Fields in nonaffine bundles. III. Effective symmetries and conserved currents in strings and higher branes

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The principles of a previously developed formalism for the covariant treatment of multiscalar fields for which (as in a nonlinear sigma model) the relevant target space is not of affine type—but curved—are recapitulated. Their application is extended from ordinary harmonic models to a more general category of *harmonious* field models, with emphasis on cases in which the field is confined to a string or higher brane world sheet, and for which the relevant internal symmetry group is non-Abelian, so that the conditions for conservation of the corresponding charge currents become rather delicate, particularly when the symmetry is gauged. Attention is also given to the conditions for conservation of currents of a different kind—representing surface fluxes of generalized momentum or energy—associated with symmetries not of the internal target space but of the underlying space-time background structure, including the metric and any relevant gauge field. For the corresponding current to be conserved the latter need not be manifestly invariant: preservation modulo a gauge adjustment will suffice. The simplest case is that of ''strong'' symmetry, meaning invariance under the action of an *effective* Lie derivative (an appropriately gauge adjusted modification of an ordinary Lie derivative). When the effective symmetry is of the more general ''weak'' kind, the kinetic part of the current is not conserved by itself but only after being supplemented by a suitable contribution from the background.

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I. INTRODUCTION

The nonlinearities most frequently encountered in classical field theories are broadly describable as being of three types, of which the most common is that of coupling nonlinearity, while the second and third types are those of kinetic nonlinearity and target nonlinearity. Following an approach initiated in two preceding articles [1,2], this article will deal with f nonlinearity of the *third* type, in which the fields under consideration take values in a target space that is not of the usual affine kind but curved.

Classical field theories of the most commonly considered kinds (including the familiar Yang Mills case) are kinematically linear: the only nonlinearity in their dynamic equations is not in the kinetic (meaning differential) part, but in the purely algebraic coupling contribution (which is commonly quartic in the Lagrangian and therefor of cubic order in the field equations).

However, even if the nonlinearity of the underlying theory is only of this *first* type, various confinement mechanisms lead to configurations that can be treated approximately, at a less fundamental level by models with fewer independent degrees of freedom, but with more general types of nonlinearity. An illustration of such a mechanism is provided by the prototype model set up by Witten [3] to demonstrate the possibility of conductivity in cosmic strings. This case furnishes an example in which the effect of confinement of the support zone of the field to the neighborhood of a string or higher brane world sheet is describable [4–6] by models of a kind [7–9] characterized by nonlinearity in the gradient terms. Nonlinearity of this *second* type has long been familiar in scalar field models of the standard kind used for the treatment of irrotational perfect fluids and superfluids [10,11] as characterized by a generalized pressure function that plays the role of the Lagrangian scalar in the present work. Such kinetic nonlinearity has also been invoked [12] in some more exotic scalar field theories recently introduced in a cosmological context.

The *third* kind of nonlinearity arises when, instead of the support zone, it is the values of the field that are effectively confined—as, for example, in a reduced model [13–16] due to the effect of steeply rising potential in an underlying model involving nonlinearity of only the first kind—so that the result will be describable in the manner exemplified by non linear sigma models [17–23] in which the (differential) kinetic part enters linearly, but in which it is the target space of allowed field values that is nonlinear in the sense that the relevant structure is no longer flat but curved.

The purpose of the present work is to extend the application of previously developed machinery [1,2] for dealing with nonlinearity of this *third* type in multiscalar field models for which—although lacking an integrable affine structure—the relevant target space, $\mathcal{X}$ say, will least at least be endowed with a local affine connection. More particularly, attention will be focussed here on the Riemannian case, for which the connection is derived from a metric, with components $\hat{g}_{AB}$ say, with respect to local coordinates X^A on the target space of field values, so the corresponding components of the connection will be

$$\hat{\Gamma}_A{}^B{}_C = \hat{g}^{BD}\Big(\hat{g}_{D(A,C)} - \frac{1}{2}\hat{g}_{AC,D}\Big) \tag{1}$$

using a comma to indicate partial differentiation with respect to the coordinates, and using round brackets to denote index symmetrization.

In the preceding work [1,2], the field Φ say, under consideration was a mapping

$$\Phi\colon \mathcal{M} \mapsto \mathcal{X} \tag{2}$$

from an n dimensional support space $\mathcal{M}$ endowed with its own metric and connection, with components $g_{\mu\nu}$ and

$$\Gamma_\mu{}^\nu{}_\rho = g^{\nu\sigma}\Big(g_{\sigma(\mu,\rho)} - \frac{1}{2}g_{\mu\rho,\sigma}\Big), \tag{3}$$

with respect to local base coordinates for x^μ, $\mu = 0, 1, \ldots n-1$. The idea was that in typical applications $\mathcal{M}$ would represent ordinary space-time, with $n = 4$, or perhaps the higher dimensional space-time of superstring theory, with $n = 10$.

As well as interest in models with even higher dimension, $n = 11$, more recent developments have been particularly concerned with the ubiquitous role of p-branes of various kinds, meaning subsystems confined to a supporting world sheet of dimension $d = p + 1$, starting with the case of a cosmic string, for which $p = 1$. In view of this development, the present work will be concerned with cases in which the support of the multiscalar field under consideration does not extend over the whole of $\mathcal{M}$ but is restricted to an embedded world sheet, $\mathcal{S}$ say.

After a recapitulation in Secs. II, III, IV, and V of the necessary machinery [1,2,24], it will first be applied in Sec. VI to a previously considered category [2] of "forced-harmonic" models that are kinetically linear, involving nonlinearity of the first type in a self coupling term, as well as nonlinearity of the third kind in a kinetic term of the harmonic kind. A category of "harmonious" brane supported models involving linearity of the second as well as the third (but not the first) type will then be introduced in Sec. VII, and the conservation of charge fluxes associated with internal symmetries therein will be studied in Sec. VIII. The final Secs. IX and X will be concerned with conservation of energy-momentum fluxes associated with underlying space-time background symmetries of various weak and strong kinds, the latter referring to invariance under the action of a gauge covariant modification of a Lie derivative.

II. THE BITENSORIAL FIELD GRADIENT

To distinguish quantities pertaining to the brane world sheet $\mathcal{S}$ from their analogues with respect to the background $\mathcal{M}$, we shall use an overline, as in the example of the induced metric, which is given with respect to local brane coordinates σ^i (for $i = 0, \ldots, p-1$) by

$$\bar{g}_{ij} = g_{\mu\nu}x^\mu{}_{,i}x^\nu{}_{,j}, \tag{4}$$

and which has a contravariant inverse, $\bar{g}^{ij}$, whose projection into the background provides the (first) *fundamental tensor* of the imbedding, [24,25], namely,

$$\eta^{\mu\nu} = \bar{g}^{ij}x^\mu{}_{,i}x^\nu{}_{,j}. \tag{5}$$

The preceding work [1,2] was concerned with a multicomponent scalar field Φ defined over $\mathcal{M}$ so that in terms of local coordinates X^A on the target space $\mathcal{X}$ its—generically nontensorial—components $X^A\{x\}$ will have tensorially transforming derivatives, expressible as

$$\Phi^A{}_\mu = \nabla_\mu X^A. \tag{6}$$

However, such a bitensorial gradient tensor will not always be well defined in the contexts to be considered the present work, which will be concerned with the case of a field $\bar{\Phi}$ having support confined to a lower dimensional world sheet $\mathcal{S}$, so that it will have components $X^A\{\sigma\}$ only for $\sigma \in \mathcal{S}$. This means that instead of (6) its gradient bitensor will have the more restricted form

$$\bar{\Phi}^A{}_\mu = \bar{\nabla}_\mu X^A \tag{7}$$

using the notation

$$\bar{\nabla}_\mu = \eta_\mu{}^\nu\nabla_\nu \tag{8}$$

for the relevant surface-tangential differentiation operator. In terms of the corresponding, surface gradient operator $\bar{\nabla}_i$—as defined in terms of the surface coordinates σ with respect to the induced metric $\bar{g}_{ij}$—the formula (7) is equivalently expressible in contravariant (meaning index raised) form as the projection

$$\bar{\Phi}^{A\mu} = x^\mu{}_{,i}\bar{\Phi}^{Ai}, \tag{9}$$

where, as the world sheet confined analogue of (6), the components

$$\bar{\Phi}^A{}_i = \bar{\nabla}_i X^A \tag{10}$$

are bitensorial in the sense of being tensorial both with respect to the target-space coordinates X^A and with respect to the world sheet coordinates σ^i.

III. GAUGE CONNECTION

If there is no symmetry group action on the target space, $\mathcal{X}$, then it is evident that there will be no ambiguity in the specification of the gradient bitensors as introduced above. However, in order to obtain a gradient operator that is well defined when the target space $\mathcal{X}$ is invariant under a differential action, it will be necessary to specify an appropriate gauge connection on the corresponding fiber bundle $\mathcal{B}$, in which each fiber has the form of the target space $\mathcal{X}$, and in which the field Φ will have the status of a section over the base space $\mathcal{M}$. For this purpose—as discussed in more detail in the preceding work [1]—the underlying background space $\mathcal{M}$ needs to be endowed, not just with its own metric $g_{\mu\nu}$, but also with a gauge form $\mathbf{A}_\mu$ having values in the Lie algebra $\mathcal{A}$ of the symmetry group of the fiber space $\mathcal{X}$.

The role of the gauge form—as represented by vector field components ${A_\mu}^A$ over $\mathcal{B}$—is to express the deviation of horizontality with respect to the local fiber coordinates X^A from horizontality with respect to the connection. This means that the effect of an infinitesimal fiber coordinate change $X^A \mapsto X^A + \delta X^A$ induced by a fiber displacement field $\hat{k}^A = -\delta X^A$ will be to map the connection form to a new value given with respect to the new coordinates by an affine transformation, ${A_\mu}^A \mapsto {A_\mu}^A + \delta[\hat{k}]{A_\mu}^A$, that will be given by

$$\delta[\hat{k}]{A_\mu}^A = \hat{k}^A{}_{,\mu} - \hat{k}^A{}_{,B}{A_\mu}^B. \tag{11}$$

As the effect of the displacement on the old connection component values will be given simply by ${A_\mu}^A \mapsto {A_\mu}^A + {A_\mu}^A{}_{,B}\delta X^A$, it can be seen that, with respect to a fixed coordinate system, the net gauge change,

$$\hat{\delta}[\hat{k}]{A_\mu}^A = \delta[\hat{k}]{A_\mu}^A - {A_\mu}^A{}_{,B}\delta X^B, \tag{12}$$

induced at a fixed position in the bundle $\mathcal{B}$ by the displacement $\hat{k}^A$ will be given by

$$\hat{\delta}[\hat{k}]{A_\mu}^A = \hat{k}^A{}_{,\mu} + [A_\mu, \hat{k}]^A, \tag{13}$$

where the square bracketed term denotes the Lie derivative of ${A_\mu}^A$ with respect to $\hat{k}^A$, namely,

$$[A_\mu, \hat{k}] = -[\hat{k}, A_\mu] = {A_\mu}^A{}_{,B}\hat{k}^B - \hat{k}^A{}_{,B}{A_\mu}^A. \tag{14}$$

This infinitesimal variation formula would be valid for an arbitrary fiber tangent vector field, but for preservation of the condition that ${A_\mu}^A$ should belong to the symmetry algebra it is to be understood that $\hat{k}^A$ should also be a symmetry generator, and therefore that it should be a solution of the target-space Killing equation

$$\hat{\nabla}^{(A}\hat{k}^{B)} = 0, \tag{15}$$

in which the round brackets indicate index symmetrization, where $\hat{\nabla}_A$ is the operator of covariant differentiation with respect to the metric $\hat{g}_{AB}$ and the corresponding connection (1) on $\mathcal{X}$.

The requirement that the gauge form should be a target-space symmetry generator means that its components will be expressible as

$$A_\mu{}^A = A_\mu{}^\alpha a_\alpha{}^A \tag{16}$$

in terms of a basis $\mathbf{a}_\alpha{}^A$ of the algebra, whose vector field components $a_\alpha{}^A$ on the target space are characterized themselves by the Killing equation

$$\hat{\nabla}^{(A}a_\alpha{}^{B)} = 0. \tag{17}$$

In terms of the commutators defined, according to the specification (14), as the Lie derivative of the first with respect to the second, the corresponding structure constants ${©_{\alpha\beta}}^\gamma$ will be determined by the relations

$$[\mathbf{a}_\alpha, \mathbf{a}_\beta] = ©_{\alpha\beta}{}^\gamma \mathbf{a}_\gamma. \tag{18}$$

The simplest nontrivial example is the case of a target space $\mathcal{X}$ that is a 2-sphere of radius $\hat{R}$ say, for which, in terms of standard coordinates $X^1 = \hat{\theta}$, $X^2 = \hat{\varphi}$, the metric components will be given by the familiar prescription $\hat{g}_{11} = \hat{R}^2$, $\hat{g}_{12} = 0$, $\hat{g}_{22} = \hat{R}^2\sin^2\hat{\theta}$. The Killing vectors of the associated standard basis for the (in this case three-dimensional) symmetry algebra will have components $a_\alpha{}^A$ that are given by $\{-\sin\hat{\varphi}, -\cot\hat{\theta}\cos\hat{\varphi}\}$ for $\alpha = \mathbf{1}$, by $\{\cos\hat{\varphi}, -\cot\hat{\theta}\sin\hat{\varphi}\}$ for $\alpha = \mathbf{2}$, and finally by $\{0, 1\}$ for $\alpha = \mathbf{3}$, from which it can be seen that the corresponding structure constants will be given simply by ${©_{\mathbf{2}\mathbf{3}}}^{\mathbf{1}} = {©_{\mathbf{3}\mathbf{1}}}^{\mathbf{2}} = {©_{\mathbf{1}\mathbf{2}}}^{\mathbf{3}} = 1$.

Subject to the understanding that the basis should be *uniform* with respect to the chosen coordinates, in the sense that its realization as a fiber tangent vector field satisfies the condition

$$a_\alpha{}^A{}_{,\mu} = 0, \tag{19}$$

the curvature two form $\mathbf{F}_{\mu\nu}$ of the gauge field will have basis components

$$F_{\mu\nu}{}^A = F_{\mu\nu}{}^\alpha a_\alpha{}^A \tag{20}$$

that are given quite generally by the familiar formula

$$F_{\mu\nu}{}^\alpha = 2\partial_{[\mu}A_{\nu]}{}^\alpha + ©_{\beta\gamma}{}^\alpha A_\mu{}^\beta A_\nu{}^\gamma, \tag{21}$$

which means [1] that its representation as a fiber space Killing vector field will be given directly by

$$F_{\mu\nu}{}^A = 2A_{[\nu}{}^A{}_{,\mu]} + 2A_{[\nu}{}^B A_{\mu]}{}^A{}_{,B}. \tag{22}$$

When subject to a gauge change of the form (13) this curvature form transforms according to the simple rule

$$\hat{\delta}[\hat{k}]F_{\mu\nu}{}^A = [F_{\mu\nu}, \hat{k}]^A, \tag{23}$$

while (as a consequence of the Jacobi commutator identity) its antisymmetrized (exterior type) derivative will satisfy the Bianchi identity

$$F_{[\mu\nu,\rho]}{}^A + [A_{[\rho}, F_{\mu\nu]}]^A = 0. \tag{24}$$

As well as its induced metric $\bar{g}_{ij}$, the brane world sheet $\mathcal{S}$ will evidently inherit a corresponding induced gauge field with components

$$\bar{A}_i{}^A = \bar{A}_i{}^\alpha a_\alpha{}^A \tag{25}$$

given by

$$\bar{A}_i{}^\alpha = A_\mu{}^\alpha x^\mu{}_{,i}. \tag{26}$$

The associated curvature two form on the world sheet will have components

$$\bar{F}_{ij}{}^{\alpha} = 2\partial_{[i}\bar{A}_{j]}{}^{\alpha} + \copyright_{\beta\gamma}{}^{\alpha}\bar{A}_i{}^{\beta}\bar{A}_j{}^{\gamma}, \tag{27}$$

that are equivalently obtainable by the pullback formula

$$\bar{F}_{ij}{}^{\alpha} = F_{\mu\nu}{}^{\alpha}x^{\mu}{}_{,i}x^{\nu}{}_{,j}. \tag{28}$$

IV. EFFECTIVE GRADIENTS IN BUNDLE

The introduction of a coordinate independent notion of horizontality via the specification of the connection form $A_\mu{}^A$ in the fiber bundle $\mathcal{B}$ enable us to reduce the degree of dependence on the fiber coordinates X^A that is involved in partial derivation with respect to the base coordinates x^μ by subtracting off the part of the gradient that is merely attributable to the associated gauge adjustment. We thereby obtain the corresponding *effective gradient* operator, which will be denoted by a curly $\mathcal{D}$ symbol, or more compactly by a curly vertical bar $\wr$ before the relevant index, in the manner illustrated as follows in the case of an ordinary fiber tangent vector field with components k^A, for which the effective gradient components

$$\hat{k}^A{}_{\wr\mu} = \mathcal{D}_\mu\hat{k}^A \tag{29}$$

will be defined by the prescription

$$\mathcal{D}_\mu\hat{k}^A = \hat{k}^A{}_{,\mu} - \hat{\delta}[A_\mu]\hat{k}^A. \tag{30}$$

The gauge adjustment term here will simply be minus the Lie derivative of $\hat{k}^A$ with respect to the fiber tangent vector A_μ, which means that, using the notation scheme introduced in (14), it will be given by

$$\hat{\delta}[A_\mu]\hat{k}^A = [\hat{k}, A_\mu]^A, \tag{31}$$

so that the result will be expressible according to (13) as

$$\mathcal{D}_\mu\hat{k}^A = \hat{\delta}[\hat{k}]A_\mu{}^A. \tag{32}$$

A noteworthy application of the forgoing formula is to the gauge curvature, for which the definition

$$\mathcal{D}_\rho F_{\mu\nu}{}^A = F_{\mu\nu}{}^A{}_{,\rho} - \hat{\delta}[A_\mu]F_{\mu\nu}{}^A \tag{33}$$

can be seen by (23) to give

$$F_{\mu\nu\wr\rho}{}^A = F_{\mu\nu}{}^A{}_{,\rho} + [A_\rho, F_{\mu\nu}]^A, \tag{34}$$

from which it can be seen that the Bianchi identity (23) will be expressible in this terminology simply as

$$F_{[\mu\nu\wr\rho]}{}^A = 0. \tag{35}$$

A more remarkable application of this formalism is to the case of the gauge form itself, for which the defining prescription,

$$\mathcal{D}_\mu A_\nu{}^A = A_\nu{}^A{}_{,\mu} - \hat{\delta}[A_\mu]A_\nu{}^A, \tag{36}$$

is to be evaluated using the formula (13), which gives

$$\hat{\delta}[A_\mu]A_\nu{}^A = A_\mu{}^A{}_{,\nu} + [A_\nu, A_\mu]^A. \tag{37}$$

As the outcome, we obtain the memorable but not so well-known theorem to the effect that the gauge curvature is simply the effective gradient of the gauge form, which is automatically antisymmetric:

$$F_{\mu\nu}{}^A = A_\nu{}^A{}_{\wr\mu} = -A_\mu{}^A{}_{\wr\nu}. \tag{38}$$

Bearing in mind the convention (19), it can be seen that the foregoing concept of effective differentiation can be taken over directly into terms of basis indices, so that we obtain

$$\mathcal{D}_\mu A_\nu{}^\alpha = F_{\mu\nu}{}^\alpha \tag{39}$$

and

$$\mathcal{D}_\rho F_{\mu\nu}{}^\alpha = \partial_\rho F_{\mu\nu}{}^\alpha + A_\nu{}^\beta\copyright_{\beta\gamma}{}^\alpha F_{\mu\nu}{}^\gamma. \tag{40}$$

The Bianchi identity is thereby expressible as

$$\mathcal{D}_{[\rho}F_{\mu\nu]}{}^\alpha = 0. \tag{41}$$

V. GAUGE COVARIANT BITENSORIAL DERIVATIVES

Whereas Secs. III and IV were mainly considered with fields (such as the fiber space symmetry generator with components $\hat{k}^A$) that were defined throughout at least an open neighborhood of the bundle $\mathcal{B}$, we shall now concentrate rather on fields (such as the basis components $\hat{k}^\alpha$) that are defined just over the relevant base space $\mathcal{M}$ (or over a world sheet $\mathcal{S}$ therein). In particular, we shall be concerned with base space supported fields that are obtained as the restriction of bundle supported fields to some particular bundle section as specified by the target value of a multi-scalar field mapping of the form $\Phi: \{x^\mu\} \mapsto \{X^A\}$ (or in the world sheet case $\bar{\Phi}: \{\sigma^i\} \mapsto \{X^A\}$) for which it is necessary to distinguish the net gradient operator, for which we shall use the symbol ∂, from the corresponding operator of partial derivation with respect to the bundle coordinates, for which we use a comma, in the manner illustrated for the fiber tangent vector field $\hat{k}$ by the relation

$$\partial_\mu\hat{k}^A = \hat{k}^A{}_{,\mu} + \hat{k}^A{}_{,B}\nabla_\mu X^B. \tag{42}$$

Proceeding in the same spirit as in the preceding section, it is useful—for reducing the degree of fiber coordinate dependence—to replace such a base space gradient operator by an effective gradient operator, from which the corresponding gauge adjustment has been subtracted off, so that it measures the deviation from horizontality with respect to the connection. Using the notation

$$D_\mu\Phi^A = \Phi^A{}_{|\mu} \tag{43}$$

for the ensuing gauge covariant derivative of the section Φ itself, the definition

$$D_\mu \Phi^A = \nabla_\mu \Phi^A - \delta[A_\mu]\Phi^A \qquad (44)$$

in which the first term is the bitensorial quantity $\nabla_\mu \Phi^A = \partial_\mu \Phi^A$ and the second term is given simply by

$$\delta[A_\mu]\Phi^A = -A_\mu{}^A, \qquad (45)$$

the effective gradient of the section is obtained, using the notation (6), simply as

$$\Phi^A{}_{|\mu} = \Phi^A{}_\mu + A_\mu{}^A. \qquad (46)$$

For the analogous case of a field $\bar{\Phi}$ with support confined to the brane world sheet $\mathcal{S}$, the surface gauge form (25) gives the correspondingly restricted covariant derivative

$$\bar{D}_i \bar{\Phi}^A = \bar{\Phi}^A{}_{|i} \qquad (47)$$

in the analogous form

$$\bar{\Phi}^A{}_{|i} = \bar{\Phi}^A{}_i + \bar{A}_i{}^A, \qquad (48)$$

which is equivalently obtainable as the pullback

$$\bar{\Phi}^A{}_{|i} = x^\mu{}_{,i} \bar{D}_\mu \bar{\Phi}^A = x^\mu{}_{,i} \Phi^A{}_{|\mu}, \qquad (49)$$

where

$$\bar{D}_\mu \bar{\Phi}^A = \eta_\mu{}^\nu D_\nu \bar{\Phi}^A. \qquad (50)$$

When the concept of gauge covariant differentiation is extended from the scalar field Φ to the vector field $\hat{k}$ on the section, it is necessary to include an extra term to take account of the fiber connection $\hat{\Gamma}$, so the ensuing covariant derivative takes the form

$$D_\mu \hat{k}^A = \partial_\mu \hat{k}^A + \hat{\Gamma}_C{}^A{}_B \Phi^C{}_{|\mu} \hat{k}^B - \delta[A_\mu]\hat{k}^A, \qquad (51)$$

in which the relevant gauge adjustment has the simple tensorial form

$$\delta[A_\mu]\hat{k}^A = -A_\mu{}^A{}_{,B} \hat{k}^B. \qquad (52)$$

(This formula for the adjustment of $\hat{k}$ by A_μ is to be contrasted with the formula (11) for the nontensorial but affine adjustment of A_μ by $\hat{k}$.)

In the strictly Riemannnian case (meaning absence of torsion) to which this work and its immediate predecessor [2] is restricted, the outcome of the forgoing prescription can be conveniently expressed in the form

$$D_\mu \hat{k}^A = \nabla_\mu \hat{k}^A + \hat{k}^B \hat{\nabla}_B A_\mu{}^A, \qquad (53)$$

in which the part involving the connection has been separated out in the second term, while the first term is given by the ordinarily covariant (not gauge covariant) differentiation operation

$$\nabla_\mu \hat{k}^A = \partial_\mu \hat{k}^A + \Gamma_\nu{}^A{}_B \hat{k}^B, \qquad (54)$$

with

$$\Gamma_\nu{}^A{}_B = \Phi^C{}_\nu \hat{\Gamma}_C{}^A{}_B. \qquad (55)$$

If $\hat{k}^A$ is defined not just on the section Φ but throughout an open neighborhood on the bundle $\mathcal{B}$—as was supposed in Sec. IV—then it can be seen that the outcome of the prescription (51) will also be expressible in terms of the effective gradient (30) as

$$D_\mu \hat{k}^A = \mathcal{D}_\mu \hat{k}^A + \Phi^B{}_{|\mu} \hat{\nabla}_B \hat{k}^A. \qquad (56)$$

However, that may be—whether or not the vector $\hat{k}^A$ extends to a bundle field off the section—the gauge covariant derivative will always be expressible in the form

$$D_\mu \hat{k}^A = \partial_\mu \hat{k}^A + \omega_\mu{}^A{}_B \hat{k}^B \qquad (57)$$

using the new connector field

$$\omega_\mu{}^A{}_B = \Phi^C{}_{|\mu} \hat{\Gamma}_C{}^A{}_B + A_\mu{}^A{}_{,B} \qquad (58)$$

that was introduced in the preceding work [1,2].

This connector field is to be used for the construction [1,2] of gauge covariant bitensorial derivatives in the manner illustrated by the case of the second gauge covariant derivative of the field Φ, namely,

$$D_\nu \Phi^A{}_{|\mu} = \Phi^A{}_{|\mu|\nu} \qquad (59)$$

by the formula

$$\Phi^A{}_{|\mu|\nu} = \Phi^A{}_{|\mu;\nu} + \omega_\nu{}^A{}_B \Phi^B{}_{|\mu}, \qquad (60)$$

in which a semicolon is used to indicate covariant derivation of the ordinary kind, as given in terms of the background space connection by an expression of the familiar form

$$\Phi^A{}_{|\mu;\nu} = \partial_\nu \Phi^A{}_{|\mu} - \Gamma_\nu{}^\rho{}_\mu \Phi^A{}_{|\rho}. \qquad (61)$$

In this case, there is no analogue of (56), because $\Phi^A{}_{|\mu}$ is well defined only on the section Φ, but, as the analogue of (53), it is of course possible to rewrite (60) in the alternative form

$$D_\nu \Phi^A{}_{|\mu} = \nabla_\nu \Phi^A{}_{|\mu} + \Phi^B{}_{|\mu} \hat{\nabla}_B A_\nu{}^A, \qquad (62)$$

where the operation of bitensorially covariant (but not gauge covariant) differentiation is specified as

$$\nabla_\nu \Phi^A{}_{|\mu} = \Phi^A{}_{|\mu;\nu} + \Gamma_\nu{}^A{}_B \Phi^B{}_{|\mu}. \qquad (63)$$

When acting on the tangentially projected field

$$\bar{\Phi}^A{}_{|\mu} = \eta_\mu{}^\nu D_\nu \Phi^A \qquad (64)$$

on a brane world sheet $\mathcal{S}$, one must take care to distinguish its tangential derivative, with components given, using the notation introduced in (10), by

$$\bar{\nabla}_\nu \bar{\Phi}^A{}_{|\mu} = \eta_\nu{}^\rho \nabla_\rho \bar{\Phi}^A{}_{|\mu} \qquad (65)$$

from its tangentially projected derivative with components

$$\overline{\nabla_\nu \Phi^A{}_{|\mu}} = \eta_\mu{}^\sigma \eta_\nu{}^\rho \nabla_\rho \bar{\Phi}^A{}_{|\sigma}, \tag{66}$$

which will be the same only if the embedding is flat. In general one must allow for the gradient of the first fundamental tensor, which will be given by the formula

$$\bar{\nabla}_\mu \eta_\nu{}^\rho = K_{\mu\nu}{}^\rho + K_\mu{}^\rho{}_\nu \tag{67}$$

in which the second fundamental tensor of the world sheet is defined [24,25] as

$$K_{\mu\nu}{}^\rho = \eta_\nu{}^\sigma \bar{\nabla}_\mu \eta_\sigma{}^\rho. \tag{68}$$

In view of its symmetry and projection properties, namely,

$$K_{[\mu\nu]}{}^\rho = 0, \qquad K_{\mu\nu}{}^\sigma \eta_\sigma{}^\rho = 0, \qquad K_{\sigma\nu}{}^\rho \perp^\sigma{}_\mu = 0, \tag{69}$$

where the orthogonal projection tensor is given by

$$\perp^\nu{}_\mu = g^\nu{}_\mu - \eta^\nu{}_\mu, \tag{70}$$

it can be seen that the difference between (65) and (66) will be given by

$$\overline{\nabla_\nu \Phi^A{}_{|\mu}} = \bar{\nabla}_\nu \bar{\Phi}^A{}_{|\mu} - K_\nu{}^\rho{}_\mu \bar{\Phi}^A{}_{|\rho}. \tag{71}$$

This distinction does not matter for the pullback onto the world sheet, which will be given by

$$\bar{\nabla}_j \bar{\Phi}^A{}_{|i} = x^\mu{}_{,i} x^\nu{}_{,j} \bar{\nabla}_\nu \bar{\Phi}^A{}_{|\mu} = x^\mu{}_{,i} x^\nu{}_{,j} \overline{\nabla_\nu \Phi^A{}_{|\mu}}, \tag{72}$$

in agreement with what is obtained directly from the analogue of (63), namely,

$$\bar{\nabla}_j \bar{\Phi}^A{}_{|i} = \bar{\Phi}^A{}_{|i;j} + \bar{\Gamma}_j{}^A{}_B \bar{\Phi}^B{}_{|i}, \tag{73}$$

with

$$\bar{\Gamma}_j{}^A{}_B = \bar{\Phi}^C{}_j \hat{\Gamma}_C{}^A{}_B. \tag{74}$$

The same considerations apply to the corresponding fully gauge covariant derivative as given by

$$\bar{D}_\nu \bar{\Phi}^A{}_{|\mu} = \eta_\nu{}^\rho D_\rho \bar{\Phi}^A{}_{|\mu} = \bar{\nabla}_\nu \bar{\Phi}^A{}_{|\mu} + \eta_\nu{}^\rho \bar{\Phi}^B{}_{|\mu} \hat{\nabla}_B A_\rho{}^A, \tag{75}$$

and its tangental projection

$$\overline{D_\nu \Phi^A{}_{|\mu}} = \eta_\mu{}^\sigma \bar{D}_\nu \bar{\Phi}^A{}_{|\sigma} = \bar{D}_\nu \bar{\Phi}^A{}_{|\mu} - K_\nu{}^\rho{}_\mu \bar{\Phi}^A{}_{|\rho}, \tag{76}$$

whose pullback

$$\bar{D}_j \bar{\Phi}^A{}_{|i} = x^\mu{}_{,i} x^\nu{}_{,j} \bar{D}_\nu \bar{\Phi}^A{}_{|\mu} = x^\mu{}_{,i} x^\nu{}_{,j} \overline{D_\nu \Phi^A{}_{|\mu}} \tag{77}$$

agrees with what is obtained directly from the analogue of (60), namely,

$$\bar{D}_j \bar{\Phi}^A{}_{|i} = \partial_j \bar{\Phi}^A{}_{|i} - \bar{\Gamma}_j{}^k{}_i \bar{\Phi}^A{}_{|k} + \bar{\omega}_j{}^A{}_B \bar{\Phi}^B{}_{|i}, \tag{78}$$

in which the induced connection on the world sheet is given by the usual Christoffel formula,

$$\bar{\Gamma}_i{}^k{}_j = \bar{g}^{jh}\left(\bar{g}_{h(i,k)} - \frac{1}{2}\bar{g}_{ik,h}\right), \tag{79}$$

and the world sheet connection form for the fiber space is given by

$$\bar{\omega}_i{}^A{}_B = x^\mu{}_{,i} \omega_\mu{}^A{}_B = \bar{\Phi}^C{}_{|i} \hat{\Gamma}_C{}^A{}_B + \bar{A}_i{}^A{}_B. \tag{80}$$

VI. CONSERVED CURRENTS FOR FORCED-HARMONIC FIELDS

Before going on to the investigation of more general cases, let us consider the conservation of charge currents associated with internal symmetries in the prototype application of the foregoing formalism, as presented in the preceding article [2]. That application was to a class of models that includes the ordinary harmonic type, but that is generalized by allowance for two kinds of force, of which the first is an internal bias provided in the action by the inclusion of a scalar self-coupling term, which can partially or completely break the symmetry—if any—of the target space. The other kind is an external force from gauge coupling of whatever target-space symmetry may remain unbroken.

To be explicit, it is to be recapitulated that (in the absence of background weighting fields) such a *biased-harmonic* system is characterized by an action integral of the form

$$\mathcal{I} = \int L\{\Phi, D\Phi\} \|g\|^{1/2} d^n x \tag{81}$$

over the base space $\mathcal{M}$, in which the Lagrangian scalar function L is taken to be a quadratic function of the gradients of the field section Φ, with the gauge invariant form

$$L = -\frac{1}{2}\Phi_A{}^{|\mu}\Phi^A{}_{|\mu} - \hat{\mathcal{V}}\{\Phi\}, \tag{82}$$

where, like the metric $\hat{g}_{AB}$ that is used for target-space index lowering, , the potential $\hat{\mathcal{V}}$ is given as a fixed field on the space $\mathcal{X}$ in which the values of Φ are located. Nonlinear σ models belong to the special category for which the structure of the target-space is homogeneous, not just geometrically (as in the spherical example mentioned above) but also for the algebraic potential function $\hat{\mathcal{V}}$ which in that case must be just a constant that can be ignored as far as the field equations are concerned. Quite generally, the field $\hat{\mathcal{V}}$ must be invariant under the action of the generators $\mathbf{a}_\alpha$ of the relevant symmetry algebra (if any) which as well as satisfying the Killing Eqs. (17) must also satisfy the conditions

$$a_\alpha{}^A \hat{\mathcal{V}}_{,A} = 0. \tag{83}$$

Whether or not any such symmetry algebra exists, the requirement that the integral $\mathcal{I}$ be unaffected by infinitesimal local variations of the field Φ can be seen [2] to give field equations of the form

$$\Phi_A{}^{|\mu}{}_{|\mu} = \hat{\mathcal{V}}_{,A}. \tag{84}$$

For any symmetry algebra element $\hat{\mathbf{k}}$ say, with fiber space Killing vector components

$$\hat{k}^A = \hat{k}^\alpha a_\alpha{}^A, \tag{85}$$

one can construct a corresponding current vector with components

$$J^\mu = \hat{k}^A \Phi_A{}^\mu = \hat{k}^\alpha J_\alpha{}^\mu, \qquad J_\alpha{}^\mu = a_\alpha{}^A \Phi_A{}^\mu, \tag{86}$$

whose divergence,

$$J^\mu{}_{;\mu} = J^\mu{}_{|\mu} \tag{87}$$

can easily be evaluated using the field Eqs. (84). Using these in conjunction with (83) and the Killing condition (17), the current divergence can be seen to be given by

$$J^\mu{}_{;\mu} = \Phi_A{}^\mu(\hat{k}^A{}_{,\mu} - \hat{k}^A{}_{,B} A_\mu{}^B + \hat{k}^B A_\mu{}^A{}_{,B}). \tag{88}$$

It follows that in order for the current to be conserved,

$$J^\mu{}_{;\mu} = 0, \tag{89}$$

the variation of the symmetry generator $\hat{\mathbf{k}}$ over the base space $\mathcal{M}$ must be restricted to satisfy the condition

$$\hat{k}^A{}_{,\mu} = [\hat{k}, A_\mu]^A, \tag{90}$$

of which the right-hand side is the Lie derivative of the symmetry generator $\hat{\mathbf{k}}$ with respect to the gauge form $\mathbf{A}_\mu$. This condition can be seen to be interpretable as the obviously natural requirement that the Killing vector $\hat{\mathbf{k}}$ should be transported horizontally with respect to the gauge connection, or equivalently as the requirement that it should preserve the gauge field in the sense that associated gauge adjustment (13) should vanish,

$$\hat{\delta}[\hat{k}] A_\mu{}^A = 0. \tag{91}$$

In cases for which the relevant bundle structure is that of a trivial direct product, for which there is a preferred gauge in which $\mathbf{A}_\mu = 0$, this horizontality requirement will be achievable in the obvious way, by simply taking $\hat{\mathbf{k}}$ to be *uniform* over $\mathcal{M}$, so that $\hat{k}^A{}_{,\mu} = 0$ in that gauge. However, in general Eq. (90) will be soluble only if the gauge field is such as to satisfy an integrability condition, which can be seen to be expressible in terms of the gauge curvature two-form $\mathbf{F}_{\mu\nu}$ as

$$[\hat{k}, F_{\mu\nu}]^A = 0, \tag{92}$$

or equivalently, by (23), as

$$\hat{\delta}[\hat{k}] F_{\mu\nu}{}^A = 0. \tag{93}$$

What this means is that—as could have been anticipated—in order for the corresponding current (86) to be conserved, $\hat{\mathbf{k}}$ must generate a symmetry not just of the fiber metric $\hat{g}_{AB}$, and of the scalar potential function $\hat{\mathcal{V}}$, but also of the gauge field $\mathbf{F}_{\mu\nu}$.

If the symmetry group is Abelian—as in the familiar case of ordinary Maxwellian electromagnetism—the requirement (92) will evidently entail no further restriction, so that for any generator $\hat{\mathbf{k}}$ chosen uniformly over $\mathcal{M}$—meaning such that $\hat{k}^A{}_{,\mu} = 0$—the corresponding current (86) will automatically satisfy the conservation law (89), as it does even in the non-Abelian case if the bundle $\mathcal{B}$ has a trivial direct product structure $\mathcal{B} = \mathcal{M} \times \mathcal{X}$ characterized by a preferred gauge in which the connection form vanishes.

VII. MODELS WITH HARMONIOUS FIELDS ON BRANES

Let us now consider cases involving a field $\bar{\Phi}$ that has support restricted to a brane world sheet $\mathcal{S}$ of dimension $d = p + 1$ say, so that as the analogue of (81) the relevant action integral is given by an expression of the form

$$\bar{\mathcal{I}} = \int_{\mathcal{S}} \bar{L}\{\bar{\Phi}, \overline{D\Phi}\} \|\bar{g}\|^{1/2} d^d\sigma. \tag{94}$$

As well as the allowance for gauge coupling, the kind of Lagrangian considered in the preceding section generalized the usual harmonic kind [26] by including the nonlinearity of the first type embodied in the algebraic self coupling term $\hat{\mathcal{V}}$ in (82). However, in the present section we shall consider a generalization of a different kind that will be referred to as *harmonious*, involving nonlinearity of the second—meaning kinetic—type as well as the nonlinearity of the third type that is embodied in the curvature of the target space $\mathcal{X}$. Specifically, we shall use this term for cases for which the surface Lagrangian depends only on the target-space metric $\hat{g}_{AB}$ and the symmetric target-space tensor defined as

$$\hat{\mathfrak{w}}^{AB} = \eta^{\mu\nu} \hat{\Phi}^A{}_{|\mu} \bar{\Phi}^B{}_{|\nu} = \bar{\Phi}^A{}_{|i} \bar{\Phi}^{B|i}. \tag{95}$$

In a gauge such that the gauge form vanishes at a particular point under consideration, this tensor $\hat{\mathfrak{w}}^{AB}$ will be identifiable simply as the induced metric on the target space $\bar{\mathcal{X}}$. In the absence of a gauge field, a harmonious model will therefor be of the of the ordinarily elastic type in cases for which the target space is of dimension p, and thus identifiable as the quotient with respect to a congruence of timelike idealized particle worldlines on the world sheet. However, it will not be an elastic model of the most general kind, for which [27,28] the specification of the elastic structure on $\bar{\mathcal{X}}$ would involve, not just the metric $\hat{g}_{AB}$, but other predetermined vectorial or tensorial fields as well.

For a model that is harmonious in the forgoing sense, the generic variation of the Lagrangian will have the form

$$\delta\bar{L} = \frac{\partial\bar{L}}{\partial\hat{g}_{AB}}\delta\hat{g}_{AB} + \frac{\partial\bar{L}}{\partial\hat{\mathfrak{w}}^{AB}}\delta\hat{\mathfrak{w}}^{AB}, \qquad (96)$$

in which the coefficients will be related by a Noether identity [24] that is obtainable by considering the effect of an arbitrary displacement field, $\hat{\xi}^A$ say, in the target space. Since the ensuing variation $\delta\bar{L} = \hat{\xi}^A\hat{\nabla}_A L$ will be equivalently determined by corresponding variations of the form $\delta\hat{g}_{AB} = \hat{\mathcal{L}}[\hat{\xi}]\hat{g}_{AB}$ and $\delta\hat{\mathfrak{w}}^{AB} = \hat{\mathcal{L}}[\hat{\xi}]\hat{\mathfrak{w}}^{AB}$ in which the relevant Lie derivatives are $\hat{\mathcal{L}}[\hat{\xi}]\hat{g}_{AB} = 2\hat{\nabla}_{(A}\hat{\xi}_{B)}$ and $\hat{\mathcal{L}}[\hat{\xi}]\hat{\mathfrak{w}}^{AB} = \hat{\xi}^C\hat{\nabla}_C\hat{\mathfrak{w}}^{AB} - 2\hat{\mathfrak{w}}^{C(A}\hat{\nabla}_C\hat{\xi}^{B)}$, the identity

$$\hat{\xi}^C\Big(\hat{\nabla}_C L - \frac{\partial\bar{L}}{\partial\hat{\mathfrak{w}}^{AB}}\hat{\nabla}_C\hat{\mathfrak{w}}^{AB}\Big) = 2\Big(\frac{\partial\bar{L}}{\partial\hat{g}_{CD}} - \frac{\partial\bar{L}}{\partial\hat{\mathfrak{w}}^{AB}}\hat{\mathfrak{w}}^{AC}\hat{g}^{BD}\Big) \times \hat{\nabla}_C\hat{\xi}_D$$

must hold. Since (at a given target-space position) the vector $\hat{\xi}^C$ and the gradient components $\hat{\nabla}_C\hat{\xi}_D$ can be chosen arbitrarily and independently, the coefficients on the left and right hand sides must vanish separately, so one obtains two distinct Noether identities which are expressible as

$$\hat{\nabla}_C L = \frac{\partial\bar{L}}{\partial\hat{\mathfrak{w}}^{AB}}\hat{\nabla}_C\hat{\mathfrak{w}}^{AB}, \qquad \frac{\partial\bar{L}}{\partial\hat{g}_{AC}}\hat{g}_{AB} = \frac{\partial\bar{L}}{\partial\hat{\mathfrak{w}}^{AB}}\hat{\mathfrak{w}}^{BC}.$$

Using the notation

$$\kappa_{AB} = \kappa_{BA} = -2\frac{\partial\bar{L}}{\partial\hat{\mathfrak{w}}^{AB}}, \qquad (97)$$

the Noether identities enable us to derive the symmetry property

$$\kappa_C{}^A\hat{\mathfrak{w}}^{BC} = \kappa_C{}^B\hat{\mathfrak{w}}^{AC} = -2\frac{\partial\bar{L}}{\partial\hat{g}_{AB}}, \qquad (98)$$

and to express the generic variation (96) in the compact form

$$\delta\bar{L} = -\frac{1}{2}\kappa_A{}^B\delta\hat{\mathfrak{w}}_B{}^A. \qquad (99)$$

The category of harmonious models defined in this way will evidently include ordinary harmonic models [26], which belong to the special subcategory for which $\bar{L}$ is *linearly* dependent just on the scalar trace

$$\hat{\mathfrak{w}} = \hat{\mathfrak{w}}_A\hat{\mathfrak{w}}^A = \hat{g}_{AB}\hat{\mathfrak{w}}^{AB}. \qquad (100)$$

A rather more extended but still relatively simple subcategory that is of special interest in various physical contexts consists of models for which the dependence of the Lagrangian on $\hat{\mathfrak{w}}_B{}^A$ is just quadratic, so that it will be expressible in terms of fixed parameters m, $\kappa_\star$, $\alpha_\star$ $\beta_\star$ in the form

$$\bar{L} = -m - \frac{1}{2}\kappa_\star\hat{\mathfrak{w}} - \frac{1}{4}\alpha_\star\hat{\mathfrak{w}}^2 + \frac{1}{4}\beta_\star\hat{\mathfrak{w}}_A{}^B\hat{\mathfrak{w}}_B{}^A,$$

which gives

$$\kappa_{AB} = (\kappa_\star + \alpha_\star\hat{\mathfrak{w}})g_{AB} - \beta_\star\hat{\mathfrak{w}}_{AB}.$$

The symmetry condition (98) is thereby made manifest in the expression

$$\kappa_C{}^A\hat{\mathfrak{w}}^{BC} = (\kappa_\star + \alpha_\star\hat{\mathfrak{w}})\hat{\mathfrak{w}}^{AB} - \beta_\star\hat{\mathfrak{w}}_C{}^A\hat{\mathfrak{w}}^{BC}.$$

Within this quadratic subcategory the harmonic case is evidently obtained by setting $\alpha_\star = 0$ and $\beta_\star = 0$. The less restrictive condition $\alpha_\star = \beta_\star$ characterizes other harmonious but not harmonic cases having a particular physical interest, of which the simplest non trivial example is that of what is known as a baby Skyrme model [29], for which the target space is just a 2-sphere.

A motivation for considering cases of more general kinds, starting with that of what will be referred to as *simply harmonious* models, namely, those for which $\bar{L}$ has arbitrary nonlinear dependence just on $\hat{\mathfrak{w}}$ (as exemplified by the quadratic subcase with $\beta_\star = 0$) is that they can arise naturally—for an underlying model with a kinetic part of the ordinary linear type—from the effect of confining mechanisms of the kind commonly considered in the theory of topological defects.

A prototypical example [4–6] is provided by the bosonic field model proposed by Witten [3], as a simple example of the way currents can be confined to the world sheets of cosmic strings. Such arise from spontaneous symmetry breaking by string or higher $(d-1)$-brane type solutions that are longitudinally symmetric in the strong sense—meaning that the relevant fields are preserved by the action of the Killing vector generators k_i^μ say of longitudinal (world sheet parallel) translations. Spontaneous symmetry breaking means that the solutions is not unique but belongs to a family of configurations mapped onto each other by the action of the relevant internal symmetry group $\mathcal{G}$. It is this family of configurations—as labeled by the central value $\bar{\Phi}$ of the field Φ—that forms the (typically curved) target space $\bar{\mathcal{X}}$ of the world-sheet confined effective model. The Lagrangian action for the effective model is obtained for such (current free) configurations simply by integrating the local action density over a transverse section of dimension $(n-d)$.

The general idea is that, starting from such a family of strongly symmetric nonconducting configurations, a more extensive family of current carrying configurations will be obtainable by relaxing the condition that the fields be strictly invariant under the action of longitudinal translations but allowing them to have changes generated by elements of the algebra $\mathcal{A}$ say of the symmetry. The effective action for current carrying states is to be obtained by integrating the result obtained by solving the field equations on a particular transverse $(n-d)$ dimensional section with values of the gradients in the longitudinal directions orthogonal to the section given by the action of the corresponding algebra elements as represented by corresponding central values of the longitudinal gradients $D_i\Phi$ with values in the tangent space of $\mathcal{X}$.

In simple cases such as that of the string configurations (with $d = 2$) obtained [4–6] from the Abelian model introduced by Witten, the result will depend only on the trace $\hat{\mathfrak{w}}$, albeit non linearly, (contrary to the oversimplified ansatz originally proposed by Witten himself [3]). The result will still depend only on the trace $\hat{\mathfrak{w}}$ in the non-Abelian case obtained from the minimal extension of the Witten model that will be presented in a following article [30] in which it will be shown that this extension will give rise to current carrying strings supporting fields for which the target space $\bar{\mathcal{X}}$ will have the 2-spherical form enviseaged in Sec. III.

VIII. CONSERVED CURRENTS FOR HARMONIOUS FIELDS ON BRANES

Whatever its physical origin may be, a Lagrangian of the harmonious kind under consideration will have an Eulerian (fixed point) variation that will be given, according to (96), by

$$\delta\bar{L} = -\frac{1}{2}\kappa_D{}^C\mathfrak{w}^{BD}\hat{g}_{BC,A}\delta X^A - \kappa_{AB}\bar{\Phi}^{B|i}\delta(X^A{}_{,i} + \bar{A}_i{}^A), \tag{101}$$

which is expressible in the convenient form

$$\delta L = \frac{\delta\bar{L}}{\delta X^A}\delta X^A - (\kappa_{AB}\bar{\Phi}^{B|i}\delta X^A)_{;i}, \tag{102}$$

with

$$\frac{\delta\bar{L}}{\delta X^A} = (\kappa_{AB}\bar{\Phi}^{B|i})_{|i}. \tag{103}$$

It evidently follows that, in terms of bitensorial surface current components defined by

$$\bar{J}_A{}^i = \kappa_{AB}\bar{\Phi}^{B|i}, \tag{104}$$

the ensuing field equations will be expressible in the neatly succinct form

$$\bar{J}_A{}^i{}_{|i} = 0. \tag{105}$$

However, it is to be observed that this will not in general be directly interpretable as a current conservation law, because (unlike the last term in (102), which is removable by integration over the base space) the left hand side of (105) is not an ordinary divergence. Nevertheless, as before, when there is an internal symmetry group we can obtain something that actually is an ordinary divergence and that will vanish under appropriate conditions, by using the fact that any fiber space symmetry generating vector field with components $\hat{k}^A$ will define a corresponding surface charge current with components

$$\bar{J}^{\,\nu} = x^\nu{}_{,i}\bar{J}^i, \qquad \bar{J}^i = \hat{k}^A\bar{J}_A{}^i \tag{106}$$

whose surface divergence will be given—when the field equations are satisfied—by

$$\bar{\nabla}_\nu\bar{J}^\nu = \bar{J}^i{}_{;i} = \bar{J}_A{}^i\hat{k}^A{}_{|i}, \tag{107}$$

where by definition (51) we have

$$\hat{k}^A{}_{|i} = \hat{k}^A{}_{,i} + \hat{k}^A{}_{,B}X^B{}_{,i} + \hat{k}^B(\bar{\Phi}^C{}_{|i}\hat{\Gamma}_C{}^A{}_B + \bar{A}_i{}^A{}_{,B}). \tag{108}$$

By rewriting the latter in the form

$$\hat{k}^A{}_{|i} = \hat{k}^A{}_{,i} + \hat{k}^B\bar{A}_i{}^A{}_{,B} - \hat{k}^A{}_{,B}\bar{A}_i{}^B + \bar{\Phi}^B{}_{|i}\hat{\nabla}_B\hat{k}^A, \tag{109}$$

and using the symmetry property (98), one can see, as before, that provided the fiber tangent vector field is chosen so as to satisfy the target-space Killing Eq. (15), as well as the surface analogue of the horizontal transport condition (90), namely,

$$\hat{k}^A{}_{,i} = [\hat{k}, \bar{A}_i]^A, \tag{110}$$

we shall finally obtain a genuine surface current conservation law of the required kind, namely,

$$\bar{J}^i{}_{;i} = 0. \tag{111}$$

In the absence of any gauge field, this can always be done for any element of the target-space symmetry algebra. However, as before, if a gauge field is present, the condition (110) for (111) can only be fulfilled if the relevant integrability condition is satisfied, namely, the requirement

$$[\hat{k}, \bar{F}_{ij}]^A = 0, \tag{112}$$

which is interpretable as the condition that $\hat{k}$ should generate a symmetry not just of the fiber metric $\hat{g}$ but also of the gauge field.

IX. ENERGY-MOMENTUM FLUX ON BRANES

The preceding section was concerned just with variations of the Lagrangian integral (94) for *fixed* values of the world sheet location (and hence of the projection bitensor with components $x^\mu{}_{,i}$ and of the relevant background fields, namely, the n-dimensional space-time metric with components $g_{\mu\nu}$ and the gauge field as specified according to (16) in terms of components $A_\mu{}^\alpha$ with respect to some uniform algebra basis. Within that scheme, it can be seen that, when the ensuing variational field equations are satisfied, the divergences of the corresponding basis current vectors with components

$$\bar{J}^{\,\nu}_\alpha = x^\nu{}_{,i}a_\alpha{}^A\bar{J}_A{}^i \tag{113}$$

will be given, in accordance with (107), by

$$\bar{\nabla}_\nu\bar{J}^\nu_\alpha = A_\nu{}^\beta\copyright_{\beta\alpha}{}^\gamma\bar{J}^\nu_\gamma. \tag{114}$$

In the present section—generalizing an approach developed for the Abelian case in preceding work [8,24]—we shall consider the effect of background field variations of the form induced by *world sheet displacements*, as generated by Lie transport with respect to a vector field ξ^μ, which gives

$$\delta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}, \tag{115}$$

and

$$\delta A_\nu{}^\alpha = \xi^\mu \nabla_\mu A_\nu{}^\alpha + A_\mu{}^\alpha \nabla_\nu \xi^\mu. \tag{116}$$

It can be seen that effect of this on the integrand in (94) will be given by an expression of the form

$$\|\bar g\|^{-1/2}\delta(\bar L\|\bar g\|^{1/2}) = \frac{1}{2}\bar T^{\mu\nu}\delta g_{\mu\nu} - \bar J^\nu_\alpha \delta A_\nu{}^\alpha, \tag{117}$$

in which it can be seen that the relevant surface current coefficients will be as given by the formula (113), which can be written more explicitly as

$$\bar J^{\,\nu}_{\,\alpha} = a_\alpha{}^A \kappa_{AB}\bar\Phi^{B|\nu}, \tag{118}$$

while the corresponding surface stress energy-momentum tensor components can be read out as

$$\bar T^{\,\mu\nu} = \kappa_{AB}\bar\Phi^{A|\mu}\bar\Phi^{B|\nu} + \bar L\eta^{\mu\nu}, \tag{119}$$

in which it is to be recalled that κ_{AB} will simply be proportional to the fiber metric $\hat g_{AB}$ in the ordinary harmonic case, but that it will in general depend also on $\mathfrak{w}_{AB}$, as defined by (95).

Up to this point we have been treating the world sheet location as something given in advance, but we shall now postulate that its motion is governed by dynamical equations of the usual variational type, meaning that the action (94) is required to be preserved, not just by infinitesimal variations of the multiscalar surface field $\bar\Phi$, but also by the arbitrary infinitesimal displacements generated by ξ^μ. The contribution of the latter to the action variation can be seen—using preceding Lie transport equations—to be obtainable from (117) in the form

$$\begin{aligned}\|\bar g\|^{-1/2}\delta(\bar L\|\bar g\|^{1/2}) = {}& \bar\nabla_\nu(\xi^\mu(\bar T_\mu{}^\nu - A_\mu{}^\alpha \bar J^\nu_\alpha)) \\ & - \xi^\mu(\bar\nabla_\nu \bar T_\mu{}^\nu - A_\mu{}^\alpha\bar\nabla_\nu \bar J^\nu_\alpha \\ & - 2\bar J^\nu_\alpha \nabla_{[\nu}A^\alpha_{\mu]})\end{aligned} \tag{120}$$

of which the first part is a surface divergence that is removable by integration. Thus, when the internal field Eqs. (105) for $\bar\Phi$ are satisfied, the only remaining contribution to the action variation will be the last, namely, the contraction with ξ^μ whose coefficient must therefore vanish. We thus obtain a dynamical equation of the form

$$\bar\nabla_\nu \bar T_\mu{}^\nu - A_\mu{}^\alpha\bar\nabla_\nu \bar J^\nu_\alpha - 2\bar J^\nu_\alpha\nabla_{[\nu}A^\alpha_{\mu]} = 0. \tag{121}$$

This can be conveniently rewritten in the standard form

$$\bar\nabla_\nu \bar T_\mu{}^\nu = f_\mu \tag{122}$$

in which the force density f_μ is a well-behaved (algebra basis independent) covector that can be seen from (114) to be expressible, using the definition (21), in the form

$$f_\mu = \bar J^\nu_\alpha F_{\nu\mu}{}^\alpha = \bar J_A{}^\nu F_{\nu\mu}{}^A. \tag{123}$$

This expression generalizes the formula that is already familiar in the ordinary electromagnetic case, for which the gauge algebra is that of a U(1) action on the unit circle. Subject to the usual understanding that the latter is parametrized by the angle coordinate $X^1 = \varphi$, our previous treatment of this Maxwellian case [8,24] can be expressed in terms of the formalism used here by setting $A_\mu{}^1 = -eA_\mu$ so that $F_{\mu\nu}{}^1 = -eF_{\mu\nu}$ and $\bar j^\mu = -e\bar J^\nu_1$, where e is the relevant charge coupling constant. (In typical applications using unrationalized Planck units, the latter will be taken to be given approximately by $e = 1/\sqrt{137}$, while the presence of the negative sign is attributable to the unfortunate but historically entrenched convention that for ordinary electrons the electromagnetic current direction is *opposite* to that of the particles themselves).

As in the familiar Abelian case [8,24], it is to be noticed that the tangentially projected part of the force Eq. (122) provides no new information, being merely an automatic consequence of the internal field Eqs. (105) on the world sheet, whereas the orthogonally projected part provides the extra information needed to determine the evolution of the string world sheet, whose equation of motion is thereby obtained in the standard form

$$\bar T^{\,\nu\rho}K_{\nu\rho}{}^\mu = \perp^{\mu\nu} f_\nu. \tag{124}$$

As was done for the charge currents considered in the preceding section, we can again specify a current that may be conserved by contracting the relevant free index with symmetry generating vector field, but this time not on the target space but on the base $\mathcal{M}$, where the relevant Killing equation for preservation of the metric $g_{\mu\nu}$ by the vector field k^μ in question takes the form

$$\nabla^{(\mu}k^{\nu)} = 0. \tag{125}$$

The corresponding current,

$$\bar\Pi^{\,\mu} = k^\nu \bar T_\nu{}^\mu, \tag{126}$$

will be interpretable as a flux of momentum when k^μ is the generator of a spacelike translation, while corresponding to a flux of energy in the timelike case for which (with the sign convention used here) $k^\mu k_\mu$ is negative. It evidently follows from (122) that its surface divergence will be given by

$$\bar\nabla_\nu \bar\Pi^{\,\nu} = k^\mu f_\mu, \tag{128}$$

and thus that it will be conserved,

$$\bar{\nabla}_\nu \bar{\Pi}^\nu = 0, \tag{127}$$

when the force does no work, which by (123) will be the case if and only if the gauge field is such that

$$k^\mu F_{\mu\nu}{}^\alpha \bar{J}_\alpha{}^\nu = 0. \tag{129}$$

It is to be remarked that this requirement will always be satisfied if the current (and hence also the world sheet in which it is contained) happens to be entirely aligned with the Killing vector,

$$k^{[\mu} \bar{J}_\alpha{}^{\nu]} = 0, \tag{130}$$

a condition that is describable as *staticity* in the case for which the Killing vector is timelike so that the ensuing conservation law is that of an energy flux. It is evident that the requirement (129) will also hold if, instead of the current, it is the gauge field itself that has the property describable, if the Killing vector is timelike, as staticity, meaning vanishing of its "electric" (as opposed to "magnetic") part, namely,

$$k^\mu F_{\mu\nu}{}^\alpha = 0. \tag{131}$$

X. WEAK, EFFECTIVE, STRICT, AND STRONG SYMMETRIES

A field over the base space $\mathcal{M}$ is describable as *manifestly symmetric* [31] with respect to the continuous transformation group generated by a vector field with components k^μ on $\mathcal{M}$ if is invariant under the corresponding Lie transport operation, that is to say if it is mapped to zero by the corresponding Lie differentiation operator $\mathcal{L}[k]$, which will be given for the section Φ simply by

$$\mathcal{L}[k]\Phi^A = k^\mu \Phi^A_{,\mu}. \tag{132}$$

For the relevant independent background fields, namely, the metric and the basis components of the gauge field, it will be given by

$$\mathcal{L}[k] g_{\mu\nu} = 2\nabla_{(\mu} k_{\nu)}, \tag{133}$$

and

$$\mathcal{L}[k] A_\mu{}^\alpha = k^\nu A_\mu{}^\alpha{}_{,\nu} + k^\nu{}_{,\mu} A_\nu{}^\alpha, \tag{134}$$

while for the basis components of the gauge curvature it will be given by

$$\mathcal{L}[k] F_{\mu\nu}{}^\alpha = k^\rho F_{\mu\nu}{}^\alpha{}_{,\rho} + 2k^\rho{}_{,[\nu} F_{\mu]\rho}{}^\alpha. \tag{135}$$

The apparent variation measured in this way is however highly gauge dependent. A more meaningful measure of actual physical variation is obtainable—as for the bitensorially gauge covariant differentation procedure described above—by subtracting off the relevant gauge adjustment as generated by the corresponding fiber tangent field, with components $\hat{k}_\mu{}^A = \hat{k}_\mu{}^\alpha a_\alpha{}^A$ given by

$$\hat{k}^\alpha = k^\mu A^\alpha_\mu. \tag{136}$$

This provides what we shall refer to as the *effective Lie derivative*, which we shall distinguish from its ordinary analogue by use of the financial euro symbol in place of the traditional Libra symbol according to the prescription

$$\mathbf{C}[k] = \mathcal{L}[k] - \hat{\delta}[\hat{k}]. \tag{137}$$

The required gauge adjustment operator $\hat{\delta}[\hat{k}]$ will be given for the section and the metric simply by

$$\hat{\delta}[\hat{k}]\Phi^A = -\hat{k}^A, \qquad \hat{\delta}[\hat{k}] g_{\mu\nu} = 0, \tag{138}$$

so for the latter there is no difference between ordinary and effective Lie differentiation while for the section, as the analogue of (132), in the notation of (43) we simply get

$$\mathbf{C}[k]\Phi^A = k^\mu \Phi^A{}_{|\mu}, \tag{139}$$

For the gauge field, according to (13), we have the less trivial adjustment

$$\hat{\delta}[\hat{k}] A_\mu{}^\alpha = \hat{k}^\alpha{}_{,\mu} + A_\mu{}^\beta ©_{\beta\gamma}{}^\alpha \hat{k}^\gamma, \tag{140}$$

which leads however to the neat and memorable result

$$\mathbf{C}[k] A_\mu{}^\alpha = k^\nu \mathcal{D}_\nu A_\mu{}^\alpha = k^\nu F_{\nu\mu}{}^\alpha, \tag{141}$$

while for the gauge curvature we have

$$\delta[\hat{k}] F_{\mu\nu}{}^\alpha = F_{\mu\nu}{}^\beta © {}_{\beta\gamma}{}^\alpha \hat{k}^\gamma, \tag{142}$$

which leads, via the Bianchi identity (41), to

$$\mathbf{C}[k] F_{\mu\nu}{}^\alpha = 2\mathcal{D}_{[\nu}(k^\rho F_{\mu]\rho}). \tag{143}$$

Just as a field configuration may be said to be manifestly symmetric, with respect to a displacement generator k^μ, if the corresponding Lie derivative vanishes, the configuration will be similarly describable as *strongly symmetric* with respect to k^μ if the corresponding *effective* Lie derivative is zero. However, it will be describable as merely *weakly symmetric* if this effective Lie derivative does not vanishes absolutely, but only modulo the action of some internal symmetry generator with base components V^α say, or equivalently if the *ordinary* Lie derivative vanishes modulo the action of the difference $V^\alpha - \hat{k}^\alpha$, with $\hat{k}^\alpha$ as defined by (136). It is to be remarked that manifest symmetry need only be of the weak kind when a nonintegrable gauge field is present, but that it will be of the strong kind when such a field is absent.

When applied to something as simple as a scalar section Φ, the weak symmetry condition,

$$\mathbf{C}[k]\Phi^A + \hat{\delta}[V]\Phi^A = 0, \tag{144}$$

can be seen, from the formula $\hat{\delta}[V]\Phi^A = -V^A$, to reduce to an equation of the form

$$k^\nu \Phi^A{}_{|\nu} = V^A. \tag{145}$$

However, this will entail no restriction at all if the symmetry group is transitive over the target space (as, for example, when the latter is spherically symmetric) as it will be trivially soluble for V^A as a space-time position dependent target-space Killing vector on the section.

A more meaningful condition that may appropriately be described as *strict* symmetry is that of a weak symmetry for which the relevant adjustment is restricted to be such as to preserve the connection. In other words a configuration will be describable as *strictly symmetric* with respect to k^μ if the effect on it of the corresponding *effective* Lie derivative can be cancelled by the action of some internal symmetry generator with base components V^α such that $\hat\delta[V]A_\mu{}^\alpha$ vanishes, which, according to (13), means that it must satisfy the horizontal transport equation

$$\partial_\mu V^\alpha + A_\mu{}^\beta \copyright_{\beta\gamma}{}^\alpha V^\gamma = 0, \tag{146}$$

which, as discussed in Sec. VI, will be integrable only if the curvature satisfies the corresponding condition

$$F_{\mu\nu}{}^\beta \copyright_{\beta\gamma}{}^\alpha V^\gamma = 0. \tag{147}$$

A less restrictive but still meaningful condition that may be described as *effective symmetry* is obtained by relaxing the foregoing condition of horizontal transport in all directions to that of horizontal transport just in the direction of the Killing vector. In other words a configuration will be describable as *effectively symmetric* with respect to k^μ if the effect on it of the corresponding *effective* Lie derivative can be cancelled by the action of some internal symmetry generator that is itself *strongly* symmetric, meaning that its base components V^α satisfy the requirement

$$\mathbf{C}[k]V^\alpha = 0 \tag{148}$$

in which it is to be recalled that, by definition, we shall have

$$\mathbf{C}[k]V^\alpha = k^\nu(\partial_\nu V^\alpha + A_\mu{}^\beta \copyright_{\beta\gamma}{}^\alpha V\gamma) = k^\nu \hat\delta[V]A_\nu{}^\alpha. \tag{149}$$

In the particular case of the section Φ it is to be remarked that effective symmetry in the foregoing sense is equivalent to the postulate of strong symmetry of its gauge covariant derivative $\Phi^A{}_{|\nu}$.

Various kinds of symmetry in the categories defined above were studied in work by Forgacs and Manton [32], albeit with limited generality, in that these authors considered only target-space symmetries that were "gauged" in the sense that the physical presence of a nonintegrable connection field was admitted by the theoretical model under consideration, whereas for strict symmetry of the most general kind [31] a target-space symmetry that is not in the gauged subalgebra but merely "global" will also be perfectly acceptable.

The most important application of these successively more restrictive notions of weak, effective, strict, and strong symmetry is of course to the gauge field itself. In this particular case the distinction between strict and strong symmetry disappears, as the former condition will automatically entail the latter, namely,

$$\mathbf{C}[k]A_\nu{}^\alpha = 0. \tag{150}$$

It can be seen from (141) that this strong symmetry condition is equivalent to the sufficient condition (131) for the generalized surface momentum flux conservation property (128). This sufficient condition for conservation of the current characterized by k^μ is thus interpretable as the requirement that, as well as satisfying the Killing Eq. (125), this vector field should generate a strong symmetry of the gauge field.

In the case of the gauge field (unlike that of the section Φ) symmetry of even the weak type has non trivial consequences. The meaning of weak symmetry for the gauge field is the possibility of constructing what is describable as a generalized voltage field, consisting of some fiber space symmetry generator, with basis components V^α such that

$$\mathbf{C}[k]A_\nu{}^\alpha + \hat\delta[V]A_\nu{}^\alpha = 0. \tag{151}$$

As a necessary integrability condition for this, it can be seen that a weak symmetry condition of the same form with the same voltage field V^α must also be satisfied by the gauge curvature, for which we thus obtain the requirement

$$\mathbf{C}[k]F_{\mu\nu}{}^\alpha = V^\beta \copyright_{\beta\gamma}{}^\alpha F_{\mu\nu}{}^\gamma. \tag{152}$$

The weak symmetry condition (151) can evidently be rewritten in the form

$$\mathcal{L}[k]A_\nu{}^\alpha = \hat\delta[\hat k - V]A_\nu{}^\alpha, \tag{153}$$

which makes it apparent how, as remarked above, manifest symmetry is interpretable as the special case of weak symmetry for which V^α is equal to $\hat k^\alpha$ as given by (136), whereas strong symmetry is the special case for which the relevant voltage field V^α simply vanishes.

By writing out the condition (151) of weak symmetry of the gauge field in the explicit form

$$k^\mu F_{\mu\nu}{}^\alpha + V^\alpha{}_{,\nu} + A_\nu{}^\beta \copyright_{\beta\gamma}{}^\alpha V^\gamma = 0, \tag{154}$$

it can be seen to imply that the surface current divergence condition (127) will be expressible as

$$\bar\nabla_\nu \bar\Pi^\nu = -(V^\alpha{}_{,\nu} + A_\nu{}^\beta \copyright_{\beta\gamma}{}^\alpha V^\gamma)\bar J_\alpha{}^\nu. \tag{155}$$

Under these circumstances it can be seen from the generally valid current divergence formula (114) that we shall obtain a strict surface current conservation law, of the form

$$\bar\nabla_\nu \bar{\mathcal{P}}^\nu = 0, \tag{156}$$

by setting

$$\bar{\mathcal{P}}^\nu = \bar\Pi^\nu + V^\alpha \bar J^\nu_\alpha, \tag{157}$$

in which both $\bar{\Pi}^{\nu}$ and V^{α} depend on the choice of the Killing field k^{ν}. In the case for which this Killing vector is a time translation generator, so that the current $\bar{\Pi}^{\nu}$ will be interpretable as a flux of kinetic energy, the extra term $V^{\alpha}\bar{J}^{\nu}_{\alpha}$ in (157) will be interpretable as a corresponding flux of potential energy, while the voltage field V^{α} can be seen to be the natural non-Abelian generalization of an ordinary electrostatic potential field in Maxwellian electromagnetism. In the special case for which the section itself satisfies the weak symmetry condition (145), this conserved total energy flux will simply be $\bar{\mathcal{P}}^{\nu} = -\bar{L}k^{\nu}$, and if the symmetry thus generated by k^{ν} is not merely weak but *strict* the right-hand side of (155) will vanish, which means that both the kinetic contribution $-\bar{\Pi}^{\nu}$ and the potential contribution $V^{\alpha}\bar{J}^{\nu}_{\alpha}$ will be separately conserved.

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