta^2}{8}\overleftarrow{\partial}_x^2(\gamma^\lambda\gamma^5 A_\lambda)\bigg)\wedge d(\partial_x\psi) + \mathcal{O}(e^2,\theta^3), \\
\hat{Y}^{(1)} &= d\dot{\bar{\psi}}\wedge\bigg(\frac{e\theta^2}{8}(\gamma^\lambda A_\lambda)\vec{\partial}_x^2\bigg)d\psi \\
&= d(\partial_x\bar{\psi})\wedge\bigg(-\frac{e\theta^2}{8}(\gamma^\lambda\gamma^5 A_\lambda)\vec{\partial}_x^2\bigg)d\psi + \mathcal{O}(e^2,\theta^3).
\end{aligned} \tag{3.23}$$

Combining these results, the modified coefficients of the fermionic part of symplectic two-form $\hat{W}_{\psi\bar{\psi}}$ and $\hat{W}_{\bar{\psi}\psi}$ from (2.16) can be determined. As we have also mentioned in Sec. II, in $\hat{W}_{\psi\bar{\psi}}$ all derivatives act on ψ, whereas in $\hat{W}_{\bar{\psi}\psi}$ they act on $\bar{\psi}$. After performing appropriate partial differentiations in (3.23) and neglecting the resulting surface terms, we arrive at

$$\begin{aligned}
\hat{W}_{\bar{\psi}\psi}(t;x,x') &= \bigg\{i\gamma^0 + e\bigg(\frac{i\theta}{2}(\gamma^\lambda\partial_x A_\lambda) + \frac{\theta^2}{8}(\gamma^\lambda\gamma^5\partial_x^3 A_\lambda) \\
&\quad + \frac{3\theta^2}{8}(\gamma^\lambda\gamma^5\partial_x^2 A_\lambda)\partial_x + \frac{\theta^2}{8}(\gamma^\lambda\partial_x^2\partial_0 A_\lambda) \\
&\quad + \frac{\theta^2}{4}(\gamma^\lambda\partial_x\partial_0 A_\lambda)\partial_x + \frac{3\theta^2}{8}(\gamma^\lambda\gamma^5\partial_x A_\lambda)\partial_x^2 \\
&\quad + \frac{\theta^2}{4}(\gamma^\lambda\gamma^5 A_\lambda)\partial_x^3\bigg)\bigg\}\delta(x-x') + \mathcal{O}(e^2,\theta^3), \\
\hat{W}_{\psi\bar{\psi}}(t;x,x') &= \bigg\{i\gamma^0 + e\bigg(\frac{i\theta}{2}(\gamma^\lambda\partial_x A_\lambda) - \frac{\theta^2}{8}(\gamma^\lambda\partial_x^2\partial_0 A_\lambda) \\
&\quad - \frac{\theta^2}{8}(\gamma^\lambda\gamma^5\partial_x^3 A_\lambda) - \frac{\theta^2}{4}(\gamma^\lambda\partial_x\partial_0 A_\lambda)\partial_x \\
&\quad - \frac{3\theta^2}{8}(\gamma^\lambda\gamma^5\partial_x^2 A_\lambda)\partial_x - \frac{3\theta^2}{8}(\gamma^\lambda\gamma^5\partial_x A_\lambda)\partial_x^2 \\
&\quad - \frac{\theta^2}{4}(\gamma^\lambda\gamma^5 A_\lambda)\partial_x^3\bigg)\bigg\}\delta(x-x') + \mathcal{O}(e^2,\theta^3).
\end{aligned} \tag{3.24}$$

They are elements of the matrix

$$\hat{W}_{ij} = \begin{pmatrix} 0 & \hat{W}_{\psi\bar{\psi}} \\ \hat{W}_{\bar{\psi}\psi} & 0 \end{pmatrix},$$

with $(i,j)\in(\psi,\bar{\psi})$, whose inverse leads, similar to the bosonic case, to the modified fermionic Poisson brackets up to $\mathcal{O}(e^2,\theta^3)$,

$$\begin{aligned}
\{\psi(t;x),\bar{\psi}(t;x')\}_{\text{PB}} &= \hat{W}^{\psi\bar{\psi}}(t;x,x') \\
&\simeq \bigg\{-i\gamma^0 + e\bigg(\frac{i\theta}{2}(\Gamma^\lambda\partial_x A_\lambda) + \frac{\theta^2}{8}(\Gamma^\lambda\partial_x^2\partial_0 A_\lambda) - \frac{\theta^2}{8}(\Gamma^\lambda\gamma^5\partial_x^3 A_\lambda) + \frac{\theta^2}{4}(\Gamma^\lambda\partial_x\partial_0 A_\lambda)\partial_x \\
&\quad - \frac{3\theta^2}{8}(\Gamma^\lambda\gamma^5\partial_x^2 A_\lambda)\partial_x - \frac{3\theta^2}{8}(\Gamma^\lambda\gamma^5\partial_x A_\lambda)\partial_x^2 - \frac{\theta^2}{4}(\Gamma^\lambda\gamma^5 A_\lambda)\partial_x^3\bigg)\bigg\}\delta(x-x'), \\
\{\bar{\psi}(t;x),\psi(t;x')\}_{\text{PB}} &= \hat{W}^{\bar{\psi}\psi}(t;x,x') \\
&\simeq \bigg\{-i\gamma^0 + e\bigg(\frac{i\theta}{2}(\Gamma^\lambda\partial_x A_\lambda) - \frac{\theta^2}{8}(\Gamma^\lambda\partial_x^2\partial_0 A_\lambda) + \frac{\theta^2}{8}(\Gamma^\lambda\gamma^5\partial_x^3 A_\lambda) - \frac{\theta^2}{4}(\Gamma^\lambda\partial_x\partial_0 A_\lambda)\partial_x \\
&\quad + \frac{3\theta^2}{8}(\Gamma^\lambda\gamma^5\partial_x^2 A_\lambda)\partial_x + \frac{3\theta^2}{8}(\Gamma^\lambda\gamma^5\partial_x A_\lambda)\partial_x^2 + \frac{\theta^2}{4}(\Gamma^\lambda\gamma^5 A_\lambda)\partial_x^3\bigg)\bigg\}\delta(x-x'),
\end{aligned} \tag{3.25}$$

and $\{\psi(t;x),\psi(t;x')\}_{\text{PB}} = \{\bar{\psi}(t;x),\bar{\psi}(t;x')\}_{\text{PB}} = 0$. In (3.25), Γ^λ is defined by $\Gamma^\lambda \equiv \gamma_0\gamma^\lambda\gamma_0 = 2g^{\lambda 0}\gamma^0 - \gamma^\lambda$. Note that the modified matrix elements $\hat{W}_{\psi\bar{\psi}}$ and $\hat{W}_{\bar{\psi}\psi}$ from (3.24) as well as their inverse operators $\hat{W}^{\bar{\psi}\psi}$ and $\hat{W}^{\psi\bar{\psi}}$ from (3.25) satisfy the relation (2.15) and (2.19), respectively.

IV. EFFECTIVE LAGRANGIAN DENSITY OF TWO-DIMENSIONAL SPACE-TIME NONCOMMUTATIVE QED

In the perturbative approach introduced in [16–18], one redefines at this stage the field variables ψ, $\bar{\psi}$, and A_μ so that the Poisson brackets of the redefined fields and their corresponding conjugate momenta are order by order the same as in the ordinary commutative theory consisting of first order time derivatives. In Sec. II, denoting the modified fields by $\tilde{\psi}$, $\tilde{\bar{\psi}}$, and $\tilde{A}_\mu$, we have performed a perturbative expansion for a generic theory described by (2.1) and (2.2) up to second order in the coupling constant e and arrived at the following modified Poisson brackets:

$$\{\tilde{A}_\mu(t;x), \tilde{\Pi}_\nu(t;x')\}_{\text{PB}} \approx g_{\mu\nu}\delta(x-x'), \quad \text{and}$$
$$\{\tilde{\bar{\psi}}(t;x), \tilde{\psi}(t;x')\}_{\text{PB}} \approx -i\gamma^0\delta(x-x'),$$

[see (2.32) and (2.37)]. In this section, we will follow the same method and will first redefine the fermionic and bosonic field variables for two-dimensional space-time noncommutative QED described by (3.11), the bosonic part, and (3.18), the fermionic part. Eventually, the redefined fields and the corresponding conjugate momenta will be used to derive an appropriate effective Lagrangian density up to $\mathcal{O}(e^2, \theta^3)$. To this purpose, we will consider the fermionic and bosonic Poisson parts separately.

A. The bosonic part

The Lagrangian density of two-dimensional space-time noncommutative QED up to $\mathcal{O}(e^2, \theta^3)$ is given in (3.11). To determine the effective Lagrangian, we will first modify the gauge field A_μ and its corresponding conjugate momentum Π_μ using the general Ansatz (2.29)

$$\tilde{A}_\mu \equiv A_\mu - e\xi^{\dot{A}}_\mu, \qquad \tilde{\Pi}_\mu \equiv -\dot{A}_\mu - e\xi^{A}_\mu.$$

Since the Lagrangian (3.11) includes only a first order time derivative in the order $\mathcal{O}(e^2, \theta^3)$, we set $\xi^{\dot{A}}_\mu = 0$ for $\mu = 0, 1$. Using further Π^A_μ from (3.13), we get

$$\begin{aligned}\xi^A_0 &= \theta(\partial_x A_1)(\dot{A}_1 - \partial_x A_0) + \mathcal{O}(e^2, \theta^3),\\ \xi^A_1 &= \theta(2\dot{A}_1(\partial_x A_0) - (\partial_x A_0)^2 - \dot{A}_0(\partial_x A_1)) + \mathcal{O}(e^2, \theta^3).\end{aligned} \tag{4.1}$$

It can be easily shown that, the modified bosonic field and its canonical conjugate momentum satisfy the canonical Poisson bracket up to order $\mathcal{O}(e^2, \theta^3)$,

$$\{\tilde{A}_\mu(t;x), \tilde{\Pi}_\nu(t;x')\}_{\text{PB}} \approx g_{\mu\nu}\delta(x-x').$$

In the next step, we will determine the effective bosonic Lagrangian density in terms of the modified fields. Performing a Legendre transformation of the Lagrangian density $\mathcal{L}_g$ from (3.11), the original Hamiltonian is given by

$$\begin{aligned}\mathcal{H}_g(A_0, A_1, \dot{A}_0, \dot{A}_1) &= \Pi^\mu_A \dot{A}_\mu + \Pi^\mu_{\dot{A}}\ddot{A}_\mu - \mathcal{L}_g\\ &= -\tfrac{1}{2}\dot{A}_0^2 + \tfrac{1}{2}\dot{A}_1^2 + \tfrac{1}{2}(\partial_x A_1)^2\\ &\quad - \tfrac{1}{2}(\partial_x A_0)^2 + e\theta((\partial_x A_0)\dot{A}_1^2\\ &\quad - (\partial_x A_1)\dot{A}_0\dot{A}_1) + \mathcal{O}(e^2, \theta^3).\end{aligned} \tag{4.2}$$

To determine the Hamiltonian $\mathcal{H}_g$ in terms of the modified variables $\tilde{A}_\mu$ and $\tilde{\Pi}_\mu$, we will use

$$\begin{aligned}\tilde{\Pi}_0 &= -\dot{A}_0 - e\theta(\partial_x A_1)(\dot{A}_1 - \partial_x A_0),\\ \tilde{\Pi}_1 &= -\dot{A}_1 - e\theta(2\dot{A}_1(\partial_x A_0) - (\partial_x A_0)^2 - \dot{A}_0(\partial_x A_1)),\end{aligned} \tag{4.3}$$

and determine $\dot{A}_\mu$, $\mu = 0, 1$ in terms of $\partial_x A_1 = \partial_x \tilde{A}_1$ and $\tilde{\Pi}_\mu$. We arrive at

$$\begin{aligned}\dot{A}_0 &= -\tilde{\Pi}_0 + e\theta[(\partial_x\tilde{A}_1)\tilde{\Pi}_1 + (\partial_x\tilde{A}_0)(\partial_x\tilde{A}_1)],\\ \dot{A}_1 &= -\tilde{\Pi}_1 + e\theta((\partial_x\tilde{A}_0)^2 + 2(\partial_x\tilde{A}_0)\tilde{\Pi}_1 - (\partial_x\tilde{A}_1)\tilde{\Pi}_0).\end{aligned} \tag{4.4}$$

Plugging (4.4) in the Hamiltonian $\mathcal{H}_g$ from (4.2), we arrive at the modified Hamiltonian

$$\begin{aligned}\tilde{\mathcal{H}}_g(\tilde{A}_\mu, \tilde{\Pi}_\mu) &= \tfrac{1}{2}\tilde{\Pi}_1^2 - \tfrac{1}{2}\tilde{\Pi}_0^2 + \tfrac{1}{2}(\partial_x\tilde{A}_1)^2 - \tfrac{1}{2}(\partial_x\tilde{A}_0)^2\\ &\quad + e\theta[(\partial_x\tilde{A}_0)(\partial_x\tilde{A}_1)\tilde{\Pi}_0 - (\partial_x\tilde{A}_0)\tilde{\Pi}_1^2\\ &\quad - (\partial_x\tilde{A}_0)^2\tilde{\Pi}_1 + (\partial_x\tilde{A}_1)\tilde{\Pi}_0\tilde{\Pi}_1]\\ &\quad + \mathcal{O}((e\theta)^2).\end{aligned} \tag{4.5}$$

The effective Lagrangian of the modified gauge fields is then defined as

$$\tilde{\mathcal{L}}_g \equiv \tilde{\Pi}^\mu\dot{\tilde{A}}_\mu - \tilde{\mathcal{H}}_g(\tilde{A}_\mu, \tilde{\Pi}_\mu). \tag{4.6}$$

To determine $\dot{\tilde{A}}_\mu$ we use the Heisenberg equation of motion

$$\dot{\tilde{A}}_\mu = \{\tilde{A}_\mu, \tilde{\mathcal{H}}_g\}. \tag{4.7}$$

Using further the modified equal-time Poisson brackets $\{\tilde{A}_\mu(t;x), \tilde{\Pi}_\nu(t;x')\}_{\text{PB}} = g_{\mu\nu}\delta(x-x') + \mathcal{O}(e^2, \theta^3)$ from (2.32), it can easily be shown that $\dot{\tilde{A}}_\mu \approx \dot{A}_\mu$ up to order $\mathcal{O}(e^2, \theta^3)$. Plugging this result back in (4.6), the effective Lagrangian density of the gauge fields is given by

$$\begin{aligned}\tilde{\mathcal{L}}_g &= \frac{1}{2}\dot{\tilde{A}}_1^2 - \frac{1}{2}\dot{\tilde{A}}_0^2 - \frac{1}{2}(\partial_x\tilde{A}_1)^2 + \frac{1}{2}(\partial_x\tilde{A}_0)^2\\ &\quad + e\theta[(\partial_x\tilde{A}_0)\dot{\tilde{A}}_1 - (\partial_x\tilde{A}_1)\dot{\tilde{A}}_0]\tilde{\mathcal{F}}_{01} + \mathcal{O}((e\theta)^2)\\ &= -\frac{1}{2}(\partial_\mu\tilde{A}_\nu)(\partial^\mu\tilde{A}^\nu) + \frac{e\theta}{2}\epsilon^{\alpha\beta}\partial_\alpha\tilde{A}^\mu\partial_\beta\tilde{A}^\nu\tilde{\mathcal{F}}_{\mu\nu}\\ &\quad + \mathcal{O}((e\theta)^2),\end{aligned} \tag{4.8}$$

where $\tilde{\mathcal{F}}_{\mu\nu} \equiv \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu$. Note that the θ dependent part of (4.8) is, as expected, exactly the same as the original Lagrangian $\mathcal{L}_g$ from (3.11).

B. The fermionic part

First let us modify the fermionic fields using the Ansatz

$$\tilde{\psi} = (1 + ie\xi_{\bar{\psi}}\gamma^0)\psi, \quad \text{and} \quad \tilde{\bar{\psi}} = \bar{\psi}(1 - ie\xi_{\psi}\gamma^0), \tag{4.9}$$

where compared to (2.36), $\xi_{\bar{\psi}} = -i\overrightarrow{\mathcal{J}}_{\bar{\psi}}\gamma^0$ and $\xi_{\psi} = i\overleftarrow{\mathcal{J}}_{\psi}\gamma^0$. Choosing $\xi_{\bar{\psi}}$ and ξ_{ψ} as

$$\begin{aligned}\xi_{\bar{\psi}} &= 0,\\ \xi_{\psi} &= \frac{i\theta}{2}\gamma^\lambda\partial_x A_\lambda + \frac{\theta^2}{8}\gamma^\lambda\gamma^5\partial_x^3 A_\lambda + \frac{\theta^2}{8}\gamma^\lambda\partial_x^2\partial_0 A_\lambda\\ &\quad + \frac{\theta^2}{4}\gamma^\lambda\overleftarrow{\partial}_x(\partial_x\partial_0 A_\lambda) + \frac{3\theta^2}{8}\gamma^\lambda\gamma^5\overleftarrow{\partial}_x(\partial_x^2 A_\lambda)\\ &\quad + \frac{3\theta^2}{8}\gamma^\lambda\gamma^5\overleftarrow{\partial}_x^2(\partial_x A_\lambda) + \frac{\theta^2}{4}\gamma^\lambda\gamma^5\overleftarrow{\partial}_x^3 A_\lambda + \mathcal{O}(e, \theta^3),\end{aligned} \tag{4.10}$$

the canonical fermionic Poisson brackets,

$$\{\tilde{\bar{\psi}}(t;x), \tilde{\psi}(t;x')\}_{\rm PB} \approx -i\gamma^0\delta(x - x'),$$

can be shown to be valid up to $\mathcal{O}(e^2, \theta^3)$, as expected. In what follows, we will determine the fermionic part of the Hamiltonian density as a function of $\tilde{\psi}$ and $\tilde{\bar{\psi}}$. The original Hamiltonian in terms of ψ and $\bar{\psi}$ is given by the Legendre transformation of the Lagrangian density $\mathcal{L}_f$ from (3.18)

$$\begin{aligned}\mathcal{H}_f(\psi, \bar{\psi}) &= \Pi_{\psi}^{\beta}\dot{\psi}_\beta + \Pi_{\dot{\psi}}^{\beta}\ddot{\psi}_\beta + \dot{\bar{\psi}}_\beta\Pi_{\bar{\psi}}^{\beta} + \ddot{\bar{\psi}}_\beta\Pi_{\dot{\bar{\psi}}}^{\beta}\\ &\quad - \mathcal{L}_f(\psi, \bar{\psi})\\ &= -i\bar{\psi}\gamma^1\partial_x\psi + e\bar{\psi}\gamma^\lambda\psi A_\lambda\\ &\quad + \frac{e\theta^2}{8}\partial_x\bar{\psi}(\gamma^\lambda\partial_x A_\lambda + \gamma^\lambda\gamma^5\partial_0 A_\lambda)\partial_x^2\psi\\ &\quad + \frac{e\theta^2}{8}\partial_x^2\bar{\psi}(\gamma^\lambda\partial_x A_\lambda + \gamma^\lambda\gamma^5\partial_0 A_\lambda)\partial_x\psi\\ &\quad + \frac{e\theta^2}{4}\partial_x^2\bar{\psi}\gamma^\lambda\partial_x^2\psi A_\lambda + \mathcal{O}(e^2, \theta^3),\end{aligned} \tag{4.11}$$

To formulate $\mathcal{H}_f$ in terms of redefined fields $\tilde{\psi}$ and $\tilde{\bar{\psi}}$, we will first invert (4.9) to get

$$\psi = \tilde{\psi}, \quad \text{and} \quad \bar{\psi} = \tilde{\bar{\psi}}(1 + ie\xi_{\psi}\gamma^0) + \mathcal{O}(e^2, \theta^3), \tag{4.12}$$

where $\xi_{\bar{\psi}}$ and ξ_{ψ} can be read from (4.10). Then replacing (4.12) in (4.11), we arrive at

$$\begin{aligned}\tilde{\mathcal{H}}_f &\equiv \mathcal{H}_f(\bar{\psi}(\tilde{\bar{\psi}}, A_\mu), \psi(\tilde{\psi}, A_\mu))\\ &= -i\tilde{\bar{\psi}}(1 + ie\xi_{\psi}\gamma^0)\gamma^1\partial_x\tilde{\psi} + e\tilde{\bar{\psi}}\gamma^\lambda\tilde{\psi}A_\lambda\\ &\quad + \frac{e\theta^2}{8}\partial_x\tilde{\bar{\psi}}(\gamma^\lambda\partial_x A_\lambda + \gamma^\lambda\gamma^5\partial_0 A_\lambda)\partial_x^2\tilde{\psi}\\ &\quad + \frac{e\theta^2}{8}\partial_x^2\tilde{\bar{\psi}}(\gamma^\lambda\partial_x A_\lambda + \gamma^\lambda\gamma^5\partial_0 A_\lambda)\partial_x\tilde{\psi}\\ &\quad + \frac{e\theta^2}{4}\partial_x^2\tilde{\bar{\psi}}\gamma^\lambda\partial_x^2\tilde{\psi}A_\lambda + \mathcal{O}(e^2, \theta^3),\end{aligned} \tag{4.13}$$

where ξ_{ψ} is given in (4.10). Because of the special form of redefined fields in (4.9) and (4.12), the above modified Hamiltonian is not Hermitian. The Hermitian conjugate of $-i\tilde{\bar{\psi}}\gamma^1\partial_x\tilde{\psi}$ on the first line is to be added to $\tilde{\mathcal{H}}_f$ to build a Hermitian Hamiltonian. We denote it by $\mathcal{C} \equiv i(\tilde{\bar{\psi}}\gamma^1\partial_x\tilde{\psi})^\dagger$. Adding $\mathcal{C}$ to $\tilde{\mathcal{H}}_f$ and using the standard definition

$$\tilde{\mathcal{L}}_f = \tilde{\mathcal{L}}_f(\tilde{\bar{\psi}}, \tilde{\psi}) = i\tilde{\bar{\psi}}\gamma^0\dot{\tilde{\psi}} - \tilde{\mathcal{H}}_f, \tag{4.14}$$

the modified Lagrangian density $\tilde{\mathcal{L}}_f$ in terms of the redefined fields is given as

$$\begin{aligned}\tilde{\mathcal{L}}_f &= i\tilde{\bar{\psi}}\gamma^\mu\partial_\mu\tilde{\psi} - e\tilde{\bar{\psi}}\gamma^\lambda\tilde{\psi}A_\lambda - i\partial_x\tilde{\bar{\psi}}\gamma^1\gamma^0\tilde{\bar{\psi}}^\dagger - e\bigg\{\tilde{\bar{\psi}}\bigg(\frac{i\theta}{2}\gamma^\lambda\gamma^5\partial_x A_\lambda\partial_x - \frac{\theta^2}{8}\gamma^\lambda\gamma^5\partial_x\partial_0 A_\lambda\partial_x^2 + \frac{\theta^2}{8}\gamma^\lambda\partial_x A_\lambda\partial_x^3\bigg)\tilde{\psi}\\ &\quad + \partial_x\tilde{\bar{\psi}}\bigg(-\frac{i\theta}{2}\gamma^\lambda\gamma^5\partial_x A_\lambda + \frac{\theta^2}{8}\gamma^\lambda\partial_x^3 A_\lambda + \frac{3\theta^2}{8}\gamma^\lambda\partial_x^2 A_\lambda\partial_x + \frac{\theta^2}{8}\gamma^\lambda\gamma^5\partial_x^2\partial_0 A_\lambda + \frac{\theta^2}{4}\gamma^\lambda\gamma^5\partial_x\partial_0 A_\lambda\partial_x + \frac{\theta^2}{4}\gamma^\lambda A_\lambda\partial_x^3\\ &\quad + \frac{3\theta^2}{8}\gamma^\lambda\partial_x A_\lambda\partial_x^2\bigg)\gamma^0\tilde{\bar{\psi}}^\dagger\bigg\} + \mathcal{O}(e^2, \theta^3).\end{aligned} \tag{4.15}$$

To simplify the above modified Lagrangian, we will use the following relations:

$$\begin{aligned}
\bar{\tilde{\psi}} &= \bar{\tilde{\psi}}(1 + ie\xi_\psi\gamma^0) \\
&= \bar{\tilde{\psi}} + ie\bar{\tilde{\psi}}\Big(\frac{i\theta}{2}\gamma^\lambda\gamma^0\partial_x A_\lambda + \frac{\theta^2}{8}\gamma^\lambda\gamma^0\partial_x^2\partial_0 A_\lambda + \frac{\theta^2}{8}\gamma^\lambda\gamma^5\gamma^0\partial_x^3 A_\lambda + \frac{3\theta^2}{8}\gamma^\lambda\gamma^5\gamma^0\overleftarrow{\partial}_x(\partial_x^2 A_\lambda) + \frac{\theta^2}{4}\gamma^\lambda\gamma^0\overleftarrow{\partial}_x(\partial_x\partial_0 A_\lambda) \\
&\quad + \frac{\theta^2}{4}\gamma^\lambda\gamma^5\gamma^0\overleftarrow{\partial}_x^3 A_\lambda + \frac{3\theta^2}{8}\gamma^\lambda\gamma^5\gamma^0\overleftarrow{\partial}_x^2(\partial_x A_\lambda)\Big) + \mathcal{O}(e^2,\theta^3), \\
\bar{\tilde{\psi}}^\dagger &= (1 + ie\gamma_0\xi_\psi^\dagger)\gamma_0\tilde{\psi} \\
&= \gamma_0\tilde{\psi} - ie\Big(\frac{i\theta}{2}\gamma^\lambda\partial_x A_\lambda - \frac{\theta^2}{8}\gamma^\lambda\gamma^5\partial_x^3 A_\lambda - \frac{\theta^2}{8}\gamma^\lambda\partial_x^2\partial_0 A_\lambda - \frac{3\theta^2}{8}\gamma^\lambda\gamma^5\partial_x^2 A_\lambda\partial_x - \frac{\theta^2}{4}\gamma^\lambda\partial_x\partial_0 A_\lambda\partial_x - \frac{3\theta^2}{8}\gamma^\lambda\gamma^5\partial_x A_\lambda\partial_x^2 \\
&\quad - \frac{\theta^2}{4}\gamma^\lambda\gamma^5 A_\lambda\partial_x^3\Big)\tilde{\psi} + \mathcal{O}(e^2,\theta^3),
\end{aligned} \tag{4.16}$$

where

$$\begin{aligned}
\xi_\psi^\dagger &= -\frac{i\theta}{2}\Gamma^\lambda\partial_x A_\lambda + \frac{\theta^2}{8}\gamma^5\Gamma^\lambda\partial_x^3 A_\lambda + \frac{3\theta^2}{8}\gamma^5\Gamma^\lambda\partial_x^2 A_\lambda\partial_x + \frac{\theta^2}{8}\Gamma^\lambda\gamma^0\partial_x^2\partial_0 A_\lambda + \frac{\theta^2}{4}\Gamma^\lambda\partial_x\partial_0 A_\lambda\partial_x + \frac{\theta^2}{4}\gamma^5\Gamma^\lambda A_\lambda\partial_x^3 \\
&\quad + \frac{3\theta^2}{8}\gamma^5\Gamma^\lambda\partial_x A_\lambda\partial_x^2 + \mathcal{O}(e^2,\theta^3),
\end{aligned} \tag{4.17}$$

with $\Gamma^\lambda = \gamma^0\gamma^\lambda\gamma^0$ is used. We arrive finally at the effective Lagrangian density including only the first order time derivative of bosonic and fermionic fields

$$\tilde{\mathcal{L}}_f = \bar{\tilde{\psi}}\Big\{i\gamma^\mu\partial_\mu - e\Big(\gamma^\lambda A_\lambda + \frac{i\theta}{2}\gamma^\lambda\gamma^5(\partial_x A_\lambda)\partial_x - \frac{\theta^2}{8}\gamma^\lambda\gamma^5(\partial_x\partial_0 A_\lambda)\partial_x^2 + \frac{\theta^2}{8}\gamma^\lambda(\partial_x A_\lambda)\partial_x^3\Big)\Big\}\tilde{\psi} + \mathcal{O}(e^2,\theta^3). \tag{4.18}$$

Combining at this stage the bosonic and fermionic parts of the Lagrangian density from (4.8) and (4.18), we arrive at the total effective Lagrangian density in terms of the redefined fields $\tilde{\psi}$, $\bar{\tilde{\psi}}$, and $\tilde{A}_\mu$,

$$\begin{aligned}
\tilde{\mathcal{L}}[\tilde{\psi},\bar{\tilde{\psi}},\tilde{A}_\mu] &= -\frac{1}{2}(\partial_\mu\tilde{A}_\nu)(\partial^\mu\tilde{A}^\nu) + \frac{e\theta}{2}\epsilon^{\alpha\beta}\partial_\alpha\tilde{A}^\mu\partial_\beta\tilde{A}^\nu\tilde{\mathcal{F}}_{\mu\nu} + \bar{\tilde{\psi}}\Big\{i\gamma^\mu\partial_\mu - e\Big(\gamma^\lambda\tilde{A}_\lambda + \frac{i\theta}{2}\gamma^\lambda\gamma^5(\partial_x\tilde{A}_\lambda)\partial_x \\
&\quad - \frac{\theta^2}{8}\gamma^\lambda\gamma^5(\partial_x\partial_0\tilde{A}_\lambda)\partial_x^2 + \frac{\theta^2}{8}\gamma^\lambda(\partial_x\tilde{A}_\lambda)\partial_x^3\Big)\Big\}\tilde{\psi} + \mathcal{O}(e^2,\theta^3).
\end{aligned} \tag{4.19}$$

The final Lagrangian includes a first order time derivative and higher order space derivatives. The Lorentz covariance of the modified Lagrangian is broken by the procedure of perturbative quantization, where higher order time derivatives are replaced by corresponding space derivatives. The above Lagrangian can be nevertheless regarded as the starting point for further perturbative and nonperturbative study of two-dimensional space-time noncommutative QED up to order $\mathcal{O}(e^2,\theta^3)$.

V. THE ALGEBRA OF CURRENTS OF THE MODIFIED NONCOMMUTATIVE THEORY

In this section, as a possible application of our previous results, we will use the Dirac brackets of the modified fermionic and bosonic fields to determine the current algebra of the global $U(1)$ vector current (3.6)

$$J^\mu(x) = -\psi_\beta(x)\star\bar{\psi}_\alpha(x)(\gamma^\mu)^{\alpha\beta},$$

corresponding to the original two-dimensional noncommutative QED described by (3.4). Using the definition of the $\star$ product from (3.1), $J^\mu(x)$ can be written as

$$\begin{aligned}
J^\mu(x) &= \bar{\psi}\gamma^\mu\psi + \frac{i\theta}{2}\epsilon^{\rho\sigma}\partial_\sigma\bar{\psi}\gamma^\mu\partial_\rho\psi \\
&\quad - \frac{\theta^2}{8}\epsilon^{\rho\sigma}\epsilon^{\lambda\eta}\partial_\sigma\partial_\eta\bar{\psi}\gamma^\mu\partial_\rho\partial_\lambda\psi + \mathcal{O}(\theta^3).
\end{aligned} \tag{5.1}$$

Replacing ψ and $\bar{\psi}$ with the modified fermionic fields $\tilde{\psi}$ and $\bar{\tilde{\psi}}$ using the relations (4.12), we arrive after some lengthy but straightforward manipulations at the corresponding modified currents

$$\tilde{J}^0(x) = \bar{\tilde{\psi}}\gamma^0\tilde{\psi} + ie\bar{\tilde{\psi}}\bigg(\frac{i\theta}{2}\gamma^\lambda\partial_x A_\lambda + \frac{\theta^2}{8}\gamma^\lambda\gamma^5\partial_x^3 A_\lambda + \frac{\theta^2}{8}\gamma^\lambda\partial_x^2\partial_0 A_\lambda + \frac{3\theta^2}{8}\gamma^\lambda\gamma^5\overleftarrow{\partial}_x(\partial_x^2 A_\lambda) + \frac{\theta^2}{4}\gamma^\lambda\overleftarrow{\partial}_x(\partial_x\partial_0 A_\lambda)$$
$$+ \frac{3\theta^2}{8}\gamma^\lambda\gamma^5\overleftarrow{\partial}_x^2(\partial_x A_\lambda) + \frac{\theta^2}{4}\gamma^\lambda\gamma^5\overleftarrow{\partial}_x^3 A_\lambda\bigg)\tilde{\psi} + \frac{i\theta}{2}\epsilon^{\rho\sigma}\partial_\sigma\bar{\tilde{\psi}}\gamma^0\partial_\rho\tilde{\psi} - \frac{\theta^2}{8}\epsilon^{\rho\sigma}\epsilon^{\lambda\eta}\partial_\sigma\partial_\eta\bar{\tilde{\psi}}\gamma^0\partial_\rho\partial_\lambda\tilde{\psi}$$
$$- \frac{ie\theta^2}{4}\epsilon^{\rho\sigma}\partial_\sigma(\bar{\tilde{\psi}}\partial_x A_\lambda)\gamma^\lambda\partial_\rho\tilde{\psi} + \mathcal{O}(e^2,\theta^3), \tag{5.2}$$

$$\tilde{J}^1(x) = \bar{\tilde{\psi}}\gamma^1\tilde{\psi} + ie\bar{\tilde{\psi}}\bigg(\frac{i\theta}{2}\gamma^\lambda\gamma^5\partial_x A_\lambda + \frac{\theta^2}{8}\gamma^\lambda\partial_x^3 A_\lambda + \frac{\theta^2}{8}\gamma^\lambda\gamma^5\partial_x^2\partial_0 A_\lambda + \frac{3\theta^2}{8}\gamma^\lambda\overleftarrow{\partial}_x(\partial_x^2 A_\lambda) + \frac{\theta^2}{4}\gamma^\lambda\gamma^5\overleftarrow{\partial}_x(\partial_x\partial_0 A_\lambda)$$
$$+ \frac{3\theta^2}{8}\gamma^\lambda\overleftarrow{\partial}_x^2(\partial_x A_\lambda) + \frac{\theta^2}{4}\gamma^\lambda\overleftarrow{\partial}_x^3 A_\lambda\bigg)\tilde{\psi} + \frac{i\theta}{2}\epsilon^{\rho\sigma}\partial_\sigma\bar{\tilde{\psi}}\gamma^1\partial_\rho\tilde{\psi} - \frac{\theta^2}{8}\epsilon^{\rho\sigma}\epsilon^{\lambda\eta}\partial_\sigma\partial_\eta\bar{\tilde{\psi}}\gamma^1\partial_\rho\partial_\lambda\tilde{\psi}$$
$$- \frac{ie\theta^2}{4}\epsilon^{\rho\sigma}\partial_\sigma(\bar{\tilde{\psi}}\partial_x A_\lambda)\gamma^\lambda\gamma^5\partial_\rho\tilde{\psi} + \mathcal{O}(e^2,\theta^3). \tag{5.3}$$

Using now the equal-time commutation relations (2.38) between redefined field operators $\tilde{\psi}$ and $\bar{\tilde{\psi}}$, we arrive at the following algebra of currents:

$$[\tilde{J}^0(t;x),\tilde{J}^1(t;x')]|_{\text{canonical}} = -i\theta\partial_0(\bar{\tilde{\psi}}\gamma^1\tilde{\psi})\partial_x\delta(x-x') + ie\theta^2(\partial_x A_\lambda)\partial_0(\bar{\tilde{\psi}}\gamma^\lambda\gamma^5\tilde{\psi})\partial_x\delta(x-x')$$
$$+ \frac{ie\theta^2}{4}(\partial_0\partial_x A_\lambda)(\bar{\tilde{\psi}}\gamma^\lambda\gamma^5\tilde{\psi})\partial_x\delta(x-x') + \mathcal{O}(e^2,\theta^3), \tag{5.4}$$

$$[\tilde{J}^0(t;x),\tilde{J}^0(t;x')]|_{\text{canonical}} = [\tilde{J}^1(t;x),\tilde{J}^1(t;x')]|_{\text{canonical}}$$
$$= -i\theta\partial_0(\bar{\tilde{\psi}}\gamma^0\tilde{\psi})\partial_x\delta(x-x') + ie\theta^2(\partial_x A_\lambda)\partial_0(\bar{\tilde{\psi}}\gamma^\lambda\tilde{\psi})\partial_x\delta(x-x')$$
$$+ \frac{ie\theta^2}{4}(\partial_0\partial_x A_\lambda)(\bar{\tilde{\psi}}\gamma^\lambda\tilde{\psi})\partial_x\delta(x-x') + \mathcal{O}(e^2,\theta^3), \tag{5.5}$$

that contains Schwinger terms on the right-hand sides.

VI. CONCLUDING REMARKS

In the first part of the paper, we have presented the general framework of perturbative quantization for a $D+1$ dimensional QED-like theory, that includes bosons and fermions whose interactions are described by terms containing higher order space-time derivatives. According to the general procedure described in [16,17], the equations of motion of the original field theory are used to define time derivatives as a function of space derivatives in the lowest order of perturbative expansion in the order of the QED coupling constant e. Then, the fermionic and bosonic field variables are appropriately modified, so that they satisfy the ordinary fundamental Poisson brackets in this first order approximation. Using the standard Dirac quantization procedure, the equal-time commutation relations corresponding to fermions and bosons are determined up to $\mathcal{O}(e^2)$.

In the second part of the paper, two-dimensional space-time NC-QED is perturbatively quantized up to $\mathcal{O}(e^2,\theta^3)$, where θ is the space-time noncommutativity parameter. Noncommutative field theories, in general, are characterized by a noncommutative Moyal product that replaces the ordinary product of functions in commutative field theory. In two dimensions, in particular, the Moyal product involves an infinite number of space-time derivatives. Appearing in the interaction part of the theory, the space-time noncommutativity renders the theory acausal and inconsistent with conventional Hamiltonian evolution [11]. The S matrix of the theory is also nonunitary [12]. Different attempts are performed to cure space-time NC-QED [13], that in two dimensions are by themselves interesting to study not only because they are the noncommutative counterpart of the well-known Schwinger model [22].

Following the procedure of perturbative quantization, we have determined the Lagrangian density of two-dimensional space-time QED in terms of modified field variables that satisfy the ordinary equal-time commutation relations up to $\mathcal{O}(e^2,\theta^3)$. In this lowest approximation, although the Poisson algebra of the bosonic field variables are modified by terms proportional to the noncommutativity parameter θ, the bosonic part of the Lagrangian density remains unchanged. The fermionic part consists of first order time derivatives and higher order space derivatives of bosonic and fermionic field variables. The modified Lagrangian density (4.19) has lost, due to the special feature of perturbative quantization, the relativistic covariance of space and time coordinates. Using the canonical equal-time commutation relations of the modified field variables, the algebra of global NC-$U_V(1)$ currents of the original NC-QED is also determined.

In summary, the modified Lagrangian density (4.19) can be regarded as the starting point for further perturbative and nonperturbative study of two-dimensional space-time noncommutative QED up to order $\mathcal{O}(e^2, \theta^3)$. In particular, the most important problem of the unitarity of the S matrix of a theory described by the modified Lagrangian density (4.19), is to be checked explicitly. The problem of one-loop unitarity of noncommutative $\lambda\varphi^3$ scalar field theory is addressed in [16]. It is claimed that "the one-loop unitarity is not affected by introducing higher spatial derivatives, but is at stake when higher time derivatives are present" [16]. Indeed, the main feature of the present perturbative quantization procedure is to replace higher time derivatives with higher space derivatives, [see (2.21)]. According to the calculation in [16], it is expected that the one-loop unitarity of the S matrix of noncommutative two-dimensional QED is also preserved after the presented perturbative quantization procedure. This shall be checked explicitly in future publications.

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