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Renormalization of gauge theories in curved space-time

Peter M. Lavrov[1,*] and Ilya L. Shapiro[2,†]

[1]*Department of Mathematical Analysis, Tomsk State Pedagogical University, 634061, Kievskaya St. 60, Tomsk, Russia*
[2]*Departamento de Física, ICE, Universidade Federal de Juiz de Fora, Juiz de Fora, CEP: 36036-330, MG, Brazil*
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We consider the renormalization of general gauge theories on curved space-time background, with the main assumption being the existence of a gauge-invariant and diffeomorphism invariant regularization. Using the Batalin-Vilkovisky formalism, one can show that the theory possesses gauge invariant and diffeomorphism invariant renormalizability at quantum level, up to an arbitrary order of the loop expansion. Starting from this point, we discuss the locality of the counterterms and the general prescription for constructing the power-counting renormalizable theories on curved background.

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I. INTRODUCTION

The quantum field theory (QFT) in curved space is an important ingredient of our general understanding of the quantum description of nature. The reason for this is that, according to general relativity, our space-time is likely to be curved. Therefore, as far as we think that the QFT approach is a fundamental one in the description of the interaction of elementary particles and fields, it must be considered on a curved space-time background. The consideration of QFT on classical curved background does not rule out the quantization of gravity, but, in some sense, is at least equally important. The reason is that we do not know which one of the existing ways to quantize gravity is close to reality, while the QFT of matter fields definitely deals with reality, as the concept of a classical curved space does.

One of the most important aspects of the modern QFT is the theory of gauge fields and their perturbative renormalizations. The gauge invariant renormalizability is the cornerstone in the construction of the very important theories including the standard model of particle physics. Hence it is quite interesting to know whether the existing methods to analyze renormalizability of gauge theories are working well in curved space. In the previous considerations of the problem [1,2] (see, also, [3]), it has been assumed that the gauge-invariant renormalization of the theory is, indeed, possible due to the existence of both gauge-invariant and diffeomorphism invariant regularization such as a dimensional one. Starting from this point, it is possible to establish the prescription for constructing the renormalizable theories of interacting matter fields on curved background [1,4] (see, also, [5] for a recent review and for a somehow more simple treatment of the issue).

The present work is intended to explore, in a more formal way than was done before, the issue of gauge-invariant renormalizability in curved space-time. To this end, we are going to apply the Batalin-Vilkovisky (BV) formalism. It is well known that this formalism enables one to prove the gauge-invariant renormalizability of general gauge theories in a situation when all fields under consideration are quantum ones [6,7] (see, also, [8,9] for an extensive review and further references). It is, of course, important to generalize these considerations to the case when the QFT is defined in the presence of external conditions, in particular, in curved space-time. In this case, one has to take care about both gauge symmetries and general covariance. The last symmetry involves both quantum and external fields, making the consideration more complicated. Our main purpose is to consider the general features of renormalization of the theory of quantum matter fields in curved space-time, using the powerful BV formalism. On the top of that, we will discuss the construction of multiplicatively renormalizable theories in curved space, the subject which was already considered previously (see, e.g. [4,5] and references therein) in a slightly different manner.

The paper is organized as follows. In the next section, we present a very brief review of the antibracket (BV) formalism in gauge theories. In Sec. III, we consider the same formalism for gauge theories in curved space. The gauge-invariant renormalization in curved space-time is considered in Sec. IV. An important aspect of the theory is the possibility to use the noncovariant gauge-fixing conditions, which is discussed in Sec. V. In Sec. VI, we introduce the quantum gravity completion of the theory to get some strong arguments supporting the locality of the counterterms of the quantum theory in curved space. The power-counting renormalizability and the receipt for constructing renormalizable theories in curved space are discussed in Sec. VII. Finally, in Sec. VIII, we draw our conclusions.

II. GAUGE THEORIES IN BV FORMALISM

In this section, we present a very brief review of the BV formalism [10], which will be used in the rest of the paper

*lavrov@tspu.edu.ru
†Also at Tomsk State Pedagogical University, Tomsk, Russia. shapiro@fisica.ufjf.br

to prove the gauge-invariant and general covariant renormalizability of the quantum field theory on curved background. An extensive review of the formalism can be found in [8,9], where we mainly collect information (and also fix notations) which will be needed in further consideration.

A. Preliminaries and terminology

The need for the advanced version of the Lagrangian quantization formalism was inspired by the discovery of supergravity theories in 1970s [11]. The gauge transformations possess linearly-dependent generators and, as a consequence, direct application of the Faddeev-Popov procedure leads to the violation of unitarity of the physical S-matrix. Moreover, attempts of covariant quantization of gauge theories with linearly-dependent generators of gauge transformations result in the understanding of the fact that it is impossible to use the Faddeev-Popov rules to construct a suitable quantum theory [12]. The quantization of general gauge theories requires taking into account such aspects as the existence of open algebras and reducible generators. The quantization can be performed only by introducing different types of ghosts, antighosts, ghosts for ghosts (Nielsen, Kallosh ghosts, etc.) [13]. A unique closed approach to the problem of covariant quantization that summarized all these attempts was proposed by Batalin and Vilkovisky [10]. The BV formalism gives the rules for the quantization of general gauge theories.

The starting point of the BV method is a theory of fields $A^i (i = 1, 2, \ldots, n)$ with Grassmann parities $\varepsilon(A^i) = \varepsilon_i$, for which the initial classical action $S_0(A)$ is assumed to have at least one stationary point A_0^i

$$S_{0,i}(A)|_{A_0} = 0, \tag{1}$$

and to be regular in the neighborhood of A_0. Here, we are using the notations

$$A_0 = \{A_0^i\} \quad \text{and} \quad F_{,i}(A) = \frac{\partial_r F(A)}{\partial A^i},$$

where the label "r" denotes the right derivative.

Geometrically, Eq. (1) defines a surface Σ in the space of functions A^i. We assume the invariance of the action $S_0(A)$ under the gauge transformations $\delta A^i = R^i_\alpha(A)\xi^\alpha$ in the neighborhood of the stationary point,

$$S_{0,i}(A)R^i_\alpha(A) = 0, \qquad \alpha = 1, 2, \ldots, m, \\ 0 < m < n, \qquad \varepsilon(\xi^\alpha) = \varepsilon_\alpha. \tag{2}$$

Here, ξ^α are arbitrary functions of space-time coordinates, and $R^i_\alpha(A)$ are generators of gauge transformations. We have also used DeWitt's condensed notations [14], such that any index includes space-time coordinates, an index of internal group, a Lorentz index, and so on. Consequently, a summation over repeated indices includes, along with summation over internal and Lorentz indices, also an integration over continuous variables such as space-time coordinates

It follows from the Noether identities (2) that, first, the equations of motion are not independent and, second, (some) propagators do not exist because the Hessian matrix $H_{ij} = S_{0,ij}$, corresponding to the action S_0, is degenerate at any point on the stationary surface Σ,

$$S_{0,i}(A)R^i_{\alpha,j}(A) + S_{0,ji}(A)R^i_\alpha(-1)^{\varepsilon_\alpha\varepsilon_j} = 0 \\ \Rightarrow S_{0,ji}R^i_\alpha|_{A_0} = 0.$$

The generators R^i_α are on shell zero-eigenvalue vectors of the Hessian matrix $S_{0,ij}$.

The structure of gauge algebra can be found by studying the commutator of gauge transformations and some consequences from the relations (2). We assume that the set of generators $R^i_\alpha(A)$ is complete. In this case, one can prove that the generators algebra has the following general form (see [15–17]):

$$R^i_{\alpha,j}(A)R^j_\beta(A) - (-1)^{\varepsilon_\alpha\varepsilon_\beta}R^i_{\beta,j}(A)R^j_\alpha(A) \\ = -R^i_\gamma(A)F^\gamma_{\alpha\beta}(A) - S_{0,j}(A)M^{ij}_{\alpha\beta}(A), \tag{3}$$

where $F^\gamma_{\alpha\beta}(A)$ are structure functions with the following symmetry properties:

$$F^\gamma_{\alpha\beta}(A) = -(-1)^{\varepsilon_\alpha\varepsilon_\beta}F^\gamma_{\beta\alpha}(A)$$

and $M^{ij}_{\alpha\beta}(A)$ are satisfying the conditions

$$M^{ij}_{\alpha\beta}(A) = -(-1)^{\varepsilon_i\varepsilon_j}M^{ji}_{\alpha\beta}(A) = -(-1)^{\varepsilon_\alpha\varepsilon_\beta}M^{ij}_{\beta\alpha}(A).$$

In case $M^{ij}_{\alpha\beta}(A) = 0$, one meets a gauge theory with a *closed* gauge algebra. If $M^{ij}_{\alpha\beta}(A) \neq 0$, then the gauge algebra is called *open*. In this case, due to the symmetry properties of $M^{ij}_{\alpha\beta}(A)$, the quantities

$$R^i_{\alpha\beta,\mathrm{triv}}(A) = S_{0,j}(A)M^{ij}_{\alpha\beta}(A)$$

are symmetry generators of the initial action $S_0(A)$ which can be called trivial. They vanish at the extremals of $S_0(A)$,

$$R^i_{\alpha\beta,\mathrm{triv}}(A)|_{S_{0,i}=0} = 0$$

and leave the action invariant. At the same time, they are not connected with an additional degeneration of the initial action $S_0(A)$ because the rank of the Hessian matrix describing the degeneracy of the initial action is defined at the extremals $S_{0,i} = 0$.

Finally, if $M^{ij}_{\alpha\beta}(A) = 0$ and $F^\gamma_{\alpha\beta}$ do not depend on the fields A, the gauge transformations form a gauge group and define a *Lie algebra*.

B. BV quantization: The general procedure

The procedure of the BV quantization for a general gauge theory involves the following steps. First, the total configuration space of the fields ϕ^A is introduced. For irreducible theories, the fields ϕ^A include A^i, ghost and antighost fields C^α, and $\bar{C}^\alpha$ and auxiliary (Nakanishi-Lautrup) fields B^α

$$\phi^A = (A^i, B^\alpha, C^\alpha, \bar{C}^\alpha), \qquad \varepsilon(\phi^A) = \varepsilon_A, \tag{4}$$

with the following distribution of the Grassmann parities and ghost numbers

$$\varepsilon(A^i) = \varepsilon_i, \quad \varepsilon(B^\alpha) = \varepsilon_\alpha, \quad \varepsilon(C^\alpha) = \varepsilon(\bar{C}^\alpha) = \varepsilon_\alpha + 1,$$
$$gh(A^i) = gh(B^\alpha) = 0, \quad gh(C^\alpha) = 1, \quad gh(\bar{C}^\alpha) = -1.$$

To each field ϕ^A of the total configuration space, one introduces corresponding *antifield* ϕ^*_A,

$$\phi^*_A = (A^*_i, \quad B^*_\alpha, \quad C^*_\alpha, \quad \bar{C}^*_\alpha). \tag{5}$$

The statistics of ϕ^*_A is opposite to the statistics of the corresponding fields ϕ^A

$$\varepsilon(\phi^*_A) = \varepsilon_A + 1$$

and ghost numbers of fields and corresponding antifields are connected by the rule

$$gh(\phi^*_A) = -1 - gh(\phi^A).$$

On the space of the fields ϕ^A and antifields ϕ^*_A, one defines an odd symplectic structure (,) called the antibracket

$$(F, G) \equiv \frac{\delta F}{\delta \phi^A}\frac{\delta G}{\delta \phi^*_A} - (F \leftrightarrow G)(-1)^{[\varepsilon(F)+1]\cdot[\varepsilon(G)+1]}. \tag{6}$$

Here, the derivatives with respect to fields are understood as the right ones and those with respect to antifields as the left ones.

One can easily verify that the following properties of the antibracket follow from the definition (6):

(1) Grassmann parity relations $\varepsilon((F, G)) = \varepsilon(F) + \varepsilon(G) + 1 = \varepsilon((G, F))$;
(2) Generalized antisymmetry $(F, G) = -(G, F) \times (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}$;
(3) Leibniz rule $(F, GH) = (F, G)H + (F, H)G(-1)^{\varepsilon(G)\varepsilon(H)}$;
(4) Generalized Jacobi identity $((F, G), H) \times (-1)^{(\varepsilon(F)+1)(\varepsilon(H)+1)} + \mathrm{cycle}(F, G, H) \equiv 0$.

Furthermore, one can readily check that the antibracket (6) is invariant under the *anticanonical* transformation of variables ϕ, ϕ^* with the generating functional $X = X(\phi, \phi^*)$, $\varepsilon(X) = 1$,

$$\phi'^A = \frac{\delta X(\phi, \phi^{*'})}{\delta \phi^{*'}_A}, \qquad \phi^*_A = \frac{\delta X(\phi, \phi^{*'})}{\delta \phi^A}. \tag{7}$$

This property of the odd symplectic structure (6) on the space of ϕ, ϕ^* is a counterpart to the invariance property of the even symplectic structure (the Poisson bracket) under a canonical transformation of canonical variables (p, q). For the first time, the importance of anticanonical transformations (7) in the formulation of the BV-method was realized in [6].

As a second step, the nilpotent generating operator Δ is introduced according to

$$\Delta = (-1)^{\varepsilon_A}\frac{\delta_l}{\delta \phi^A}\frac{\delta}{\delta \phi^*_A}, \qquad \Delta^2 = 0, \qquad \varepsilon(\Delta) = 1. \tag{8}$$

We will always assume that formal manipulations with operators such as Δ can be supported by suitable regularization scheme. This is a nontrivial requirement since the operator (8) is not well defined on local functionals. The reason is that for any local functional S, $\Delta S \sim \delta(0)$ and one faces the so-called problem of $\delta(0)$. The usual way to deal with this problem is to use the dimensional regularization [18], where $\delta(0)$ is equal to zero. Recently, a new calculus for local variational differential operators in local quantum field theory has been proposed by Shahverdiev, Tyutin, and Voronov [19], where $\delta(0)$ does not arise at all.

Note that acting by Δ on the product of two functionals, F and G reproduce the antibracket

$$\Delta[F \cdot G] = (\Delta F) \cdot G + F \cdot (\Delta G)(-1)^{\varepsilon(F)} + (F, G)(-1)^{\varepsilon(F)}.$$

As a third step, the quantum master equation is defined according to

$$\frac{1}{2}(S, S) = i\hbar \Delta S \tag{9}$$

or, equivalently,

$$\Delta \exp\left\{\frac{i}{\hbar}S\right\} = 0, \tag{10}$$

where $S = S(\phi, \phi^*)$ is a bosonic functional satisfying the boundary condition

$$S|_{\phi^* = \hbar = 0} = S_0(A). \tag{11}$$

The bosonic functional S is the fundamental object of the BV-quantization scheme.

The generating functional of the Green functions $Z(J)$ is defined as

$$Z(J) = \int d\phi \exp\left\{\frac{i}{\hbar}[S_{\mathrm{eff}}(\phi) + J_A \phi^A]\right\}, \qquad S_{\mathrm{eff}}(\phi) = S\left(\phi, \phi^* = \frac{\delta \Psi}{\delta \phi}\right). \tag{12}$$

Here, $\Psi = \Psi(\phi)$ is a fermionic gauge functional. For instance, if the gauge-fixing condition in the Yang-Mills theory is chosen to be $\chi_\alpha = 0$, the fermionic gauge functional has the form $\Psi = \bar{c}^\alpha \chi_\alpha$, where $\bar{c}^\alpha$ is the FP antighost. Furthermore, in the Eq. (12), J_A are the usual external sources to the fields ϕ^A. The Grassmann parities of these sources are defined in a natural way, $\varepsilon(J_A) = \varepsilon_A$.

Note [6], that the gauge-fixing procedure (12) in the BV quantization can be described in terms of anticanonical transformation of the variables ϕ, ϕ^* (7) in $S(\phi, \phi^*)$ with the generating functional X

$$X(\phi, \phi^*) = \phi^*_A \phi^A + \Psi(\phi).$$

To discuss some features of the BV quantization, it is convenient to rewrite the expression for the generating functional $Z(J)$ in the equivalent form

$$\begin{aligned} Z(J) &= \int d\phi d\phi^* \delta\Big(\phi^* - \frac{\delta\Psi}{\delta\phi}\Big)\exp\Big\{\frac{i}{\hbar}[S(\phi, \phi^*) + J_A\phi^A]\Big\} \\ &= \int d\phi d\phi^* d\lambda \exp\Big\{\frac{i}{\hbar}\Big[S(\phi, \phi^*) + \Big(\phi^*_A - \frac{\delta\Psi}{\delta\phi^A}\Big)\lambda^A \\ &\quad + J_A\phi^A\Big]\Big\}, \end{aligned} \tag{13}$$

where we have introduced the auxiliary (Nakanishi-Lautrup) fields λ^A with $\varepsilon(\lambda^A) = \varepsilon_A + 1$.

Note, first of all, that the integrand in (13) for $J_A = 0$ is invariant under the following global transformations:

$$\delta\phi^A = \lambda^A\mu, \qquad \delta\phi^*_A = \mu\frac{\delta S}{\delta\phi^A}, \qquad \delta\lambda^A = 0. \tag{14}$$

It is very important to remember that the existence of this symmetry follows from the fact that the bosonic functional S satisfies the generating Eq. (9). The transformations (14) represent the Becchi, Rouet, Stora, and Tyutin transformations in the space of variables ϕ, ϕ^*, λ.

The symmetry of the vacuum functional $Z(0)$ under the BRST transformations (14) paves the way for establishing an independence of the S matrix on the choice of gauge in the BV quantization. Indeed, suppose $Z_\Psi \equiv Z(0)$. We shall change the gauge $\Psi \to \Psi + \delta\Psi$. In the functional integral for $Z_{\Psi+\delta\Psi}$, we make the change of variables, choosing for μ

$$\mu = -\frac{i}{\hbar}\delta\Psi.$$

After simple algebraic calculations, we find that

$$Z_{\Psi+\delta\Psi} = Z_\Psi. \tag{15}$$

In order to derive the Ward identity corresponding to the BRST-symmetry, it is convenient to consider the extended generating functional of the Green functions

$$\mathcal{Z}(J, \phi^*) = \int d\phi \exp\Big\{\frac{i}{\hbar}[S_\psi(\phi, \phi^*) + J_A\phi^A]\Big\}, \tag{16}$$

where

$$S_\psi(\phi, \phi^*) = S\Big(\phi, \phi^* + \frac{\delta\Psi}{\delta}\phi\Big). \tag{17}$$

From the above definition, it follows that

$$\mathcal{Z}(J, \phi^*)|_{\phi^*=0} = Z(J),$$

where $Z(J)$ has been defined in (12). From BRST symmetry follows the Ward identity for the extended generating functional of the Green functions

$$J_A\frac{\delta\mathcal{Z}}{\delta\phi^*_A} = 0. \tag{18}$$

Introducing the generating functional of connected the Green functions $\mathcal{W} = \mathcal{W}(J, \phi^*) = -i\hbar\ln\mathcal{Z}$, the identity (18) can be rewritten as

$$J_A\frac{\delta\mathcal{W}}{\delta\phi^*_A} = 0. \tag{19}$$

The generating functional of the vertex functions (effective action) $\Gamma = \Gamma(\phi, \phi^*)$ is introduced in a standard way, through the Legendre transformation of $\mathcal{W}$,

$$\begin{aligned} &\Gamma(\phi, \phi^*) = \mathcal{W}(J, \phi^*) - J_A\phi^A, \\ &\phi^A = \frac{\delta\mathcal{W}}{\delta J_A}, \qquad \frac{\delta\Gamma}{\delta\phi^A} = -J_A. \end{aligned} \tag{20}$$

Finally, the Ward identity for the generating functional of the vertex functions can be obtained directly from (19) and (20), in the form

$$(\Gamma, \Gamma) = 0. \tag{21}$$

The Ward identity (21) has a universal form and plays a very important role in proof of gauge-invariant renormalizability of general gauge theories [6]. In deriving this identity all fields under consideration have been assumed to be quantized. However, it looks evident that the form of Eq. (21) will be the same in presence of external background (for example, a gravitational background) fields as well (see below). In the next section, we will see that this equation represents a suitable basis for the consideration of quantum field theory in curved space.

III. GENERAL GAUGE THEORIES IN CURVED SPACE

Let us consider a theory of gauge fields A^i in an external gravitational field $g_{\mu\nu}$. The classical theory is described by the action which depends on both dynamical fields and external metric

$$S_0 = S_0(A, g). \tag{22}$$

Here, and below, we use the condensed notation $g \equiv g_{\mu\nu}$ for the metric, when it is an argument of some functional or function. The action (22) is assumed to be gauge invariant,

$$\begin{aligned} &S_{0,i}R^i_a = 0, \qquad \delta A^i = R^i_a(A, g)\lambda^a, \\ &\lambda^a = \lambda^a(x), \qquad (a = 1, 2, \ldots, n), \end{aligned} \tag{23}$$

as well as covariant

$$\delta_g S_0 = \frac{\delta S_0}{\delta A^i}\delta_g A^i + \frac{\delta S_0}{\delta g_{\mu\nu}}\delta_g g_{\mu\nu} = 0, \tag{24}$$

where λ^a are independent parameters of the gauge transformation, corresponding to the symmetry group of the theory. The diffeomorphism transformation of the metric in Eq. (24) has the form

$$\delta_g g_{\mu\nu} = -g_{\mu\alpha}\partial_\nu \xi^\alpha - g_{\nu\alpha}\partial_\mu \xi^\alpha - \partial_\alpha g_{\mu\nu}\xi^\alpha = -g_{\mu\alpha}\nabla_\nu \xi^\alpha - g_{\nu\alpha}\nabla_\mu \xi^\alpha = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu. \quad (25)$$

Here, ξ^α are the parameters of the coordinates transformation

$$\xi^\alpha = \xi^\alpha(x) \qquad (\alpha = 1, 2, \ldots, d). \quad (26)$$

As usual, an explicit expression for $\delta_g A^i$ depends on tensor (or spinor) properties of A^i. For example, in the case of a scalar field A, one has $\delta_g A = -\partial_\alpha A \xi^\alpha$ while in the case of a vector field A^μ, the transformation rule is $\delta_g A^\mu = A^\nu \nabla_\nu \xi^\mu - \xi^\nu \nabla_\nu A^\mu$, etc. In general, our interest is to explore the renormalization properties of the theories which include all three kind of fields (fermions, vectors, and scalars), for instance the standard model and its extensions, including grand unified theories (GUTs), would be covered. Therefore the notation A^i in (23) and (24) means the set of fields with the different transformation rules.

The generating functional $Z(J, \phi^*, g)$ of the Green functions can be constructed in the form of the functional integral

$$\mathcal{Z}(J, \phi^*, g) = \int d\phi \exp\left\{\frac{i}{\hbar}[S_\psi(\phi, \phi^*, g) + J_A \phi^A]\right\}. \quad (27)$$

Here,[1] $\phi^A = (A^i, B^a, C^a, \bar{C}^a)$ represents the full set of fields of the complete configuration space of the theory under consideration and $\phi^*_A = (A^*_i, B^*_a, C^*_a, \bar{C}^*_a)$ are corresponding antifields. Finally, $S_\psi(\phi, \phi^*, g)$ is the quantum action constructed with the help of the solution $S = S(\phi, \phi^*, g)$ of the master equation

$$(S, S) = 0, \qquad S(\phi, \phi^*, g)|_{\phi^*=0} = S_0(A, g) \quad (28)$$

in the form

$$S_\psi(\phi, \phi^*, g) = S\left(\phi, \phi^* + \frac{\delta\Psi(\phi, g)}{\delta\phi}, g\right). \quad (29)$$

In the last Eq. (29), $\Psi(\phi, g)$ is a gauge-fixing functional. Note that S_ψ satisfies the master equation

$$(S_\psi, S_\psi) = 0. \quad (30)$$

[1]We restrict ourself to the case of irreducible close gauge theories only, in order to simplify the description of the configuration space.

From the gauge invariance of initial action (23), in the usual manner, one can derive the BRST symmetry and the Ward identities for generating functionals $\mathcal{Z}$, $\mathcal{W}$, and Γ in the form (18), (19), and (21), respectively.

A solution to the master Eq. (28) can be always found in the form of a series in antifields ϕ^* (see [10]),

$$S(\phi, \phi^*, g) = S_0(A, g) + A^*_i R^i_a(A, g) C^a + \bar{C}^*_a B^a + \cdots, \quad (31)$$

where dots mean higher order terms in fields B^a, C^a. We assume that every term in (31) is transformed as a scalar under arbitrary local transformations of coordinates $x^\mu \to x^\mu + \xi^\mu(x)$. It means the general covariance of $S = S(\phi, \phi^*, g)$,

$$\delta_g S(\phi, \phi^*, g) = \frac{\delta S}{\delta \phi^A}\delta_g \phi^A + \delta_g \phi^*_A \frac{\delta S}{\delta \phi^*_A} + \frac{\delta S}{\delta g_{\mu\nu}}\delta_g g_{\mu\nu} = 0. \quad (32)$$

Let us choose the gauge-fixing functional $\Psi = \psi(\phi, g)$ in a covariant form

$$\delta_g \Psi = 0, \quad (33)$$

and then the quantum action $S_\psi = S_\psi(\phi, \phi^*, g)$ obeys the general covariance too

$$\delta_g S_\psi = 0. \quad (34)$$

From the Eq. (34) and the assumption that the term with the sources J_A in (27) is covariant,

$$\delta_g(J_A \phi^A) = (\delta_g J_A)\phi^A + J_A(\delta_g \phi^A) = 0, \quad (35)$$

follows the general covariance of $\mathcal{Z} = \mathcal{Z}(J, \phi^*, g)$. Indeed,

$$\delta_g \mathcal{Z}(J, \phi^*, g) = \frac{i}{\hbar}\int d\phi\left[\delta_g \Phi^*_A \frac{\delta S_\psi(\phi, \phi^*, g)}{\delta\phi^*_A} + \frac{\delta S_\psi(\phi, \phi^*, g)}{\delta g_{\mu\nu}}\delta_g g_{\mu\nu} + (\delta_g J_A)\phi^A\right] \times \exp\left\{\frac{i}{\hbar}[S_\psi(\phi, \phi^*, g) + J_A\phi^A]\right\}. \quad (36)$$

Making change of integration variables in the functional integral, (36),

$$\phi^A \to \phi^A + \delta_g \phi^A, \quad (37)$$

we arrive at the relation

$$\delta_g Z(J,\phi^*,g) = \frac{i}{\hbar}\int d\phi\Big[\frac{\delta S_\psi}{\delta\phi^A}\delta_g\phi^A + \delta_g\phi^*_A\frac{\delta S_\psi}{\delta\phi^*_A} + \frac{\delta S_\psi}{\delta g_{\mu\nu}}\delta_g g_{\mu\nu} + (\delta_g J_A)\phi^A + J_A(\delta_g\phi^A)\Big] \times \exp\Big\{\frac{i}{\hbar}[S_\psi(\phi,\phi^*,g) + J_A\phi^A]\Big\} = \frac{i}{\hbar}\int d\phi[\delta_g S_\psi + \delta_g(J_A\phi^A)]\exp\Big\{\frac{i}{\hbar}[S_\psi(\phi,\phi^*,g) + J_A\phi^A]\Big\} = 0. \tag{38}$$

From (38), it follows that the generating functional of connected Green functions $\mathcal{W}(J,\phi^*,g))$

$$\mathcal{W}(J,\Phi^*,g) = \frac{i}{\hbar}\ln Z(J,\phi^*,g) \tag{39}$$

obeys the property of the general covariance as well as

$$\delta_g\mathcal{W}(J,\phi^*,g) = 0. \tag{40}$$

Consider now the generating functional of vertex functions $\Gamma = \Gamma(\Phi,\Phi^*,g)$

$$\Gamma(\phi,\phi^*,g) = \mathcal{W}(J,\phi^*,g) - J_A\phi^A, \tag{41}$$

where

$$\phi^A = \frac{\delta\mathcal{W}(J,\phi^*,g)}{\delta J_A}, \qquad J_A = -\frac{\delta\Gamma(\phi,\phi^*,g)}{\delta\phi^A}. \tag{42}$$

From the definition of ϕ^A (42) and the general covariance of $W(J,\phi^*,g)$, we can conclude the general covariance of $J_A\phi^A$. Therefore

$$\delta_g\Gamma(\phi,\phi^*,g) = \delta_g\mathcal{W}(J,\phi^*,g) = 0. \tag{43}$$

IV. GAUGE-INVARIANT RENORMALIZATION IN CURVED SPACE-TIME

Up to now, we have considered nonrenormalized generating functionals of the Green functions. The next step is to prove the general covariance for renormalized generating functionals. To this end, let us first consider the one-loop approximation for $\Gamma = \Gamma(\phi,\phi^*,g)$,

$$\Gamma = S_\psi + \bar\Gamma^{(1)} = S_\psi + \hbar[\bar\Gamma^{(1)}_{\rm div} + \bar\Gamma^{(1)}_{\rm fin}] + O(\hbar^2), \tag{44}$$

where $\bar\Gamma^{(1)}_{\rm div}$ and $\bar\Gamma^{(1)}_{\rm fin}$ denote the divergent and finite parts of the one-loop approximation for Γ. The divergent local[2] term $\bar\Gamma^{(1)}_{\rm div}$ gives the first counterpart in the one-loop renormalized action $S_{\psi 1}$

$$S_\psi \to S_{\psi 1} = S_\psi - \hbar\bar\Gamma^{(1)}_{\rm div}. \tag{45}$$

From (34) and (43), it follows that in one-loop approximation we have

$$\delta_g[\bar\Gamma^{(1)}_{\rm div} + \bar\Gamma^{(1)}_{\rm fin}] = 0 \tag{46}$$

and therefore $\bar\Gamma^{(1)}_{\rm div}$ and $\bar\Gamma^{(1)}_{\rm fin}$ obey the general covariance independently

$$\delta_g\bar\Gamma^{(1)}_{\rm div} = 0, \qquad \delta_g\bar\Gamma^{(1)}_{\rm fin} = 0. \tag{47}$$

In its turn, the one-loop renormalized action $S_{\psi 1}$ (i.e., classical action, renormalized at the one-loop level) is covariant

$$\delta_g S_{\psi 1} = 0. \tag{48}$$

Constructing the generating functional of one-loop renormalized Green functions $\mathcal{Z}_1(J,\phi^*,g)$, with the action $S_{\psi 1} = S_{\psi 1}(\phi,\phi^*,g)$, and repeating arguments given above, we arrive at the relation

$$\delta_g\mathcal{Z}_1 = 0, \qquad \delta_g W_1 = 0, \qquad \delta_g\Gamma_1 = 0. \tag{49}$$

In the last equation, we have introduced the new useful notation for the renormalized up to the one-loop order effective action Γ_1. This functional includes the contributions of one-loop and also higher loop orders, however, only the one-loop divergences are removed by renormalization. This means that Γ_1 is finite in the $\mathcal{O}(\hbar)$ order, but may be divergent starting from $\mathcal{O}(\hbar^2)$ and beyond.

The generating functional of vertex functions $\Gamma_1 = \Gamma_1(\phi,\phi^*,g)$ which is finite in the one-loop approximation, can be presented in the form

$$\Gamma_1 = S_\psi + \hbar\bar\Gamma^{(1)}_{\rm fin} + \hbar^2[\bar\Gamma^{(2)}_{1,\rm div} + \bar\Gamma^{(2)}_{1,\rm fin}] + O(\hbar^3). \tag{50}$$

Indeed, this functional contains a divergent part $\bar\Gamma^{(2)}_{1,\rm div}$ and defines renormalization of the action S_ψ in the two-loop approximation

$$S_\psi \to S_{\psi 2} = S_{\psi 1} - \hbar^2\bar\Gamma^{(2)}_{1,\rm div}. \tag{51}$$

Starting from (47)–(49), we derive

$$\delta_g\bar\Gamma^{(2)}_{1,\rm div} = 0, \qquad \delta_g\bar\Gamma^{(2)}_{1,\rm fin} = 0. \tag{52}$$

The last equation means that the general covariance condition is satisfied separately for the divergent and finite parts of $\bar\Gamma_1$ in two-loop approximation. As a consequence, the two-loop renormalized action $S_{\psi 2} = S_{\psi 2}(\phi,\phi^*,g)$ is a covariant functional

$$\delta_g S_{\psi 2} = 0. \tag{53}$$

[2]The discussion of locality of the divergent part of effective action will be given in the next section.

Applying the induction method, we can repeat the procedure to an arbitrary order of the loop expansion. In this way, we arrive at the followings results:

(a) The full renormalized action $S_{\psi R} = S_{\psi R}(\Phi, \Phi^*, g)$,

$$S_{\psi R} = S_\psi - \sum_{n=1}^{\infty} \hbar^n \bar{\Gamma}^{(n)}_{n-1,\mathrm{div}}, \tag{54}$$

which is local in each finite order in $\hbar$, obeys the general covariance

$$\delta_g S_{\psi R} = 0; \tag{55}$$

(b) The renormalized generating functional of vertex functions $\Gamma_R = \Gamma_R(\Phi, \Phi^*, g)$,

$$\Gamma_R = S_\psi + \sum_{n=1}^{\infty} \hbar^n \bar{\Gamma}^{(n)}_{n-1,\mathrm{fin}}, \tag{56}$$

which is finite in each finite order in $\hbar$, is covariant

$$\delta_g \Gamma_R = 0. \tag{57}$$

It was proved in [6] that the renormalized action $S_{\psi R}$ satisfies the master equation

$$(S_{\psi R}, S_{\psi R}) = 0 \tag{58}$$

and the Ward identities for nonrenormalized and renormalized generating functionals of vertex functions have the form

$$(\Gamma, \Gamma) = 0, \qquad (\Gamma_R, \Gamma_R) = 0. \tag{59}$$

The last equations mean that the gauge-invariant renormalizability (59) of a quantum field theory takes place in the presence of an external gravitational field, such that the general covariance of effective action (57) is also preserved. In order to use this important result, we have to perform an additional consideration and check how the covariance is preserved in case when we use apparently noncovariant techniques, e.g., related to the representation of the metric as a sum of the flat one and perturbation. This subject will be treated in the next section.

V. NONCOVARIANT GAUGES

In many cases, it is interesting to consider the renormalization of quantum field theory in curved space using the noncovariant gauge-fixing functionals. One important example of such consideration can be found in Sec. VII of the present article, where we discuss power-counting renormalizability in curved space. Let us see how the noncovariant gauge-fixing can be implemented in the quantum theory.

Our purpose is to investigate the problem of general covariant renormalizability for general gauge theories in the presence of an external gravitational field, when one uses noncovariant gauge-fixing functional $\Psi = \Psi(\phi, g)$,

$$\delta_g \Psi \neq 0. \tag{60}$$

As before, we assume that the classical action of the theory $S = S(\phi, \phi^*, g)$ is covariant, i.e. $\delta_g S = 0$, but now the action $S_\psi = S_\psi(\phi, \phi^*, g) = S(\phi, \phi^* + \delta\Psi/\delta\phi, g)$ is not covariant, $\delta_g S_\psi \neq 0$. Our consideration will be essentially based on the known formalism for investigating the gauge dependence in general gauge theories, given in [6]. Noncovariance of S_ψ can be described in the form of anticanonical infinitesimal transformation with the odd generating functional

$$X(\phi, \phi^*, g) = \phi^*_A \phi^A + \delta_g \Psi(\phi, g), \tag{61}$$

$$\Phi^A = \frac{\delta X(\phi', \phi^*, g)}{\delta \phi^*_A} = \Phi^{A'}, \qquad \phi^{*\prime}_A = \frac{\delta X(\phi', \phi^*, g)}{\delta \phi'_A} = \phi^*_A + \frac{\delta \delta_g \Psi}{\delta \phi^A}, \tag{62}$$

when

$$\delta_g S_\psi = \frac{\delta \delta_g \Psi}{\delta \phi^A} \frac{\delta S_\psi}{\delta \phi^*_A} = (\delta_g \Psi, S_\psi). \tag{63}$$

The variation of S_ψ (63) leads to the variations of generating functionals of the Green functions $Z = Z(J, \phi^*, g)$, connected Green functions $\mathcal{W} = \mathcal{W}(J, \phi^*, g)$, and vertex functions $\Gamma = \Gamma(\phi, \phi^*, g)$ in the form

$$\delta_g Z = \frac{i}{\hbar} J_A \frac{\delta}{\delta \phi^*_A} \delta_g \Psi\Big(\frac{\hbar}{i}\frac{\delta}{\delta J}, g\Big) Z, \tag{64}$$

$$\delta_g \mathcal{W} = J_A \frac{\delta}{\delta \phi^*_A} \langle \delta_g \Psi \rangle, \tag{65}$$

$$\delta_g \Gamma = (\langle\langle \delta_g \Psi \rangle\rangle, \Gamma), \tag{66}$$

where the notations

$$\langle \delta_g \Psi \rangle = \delta_g \Psi\Big(\frac{\delta \mathcal{W}}{\delta J} + \frac{\hbar}{i}\frac{\delta}{\delta J}, g\Big), \qquad \langle\langle \delta_g \Psi \rangle\rangle = \delta_g \Psi\Big(\phi + i\hbar (\Gamma'')^{-1} \frac{\delta_l}{\delta \phi}, g\Big), \qquad \Gamma''_{AB} = \frac{\delta_l}{\delta \phi^A} \frac{\delta}{\delta \phi^B} \Gamma, \tag{67}$$

were used. These results can be immediately reproduced in the renormalized theory [6]. Namely, for the variation (63), the corresponding variation of renormalized action $\delta_g S_{\psi R}$ can be presented in the form

$$\delta_g S_{\psi R} = (\delta_g \Psi_R, S_{\psi R}) \tag{68}$$

of the anticanonical transformation with local generating functional $X = \phi^*_A \phi^A + \delta_g \Psi_R$,

$$\delta_g \Psi_R(\phi, \phi^*, g) = \delta_g \Psi(\phi, g) - \sum_{n=1}^{\infty} \hbar^n \delta_g \Psi^{(n)}_{n-1,\mathrm{div}}(\phi, \phi^*, g), \quad (69)$$

while the variation of renormalized vertex generating functional $\delta_g \Gamma_R$ has the form

$$\delta_g \Gamma_R = (\langle\langle \delta_g \Psi_R \rangle\rangle_R, \Gamma_R), \quad (70)$$

which corresponds to finite anticanonical transformation with generating function

$$X = \phi^*_A \phi^A + \langle\langle \delta_g \Psi_R \rangle\rangle_R,$$
$$\langle\langle \delta_g \Psi_R \rangle\rangle_R = \delta_g \Psi(\phi, g) + \sum_{n=1}^{\infty} \hbar^n \delta_g \Psi^{(n)}_{n-1,\mathrm{fin}}. \quad (71)$$

In the formulas presented above, we have used the notations $\delta_g \Psi^{(n)}_{n-1,\mathrm{div}}$ and $\delta_g \Psi^{(n)}_{n-1,\mathrm{fin}}$ for the divergent and finite terms, respectively, of the n-loop approximation for the generating function of an anticanonical transformation which is finite in $(n-1)$th order approximation and is constructed on the basis of the theory with the action $S_{\psi(n-1)}$.

The interpretation of the relations (70) and (71) is that the theory with external gravitational field may have noncovariance in the renormalized effective action, but it comes only from the possible noncovariance of the arguments. Here, the expression arguments is used to denote the full set of the mean fields from which the effective action depends, as defined in (20). Therefore the violation of the general coordinate symmetry which can occur because of the noncovariant gauge fixing can be always included into the arguments. As a consequence, one can always define some special set of arguments in terms of which the quantum dynamics is described in a completely covariant way. One important aspect of this feature is that we can actually perform general considerations or make practical calculations in a noncovariant gauges. After that, we can always restore the covariance using those parts of effective action which are not affected by gauge transformation. A practical examples of this technique can be found in many publications, but here we constructed a theoretical background for its consistent use. In the next sections, we will see, also, that this result opens the way for a practical construction of renormalizable gauge theories in curved space-time.

Note that there exists another interpretation of the gauge dependence of effective action (see [20]). Namely, it can be proved that dependence on the gauge of effective action is proportional to its extremals, i.e. physical quantities calculated on shell do not depend on the gauge.

VI. ON THE LOCALITY OF THE COUNTERTERMS

In most cases, the general consideration of renormalizability is based on the hypothesis of locality of all necessary counterterms. This statement was first proved in general form in [21] and is known as Weinberg theorem. One can find a more pedagogical consideration of this theorem in the book [22]. It is important for us to understand whether the locality of the counterterms holds for the case when the external gravitational field is present. It is easy to see that the arguments of [22] can be taken carefully in this case and, in principle, some special attention to this issue is in order. Here, we present a qualitative consideration which shows that the locality of the counterterms still holds in the presence of external gravity.

Let us consider the theory of the matter fields $A \equiv A^i$ with the action (22), which depends also on the external metric $g \equiv g_{\mu\nu}$, $S_0(A, g)$. In order to discuss the locality of the counterterms, it proves useful to parametrize the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (72)$$

where we do not need to make special assumptions about the field $h_{\mu\nu}$. Starting from the parametrization (72) of the metric, one can construct the diagrammatic representation of the path integral (27). The relevant Feynman diagrams include external lines of the fields $\tilde{\Phi}$ only and the external lines of both quantum fields (given by sources in the Schwinger formalism) and the classical background field $h_{\mu\nu}$.

How can we know that the presence of the background field $h_{\mu\nu}$ does not lead to the nonlocal counterterms at higher orders of the loop expansion? In order to address this question, let us consider the quantum gravity completion of the theory. This means, we start from the extended classical action

$$S_0^{\mathrm{ext}} = S_0(A, g) + S_{\mathrm{QG}}, \quad (73)$$

where S_{QG} is an action of a quantum gravitational field. As far as we do not care about power-counting renormalizability of the theory at this stage (see the next section for the corresponding discussion), S_{QG} can be just the Einstein-Hilbert action. Another possibility is to include the higher derivative terms. In fact, as we shall see in a moment, the result does not depend on the choice of the action S_{QG}. Let us also remark that the path integral representation of the quantum gravitational theory includes a set of ghost and antighost fields (see, e.g., [4,23,24] for the higher derivative case). For the sake of simplicity, we will not write these extra fields here, or assume they are included automatically into Φ^*.

One can note that the new theory, based on the action (73), includes internal lines of the metric field $h_{\mu\nu}$ and does not include external fields. Therefore the Weinberg theorem can be applied, and we can use the result for the locality of the counterterms at any loop order in the complete theory. In particular, one can prove that only local solutions of the master equations can be relevant for the divergences in the case of the fourth derivative quantum gravity [24]. Moreover, the proof presented in [24] does not require the details of the action of quantum gravity and indeed can be generalized for other cases, including the quantum general relativity.

On the other hand, the theory with the quantum gravity completion includes all those Feynman diagrams which give contribution to divergences of the theory with external metric. Therefore, since the complete theory does not have nonlocal divergences, the reduced one with external metric does not have them either. Hence, for the usual quantum field theory on curved background, we have strong reasons to assume the locality of the necessary counterterms to all orders in the loop expansion.

One more observation is in order. All arguments presented above correspond to the usual quantum field theory on curved background and can be violated in the case that we consider the theory with spontaneous symmetry breaking [25]. In this case, the nonlocalities show up already at the classical level, in the induced action of gravity. At the quantum level, the nonlocal structures get renormalized, and hence we are forced to introduce an infinite set of nonlocal counterterms. However, the details of the consideration presented in [25] show that the mentioned nonlocalities are always related to the scalar (Higgs) field, such that the corresponding renormalization becomes local if this field is treated as an independent one.

VII. POWER-COUNTING RENORMALIZABILITY AND CONSTRUCTION OF RENORMALIZABLE THEORIES

In the previous sections, we have shown that the non-anomalous gauge theory in curved space-time is renormalizable in a sense that the necessary counterterms, in all orders of the loop expansion, are given by the local, covariant, and gauge-invariant expressions. This fact enables one to prepare the receipt of constructing the renormalizable theories in curved space.

Let us consider the $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ parametrization of the external metric, which enables one to deal with the usual flat-space Feynman diagrams. Compared to the diagrams of the flat-space-time theory, these diagrams have external lines of the metric field $h_{\mu\nu}$. As far as gravity is a nonpolynomial interaction, there may be, in principle, an unrestricted amount of such external lines coming to any vertex of the diagram. However, the covariance of the counterterms which we have proven in Sec. V, enables one to establish the general form of the counterterms.

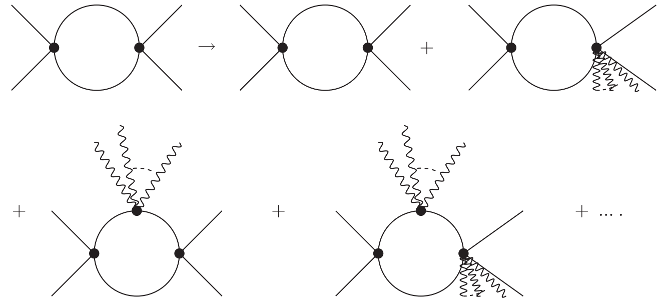

FIG. 1. The single diagram with quadratic divergences in flat space generates an infinite set of diagrams with external lines of $h_{\mu\nu}$. Some of those diagrams have quadratic or logarithmic divergences, others are finite.

We start from the case of a scalar field φ with the $\lambda\varphi^4$-interaction. The first diagram we will be interested in is the one-loop correction vertex function. The situation which occurs in curved space-time is illustrated in the Fig. 1. One can note that the lines of the field $h_{\mu\nu}$ may either produce new vertices or be connected to the existing vertex due to the expansion

$$\sqrt{-g}\lambda\varphi^4 = \lambda\varphi^4 \cdot \left[1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu} + \dots\right], \quad h = h_{\mu\nu}g^{\mu\nu}. \tag{74}$$

It is easy to see that the first kind of diagrams has more propagators in the loop that the initial flat-space diagram. The typical examples are the diagrams in the second line in Fig. 1. It is obvious that the divergence of the diagrams with larger number of propagators will be smaller. For instance, the mentioned diagrams in the second line are all finite. On the other hand, the diagrams with the lines of $h_{\mu\nu}$ connected only to the vertices will sum up to produce the logarithmic divergences which will be exactly of the form of the flat-space divergence, multiplied by the $\sqrt{-g}$, defined in (74). Any other form would enter in conflict with locality and covariance of the divergences which we have proven in the previous sections.[3]

As the next step, let us consider the one-loop contribution to the field propagator, which has quadratic divergence in the flat-space-time case. The situation which occurs in curved space-time is illustrated in the Fig. 2. Again, as in the case of the vertex diagram, one can distinguish the two kinds of diagrams. The first kind of diagrams has more propagators in the loop, compared to the initial flat-space diagram. The typical examples are the last diagram in the first line and the last two diagrams in the second line on

[3]The explicit calculations in the momentum-subtraction scheme confirm this conclusion [26]. Also, they show that the finite part of the vertex function is a nonlocal object, as it usually happens. Of course, this does not contradict the Weinberg theorem [21,22] which concerns only the UV divergences.

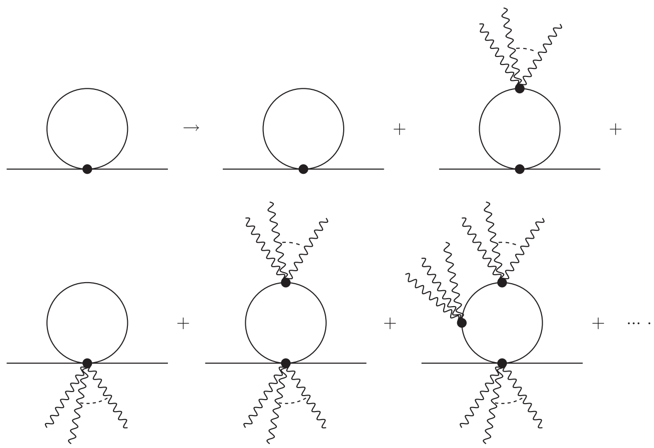

FIG. 2. The single diagram with quadratic divergences in flat space generates an infinite set of diagrams with external lines of $h_{\mu\nu}$. Some of those diagrams have quadratic or logarithmic divergences, others are finite.

Fig. 2. It is obvious that the divergence of the diagrams with larger number of propagators will be smaller. For instance, the initial flat-space diagram on Fig. 2 has quadratic divergences and the last diagram in the first line has only logarithmic divergences, exactly as all other diagrams with one extra vertex. Moreover, the diagrams with two extra vertices are all finite.

What are the counterterms needed to cancel the new logarithmic divergences, e.g., the ones produced by the last diagram in the first line of Fig. 2? As we already know, this counterterm must be covariant and local. It is obvious that there can not be derivatives of the scalar. Furthermore, the dimensional consideration shows that the correct dimension of the counterterm can be provided only by including second derivatives of $h_{\mu\nu}$ functions. As we know, the only invariant which can be constructed from the second derivatives of the metric is the scalar curvature R. Therefore the unique possible form of the counterterm is the integral of

$$\sqrt{-g}R\varphi^2, \tag{75}$$

which is called the nonminimal term.

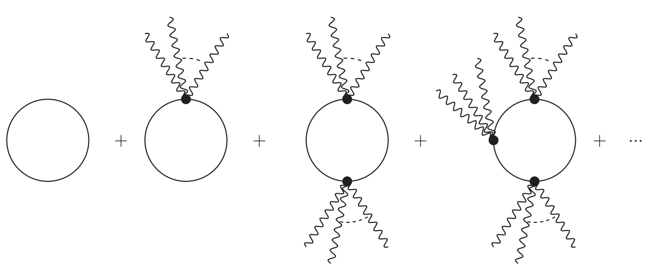

FIG. 3. The single diagram with quartic divergences in flat space leads to the diagrams with quartic, quadratic, and logarithmic divergences due to external lines of $h_{\mu\nu}$ with the new vertices. Even though there are infinitely many new diagrams, the divergences are well controlled by covariance.

Finally, let us consider the last possible source of the one-loop divergences which are the vacuum diagrams. The generalization of the single one-loop vacuum diagram in flat space to the curved space-time case is demonstrated in Fig. 3. It is obvious that the situation is similar to the one with the previous diagrams, in the sense that inserting the new vertices will produce less divergent diagrams. The divergences can be classified by a number of derivatives of the metric, and we start from the zero-derivative case. Both the initial diagram and its covariant version have only quartic divergence for the massless scalar and, also, quadratic and logarithmic divergences in the massive case. All these divergences can be removed by renormalizing the covariant cosmological constant term $\int d^4x\sqrt{-g}\rho_\Lambda$, which must be therefore included into the classical action. Let us note that the diagrams corresponding to the renormalization of the covariant cosmological constant term have only one vertex and no derivatives of the external $h_{\mu\nu}$ functions.

Since the initial diagram has quadratic divergences, the ones with one new vertex will have quadratic and (in case of massive scalar) logarithmic divergence. The analysis is pretty much the same as in the case of the diagrams from Fig. 2. It is obvious, from the dimensional reasons and covariance, that the quadratic divergence will be removed by the counterterm linear in curvature and the logarithmic ones by the counterterm proportional to

$$d^4x\sqrt{-g}Rm^2, \tag{76}$$

where m is the mass of the scalar field. All these counterterms can be removed by renormalizing the Einstein-Hilbert term, which is also (along with the cosmological term) a necessary element of renormalizable theory in curved space-time.

Finally, there are logarithmically divergent diagrams with two new vertices and with four derivatives of the external $h_{\mu\nu}$ functions. The covariance and locality show that the necessary counterterms have the following form

$$\int d^4x\sqrt{-g}\{\alpha_1 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R^2 + \alpha_4\nabla^2 R\}. \tag{77}$$

It is very important that the possible divergences listed above represent the complete set and no others can appear. Moreover, this consideration can be immediately generalized for an arbitrary renormalizable (in flat-space-time) theory including fermions, massless gauge vectors, and scalars. It is easy to see that the counterterms listed above, plus covariant generalizations of the familiar counterterms in flat-space-time, still represent the complete set. Let us note that the nonminimal term is possible only in the scalar sector of the theory. According to the consideration performed in Sec. V and VI, the described structure of divergences is compatible with the gauge invariance of the theory at quantum level.

The analysis of the one-loop divergences can be used to establish the renormalization structure at higher loops. Let us consider the two-loop divergences. The one-loop subdiagrams produce the divergences described above and can be removed by adding minimal, nonminimal, and vacuum local counterterms. As far as these counterterms have the same structure as the classical action, and the nonlocal part does not influence the second-loop counterterms, the part of the one-loop diagrams which is relevant for the divergences coming from the last integration, is essentially the same as in flat-space, plus nonminimal term. Therefore at the second-loop we meet exactly the same types of counterterms as at the one-loop level, which we have described above. The only difference will be the renormalization coefficients which will have higher powers of coupling constants.

The iteration procedure can be applied to higher loops and we will always meet the same structure of renormalization in curved space which was already described in [4] (see further references therein). All in all, we can state that the an arbitrary renormalizable in flat-space-time theory can be properly generalized into curved space-time such that it keeps its renormalizability.

VIII. CONCLUSIONS

We have considered the general scheme of gauge-invariant and covariant renormalization of the quantum gauge theory of matter fields in curved space-time. Using the Batalin-Vilkovisky formalism, we have shown that in the theory which admits gauge-invariant and diffeomorphism invariant regularization, these two symmetries hold in the counterterms to all orders of the loops expansion. The locality of the necessary counterterms can be shown by the use of the Weinberg theorem if we complete the theory of quantum matter by some version of quantum gravity theory. As a result, one can always perform renormalization of the theory in the gauge invariant and generally covariant way. Of course, this feature does not guarantee the multiplicative renormalizability of the theory, exactly as in the flat-space-time quantum theory. However, starting from a renormalizable theory in flat-space-time and using a standard prescription [1,4], one can always arrive at the theory which is renormalizable in curved space-time as well.

Let us note that the renormalizability of the theory in curved space should not be understood in such a way that the quantum theory in curved space is as successful as the one in flat space. Unfortunately, the real situation is far from this. Let us remember that the renormalization of the theory includes the following two steps: (i) removing divergences; (ii) extracting finite part of effective action (or of the Green functions, etc.). As we have shown in this paper (see, also, previous publications [1,2,4] and references therein), the (i) of the program formulated above can be completed in a consistent and covariant way, such that the gauge invariance of the theory can be preserved in the same way as in flat-space-time.

Unfortunately, the part (ii) of the above program meets very serious difficulties and here, the situation is, at present, very far from the one in flat-space-time. One can see the recent papers [5,27] for the review and discussion of this interesting and challenging issue, which we will not elaborate on here. At the same time, one can not underestimate the covariance of the renormalized effective action, which we have shown to hold in all orders in the loop expansion. This feature can be very important for it can provide an essential guide in exploring the possible forms of the quantum corrections, even if they can not be derived explicitly.

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