Gluon condensate, modified gravity, and the accelerating Universe

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It has been suggested recently to study the dynamics of a gravitating gluon condensate q in the context of a spatially flat Friedmann-Robertson-Walker universe. The expansion of the Universe (or, more generally, the presence of a nonvanishing Ricci curvature scalar R) perturbs the gluon condensate and may induce a nonanalytic term $\tilde{h}(R, q)$ in the effective gravitational action. The aim of this article is to explore the cosmological implications of a particular nonanalytic term $\tilde{h} \propto \eta |R|^{1/2} |q|^{3/4}$. With a quadratic approximation of the gravitating gluon-condensate vacuum energy density $\rho_V(q)$ near the equilibrium value q_0 and a small coupling constant η of the modified-gravity term $\tilde{h}$, an "accelerating universe" is obtained which resembles the present Universe, both qualitatively and quantitatively. The unknown component X of this model universe (here, primarily due to modified-gravity effects) has an effective equation-of-state parameter $\bar{w}_X$ which is found to evolve toward the value -1 from above.

I. INTRODUCTION

The fundamental theory of the strong interactions is nowadays taken to be quantum chromodynamics (QCD); see, e.g., Refs. [1,2] and other references therein. In the framework of this theory, there is evidence for the existence of a gluon condensate [3–6]. The question, then, is how the gluon condensate gravitates and evolves as the Universe expands. Here, a tentative answer is obtained by use of the so-called q-theory approach for the gravitational effects of vacuum energy density [7–10].

The outline of this article is as follows. In Sec. II, an example of a gluon-condensate-induced modification of gravity is presented and the corresponding field equations are derived, which are then reduced for the case of a spatially flat Friedmann-Robertson-Walker universe. In Sec. III, the resulting evolution of a simple three-component model universe is studied both analytically and numerically, in order to establish whether or not a model universe can be obtained which resembles the observed "accelerating Universe" [11,12]. In Sec. IV, concluding remarks are presented.

II. QCD-SCALE MODIFIED GRAVITY AND COSMOLOGY

A. Theory: Action and field equations

It has been argued [10] that, in a de Sitter universe with Hubble constant H, a QCD-scale vacuum energy density $\rho_V \sim |H| \Lambda_{QCD}^3$ could arise from infrared effects of the gluon propagator. Since the de Sitter universe has Ricci curvature scalar $|R| \sim H^2$ and the particular gluon condensate q has energy scale $q \sim \Lambda_{QCD}^4$, one is led to consider the following modified-gravity action ($\hbar = c = 1$):

$$S_{\text{eff}} = \int_{\mathbb{R}^4} d^4x \sqrt{-g} [K\tilde{f}(R, q) + \epsilon(q) + \mathcal{L}_M(\psi)], \quad (2.1a)$$

$$\tilde{f} \equiv R + \tilde{h} \equiv R + \eta K^{-1} |R|^{1/2} |q|^{3/4}, \quad (2.1b)$$

with gravitational coupling constant $K \equiv (16\pi G)^{-1} > 0$, dimensionless coupling constant $\eta > 0$ (standard general relativity has $\eta = 0$), energy density $\epsilon(q)$ of the gluon condensate $q(x)$, and matter field $\psi(x)$ (later on, this single matter component will be generalized to N matter components). The precise definition of the gluon-condensate variable $q(x)$ in the context of QCD has been given in Ref. [10], to which the reader is referred for details. In the following, q is simply assumed to be nonzero and is, in fact, taken to be positive. The relation between the gravitational constant G and Newton's constant G_N [13,14] will be discussed in Sec. III B. Throughout, the conventions of Ref. [15] are used, in particular, those for the Riemann tensor and the metric signature $(-+++)$.

The field equations from (2.1) are fourth order and it is worthwhile to switch to the scalar-tensor formulation which has field equations of second order. The equivalent Jordan-frame Brans-Dicke theory [15–18] has action

$$S_{\text{eff}}^{(BD)} = \int_{\mathbb{R}^4} d^4x \sqrt{-g} [K(\phi R - U(\phi, q)) + \epsilon(q) + \mathcal{L}_M(\psi)],$$
$$(2.2a)$$

$$U \equiv -(1/4)(\eta^2/K^2)|q|^{3/2}/(1 - \phi), \quad (2.2b)$$

in terms of a dimensionless scalar field ϕ restricted to values less than 1 (ϕ would be greater than 1 for the $\eta < 0$ case not considered here). The ϕ dependence of potential (2.2b) allows for the so-called chameleon effect [19], which will be briefly discussed at the end of this section.[1]

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[1]See also Ref. [20] for chameleon-type effects in a different context and Ref. [21] for recent analytic and numerical work on the scalar profiles from compact objects, extending the original analysis of Ref. [19].

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The proof of the classical equivalence of the actions (2.1) and (2.2), for $\eta \neq 0$ and $q \neq 0$, is not affected by the presence of the q field in the function $\tilde{f}$ of (2.1b). See, e.g., Refs. [22–24] for details of the proof, which is straightforward and need not be repeated here. Anyway, the classical equivalence of (2.1) and (2.2) can be verified directly by eliminating ϕ from (2.2a), using its field equation $R = \partial U / \partial \phi$ with $U(\phi)$ given by (2.2b).

At this moment, two remarks may be helpful to place the theory considered in context. First, the rigorous microscopic derivation of the effective action (2.1) remains a major outstanding problem, because only a rough argument has been given in the appendix of Ref. [10], where η was called f (see also Ref. [25] for a general discussion of the physics involved and [26] for a heuristic argument). Awaiting this derivation, the main motivation of (2.1) is that it naturally gives the correct order of magnitude for the present vacuum energy density (see Ref. [10] and Sec. IV). Just to be crystal clear: the term $\tilde{h}$ in (2.1b) is, at present, purely hypothetical and the aim of this article is to explore its cosmological consequences, leaving aside its theoretical derivation.

Second, the effective action (2.1) is only considered to be valid on cosmological length scales and additional non-standard terms in $\tilde{f}(R, q)$ can be expected to be operative at smaller length scales, relevant to solar-system tests and laboratory experiments [22,23]. Purely phenomenologically, the $\tilde{h}$ term in (2.1b) could, for example, be replaced by an extended term

$$\tilde{h}_{\text{ext}} = \eta K^{-1} |q|^{9/4} |R|^{1/2} / (|q|^{3/2} + \zeta K^2 |R|), \quad (2.3)$$

with constants $0 < \eta \ll |\zeta| \lesssim 1$. This term $\tilde{h}_{\text{ext}}$ vanishes as $|R|^{-1/2}$ at large enough curvatures and, for $\eta \sim 10^{-3}$ and $|\zeta| \sim 1$, is consistent with the relevant bound in Ref. [23] based on the Eöt-Wash laboratory experiment [27].

Returning to the action (2.2), the field equations are obtained from the variational principle for variations $\delta g_{\mu\nu}$ of the metric $g_{\mu\nu}$, variations $\delta \phi$ of the Brans-Dicke field ϕ, and variations δA of the microscopic field A responsible for q condensate (see, in particular, Refs. [8,10]). Specifically, the field equations are

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -\frac{1}{2\phi K} (T_M^{\mu\nu} - \tilde{\epsilon} g^{\mu\nu}) - \frac{1}{2\phi} \tilde{U} g^{\mu\nu}$$
$$- \frac{1}{\phi} (\nabla^\mu \nabla^\nu - g^{\mu\nu} \Box) \phi, \quad (2.4a)$$

$$R = \frac{\partial U}{\partial \phi}, \quad (2.4b)$$

$$\frac{\partial \epsilon}{\partial q} - K \frac{\partial U}{\partial q} = \mu, \quad (2.4c)$$

with the covariant derivative ∇_μ, the invariant d'Alembertian $\Box \equiv \nabla^\nu \nabla_\nu$, the energy-momentum tensor $T_M^{\mu\nu}$ of the matter field ψ, the integration constant μ, and

the effective energy densities

$$\tilde{\epsilon} \equiv \epsilon - q \frac{\partial \epsilon}{\partial q}, \quad (2.5a)$$

$$\tilde{U} \equiv U - q \frac{\partial U}{\partial q}. \quad (2.5b)$$

Two comments are in order. First, the reason of having the extra term $-q \partial \epsilon / \partial q$ in (2.5a) and $-q \partial U / \partial q$ in (2.5b) is the fact that the field q is not fundamental but contains, in addition to the microscopic field A mentioned above, the inverse metric $g^{\mu\nu}$ (see Sec. II of Ref. [10]). Second, the constant μ on the right-hand side of (2.4c) can be interpreted, for spacetime-independent q and $dU/dq = 0$, as the chemical potential corresponding to the conserved charge q (see, in particular, the detailed discussion in Secs. II A and B of Ref. [7]).

For completeness, also the generalized Klein-Gordon equation is given, which is obtained by taking the trace of (2.4a) and using (2.4b):

$$\Box \phi = \frac{1}{6K} (T_M - 4\tilde{\epsilon}) + \frac{2}{3} \tilde{U} - \frac{1}{3} \phi \frac{\partial U}{\partial \phi}, \quad (2.6)$$

with the matter energy-momentum trace $T_M \equiv T_M^{\mu\nu} g_{\mu\nu}$.

Eliminating $q \partial U / \partial q$ from (2.4a) and (2.4c), the final field equations are

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -\frac{1}{2\phi K} (T_M^{\mu\nu} - \rho_V g^{\mu\nu}) - \frac{1}{2\phi} U g^{\mu\nu}$$
$$- \frac{1}{\phi} (\nabla^\mu \nabla^\nu - g^{\mu\nu} \Box) \phi, \quad (2.7a)$$

$$R = \frac{\partial U}{\partial \phi}, \quad (2.7b)$$

$$\frac{\partial \rho_V}{\partial q} = K \frac{\partial U}{\partial q}, \quad (2.7c)$$

in terms of the gravitating vacuum energy density

$$\rho_V(q) \equiv \epsilon(q) - \mu q, \quad (2.8)$$

with the integration constant μ. Equally, the generalized Klein-Gordon equation (2.6) becomes

$$\Box \phi = \frac{1}{6K} (T_M - 4\rho_V) + \frac{2}{3} U - \frac{1}{3} \phi \frac{\partial U}{\partial \phi}, \quad (2.9)$$

where the very last term on the right-hand side, in particular, is relevant to the previously mentioned chameleon effect. With (2.7b), this last term of (2.9) becomes $(-R/3)\phi$ and corresponds to an effective mass square term for the scalar field, with a mass square of the order of ρ_M / K for the case of a pressureless perfect fluid. This is indeed one aspect of the chameleon effect, namely, an effective mass value dependent on the environment [19].

B. Differential equations for a flat FRW universe

For a spatially flat $(k = 0)$ Friedmann-Robertson-Walker (FRW) universe [15] with scale factor $a(\tau)$ and matter described by a perfect fluid, the 00 and 11 components of the generalized Einstein field equation (2.7a) can be combined to give a generalized Friedmann equation. Together with equations obtained directly from (2.7b) and (2.9), the relevant equations are then

$$H^2 \phi = \frac{1}{6K} \rho_{\rm tot} - \frac{1}{6} U - H\dot{\phi}, \tag{2.10a}$$

$$\dot{H} = -2H^2 - \frac{1}{6} \frac{\partial U}{\partial \phi}, \tag{2.10b}$$

$$\ddot{\phi} = -3H\dot{\phi} + \frac{1}{6K}(\rho_{\rm tot} - 3P_{\rm tot}) - \frac{2}{3} U + \frac{1}{3} \phi \frac{\partial U}{\partial \phi}, \tag{2.10c}$$

with the overdot standing for the derivative with respect to τ (the somewhat unusual notation τ is used for the dimensionful cosmic time, in order to reserve the letter t for the dimensionless time later on). The total energy density and pressure are given by

$$\rho_{\rm tot} \equiv \rho_V + \rho_M, \qquad P_{\rm tot} \equiv P_V + P_M, \tag{2.11a}$$

for the gravitating vacuum energy density

$$\rho_V(q) = -P_V(q) = \epsilon(q) - \mu q, \tag{2.11b}$$

as discussed in the previous section. Observe that (2.10a) reproduces the standard Friedmann equation for $U = 0$, $\phi = 1$, and $K \equiv (16\pi G)^{-1} = (16\pi G_N)^{-1} \equiv K_N$.

The last two equations in (2.10) are, respectively, first- and second-order ordinary differential equations (ODEs) for H and ϕ. Two further ODEs can be obtained as follows. First, multiplying (2.7c) by $\dot{q}$ gives an equation for the time dependence of the vacuum energy density,

$$\dot{\rho}_V = K\left(\dot{U} - \dot{\phi} \frac{\partial U}{\partial \phi}\right), \tag{2.12a}$$

which describes the energy exchange between the vacuum and the nonstandard gravitational field $(U \neq 0)$. Second, the standard energy conservation of matter gives

$$\dot{\rho}_M = -3H(\rho_M + P_M) = -3H(1 + w_M)\rho_M, \tag{2.12b}$$

where the matter equation-of-state (EOS) parameter $w_M \equiv P_M/\rho_M$ has been introduced (henceforth, w_M will be assumed to be time independent). Equation (2.12b) implies that, for the theory considered, there is no energy exchange between vacuum and matter (such an energy exchange for a different version of q-theory has been studied in Ref. [28]).

C. Dimensionless variables and ODEs

Now rewrite the cosmological equations in appropriate microscopic units. The gluon condensate q from Refs. [3,10] has the dimension of energy density, $[q] = [\epsilon]$, which implies that the corresponding integration constant μ is dimensionless, $[\mu] = [1]$. The equilibrium value q_0 of the gluon-condensate variable q is taken to be determined by a laboratory experiment in an environment with negligible spacetime curvature and has the order of magnitude $q_0 \equiv E_{\rm QCD}^4 = {\rm O}(10^9 \ {\rm eV}^4)$; see Sec. III C for further remarks. From this moment on, consider N matter components, labeled by an index $n = 1, \ldots, N$.

Specifically, the following dimensionless variables t, h, f, r, u, and s can be introduced:

$$\tau \equiv tK/q_0^{3/4}, \qquad H(\tau) \equiv h(t)q_0^{3/4}/K, \tag{2.13a}$$

$$q(\tau) \equiv f(t)q_0, \qquad \rho(\tau) \equiv r(t)q_0^{3/2}/K, \tag{2.13b}$$

$$U(\tau) \equiv u(t)q_0^{3/2}/K^2, \qquad \phi(\tau) \equiv s(t). \tag{2.13c}$$

Observe that all dimensionless quantities are denoted by lowercase Latin letters. A further rescaling $t = t'/\eta$ and $h = h'\eta$ will not be used in the present article, as the effects from the unknown coupling constant η are preferred to be kept as explicit as possible.

It is, then, straightforward to obtain the dimensionless versions of the algebraic equation (2.7c), the last two ODEs in (2.10), and the matter conservation equation (2.12b) generalized to N matter components. This gives a closed system of $4 + N$ equations for the $4 + N$ dimensionless variables $f(t)$, $h(t)$, $s(t)$, $v(t)$, and $r_{M,n}(t)$. Specifically, this system of equations consists of a single algebraic equation,

$$\frac{\partial r_V(f)}{\partial f} = \frac{\partial u(s, f)}{\partial f}, \tag{2.14}$$

and $3 + N$ ODEs,

$$\dot{h} = -2h^2 - \frac{1}{6} \frac{\partial u}{\partial s}, \tag{2.15a}$$

$$\dot{s} = v, \tag{2.15b}$$

$$\dot{v} = \frac{1}{6}(r_{\rm tot} - 3p_{\rm tot}) - 3hv - \frac{2}{3} u + \frac{1}{3} s \frac{\partial u}{\partial s}, \tag{2.15c}$$

$$\dot{r}_{M,n} = -3h(1 + w_{M,n})r_{M,n}, \tag{2.15d}$$

where, now, the overdot stands for differentiation with respect to the dimensionless cosmic time t and the dimensionless total energy density and pressure are given by

$$r_{\rm tot} = +r_V + \sum_{n=1}^{N} r_{M,n}, \tag{2.16a}$$

$$p_{\rm tot} = -r_V + \sum_{n=1}^{N} w_{M,n} r_{M,n}, \tag{2.16b}$$

with matter EOS parameters $w_{M,n}$ still to be specified. The dimensionless vacuum energy density r_V appearing in the above equations will be discussed in Sec. II D. The dimensionless potential u has already been defined by (2.2b) and (2.13c), but will be given again in Sec. II D.

With the solution of Eqs. (2.14) and (2.15) for appropriate boundary conditions, it is possible to verify *a posteriori* the Friedmann-type equation (2.10a) in dimensionless form:

$$h^2 s + h v = (r_{\text{tot}} - u)/6, \qquad (2.17)$$

which, in general, is guaranteed to hold by the contracted Bianchi identities and energy conservation (cf. Refs. [15,28]). Specifically, if the solution of Eqs. (2.14) and (2.15) satisfies (2.17) at one particular time, then (2.17) is satisfied at all the times considered. The additional constraint (2.17) will provide a valuable check on the numerical solution of the equations.

D. Ansatz for $r_V(f)$ and solution for $f(s)$

The only further input needed for the cosmological Eqs. (2.14) and (2.15) is an *Ansatz* for the gravitating vacuum energy density $\rho_V(q)$ from (2.8) or the corresponding dimensionless quantity r_V from (2.13b). In Refs. [7–10], it was argued that the vacuum variable q of the late Universe is close to its flat-spacetime equilibrium value q_0 and the quadratic approximation can be used

$$r_V = \gamma(1 - f)^2, \qquad (2.18)$$

with positive constant γ.

From the r_V definition in (2.13b), the constant γ in (2.18) can be expected to be of order Z^{-1}, with definition

$$Z \equiv q_0^{1/2} K^{-1} \sim 16\pi (E_{\text{QCD}}/E_{\text{Planck}})^2 \sim 10^{-38}, \qquad (2.19)$$

for the quantum-chromodynamics energy scale $E_{\text{QCD}} \approx 0.2$ GeV and the standard gravitational energy scale $E_{\text{Planck}} \equiv \sqrt{\hbar c^5/G_N} \approx 1.22 \times 10^{19}$ GeV (having set $G \sim G_N$; see Sec. III B). According to the discussion in Refs. [7–10], f can also be expected to be sufficiently close to 1, in order to reproduce an r_V value of order unity or less for the present Universe. For technical reasons, the value $Z = 10^{-2}$ is taken in a first numerical study (Sec. III C). Later, the proper boundary conditions and scaling behavior are considered (Sec. III D).

The dimensionless scalar potential $u(s, f)$ from (2.2b) and (2.13c) can be written as

$$u(t) \equiv U K^2 q_0^{-3/2} = -(\eta^2/4)\frac{f(t)^{3/2}}{1 - s(t)}, \qquad (2.20)$$

where a relatively small value for η appears to be indicated [10] by the measured value of the vacuum energy density; see Secs. III B and III D for further discussion on the numerical value of η.

With the specific functions (2.18) and (2.20), Eq. (2.14) is a quadratic in $\sqrt{f}$ and the positive root gives

$$\bar{f}_\pm(s) = (\sqrt{1 + D(s)^2} \pm D(s))^2, \qquad (2.21a)$$

$$D(s) \equiv \kappa/|1 - s| \geq 0, \qquad (2.21b)$$

$$\kappa \equiv (3/32)\eta^2/\gamma \geq 0, \qquad (2.21c)$$

where the minus sign inside the outer parentheses on the right-hand side of (2.21a) holds for $s < 1$ (the plus sign appears for the $s > 1$ case not considered here). Expression (2.21a) can then be used to eliminate all occurrences of f in the $3 + N$ ODEs (2.15) for the remaining $3 + N$ variables $h(t)$, $s(t)$, $v(t)$, and $r_{M,n}(t)$. Referring to the ODEs (2.15) in the following, it will be understood that f has been replaced by $\bar{f}_-(s)$ from (2.21).

III. THREE-COMPONENT MODEL UNIVERSE

A. Preliminaries

The modified-gravity theory considered in this article has been presented in Sec. II A and the corresponding dynamical equations for a spatially flat FRW universe in Secs. II B, II C, and II D. The specific model studied in this section is a simplified version with only three components labeled $n = 0, 1, 2$:

(0) A gluon condensate (described by the dimensionless variable f) with dimensionless energy density $r_V(f)$ from (2.18) and constant equation-of-state parameter $w_V = -1$, which is taken to give rise to a nonanalytic term in the modified-gravity action (2.1).

(1) A perfect fluid of ultrarelativistic matter (e.g., photons) with energy density $r_{M,1}$ and constant EOS parameter $w_{M,1} = 1/3$.

(2) A perfect fluid of nonrelativistic matter [e.g., cold dark matter (CDM) and baryons (B)] with energy density $r_{M,2}$ and constant EOS parameter $w_{M,2} = 0$.

From the scalar-tensor formalism of the gluon-condensate-induced modification of gravity, there is also the auxiliary Brans-Dicke scalar $s(t)$ to consider, with the dimensionless potential $u(s, f)$ from (2.20).

The relevant ODEs follow from (2.15) by letting the matter label run over $n = 1, 2$. The ideal starting point of the calculations would be some time after the QCD crossover at $T \sim \Lambda_{\text{QCD}}$ with $r_{M,1} \gg r_{M,2}$. The physical idea is that the expansion of the Universe was standard up till that time and that, then, a type of phase transition occurred with the creation of the gluon condensate. Clearly, the gluon condensate can be expected to start out in a nonequilibrium state, $f \neq 1$ and $s \neq 1$. These issues will be discussed further in Sec. III D.

At this moment, it is useful to recall the basic equations of a standard flat FRW universe [15,29] with gravitational coupling constant $G = G_N$ or $K = K_N$. For two components, a pressureless material fluid labeled M and an unknown fluid labeled X, these equations are

$$6h^2 \equiv 6(\dot{a}/a)^2 = r_M + r_X, \qquad (3.1a)$$

$$-12\ddot{a}/a = r_M + r_X + 3p_M + 3p_X = r_M + r_X(1 + 3w_X), \qquad (3.1b)$$

where p_M in (3.1b) has been set to zero and the EOS parameter $w_X \equiv p_X/r_X$ has been introduced. The standard energy-density parameters are defined as follows:

$$\Omega_M \equiv r_M/(6h^2), \qquad \Omega_X \equiv r_X/(6h^2) = 1 - \Omega_M. \qquad (3.2a)$$

In addition, the following combination of observables can be introduced to determine the unknown EOS parameter:

$$\bar{w}_X \equiv -\frac{2}{3}\left(\frac{\ddot{a}a}{(\dot{a})^2} + \frac{1}{2}\right)\frac{1}{1 - \Omega_M} = w_X, \qquad (3.2b)$$

where the last equality holds, again, for $p_M = 0$. See, e.g., Refs. [30,31] for details on how to reconstruct the dark-energy equation of state from observations.

In order to be specific, take the following fiducial values:

$$\{\Omega_M, \Omega_X, \bar{w}_X\}_{\text{present}}^{\text{standard FRW}} = \{0.25, 0.75, -1\}, \qquad (3.3)$$

which agree more or less with the recent data compiled in Refs. [32–37]. The standard flat FRW universe with parameters (3.3) corresponds, in fact, to the basic ΛCDM model [29] with CDM energy density $r_M \propto 1/a^3$ (with constant EOS parameter $w_M = 0$) and time-independent vacuum energy density $l \equiv r_X \propto a^0$ (with constant EOS parameter $w_X = -1$ and l the dimensionless version of the cosmological constant Λ).

Returning to the modified-gravity theory (2.1) and (2.2), the same observables Ω and $\bar{w}_X$ can be identified. Specifically, the generalized Friedmann equation (2.17) gives

$$\Omega_X + \Omega_M = 1, \qquad (3.4a)$$

$$\Omega_X \equiv \Omega_{\text{grav}} + \Omega_V, \qquad (3.4b)$$

$$\Omega_{\text{grav}} \equiv 1 - s - \dot{s}/h - u/(6h^2), \qquad (3.4c)$$

$$\Omega_V \equiv r_V/(6h^2), \qquad (3.4d)$$

$$\Omega_M \equiv r_M/(6h^2), \qquad (3.4e)$$

where Ω_{grav} is the new ingredient, as it vanishes for the standard theory with $u = 0$ and $s = 1$. Similarly, the effective EOS parameter of the unknown component X can be extracted from (2.15) and (2.17) for $p_M = 0$:

$$\bar{w}_X \equiv -\frac{2}{3}\left(\frac{\ddot{a}a}{(\dot{a})^2} + \frac{1}{2}\right)\frac{1}{1 - \Omega_M}$$

$$= -\frac{r_V - u - 4h\dot{s} - 2\ddot{s}}{r_V - u - 6h\dot{s} + r_M(1 - s))}. \qquad (3.5)$$

The right-hand side of (3.5) shows that $\bar{w}_X$ of the modified-gravity model (2.2) approaches the value -1 in the limit of vanishing matter content and constant Brans-Dicke scalar s as $t \to \infty$. *A priori*, there is no reason why this approach

cannot be from below, so that $1 + \bar{w}_X$ would be negative for a while (cf. Ref. [38]).

The main goal of this section is to get a quasirealistic model for the "present universe," which is taken to be defined by a value of approximately 0.25 for the matter energy-density parameter Ω_M. This can only be done with a numerical solution of the ODEs, but, first, analytic results relevant to the asymptotic behavior at early and late times are discussed.

B. Analytic results

It is not difficult to get two types of analytic solutions of the combined ODEs (2.15) and (2.17) for the specific functions (2.18) and (2.20), having used solution (2.21) to eliminate f in favor of s. The first corresponds to a Friedmann universe with relativistic matter and without vacuum energy. The second corresponds to a de Sitter–type universe without matter and with an effective form of vacuum energy.

For $\eta = 0$, the first analytic solution of (2.15), (2.16), (2.17), (2.18), (2.19), (2.20), and (2.21) has only relativistic matter $(w_{M,1} = 1/3)$ contributing to the expansion. Specifically, this Friedmann solution (labeled "F") is given by

$$h^{(F)} = (1/2)t^{-1}, \qquad s^{(F)} = f^{(F)} = 1, \qquad (3.6a)$$

$$r_{M,1}^{(F)} = (3/2)t^{-2}, \qquad r_{M,2}^{(F)} = 0. \qquad (3.6b)$$

Remark that standard general relativity [which has, from the start, the action equal to (2.1) for $\eta = 0$ and $G = G_N$] allows for arbitrary values $r_{M,1}(1)$ and $r_{M,2}(1)$ at reference time $t = 1$.

For $\eta > 0$, the second set of analytic solutions of (2.15), (2.16), (2.17), (2.18), (2.19), (2.20), and (2.21) has only vacuum energy contributing to the expansion, together with the effects of the gluon-condensate-induced modification of gravity $(\bar{w}_X = -1)$. This type of solution has constant (time-independent) variables $h > 0$ and $s \in (0, 1)$, with f given by (2.21a). From (2.15a) and (2.15c), using (2.20), a cubic in s is obtained, which needs to be discussed first.

Specifically, the cubic in $x \equiv 1 - s$ reads

$$9x^3 - 6x^2 + (1 + 9\kappa^2)x - 6\kappa^2 = 0, \qquad (3.7)$$

with parameter κ defined by (2.21c). This cubic has three distinct real solutions for $0 < \kappa^2 < (5\sqrt{5} - 11)/18 \approx (0.100\,094)^2$. Two of these solutions (with $2/3 < s < 1$) give stationary de Sitter–type solutions of the ODEs (2.15), (2.16), (2.17), (2.18), (2.19), (2.20), and (2.21). These two roots can be written in manifestly real form by use of the Chebyshev cube root

$$C_{1/3}(t)|_{|t|<2} \equiv 2\cos[(1/3)\arccos(t/2)], \qquad (3.8a)$$

$$C_{1/3}(0) \equiv \sqrt{3}. \qquad (3.8b)$$

Defining the auxiliary parameters

$$p \equiv (1/3)(1/27 + \kappa^2), \tag{3.9a}$$
$$q \equiv (2/9)(1/82 - 2\kappa^2), \tag{3.9b}$$

the relevant roots of (3.7) are

$$s_{\text{high}} = 7/9 + \sqrt{p}C_{1/3}(-qp^{-3/2}), \tag{3.10a}$$
$$s_{\text{mid}} = 7/9 + \sqrt{p}[C_{1/3}(qp^{-3/2}) - C_{1/3}(-qp^{-3/2})], \tag{3.10b}$$

where the third solution $s_{\text{low}} = 7/3 - s_{\text{high}} - s_{\text{mid}}$ can be omitted, as it lies below $2/3$ for κ in the domain considered [the stationary limit of, e.g., Eq. (2.15c) requires $s \geq 2/3$ because r_V from (2.18) is non-negative by definition].

The first de Sitter–type solution (labeled "deS,0" because $f \sim 0$ for $|\kappa| \ll 1$) is then given by

$$s^{(\text{deS},0)} = s_{\text{high}} = 1 - 6\kappa^2 - 162\kappa^4 + O(\kappa^6), \tag{3.11a}$$
$$f^{(\text{deS},0)} = \bar{f}_-(s_{\text{high}}) = 9(\kappa^2 + 36\kappa^4) + O(\kappa^6), \tag{3.11b}$$
$$h^{(\text{deS},0)} = \eta/(4\sqrt{3})|f^{(\text{deS},0)}|^{3/4}|1 - s^{(\text{deS},0)}|^{-1}$$
$$= \sqrt{\gamma/6}[1 - (81/2)\kappa^4 + O(\kappa^6)], \tag{3.11c}$$
$$r_{M,n}^{(\text{deS},0)} = 0, \tag{3.11d}$$

in terms of the function $\bar{f}_-(s)$ defined by (2.21a) and with an integer $n = 1, 2$ to label the different matter components. Note that the expression in the middle of (3.11c) simply follows from (2.15a) for $\dot{h} = 0$ and u from (2.20).

The second solution (labeled "deS,1" because $f \sim 1$ for $|\kappa| \ll 1$) is given by

$$s^{(\text{deS},1)} = s_{\text{mid}} = 2/3 + \kappa + 3\kappa^2 + (27/2)\kappa^3$$
$$+ 81\kappa^4 + O(\kappa^5), \tag{3.12a}$$
$$f^{(\text{deS},1)} = \bar{f}_-(s_{\text{mid}}) = 1 - 6\kappa - 27\kappa^3 - 162\kappa^4 + O(\kappa^5), \tag{3.12b}$$
$$h^{(\text{deS},1)} = \eta/(4\sqrt{3})|f^{(\text{deS},1)}|^{3/4}|1 - s^{(\text{deS},1)}|^{-1}$$
$$= \sqrt{2\gamma\kappa}/1024[1024 - 1536\kappa + 1152\kappa^2 + 1728\kappa^3$$
$$+ 17496\kappa^4 + O(\kappa^5)], \tag{3.12c}$$
$$r_{M,n}^{(\text{deS},1)} = 0, \tag{3.12d}$$

where κ is non-negative according to the original definition (2.21c). Note that the last expressions of both (3.11c) and (3.12c) are proportional to $\sqrt{\gamma}$ with all further dependence on γ entering through the parameter $\kappa \propto \eta^2/\gamma$, as can be expected on general grounds from the ODEs (2.15) without matter.

It is not quite trivial that there indeed exist de Sitter–type solutions in the modified-gravity theory (2.1). The first solution (3.11) is far from the equilibrium state $f_{\text{equil}} = 1$ and the second solution (3.12) is close to it, at least for $|\kappa| \ll 1$. The scaling behavior of both solutions under the

limit $\gamma \to \infty$ for constant η is also different, with h diverging for the first solution and staying constant for the second. For fixed parameters γ and η, numerical results suggest that the first solution (3.11) is unstable and the second solution (3.12) stable (and possibly an attractor). In the following, the focus is on the second solution close to the equilibrium value $f_{\text{equil}} = 1$ (corresponding to $q = q_0$).

In fact, two remarks on the de Sitter–type solution (3.12) are in order. First, observe that local experiments in this model universe with $\phi^{(\text{deS},1)} \sim 2/3 < 1$ would have an increased effective gravitational coupling

$$\bar{G}_N \equiv G_{\text{eff}}^{\text{local exps}}|^{(\text{deS},1)} \sim (1/\phi^{(\text{deS},1)})G \sim (3/2)G, \tag{3.13}$$

where the term $G/\phi^{(\text{deS},1)}$ in the middle comes directly from the combination $K\phi = \phi/(16\pi G)$ present in the action (2.2). Here, "local experiments" denote experiments on length scales very much less than the typical length scale of de Sitter–type spacetime, the horizon distance $L_{\text{hor}} = cH^{(\text{deS},1)}$, whose numerical value will be discussed shortly. It would then appear that the quantity (3.13) must be identified with Newton's gravitational constant G_N as measured by Cavendish [13] and modern-day experimentalists [14]; see [39] for additional comments.

Second, the de Sitter–type solution (3.12) of model (2.2) or equivalently model (2.1) has the inverse Hubble constant

$$(h^{(\text{deS},1)})^{-1} = 4/\sqrt{3}\eta^{-1} \approx 2.3 \times 10^3\left(\frac{10^{-3}}{\eta}\right), \tag{3.14}$$

as follows from (3.12c) by neglecting terms suppressed by powers of $\kappa = O(1/\gamma) = O(10^{-38})$ and anticipating a particular order of magnitude for the model parameter η. With the conversion factor from (2.13a), the dimensionless quantity (3.14) corresponds to

$$(H^{(\text{deS},1)})^{-1} \sim 4/\sqrt{3}\eta^{-1}(3/2)K_N q_0^{-3/4}$$
$$\sim 8 \times 10^{17} \text{ s}\left(\frac{10^{-3}}{\eta}\right)\left(\frac{200 \text{ MeV}}{q_0^{1/4}}\right)^3, \tag{3.15}$$

where, according to (3.13), an approximate factor $3/2$ appears in going from K to the Newtonian value $K_N \equiv (16\pi G_N)^{-1}$. The time scale found in (3.15) is of the same order as the inverse Hubble constant $(H_0)^{-1} \approx 4.5 \times 10^{17}$ s $(0.70/h_0)$ for the measured value $h_0 \approx 0.70$ as reported in Refs. [32,36,37].

By equating the theoretical quantity $1/H^{(\text{deS},1)}$ from (3.15) multiplied by an *ad hoc* factor $g = \frac{1}{2}$ with the measured value $1/H_0$, a first estimate of the model parameter η in the original action (2.1) is obtained,

$$\eta \sim \sqrt{3}K_N q_0^{-3/4}H_0 \sim 10^{-3}, \tag{3.16}$$

for the q_0 and H_0 values mentioned in the previous paragraph. Admittedly, the choice of one-half for the factor g is somewhat arbitrary, but consistent with the physical pic-

ture of our present Universe entering a de Sitter phase. A more reliable estimate of η will come from the numerical study of a model universe with both vacuum and matter energies. The numerical solution found will be seen to interpolate between the analytic solutions (3.6) and (3.12).

C. Exploratory numerical results

Equation (2.15a) for the potential $u(s, f)$ from (2.20) makes clear that a model universe with an asymptotically nonvanishing Hubble constant, $h(t) \to \text{const} \neq 0$, requires a nonvanishing modified-gravity parameter, $\eta \neq 0$. The analytic de Sitter solution with $\dot{h} = \dot{s} = \dot{f} = 0$ has already been given in Sec. III B.

The numerical solution of ODEs (2.15) for $\eta \sim 10^{-3}$ is presented in Fig. 1 and several observations can be made:

(i)   The boundary conditions on the functions will be discussed in Sec. III D.
(ii) There is a transition from deceleration in the early universe to acceleration in the late universe.
(iii) The values for s, $1 - f$, and h at the largest time shown in Fig. 1 agree already at the 10% level with those of the analytic de Sitter–type solution (3.12).

(iv) The ratio $r_{M,\text{tot}}/(6h^2)$ is equal to 0.25 at the dimensionless cosmic time $t \approx 1.4 \times 10^3$.

Points (ii)–(iv) suggest that, for the model parameter values chosen, the model universe at $t_p = 1.432 \times 10^3$ resembles our own present Universe, characterized by the values (3.3).

More quantitatively, the following three estimates can be obtained. First, the product of the dimensionless age t_p of the present universe with its dimensionless expansion rate $h(t_p) \approx 0.6351 \times 10^{-3}$ gives

$$t_p h(t_p) \approx 0.91, \tag{3.17a}$$

which also holds for the product of the dimensionful quantities, $\tau_p H(\tau_p) \approx 0.91$.

Second, evaluating the particular combination (3.5) of first and second derivatives of $a(t)$ and the matter energy density ρ_M, the present effective EOS parameter of the unknown component is found to be

$$\bar{w}_X(t_p) \equiv -\frac{2}{3}\left(\frac{\ddot{a}a}{(\dot{a})^2} + \frac{1}{2}\right)\frac{1}{1 - \Omega_M}\bigg|_{t=t_p} \approx -0.66. \tag{3.17b}$$
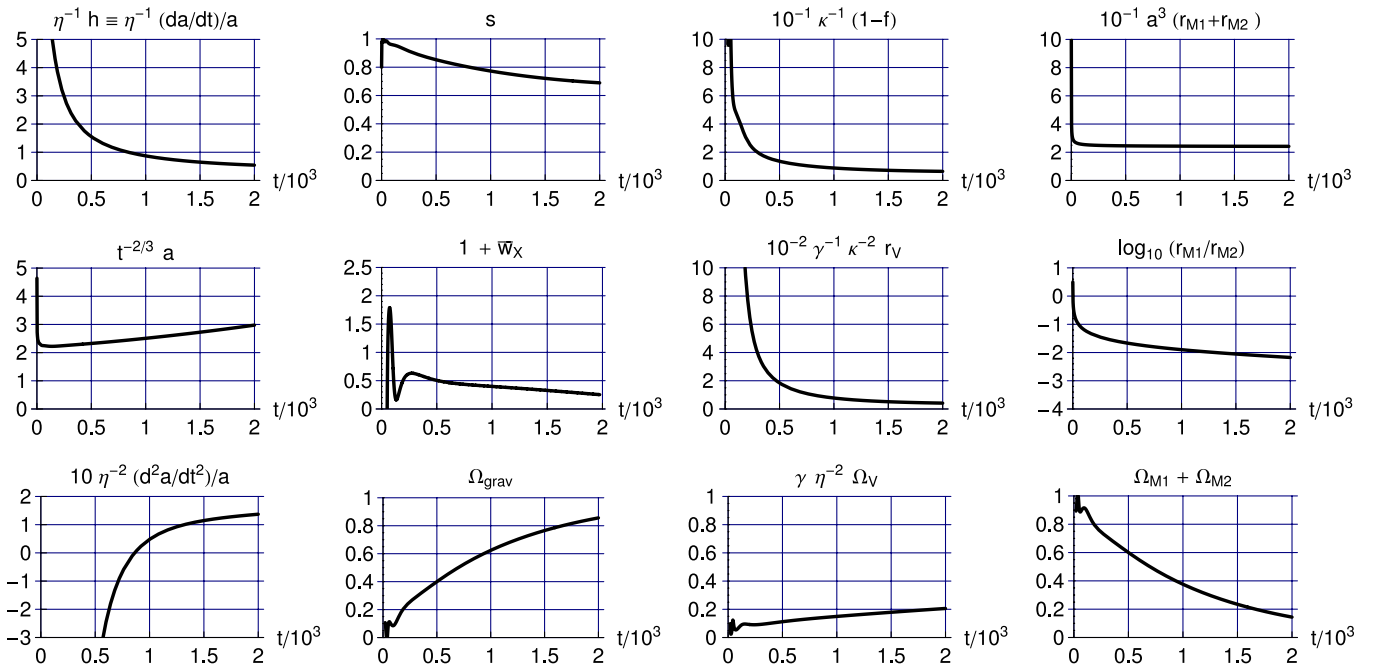

FIG. 1 (color online).   Numerical solution of ODEs (2.15), with vacuum energy density (2.18), Brans-Dicke scalar potential (2.20), and both relativistic matter (energy density $r_{M,1}$) and nonrelativistic matter (energy density $r_{M,2}$). The figure panels are organized as follows: the panels of the first column from the left concern the expansion factor $a(t)$, those of the second column the modified-gravity scalar $s(t)$, those of the third column the gluon-condensate vacuum variable $f(t)$, and those of the fourth column the matter energy densities $r_{M,n}$. The model parameters are $(\gamma, \eta^2, w_{M,1}, w_{M,2}) = (10^2, 9 \times 10^{-7}, 1/3, 0)$, with the resulting parameter $\kappa \equiv (3/32)\eta^2/\gamma = 8.4375 \times 10^{-10}$. The boundary conditions at $t_{\text{start}} = 0.1$ are $(a, h, s, v, 1 - f, r_{M,1}, r_{M,2}) = (1, 4.082483, 0.8, 0.8164966, 8.437500 \times 10^{-9}, 75.97469, 24.02531)$; see Sec. III D for details. The several energy-density parameters Ω and the effective "dark-energy" equation-of-state parameter $\bar{w}_X$ are defined in (3.4) and (3.5), respectively. With $\gamma/\eta^2 \gg 1$, the values of Ω_V are negligible compared to those of Ω_{grav} for the time interval shown.

For larger times $t \gg t_p$, this parameter $\bar{w}_X(t)$ drops to the value -1, as can be expected from the right-hand side of (3.5). Additional numerical values are $\bar{w}_X = -0.750\,82$, $-0.989\,21$, $-0.997\,80$, and $-0.999\,89$ for $t = 2000$, 4000, 8000, and $16\,000$, respectively. Observe that the particular combination of observables (3.5) is designed to be interpreted as the effective EOS parameter of the unknown component X only if matter-pressure effects are negligible ($t \gtrsim 500$ in Fig. 1).

Third, consider the transition of deceleration to acceleration mentioned in point (ii) above. In mathematical terms, this time corresponds to the nonstationary inflection point of the function $a(t)$, that is, the value t_{inflect} at which the second derivative of $a(t)$ vanishes but not the first derivative. Referring to the model universe at $t_p = 1.432 \times 10^3$, the inflection point $t_{\text{inflect}} \approx 0.863 \times 10^3$ corresponds to a redshift

$$z_{\text{inflect}} \equiv a(t_p)/a(t_{\text{inflect}}) - 1 \approx 0.5, \qquad (3.17c)$$

which implies that the acceleration is a relatively recent phenomenon in this model universe. Inspection of the lower panels of Fig. 1 shows that the acceleration sets in when the ratio of $\Omega_X = \Omega_{\text{grav}} + \Omega_V$ and $\Omega_{M,\text{tot}}$ is approximately unity, whereas the standard ΛCDM model would have $\Omega_X / \Omega_{M,\text{tot}} \sim 1/2$ according to (3.1b).

Returning to the first estimate (3.17a), note that this quantity can be interpreted as the age of the present universe in time units obtained from the present expansion rate. But it is also possible to obtain the absolute age of the model universe, using the time scale contained in (2.13a), which requires as input the experimental value of the QCD gluon condensate q_0 and the one of Newton's constant G_N, taken to be equal to the effective gravitational coupling $\bar{G}_N$ from (3.13). With the conversion factors from (2.13a) and the relation $G \sim s(t_p)G_N$ for $K \equiv 1/(16\pi G)$, the numerical results $t_p \approx 1432$, $h(t_p) \approx 1/1575$, and $s(t_p) \approx 0.7267$ give the following two dimensionful quantities of the present universe:

$$\tau_p = t_p K q_0^{-3/4} \sim 13.1 \text{ Gyr}, \qquad (3.18a)$$

$$H_p = h(t_p) K^{-1} q_0^{3/4} \sim 68 \text{ km s}^{-1} \text{ Mpc}^{-1}, \qquad (3.18b)$$

where the numerical values have been calculated with $q_0 = (210 \text{ MeV})^4$. Remark that, if the relation $G \sim G_N$ holds for Cavendish-type experiments as mentioned in [39], the same numerical values are obtained in (3.18) by taking $q_0 \approx (190 \text{ MeV})^4$ and, if $G \sim G_N/2$ holds, by taking $q_0 \approx (230 \text{ MeV})^4$. All of these three q_0 values lie below the value $q_0 \approx (330 \text{ MeV})^4$ indicated by particle physics [3], but the uncertainty in the latter value appears to be large [4–6]. In addition, it may be that certain particle-physics experiments are more appropriate than others to determine the truly homogeneous condensate q_0 relevant to cosmology.

Next to the observations [11,12,32–37], the values obtained in (3.17) and (3.18) have the correct order of magnitude, which is all that can be hoped for at the present stage. Still, it is remarkable that more or less reasonable values appear at all [40].

For comparison, the standard flat-ΛCDM model (3.1), (3.2), and (3.3) with boundary condition $r_M(t_p)/r_V = 1/3$ gives the product $\tau_p H(\tau_p) \approx 1.01$, the effective EOS parameter $\bar{w}_X = -1$, and the inflection-point redshift $z_{\text{inflect}} = (6)^{1/3} - 1 \approx 0.82$. These three numbers fit the observational data perfectly well, but the ΛCDM model is purely phenomenological and cannot explain, without further input,[2] the absolute age of the Universe as in (3.18a) or the absolute vacuum energy density as will be discussed in Sec. IV.

D. Elementary scaling analysis

In the previous section, the ODEs (2.15) have been solved numerically for certain parameter values and boundary conditions at $t = t_{\text{start}}$, which need to be discussed further. As explained in Sec. III A, t_{start} is considered to correspond to a time just after the QCD crossover has happened. This implies, in particular, that the starting value $h(t_{\text{start}})$ for the expansion rate is approximately given by the value $[(r_V + r_{M,\text{tot}})/6]^{-1/2}$ of the corresponding standard FRW universe (3.1a). The f value at t_{start} follows from (2.21) for the chosen s value (see below) and the starting value for v is obtained by solving (2.17), considered as a linear equation in v with all other quantities given.

Next, the value of t_{start} itself and the corresponding values for $r_{M,1}$ and $r_{M,2}$ need to be specified. These values depend on the physical ratio Z defined by (2.19). Following the results for the standard FRW universe, take

$$\gamma = \hat{\gamma} Z^{-1}, \qquad (3.19a)$$

$$t_{\text{start}} = \hat{t} \sqrt{Z}, \qquad (3.19b)$$

$$r_{M,1}(t_{\text{start}}) = \hat{r} Z^{-1}/(1 + Z^{1/4}), \qquad (3.19c)$$

$$r_{M,2}(t_{\text{start}}) = \hat{r} Z^{-3/4}/(1 + Z^{1/4}), \qquad (3.19d)$$

where the constants $\hat{\gamma}$, $\hat{t}$, and $\hat{r}$ are numbers of order unity (in the present elementary analysis, they are just set equal to 1). With $\hat{t} = 1$ and the particular *Ansätze* (3.19c) and (3.19d), there is equality of the relativistic (label $n = 1$) and nonrelativistic (label $n = 2$) energy densities around $t \sim 1$, which is not entirely unrealistic if the present universe has $t \sim 10^3$.

Finally, the boundary condition value $s(t_{\text{start}})$ is taken between 0 and 1. The results are, however, rather insensi-

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[2]Taking as additional input the *measured* value [32] $h_0 \approx 0.70$ of the Hubble constant $H_0 \equiv h_0 100 \text{ km s}^{-1} \text{ Mpc}^{-1} = h_0(9.778 \times 10^9 \text{ yr})^{-1}$, the ΛCDM-model result $\tau_0 H_0 \approx 1.01$ gives the dynamic age $\tau_0 \approx 14.2$ Gyr.

TABLE I. Numerical results for the "present epoch" [defined by $\Omega_M(t_p) = 0.25$] in model universes with different numerical values for the parameters Z and η, where the latter parameter controls the modified-gravity term in the action (2.1) and the former is defined by (2.19) in terms of the physical energy scales. Other parameters and boundary conditions are given by (3.19), with constants $\hat{\gamma}$, $\hat{t}$, and $\hat{r}$ set equal to 1. A further boundary condition is $s(t_{start}) = 0.8$; see Sec. III D for details. The effective equation-of-state parameter $\bar{w}_X$ and the inflection-point redshift $z_{inflect}$ are defined in (3.17b) and (3.17c), respectively. Figure 1 for $Z = 10^{-2}$ illustrates the general behavior of $h(t)$, $\bar{w}_X(t)$, and other physical quantities.

Z	$10^6 \eta^2$	$10^{-3} t_p$	$10^4 h(t_p)$	$s(t_p)$	$t_p h(t_p)$	$\bar{w}_X(t_p)$	$z_{inflect}$
10^{-1}	0.8	1.522	5.980	0.7272	0.910	-0.669	0.541
10^{-2}	0.9	1.432	6.351	0.7267	0.910	-0.662	0.538
10^{-4}	0.7	1.629	5.584	0.7259	0.910	-0.663	0.515
10^{-8}	0.8	1.523	5.967	0.7255	0.909	-0.660	0.505
10^{-16}	0.9	1.436	6.330	0.7256	0.909	-0.660	0.506

tive to the precise value of $s(t_{start})$; see [43] for selected numerical results. The explanation is that, independent of the precise starting value, $s(t)$ increases rapidly until, at $t \sim 1$, it bounces back from the $s = 1$ "wall" and, then, slowly descends towards the de Sitter value, with some initial oscillations.

Having specified the boundary conditions of the physical variables, the optimal model parameter η needs to be determined. The strategy is as follows: for a given Z value, assume an η value, determine t_p from the condition $\Omega_{M,tot}(t_p) = 0.25$, evaluate the product $t_p h(t_p)$, and, if necessary, return to a new value of η in order to get $t_p h(t_p)$ closer to the asymptotic value of approximately 0.909.

Numerical results are given in Table I. Three physical quantities, the relative age of the present universe $t_p h(t_p)$, the effective EOS parameter $\bar{w}_X$, and the inflection-point redshift $z_{inflect}$, appear to approach constant values as Z drops to zero. This nontrivial result suggests that the behavior shown in Fig. 1 and the corresponding estimates (3.17) and (3.18) also apply to the physical case with $Z \sim 10^{-38}$ as given by (2.19).

IV. CONCLUSION

The bottom-row panels of Fig. 1, if at all relevant to our Universe, suggest that the present accelerated expansion may be due primarily to the nonanalytic modified-gravity term in the action (2.1) rather than the direct vacuum energy density $\rho_V(q)$, because q is already very close to its equilibrium value q_0, making $\rho_V(q) \sim \rho_V(q_0) = 0$. Referring to the definitions in (3.4), the second panel of the bottom row shows the effective energy-density parameter Ω_{grav} due to the gluon-condensate-induced modification of gravity and the third panel the energy-density parameter Ω_V from the vacuum energy density proper (with EOS parameter $w_V = -1$), their total giving Ω_X which equals $1 - \Omega_M$ for a flat FRW universe. As discussed in Secs. III A and III C, the total unknown X component has an effective EOS parameter $\bar{w}_X$ which drops to the value -1 as the de Sitter–type universe is approached.

Remark that, in contrast to the results of, e.g., Refs. [22,23], nontrivial dark-energy dynamics has been obtained, because the effective action (2.1) is assumed to be valid only on cosmological length scales, not solar-system or laboratory length scales [see also the discussion in the paragraph of Sec. II A containing Eq. (2.3)]. As it stands, the effective action (2.1) can be viewed as an efficient way to describe the main aspects of the late evolution of the Universe, with only two fundamental energy scales, $E_{QCD} \sim 10^8$ eV and $E_{Planck} \sim 10^{28}$ eV, and a single dimensionless coupling constant, $\eta \sim 10^{-3}$. Moreover, this effective coupling constant η can, in principle, be calculated from quantum chromodynamics and general relativity, which may or may not confirm our numerical value of approximately 10^{-3}; cf. Refs. [10,25] and the third remark in the Note Added.

Elaborating on the source of the present acceleration, consider the second term on the right-hand side of (2.7a), which can be rewritten as $+(2\phi K)^{-1}(\rho_{V,BD})g_{\mu\nu}$ for the Brans-Dicke vacuum energy density $\rho_{V,BD} \equiv -KU$. The exact de Sitter–type solution (3.12) for $\kappa \ll 1$, together with the conversion factor from (2.13c) and Newton's constant from (3.13), then allows for the following estimate:

$$
\begin{aligned}
\rho_{V,BD}|^{(deS,1)} &= -uq_0^{3/2}/K|^{(deS,1)} \\
&= 12\pi\eta^2 q_0^{3/2} G \sim (\pi/8)\eta^2 K_{QCD}^3/E_{Planck}^2 \\
&\sim (2 \times 10^{-3} \text{ eV})^4 \left(\frac{\eta}{10^{-3}}\right)^2 \left(\frac{K_{QCD}}{(420 \text{ MeV})^2}\right)^3,
\end{aligned}
$$

(4.1)

where q_0 has been expressed in terms of the QCD string tension K_{QCD} [1], specifically, $q_0 = E_{QCD}^4 \approx (K_{QCD}/4)^2$. The parametric dependence of the above expression, $\rho_V \propto K_{QCD}^3/E_{Planck}^2$, is the same as that of the previous estimate (6.7) in Ref. [10], but expression (4.1) now comes from the solution of field equations. Two other dimensionful quantities, the age and expansion rate of the Universe, have already been given in (3.18).

Before the asymptotic de Sitter–type universe with effective energy density (4.1) is reached, the Brans-Dicke scalar ϕ evolves and allows for an effective EOS parameter $\bar{w}_X$ different from -1 [the scalar ϕ has no direct kinetic term in the action (2.2a), but the ϕR term does give, by partial integration, an effective kinetic term for ϕ, which, in fact, leads to the generalized Klein-Gordon equation (2.9)]. For the present Universe, the general lesson may be that the deformation of the QCD gluon condensate q by the spacetime curvature of the expanding Universe can result in an effective EOS parameter $\bar{w}_X$ which evolves with time and, for the present epoch, can still be somewhat above its asymptotic value of -1. In turn, a possible discovery of a $\bar{w}_X$ time dependence may provide an additional incentive to theoretical investigations of the physics of the gravitating gluon condensate.

Note Added.—After completion of the work reported here, we became aware of two earlier articles and a third article recently posted on the archive. The first article [44] is a systematic study of the cosmology of $f(R)$ modified-gravity models and identifies the modified-gravity term (2.1b), for constant q, as cosmologically viable (observe the different sign definition of R compared to ours). The second article [45] investigates the growth of density perturbations in $f(R)$ modified-gravity models and establishes, in Eq. (42), the effective gravitational coupling parameter for subhorizon CDM density perturbations, which turns out to be close to G_N for the model universe of Fig. 1 at times $t \lesssim 500$ (redshifts $z \gtrsim 1$). The third article [46] presents a QCD calculation for the origin of the modified-gravity term (2.1b) and may also explain the smallness of the coupling constant η, even though many conceptual and technical issues remain to be resolved.

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[39] Considering the gravitational attraction of two sufficiently small test bodies *in vacuo* (i.e., without gas between them), there would be, according to Eq. (5.38) in Ref. [17] for Brans-Dicke parameter $\omega_{BD} = 0$, a factor $4/3$ multiplying $1/\phi^{(\mathrm{deS},1)} \sim 3/2$ in the middle expression of (3.13), giving $2G$ on the far right-hand side. The extra contribution contained in this factor $4/3 = 1 + 1/3$ would originate from the attraction ("fifth force") of the dynamical scalar field $\phi(\vec{x}) = \phi^{(\mathrm{deS},1)} + \varphi(\vec{x})$, where $\varphi(\vec{x})$ obeys the Poisson equation coming from (2.9) for negligible ρ_V and U (see also Sec. 9.9 of Ref. [15]). For two sufficiently large test bodies *in vacuo*, the chameleon effect [19,22,23] can be expected to give an effective mass to the scalar degree of freedom inside the bodies, which results in a suppression of the additional long-range attraction, thereby reducing the $4/3$ factor to 1 and giving the relation $\bar{G}_N \sim (1/\phi^{(\mathrm{deS},1)})G$. For two sufficiently large test bodies with a sufficiently dense gas between them, the intermediate scalar field may be forced to be close to 1, so that the cosmological $1/\phi^{(\mathrm{deS},1)}$ factor in (3.13) is removed altogether, resulting in the relation $\bar{G}_N \sim G$. These considerations make clear that the details of the precise numerical factor in (3.13) remain to be worked out and will depend on both the physical setup and the precise form of the gravity modification $\tilde{f} = R + \tilde{h}$ [see (2.3) for a particular example].

[40] Ultimately, the constraints from big bang nucleosynthesis (BBN) on $G_{\mathrm{eff}} \equiv G/\phi \equiv G/s$ and bounds on its present time variability will need to be addressed. [The particular combination G/ϕ controls the Hubble expansion according to (2.10a) with a dominant radiative component and negligible contributions from $\dot{\phi}$ and U.] The s-panel results in Fig. 1 show that G_{eff} during the BBN epoch would be some 30% smaller than the present value and that $(dG_{\mathrm{eff}}/dt)/G_{\mathrm{eff}}|_{t=t_p}$ would be of order 10^{-11} yr^{-1}, both values being marginally consistent with the existing experimental bounds [17,18]. The same conclusion appears to hold for cosmic microwave background bounds [41,42], but a self-consistent analysis of the data remains to be performed.

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[43] Three present-universe quantities have been given in (3.17) for $s(t_{\mathrm{start}}) = 0.80$, model parameters $(Z, \eta^2) = (10^{-2}, 9 \times 10^{-7})$, and further values given by (3.19) with constants $\hat{\gamma}$, $\hat{t}$, and $\hat{r}$ set equal to 1. For $s(t_{\mathrm{start}})$ ranging over the interval [0,0.99] and all other inputs kept the same, the values of $t_p h(t_p)$ vary by approximately 1% around the central value, those of $\bar{w}_X(t_p)$ by approximately 5%, and those of z_{inflect} by approximately 25%. These results suggest that, in the framework of the modified-gravity theory (2.1), especially the quantity z_{inflect} can be used as a diagnostic of the state of the Universe after the QCD crossover.

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