Equivalence between domain walls and ‘‘noncommutative’’ two-sheeted spacetimes: Model-independent matter swapping between branes

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(Received 19 June 2009; revised manuscript received 12 October 2009; published 16 February 2010)

We report a mathematical equivalence between certain models of the Universe relying on domain walls and noncommutative geometries. It is shown that a two-braneworld made of two domain walls can be seen as a ‘‘noncommutative’’ two-sheeted spacetime under certain assumptions. This equivalence also implies a model-independent phenomenology, which is presently studied. Matter swapping between the two branes (or sheets) is predicted through fermionic oscillations induced by magnetic vector potentials. This phenomenon, which might be experimentally studied, could reveal the existence of extra dimensions in a new and accessible way.

DOI: 10.1103/PhysRevD.81.035014 PACS numbers: 11.25.Wx, 02.40.Gh, 11.27.+d, 13.40.−f

I. INTRODUCTION

During the last two decades, the possibility that our observable (3 + 1)-dimensional Universe could be a sheet (a brane) embedded in a higher-dimensional bulk spacetime has received a lot of attention. This line of thought has shown to provide nice explanations to several puzzling phenomena such as the hierarchy between the electroweak and the Planck scales [1], the dark matter origin [2], or the cosmic acceleration [3]. The domain wall was demonstrated as a believable mechanism to explain the trapping of the standard model (SM) particles on branes [4], especially fermions [5–11]. The confinement of gauge fields on lower-dimensional hypersurfaces was also investigated [12,13]. More recent models even suggest that all standard model particles could be confined on the branes [14]. Therefore, finding physical evidences of extra dimensions is a major contemporary challenge. Interesting results could arise from high energy physics (Kaluza-Klein tower states [15], for instance) or low-energy physics (deviations from the inverse square law of gravity [16], for instance).

In the present paper, we are mainly motivated by the quest of new phenomena at a nonrelativistic energy scale. We explore how the quantum dynamics of fermions is modified when the higher-dimensional bulk contains more than only one brane. Hence, we focus on a two-braneworld (related to two domain walls) and investigate the dynamics of a massive fermion in this extended framework. It is shown that such a model is formally equivalent to a two-sheeted spacetime (a product manifold $M_4 \times Z_2$) described in the formalism of the noncommutative geometry [17–20], at least as a low-energy effective theory.

In previous works [21,22], the present authors have studied the phenomenology of certain of these $M_4 \times Z_2$ two-sheeted spacetimes, but no formal proof had been given that these exotic geometries could be related to more conventional branes theories: the link between both approaches was just considered as a fairly working hypothesis. For the first time, in the present paper, a physical and mathematical proof of this link is derived. Moreover, the mathematical description of the $M_4 \times Z_2$ geometry is enlarged by contrast to the previous works [21,22]. The demonstration made in the present paper is inspired by quantum chemistry and the construction of molecular orbitals, here extended to branes. As a consequence of the bulk dimensionality extension, the quantum dynamics phenomenology is considerably enriched: for a five-dimensional bulk containing two branes, matter swapping between these two worlds is made possible (although the effect could remain difficult to observe). More important, since the obtained equations (extended Dirac and Pauli's equations) are completely model independent, we conclude that this matter swapping phenomenon might probably be shared by every model of the Universe containing at least two ‘‘worlds’’.

In Sec. II, we introduce the kink-antikink domain walls description of a two-braneworld. Section III gives the fermion eigenstates in such a braneworld setup. In Sec. IV, we introduce gauge fields in the two-braneworld model. In Sec. V, we then derive a two-level description of the fermion dynamics in a braneworld with two domain walls in the presence of an electromagnetic field. In Sec. VI, we show that the two-level description fits with that of a two-sheeted spacetime as described by noncommutative geometry. Finally, in Sec. VII we underline the basic phenomenological consequences of the present work.

II. BRANEWORLDS WITH TWO DOMAIN WALLS

The brane concept takes its origin in superstring theories [23], though earlier similar concepts were proposed in other theoretical contexts [24,25]. However, since super-

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string theories suffer from mathematical complications, several simplified approaches relying on more classical field theories [26] have been suggested. For instance, several works [5–11] are now inspired by the approach of Rubakov and Shaposhnikov [25]. These authors have suggested that elementary particles might be trapped on a defect (a domain wall) on a higher-dimensional spacetime. Such a defect would arise from a scalar field whose dynamics should be described by a soliton-kink solution in a φ^4 theory. Bosonic excitations of the scalar field are trapped and propagate along the kink. In addition, any chiral Dirac particle is also trapped on the domain wall [5,6,25]. More recently, Randall and Sundrum have suggested braneworlds models where the bulk metric is warped to ensure (3 + 1)-dimensional gravity to be reproduced as well [27].

The model considered in the present paper relies on a usual braneworld description involving domain walls in the bulk [5,6]. Two branes are here considered living in a continuous five-dimensional manifold. Branes are described by the kink (antikink) solutions of a scalar field in a φ^4 theory. Matter is then described through a five-dimensional fermionic field coupled to this scalar field. Since we are not motivated by gravitational considerations, we use a flat metric for the bulk spacetime, i.e.

$$\begin{aligned} ds^2 = g_{AB}dx^A dx^B &= g_{\mu\nu}dx^\mu dx^\nu - dz^2 \\ &= \eta_{\mu\nu}dx^\mu dx^\nu - dz^2, \end{aligned} \tag{1}$$

where g_{AB} is the five-dimensional metric tensor with signature $(+,-,-,-,-)$ with $A, B = 0, \ldots, 4$. $\eta_{\mu\nu}$ is the four-dimensional Minkowski metric tensor of signature $(+,-,-,-)$ with $\mu, \nu = 0, \ldots, 3$, and z the coordinate along the extra dimension. An improved model involving a warped metric could be considered as well [5–7], but this choice would introduce supplementary complications, which are neither relevant nor necessary to illustrate the mechanism discussed in this paper. The action S for a real scalar field Φ coupled to a matter field Ψ in a five-dimensional spacetime is then

$$\begin{aligned} S = \int\bigg[&\frac{1}{2}g^{AB}(\partial_A\Phi)(\partial_B\Phi) - V(\Phi) \\ &+ \bar{\Psi}(i\Gamma^A\partial_A - \lambda\Phi)\Psi\bigg]\sqrt{g}d^5x. \end{aligned} \tag{2}$$

We assume that the Dirac field Ψ is coupled to the scalar field Φ through a Yukawa coupling term $\lambda\bar{\Psi}\Phi\Psi$ where λ is the coupling constant. It should be pointed out that another choice for the coupling term [8] would not change the final conclusions of the paper. In addition, a convenient potential $V(\Phi)$ is given by [5,6,25]

$$V(\Phi) = \frac{\chi}{4}(\Phi^2 - \eta^2)^2, \tag{3}$$

where χ and η are constants of the potential. Though several possibilities can be considered for the potential [5,6,8], we just assume that it allows the existence of domain walls (i.e. topological defects) in accordance with the original Rubakov-Shaposhnikov concept [25]. The scalar field equations of motion can be easily derived from relation (2)

$$\Phi'' + \chi\eta^2\Phi - \chi\Phi^3 = 0 \tag{4}$$

from which domain-wall solutions can then be derived. Φ'' is the second-order derivative of Φ along the extra dimension.

For a single brane, two solutions have to be considered:

$$\Phi_{(k/(ak))}(z) = \pm\Phi(z) = \pm\eta\tanh(z/\xi), \tag{5}$$

where "+" (respectively "−") refers to the kink (k) solution [respectively, antikink (ak) solution]. ξ is the brane thickness such that $\xi^{-1} = \eta\sqrt{\chi/2}$.

Now if we shift from a single brane to a two-braneworld, the solution can be expressed as a kink-antikink pair, each wall being localized, respectively, at $z = -d/2$ and $z = +d/2$. The field solution of Eq. (4), which describes such a kink-antikink system, can be approximated by [10,11]

$$\begin{aligned} \Phi(z) &= \Phi_-(z) + \Phi_+(z) + \Delta\Phi \\ &= \Phi(z + d/2) - \Phi(z - d/2) - \eta \end{aligned} \tag{6}$$

provided that $d \gg \xi$, i.e. the distance d between branes is larger than the brane thickness. For that reason, we are now considering that both branes are independent from each other both from a scalar and a gravitational point of view.

III. FERMIONS IN BRANEWORLDS

From Eq. (2), the five-dimensional massless Dirac equation can be easily expressed. In what follows, the Dirac matrices are given by $\Gamma^\mu = \gamma^\mu$ and $\Gamma^4 = -i\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, where γ^μ and γ^5 are the usual Dirac matrices in the four-dimensional Minkowski spacetime. The Clifford algebra is verified since

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}, \tag{7}$$

where η^{AB} is the five-dimensional metric tensor of the Minkowski spacetime. The Dirac equation is therefore

$$(i\gamma^\mu\partial_\mu + \gamma^5\partial_z - \lambda\Phi)\Psi = 0. \tag{8}$$

By using the separating variable method and due to γ^5 matrix in Eq. (8) the solution Ψ can be expressed as

$$\Psi(x, z) = f_L(z)\psi_L(x) + f_R(z)\psi_R(x), \tag{9}$$

where the $\psi_{L,R}$ are left- and right-handed four-dimensional spinors such that $\gamma^5\psi_{R/L} = \pm\psi_{R/L}$. x are the four-dimensional coordinates. For a trapped fermion, we expect that the five-dimensional Dirac equation can be expressed as an effective four-dimensional massive equation such that

$$i\gamma^\mu \partial_\mu \psi_{L/R} = m\psi_{R/L}, \tag{10}$$

where m is assumed to be the apparent (four-dimensional) particle mass. Substituting Eq. (9) into Eq. (8), we get then

$$\partial_z f_R - \lambda\Phi f_R + m f_L = 0 \quad \text{and} \quad \partial_z f_L + \lambda\Phi f_L - m f_R = 0. \tag{11}$$

After a convenient rearrangement of the equations, we get

$$-\partial_{zz} f_L(z) + W_L f_L(z) = m^2 f_L(z) \quad \text{and} \quad -\partial_{zz} f_R(z) + W_R f_R(z) = m^2 f_R(z), \tag{12}$$

with

$$W_L = \lambda(\lambda\Phi^2 - (\partial_z\Phi)), \qquad W_R = \lambda(\lambda\Phi^2 + (\partial_z\Phi)) \tag{13}$$

and

$$\int f_\alpha(z) f_\beta(z) dz = \delta_{\alpha,\beta} \quad \text{and} \quad \int (\psi_L^\dagger(x)\psi_L(x) + \psi_R^\dagger(x)\psi_R(x)) d^4x = 1. \tag{14}$$

Obviously, owing to the Schrodinger-like Eqs. (12), $f_{L,R}(z)$ define the localization of the left- and right-handed states of the fermion along the extra dimension with effective potentials $W_{L/R}$. m is the effective mass of the trapped fermion and it is related to the eigenvalues of the bound states in the potentials $W_{L/R}$. Equations (13) imply that the effective potential felt by the fermion depends on its helicity state. Then, left- and right-handed states are not necessarily localized at the same place, and it is even possible that bound state cannot exist. Previous works [5,6,25] have shown that for a kink solution of the scalar field, the $m = 0$ mode is localized on the kink for the left-handed state, whereas the right-handed state cannot be localized. By contrast, the $m \neq 0$ modes are localized around the kink whatever their state. In an antikink world, the $m \neq 0$ modes are also localized around the antikink whatever their state, but in the opposite only right-handed $m = 0$ fermions can exist. Note that since the usual four-dimensional fermion wave function $\psi(x)$ can be expressed as

$$\psi(x) = \psi_L(x) + \psi_R(x), \tag{15}$$

it can then be easily shown that Eq. (9) can be rewritten as

$$\Psi(x, z) = \Pi(z)\psi(x), \tag{16}$$

where

$$\Pi(z) = f(z) + \gamma^5\kappa(z), \tag{17}$$

with

$$\kappa(z) = (1/2)(f_R(z) - f_L(z)), \quad f(z) = (1/2)(f_R(z) + f_L(z)) \tag{18}$$

and

$$\int \Pi^\dagger(z)\Pi(z) dz = \mathbf{1}_{4\times 4}, \tag{19}$$

where $\Pi(z)$ defines the localization of the fermion along the extra dimension for any helicity state. We note that $\bar{\Psi}(x, z) = \bar{\psi}(x)\bar{\Pi}(z)$ with $\bar{\Pi}(z) = f(z) - \gamma^5\kappa(z)$.

A. Fermionic wave functions in a single braneworld

A single braneworld solution for a trapped fermion can be easily derived from Eqs. (12). By first considering a single kink domain wall, the effective potential derived from Eqs. (13) and (5) becomes

$$W_{L/R}^{1kB}(z) = \frac{1}{\xi^2}\left\{\varepsilon^2 - \varepsilon(\varepsilon \pm 1)\frac{1}{\cosh^2(z/\xi)}\right\}, \tag{20}$$

where $\varepsilon = \lambda\sqrt{2/\chi} = \lambda\eta\xi$. Expression (20) corresponds to a Pöschl-Teller potential for which Eqs. (12) present well-known analytical solutions [6]. Let us recall the first two modes:

(i) For $m = m_0 = 0$

$$f_{0,L} = N_0 \cosh^{-\varepsilon}(z/\xi), \qquad f_{0,R} = 0 \tag{21}$$

(ii) For $m = m_1 = (1/\xi)\sqrt{2\varepsilon - 1}$

$$f_{1,L} = N_1 \cosh^{-\varepsilon}(z/\xi)\sinh(z/\xi), \quad f_{1,R} = N_2 \cosh^{-\varepsilon+1}(z/\xi) \tag{22}$$

with

$$N_0 = \sqrt{\frac{\Gamma(\varepsilon + 1/2)}{\xi\sqrt{\pi}\Gamma(\varepsilon)}}, \quad N_1 = \sqrt{2\varepsilon - 2}\sqrt{\frac{\Gamma(\varepsilon + 1/2)}{\xi\sqrt{\pi}\Gamma(\varepsilon)}}, \quad \text{and} \quad N_2 = \sqrt{\frac{\Gamma(\varepsilon - 1/2)}{\xi\sqrt{\pi}\Gamma(\varepsilon - 1)}}, \tag{23}$$

where $\Gamma(x)$ is the usual Gamma function. Obviously, ε behaves like a coupling constant between the brane and the fermion. The trapping mechanism becomes more and more effective when ε increases (the spatial extensions of solutions (21) and (22) decrease when ε increases). For an antikink-brane, the solutions can be easily deduced from the previous ones through a simple $L \leftrightarrow R$ substitution.

B. Fermionic wave functions in a two-braneworld

A system of two branes can be described by a two-well effective potential $W_{L,R}^{2B}$ derived from Eqs. (6) and (13). The condition $d \gg \xi$ implies that the distance between the two wells is large. When $d \to +\infty$, each well becomes a local one, and it behaves as if there was a single kink (or antikink) in the bulk. In that case, we obtain the local potentials $W^{1kB}_{L/R}(z + d/2)$ [or $W^{1akB}_{L/R}(z - d/2)$] resulting from a single kink (or antikink) distant from the antikink (or kink). If we assume that each brane should possess its own copy of the standard model, it is then legitimate to build the two-brane fermionic solutions from the local one-brane fermionic solutions. This way to proceed is similar to atomic orbital combination used in quantum chemistry to build molecular orbitals. Similarly, for a system of two branes we define the global fermion state as

$$\Psi(x, z) = \Pi_+(z)\psi_+(x) + \Pi_-(z)\psi_-(x), \qquad (24)$$

where $\pm$ denote $z = \pm d/2$, i.e. the location of each brane. The states $\Pi_\pm(z)$ correspond to the fermion eigenstates related to the branes $(+)$ and $(-)$ considered here as independent from each other. Obviously, $\psi_\pm(x)$ will differ from the solutions $\psi(x)$ obtained by solving Eq. (10) for a single braneworld. The states $\Pi_\pm(z)$ can be easily deduced from the one-brane fermionic solutions (see Sec. III A). $\Pi_-(z)$ will use kink-brane fermionic solutions with a translation $z \to z + d/2$ while $\Pi_+(z)$ will use antikink-brane fermionic solutions with a translation $z \to z - d/2$.

IV. GAUGE FIELDS IN DOMAIN WALLS

In the following, we consider the introduction of gauge fields in the model with a special emphasis on electromagnetism. Localizing gauge fields on a domain wall remains a delicate task [12–14]. Among the numerous approaches proposed to localize gauge fields on branes [12–14], the approach of Dvali, Gabadadze, Porrati, and Shifman [13] is quite generic and model independent. The basic ingredient is a bulk vectorial field (a photon-like field), which is minimally coupled to some of the matter fields localized on a brane. Thus, introducing a $U(1)$ gauge field $\mathcal{A}$ in Eq. (2) the five-dimensional action becomes then

$$S = \iint \Big[-\frac{1}{4G^2}\mathcal{F}_{AB}\mathcal{F}^{AB} + \frac{1}{2}g^{AB}(\partial_A\Phi)(\partial_B\Phi) - V(\Phi) + \bar{\Psi}(i\Gamma^A(\partial_A + i\mathcal{A}_A) - \lambda\Phi)\Psi \Big]\sqrt{g}d^5x, \qquad (25)$$

where G is a coupling constant.

Through the quantum fluctuations of the five-dimensional gauge field, the localized fermionic fields induce gauge field localization. Indeed, the gauge field propagator receives corrections from one-loop diagrams with localized matter fields running in the loops. This leads to a four-dimensional kinetic term, which results from the need of a counterterm in the five-dimensional gauge field Lagrangian. An effective four-dimensional gauge field theory results on the brane. The bulk field is then forced to propagate along the three-dimensional space of the brane, at least for distances lower than a critical cosmological distance [13]. The same procedure can be used with more complex domain wall approaches (including those relying on warped metric) where other phenomena can contribute to gauge fields confinement on branes [13].

A. Gauge field in a single braneworld

From Eq. (25) the five-dimensional interaction action between the matter field and the $U(1)$ gauge field takes the form

$$S_{\rm int} = -\int d^4x dz J_A(x, z)\mathcal{A}^A(x, z) = -\int d^4x dz \bar{\Psi}(x, z)\Gamma_A\Psi(x, z)\mathcal{A}^A(x, z). \qquad (26)$$

The four-dimensional current is $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$, with $j^\mu(x) = \int J^\mu(x, z)dz$ and $\int J^5(x, z)dz = 0$. Using Eqs. (16) to (18), it can be shown that $\int \partial_A J^A(x, z)dz = \partial_\mu j^\mu(x)$. From the five-dimensional current conservation $\partial_A J^A(x, z) = 0$, as $\int \partial_A J^A(x, z)dz = 0$, we deduce then that $\partial_\mu j^\mu(x) = 0$, i.e. the four-dimensional current is conserved. This implies the transversality of the loop. Moreover, we chose the Lorentz gauge in the bulk

$$\partial_A\mathcal{A}^A(x, z) = 0. \qquad (27)$$

From the four-dimensional transversality of currents we get $\partial_\mu\mathcal{A}^\mu(x, z) = 0$ [13] and then from Eq. (27) we deduce that $\partial_z\mathcal{A}^z(x, z) = 0$, i.e. $\mathcal{A}^z(x, z)|_{x=Cte} = Cte$. As a consequence, the remaining relevant interaction action between the localized matter field and the bulk vector field is then

$$S_{\rm int} = -\int d^4x \bar{\psi}(x)\gamma_\mu\psi(x)a^\mu(x), \qquad (28)$$

where an effective four-dimensional vector field $a^\mu(x)$ can be defined as

$$a^\mu(x) = (1/2)\int \{f_R^2(z) + f_L^2(z)\}\mathcal{A}^\mu(x, z)dz, \qquad (29)$$

$a^\mu(x)$ acts as a $U(1)$ gauge field in a four-dimensional spacetime. The interaction Lagrangian leads to a supplementary kinetic term induced by localized fermionic one-loop diagrams with two external $a_\mu(x)$ legs [13]. The low-energy action on the brane must then contain the induced term

$$-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} \qquad (30)$$

with

$$F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \qquad (31)$$

where e is the effective coupling constant. Rigorously other corrective terms should be also considered in the action, but we do not discuss them here. Moreover, we are not considering the details of the propagation of the gauge field in the bulk or onto the brane (see Ref. [13]). The matters have already been considered in details in previous works [13]. We just note that the separating variable method leads to write

$$\mathcal{A}(x,z) = \Lambda(z)\mathcal{A}(x), \tag{32}$$

with $\Lambda(z)$ a function that quickly decreases when moving away from the branes [13].

B. Gauge field in a two-braneworld

Let us now consider the introduction of gauge fields in our two-braneworld. As previously explained for fermions, we assume that each brane possesses its own copy of the standard model. The two-brane gauge field solutions can be derived from the local one-brane gauge solutions such that

$$\begin{aligned}\mathcal{A}(x,z) &= \mathcal{A}^{+}(x,z) + \mathcal{A}^{-}(x,z)\\ &= \Lambda_{+}(z)\mathcal{A}_{+}(x) + \Lambda_{-}(z)\mathcal{A}_{-}(x),\end{aligned} \tag{33}$$

where $\pm$ denote $z = \pm d/2$, i.e. the location of each brane. The states $\Lambda_{\pm}(z)$ correspond to the gauge field localized states related to the branes $(+)$ and $(-)$ considered here as independent from each other. Obviously, $\mathcal{A}_{\pm}(x)$ will differ from the solutions $\mathcal{A}(x)$ in Eq. (32) for a single braneworld. The states $\Lambda_{\pm}(z)$ can be easily deduced from the one-brane gauge field solutions. $\Lambda_{-}(z)$ $(\Lambda_{+}(z))$ is related to kink-brane (antikink-brane) gauge field solutions with a translation $z \to z + d/2$ $(z \to z - d/2)$.

On each brane, we get the local four-dimensional vector field

$$a^{\pm}_{\mu}(x) = (1/2)\int \{f^2_{R,\pm}(z) + f^2_{L,\pm}(z)\}\mathcal{A}^{\pm}_{\mu}(x,z)dz. \tag{34}$$

From Eq. (26), it must be noted that the two-brane description of the gauge field and of the fermionic field leads to specific cross terms. For instance, we get

$$(1/2)\int \{f^2_{R,+}(z) + f^2_{L,+}(z)\}\mathcal{A}^{-}_{\mu}(x,z)dz, \tag{35}$$

which can be interpreted as the four-dimensional gauge field induced in the brane $(+)$ by charges localized in the brane $(-)$. In fact, a simple analysis shows that this term is proportional to $\exp(-2(\varepsilon - 1)d/\xi)$, i.e. a charge localized in a brane acts as a kind of ''millicharged'' particle in the second brane. For instance, with $\varepsilon = 2$ and $d/\xi = 22$ (see discussion in Appendix B), a charge q_e localized in the other brane would act in our brane as an effective particle with a charge $q = 10^{-19}q_e$. For such tiny values, the effect can be neglected [28].

V. TWO-LEVEL APPROXIMATION OF FERMION DYNAMICS IN A BRANEWORLD WITH TWO DOMAIN WALLS

Let us now show that the above two-braneworld model reduces to a simple two-level quantum description. At low energy, a brane can be assumed to be an infinitely thin four-dimensional sheet where SM particles live. Therefore, for a single kink (antikink)-brane, the projection of Eq. (8) onto its $f_{L,R}$ eigenstates, will reduce the five-dimensional Dirac equation to a four-dimensional equation with a mass m particle located at $z = -d/2$, for instance, (or $z = d/2$). The projection is equivalent to a dimensional reduction leading to a single four-dimensional Dirac equation. Similarly, for a system of two thin branes, the projection of Eq. (8) on the eigenstates of each independent brane, will lead to two coupled four-dimensional Dirac equations. Although this approach is quite unusual in the present context, it is perfectly well founded. It is exactly the procedure used in quantum chemistry to approximate molecular orbitals by solving the Hamiltonian in the subspace of each atomic eigenstates. Here, the Hamiltonian for a two-braneworld is expressed by using the fermionic eigenstates of each independent branes. This approximation is valid as long as both branes are distant enough in the bulk. Let us now derive the resulting system of four-dimensional coupled Dirac equations.

Taking account of the electromagnetic gauge vector field [see Eq. (25)], the five-dimensional Dirac equation can be expressed in a Schrodinger form:

$$i\partial_0\Psi = H\Psi \tag{36}$$

with

$$\begin{aligned}H = &-i\gamma^0\gamma^\eta(\partial_\eta + i\mathcal{A}_\eta) - \gamma^0\gamma^5(\partial_z + i\mathcal{A}_z)\\ &+ \gamma^0\lambda\Phi + \mathcal{A}_0,\end{aligned} \tag{37}$$

where $\eta = 1, 2, 3$. In the following, we will consider a specific mass state, and we assume that there is no mixing, coupling or interaction between this state and other fermion states of different mass. Therefore, the states $\psi_+(x)$ and $\psi_-(x)$ exhibit the same mass. Moreover, since the terms of higher mass are neglected, our approach remains clearly an approximation of low energy. Using the expression of Ψ given by Eq. (24), the equation (36) can be projected onto the localized states $\Pi_{\pm}(z)$. Introducing then

$$h_{i,j} = \int \Pi^{\dagger}_i(z)H\Pi_j(z)dz \quad \text{and} \quad s = s^{\dagger} = \int \Pi^{\dagger}_{+}(z)\Pi_{-}(z)dz \tag{38}$$

and using a convenient matrix representation, one obtains easily

$$i\partial_0\begin{pmatrix} 1 & s\\ s & 1\end{pmatrix}\begin{pmatrix}\psi_+(x)\\ \psi_-(x)\end{pmatrix} = \begin{pmatrix} h_{+,+} & h_{+,-}\\ h_{-,+} & h_{-,-}\end{pmatrix}\begin{pmatrix}\psi_+(x)\\ \psi_-(x)\end{pmatrix}. \tag{39}$$

Considering

$$\Psi = \begin{pmatrix}\psi_+(x)\\ \psi_-(x)\end{pmatrix} \tag{40}$$

the two-level Dirac-Schrodinger equation

$$i\partial_0\Psi = \tilde{H}\Psi \tag{41}$$

can be easily deduced from Eq. (39) with the two-level Hamiltonian $\tilde{H}$ given by

$$\tilde{H} = \frac{1}{1-s^2} \otimes \begin{pmatrix} h_{+,+} - sh_{-,+} & h_{+,-} - sh_{-,-} \\ h_{-,+} - sh_{++} & h_{-,-} - sh_{+,-} \end{pmatrix}. \tag{42}$$

Using Eqs. (10), (11), (14)–(18), (37), and (38), the terms in Eq. (42) can be simplified (see Appendix A) to give

$$\tilde{H} = \begin{pmatrix} -i\gamma^0\gamma^\eta(\partial_\eta + iqA_\eta^+) + \gamma^0 m + \gamma^0\delta m + qA_0^+ & -\gamma^0\gamma^5 g + \gamma^0 m_r - \gamma^0\gamma^5\Upsilon \\ \gamma^0\gamma^5 g + \gamma^0 m_r + \gamma^0\gamma^5\bar{\Upsilon} & -i\gamma^0\gamma^\eta(\partial_\eta + iqA_\eta^-) + \gamma^0 m + \gamma^0\delta m + qA_0^- \end{pmatrix} \tag{43}$$

with $A_\mu^\pm$ the electromagnetic fields of the brane $(+)$ or $(-)$, q the electric charge of the fermion, and where

$$g = \lambda \int \{\Phi_- + \Delta\Phi\}\{f_-(z)\kappa_+(z) - \kappa_-(z)f_+(z)\}dz,$$
$$m_r = \lambda \int \{\Phi_- + \Delta\Phi\}\{f_-(z)f_+(z) - \kappa_-(z)\kappa_+(z)\}dz,$$
$$\delta m = \lambda \int \Phi_-\{f_+^2(z) - \kappa_+^2(z)\}dz \tag{44}$$

and

$$\Upsilon = \varphi + \gamma^5\phi, \qquad \bar{\Upsilon} = \varphi^* - \gamma^5\phi^*, \tag{45}$$

where φ and ϕ are the scalar components of the off-diagonal part Υ of the effective gauge field such that

$$\phi = i\int \{f_+(z)\kappa_-(z) - \kappa_+(z)f_-(z)\}\{\mathcal{A}_z^+(x,z) + \mathcal{A}_z^-(x,z)\}dz$$

and

$$\varphi = i\int \{f_+(z)f_-(z) - \kappa_+(z)\kappa_-(z)\}\{\mathcal{A}_z^+(x,z) + \mathcal{A}_z^-(x,z)\}dz. \tag{46}$$

An interpretation of those off-diagonal gauge terms as well as the way to deal with them will be discussed in the next section.

Let us apply a convenient $SU(2)$ rotation such that

$$\begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix} \rightarrow \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix}. \tag{47}$$

Back to the Dirac form, Eq. (41) then reads

$$\begin{pmatrix} i\gamma^\mu(\partial_\mu + iqA_\mu^+) - m - \delta m & ig\gamma^5 - im_r + i\gamma^5\Upsilon \\ ig\gamma^5 + im_r + i\gamma^5\bar{\Upsilon} & i\gamma^\mu(\partial_\mu + iqA_\mu^-) - m - \delta m \end{pmatrix}\Psi = 0. \tag{48}$$

As a consequence of the first order approximation (see Appendix A), g is only related to the first-order derivative ∂_z along the continuous extra dimension, while m_r is only related to the scalar field Φ. From solutions of Eqs. (11) [see Sec. III A], g, m_r, and δm can be easily estimated as shown in Appendix B.

Let us now show that Eq. (48) describing the dynamics of a fermion in a two domain-walls setup is equivalent to that of a particle embedded in a noncommutative two-sheeted spacetime.

VI. "NONCOMMUTATIVE" TWO-SHEETED SPACETIME INTERPRETATION OF THE TWO-LEVEL APPROXIMATION

A. Noncommutative two-sheeted spacetime

In Refs. [21,22], a model describing the quantum dynamics of fermions in a two-sheeted spacetime (i.e. a two-braneworld) has been proposed. Such a universe corresponds to the product of a four-dimensional continuous manifold with a discrete two-point space and can be seen as a five-dimensional universe with a fifth dimension reduced to two points with coordinates $\pm\delta/2$ (both sheets are separated by a phenomenological distance δ). Mathematically, the model relies on a bi-Euclidean space $X = M_4 \times Z_2$ in which any smooth function belong to the algebra $A = C^\infty(M) \oplus C^\infty(M)$ and can be adequately represented by a 2×2 diagonal matrix F such that

$$F = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}. \tag{49}$$

In the noncommutative formalism, the expression of the exterior derivative $D = d + Q$, where d acts on M_4 and Q on the Z_2 internal variable, has been given by A. Connes [17]: D: $(f_1, f_2) \rightarrow (df_1, df_2, g(f_2 - f_1), g(f_1 - f_2))$ with $g = 1/\delta$. Viet and Wali [19] have proposed a representation of D acting as a derivative operator and fulfilling the above requirements (see also [20]). Because of the specific geometrical structure of the bulk, this operator is given by

$$D_\mu = \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix}, \quad \mu = 0,1,2,3 \quad \text{and} \quad D_5 = \begin{pmatrix} 0 & g \\ -g & 0 \end{pmatrix}. \tag{50}$$

Where the term g acts as a finite difference operator along the discrete dimension. Using (50), one can build the Dirac operator defined as

$$\not{D} = \Gamma^N D_N = \Gamma^\mu D_\mu + \Gamma^5 D_5. \tag{51}$$

By considering the following extension of the gamma matrices (we are working in the Hilbert space of spinors, see [17])

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix} \quad \text{and} \quad \Gamma^5 = \begin{pmatrix} \gamma^5 & 0 \\ 0 & -\gamma^5 \end{pmatrix}, \tag{52}$$

where γ^μ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ are the usual Dirac matrices, it can be easily shown that the Dirac operator given by Eq. (51) has the following self adjoint realization:

$$\not{D} = \begin{pmatrix} \not{D}_+ & g\gamma^5 \\ g\gamma^5 & \not{D}_- \end{pmatrix} = \begin{pmatrix} \gamma^\mu\partial_\mu & g\gamma^5 \\ g\gamma^5 & \gamma^\mu\partial_\mu \end{pmatrix}. \tag{53}$$

In some noncommutative two-sheeted models [18], the off-diagonal terms proportional to g are often related to the particle mass through the Higgs field. As shown in previous works [21], g can also be considered as a constant geometrical field and takes the same value for each particle. We can therefore introduce a mass term as in the standard Dirac equation, but more general, i.e.

$$M = \begin{pmatrix} m\mathbf{1}_{4\times4} & m_c\mathbf{1}_{4\times4} \\ m_c^*\mathbf{1}_{4\times4} & m\mathbf{1}_{4\times4} \end{pmatrix}, \tag{54}$$

where "$*$" denotes the complex conjugate. The two-sheeted Dirac equation writes

$$\not{D}_{\text{dirac}}\Psi = (i\not{D} - M)\Psi = (i\Gamma^N D_N - M)\Psi = \begin{pmatrix} i\gamma^\mu\partial_\mu - m & ig\gamma^5 - m_c \\ ig\gamma^5 - m_c^* & i\gamma^\mu\partial_\mu - m \end{pmatrix}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0, \tag{55}$$

with

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

the two-sheeted wave function. In this notation, the indices "+" and "−" are purely conventional and simply allow to discriminate the two sheets (or branes) embedded in the five-dimensional bulk. It can be noticed that by virtue of the two-sheeted structure of spacetime, the wave function ψ of the fermion is split into two components, each component living on a distinct spacetime sheet. If one considers the (−) sheet to be our brane, the ψ_+ part of the wave function, in the (+) sheet, can be considered as a hidden particle component.

B. Gauge fields in the two-sheeted spacetime

Pursuing with the approach introduced in the Sec. IV, we are now illustrating how the electromagnetic fields behaves in the present formalism. It should be emphasized that the results presented here for electromagnetism could be extended to other interactions as well, especially electroweak interactions and chromodynamics. To be consistent with the structure of the Dirac field Ψ in Eq. (55), the usual $U(1)$ electromagnetic gauge field has to be replaced by an extended $U(1) \otimes U(1)$ gauge field. The group representation is therefore

$$G = \begin{pmatrix} \exp(-iq\Lambda_+) & 0 \\ 0 & \exp(-iq\Lambda_-) \end{pmatrix}. \tag{56}$$

We are looking for an appropriate gauge field such that the covariant derivative becomes $\not{D}_A \to \not{D} + \not{A}$ with the following gauge transformation rule:

$$\not{A}' = G\not{A}G^\dagger - iG[\not{D}_{\text{dirac}}, G^\dagger]. \tag{57}$$

A convenient choice is (see Refs. [18–20])

$$\not{A} = \begin{pmatrix} iq\gamma^\mu A_\mu^+ & \gamma^5\Upsilon \\ \gamma^5\bar{\Upsilon} & iq\gamma^\mu A_\mu^- \end{pmatrix}, \tag{58}$$

where γ^μ are the usual Dirac matrices and with

$$\Upsilon = \varphi + \gamma^5\phi, \qquad \bar{\Upsilon} = \varphi^* - \gamma^5\phi^*, \tag{59}$$

where φ and ϕ are the scalar components of the field Υ. Those terms are fully equivalent to those in Eq. (45). If Υ is different from zero, each charged particle of each brane becomes sensitive to the electromagnetic fields of both branes irrespective of their localization in the bulk. This kind of exotic interactions has been considered previously in literature within the framework of mirror matter [29] and is not covered by the present paper. Moreover, to be consistent with known physics, at least at low energies, Υ is necessarily tiny (whereas $qA_\mu^\pm$ needs not to be). This is theoretically corroborated by Eq. (57), which shows that during each gauge transformation, $|\varphi|$ (respectively, $|\phi|$) varies with an amplitude of order g (respectively, $|m_c|$) whatever Λ_+ and Λ_-.

Using the covariant derivative $\not{D}_A \to \not{D} + \not{A}$ and according to expression (58), the electromagnetic field can be easily introduced in the two-brane Dirac equation [Eq. (55)]

$$(i\not{D}_A - M)\Psi = \begin{pmatrix} i\gamma^\mu(\partial_\mu + iqA_\mu^+) - m & ig\gamma^5 - m_c + i\gamma^5\Upsilon \\ ig\gamma^5 - m_c^* + i\gamma^5\bar{\Upsilon} & i\gamma^\mu(\partial_\mu + iqA_\mu^-) - m \end{pmatrix} \times \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0. \tag{60}$$

C. Noncommutative two-sheeted spacetime vs two domain walls

It can immediately be noticed that Eqs. (48) and (60) are globally similar. In the two domain walls approximation, $m \to m + \delta m$ and $m_c = im_r$. The similarity between both equations was not expected *a priori*, and it suggests that Eq. (60) is quite generic in a context of a two-level discretization of a five-dimensional fermionic field. Note that in the model introduced in Refs. [21,22], $m_c = 0$.

One notes that in the noncommutative two-sheeted approach, the term proportional to g [see (50)] is, by definition, a first order discrete derivative along the Z_2 extra dimension. Similarly, in the domain-wall approach, g is related to the continuous extra dimension R_1 through an overlap integral (see Sec. V and Appendix A). From a pictorial perspective (see Fig. 1), the two-braneworld system "collapses" from a continuous description ($M_4 \times R_1$) to a discrete one ($M_4 \times Z_2$). Obviously the similarity between Eqs. (48) and (60) arises from the separation ansatz made in Eq. (24) where the five-dimensional fermion is restricted to the sum of two localized (i.e. four-dimensional) eigenmodes with the same mass (see Sec. III A). Thus, by construction, there are only two four-dimensional states in the model, and it seems that the continuous extra dimension has been reduced to two points. As a consequence, the continuous real extra dimension (and its continuous derivative) is replaced by an effective phenomenological discrete extra dimension (with its discrete derivative). The effective distance between branes $\delta = 1/g$ in noncommutative geometry is then related to the real extra dimension through the integral in Eqs. (44) [see Appendix B].

One also notes that the five-dimensional $U(1)$ bulk gauge field is substituted by an effective $U(1) \otimes U(1)$ gauge field acting in the $M_4 \times Z_2$ spacetime (see Fig. 1). This is clearly illustrated through the comparison between the gauge terms in Eq. (48) and in Eq. (60). This result could be expected *a priori*. Indeed, if one considers a domain wall at low energy, any valid gauge theory on the bulk should lead to retrieve an effective $U(1)$ gauge field on the brane to be consistent with known physics. As a consequence, for a set of two domain walls, it is not surprising to obtain an effective $U(1) \otimes U(1)$ theory. It is also interesting to note that the extra-dimensional component $\mathcal{A}_z$ of the bulk gauge field is related to the off-diagonal part Υ of the gauge field in the two-sheeted spacetime formalism [see Eqs. (45) and (46)].

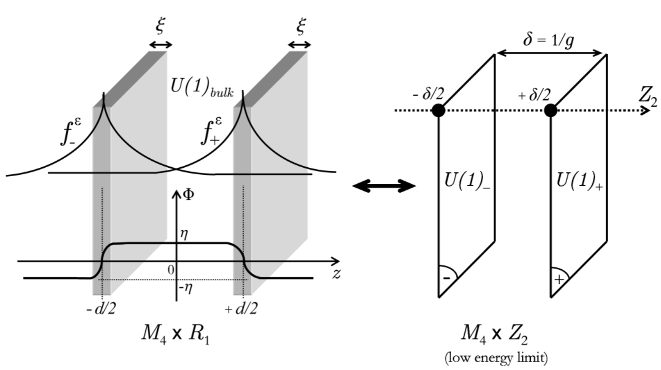

FIG. 1. The two domain walls in a $M_4 \times R_1$ geometry are approximated by a $M_4 \times Z_2$ two-sheeted spacetime with an effective distance $\delta = \delta(\xi, d, \varepsilon)$. The spatial extensions of the fermion wave functions $f_\pm^\varepsilon$ depend on ε. The bulk $U(1)_{\rm bulk}$ gauge group can be substituted by an effective $U(1)_+ \otimes U(1)_-$ gauge group where $U(1)_+$ (respectively, $U(1)_-$) acts on the brane (+) [respectively, (−)]. In this paper, we simply refer to $U(1)_{\rm bulk}$ as $U(1)$ and to $U(1)_+ \otimes U(1)_-$ as $U(1) \otimes U(1)$.

However, the link between the noncommutative two-sheeted spacetime and a system of two domain walls must be consider cautiously. The validity of such a link rests on the following conditions:

(i) As previously explained, Eq. (48) is derived for a single mass state. We have just considered the lowest massive left-, right-states only, i.e. the localized $n = 1$ fermionic states (see Sec. III A). Of course the $n > 1$ modes could be used as well. Since we have neglected the role of the heaviest fields, our model is therefore valid only for low energies. From that point of view, the two-sheeted spacetime can be seen as a simple low-energy approximation of a two domain-walls system (see Fig. 1). Nevertheless, a more general calculation would retain all mass states and would be quite different from the presently considered noncommutative model. Each domain wall in the bulk would be then approximated by a set of strongly coupled sheets, each one being related to a specific mass instead of a single one. Moreover, excepted at high energy, this would not change the main phenomelogical results of the present work (hereafter discussed).

(ii) As mentioned in Sec. V, the two-level approximation is valid as long as both branes are assumed to be distant enough in the bulk. Indeed, the approximation assumes that quadratic terms (and upper) implying overlap integrals can be neglected (see Appendices A and B).

VII. NONRELATIVISTIC LIMIT AND PHENOMENOLOGY

As explained in the introduction, we are mainly concerned by low energies phenomena occurring at a nonrelativistic scale. To derive the nonrelativistic limit of the two-brane Dirac equation, we just observe that Eq. (60) can also be written as

$$\begin{pmatrix} i\gamma^\mu(\partial_\mu + iqA_\mu^+) - m & i\tilde{g}\gamma^5 - \tilde{m}_c \\ i\tilde{g}^*\gamma^5 - \tilde{m}_c^* & i\gamma^\mu(\partial_\mu + iqA_\mu^-) - m \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0 \qquad (61)$$

with

$$\tilde{g} = g + \varphi, \qquad \tilde{m}_c = m_c - i\phi. \qquad (62)$$

We have just replaced the field Υ and the coupling parameters g and m_c by the effective fields $\tilde{g}$ and $\tilde{m}_c$ as shown by Eqs. (62). Without loss of generality, we will consider now that $\tilde{g} \approx g$ and $\tilde{m}_c \approx m_c$ since $|\varphi|$ (respectively, $|\phi|$) should not exceed g (respectively, $|m_c|$) as explained before. This choice allows to further simplify the model. It is somewhat equivalent to set the off-diagonal term Υ to zero. With such a choice, we simply assume that the electromagnetic field of a brane couples only with the particles belonging to the same brane. Each brane possesses its own current and charge density distribution as sources of the local electromagnetic fields. On the two branes live then the distinct A_μ^+ and A_μ^- electromagnetic fields. The photon fields $A_\mu^\pm$ behave independently from each other and are totally trapped in their original brane in accordance with observations: photons belonging to a given brane are not able to reach the other brane. As a noticeable consequence, the structures belonging to the branes are mutually invisible by local observers. Without loss of generality, Eq. (61) can be recast as

$$\begin{pmatrix} i\gamma^\mu(\partial_\mu + iqA_\mu^+) - m & ig\gamma^5 - m_c \\ ig\gamma^5 - m_c^* & i\gamma^\mu(\partial_\mu + iqA_\mu^-) - m \end{pmatrix} \times \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0. \tag{63}$$

Following the well-known standard procedure, a two-brane Pauli equation can then be derived

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \{\mathbf{H}_0 + \mathbf{H}_{cm} + \mathbf{H}_c + \mathbf{H}_s\}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \tag{64}$$

where ψ_+ and ψ_- correspond to the wave functions in the $(+)$ and $(-)$ branes, respectively. ψ_+ and ψ_- are here Pauli spinors. The Hamiltonian $\mathbf{H}_0$ is a block-diagonal matrix

$$\mathbf{H}_0 = \begin{pmatrix} \mathbf{H}_+ & 0 \\ 0 & \mathbf{H}_- \end{pmatrix}, \tag{65}$$

where each block is simply the classical Pauli Hamiltonian expressed in each of the branes:

$$\mathbf{H}_\pm = -\frac{\hbar^2}{2m}\left(\nabla - i\frac{q}{\hbar}\mathbf{A}_\pm\right)^2 + g_s\mu\frac{1}{2}\sigma\cdot\mathbf{B}_\pm + V_\pm \tag{66}$$

such that $\mathbf{A}_+$ and $\mathbf{A}_-$ correspond to the magnetic vector potentials in the branes $(+)$ and $(-)$, respectively. The same convention is applied to the magnetic fields $\mathbf{B}_\pm$ and to the potentials $V_\pm$. $g_s\mu$ is the magnetic moment of the particle with g_s the gyromagnetic factor and μ the magneton. In addition to these "classical" terms, the two-brane Hamiltonian comprises also new terms specific of the two-braneworld:

$$\mathbf{H}_{cm} = igg_s\mu\frac{1}{2}\begin{pmatrix} 0 & -\sigma\cdot\{\mathbf{A}_+ - \mathbf{A}_-\} \\ \sigma\cdot\{\mathbf{A}_+ - \mathbf{A}_-\} & 0 \end{pmatrix} \tag{67}$$

and

$$\mathbf{H}_c = \begin{pmatrix} 0 & m_c c^2 \\ m_c^* c^2 & 0 \end{pmatrix} \tag{68}$$

and

$$\mathbf{H}_s = \frac{g^2\hbar^2}{2m}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{69}$$

It can be noticed that $\mathbf{H}_s$ is constant and vanishes through a convenient energy rescaling. By contrast, $\mathbf{H}_c$ and $\mathbf{H}_{cm}$ are nonconventional Hamiltonian components whose effects are now discussed.

A. Spontaneous oscillations between branes?

$\mathbf{H}_c$ is a constant term resulting from the off-diagonal mass terms [Eq. (54)]. It can be responsible for free spontaneous oscillations between the two branes. Let us illustrate this. For sake of simplicity we assume that $\mathbf{B}_\pm = \mathbf{0}$ and $\mathbf{A}_\pm = \mathbf{0}$. $\mathbf{H}_\pm$ reduces then to $\mathbf{H}_\pm = V_\pm$ and $\mathbf{H}_{cm} = 0$. From Eq. (64), it can be shown that a particle initially ($t = 0$) localized in our brane will have a probability to be located in the other brane at time t given by

$$P(t) = \frac{4\Omega_c^2}{\Omega_0^2 + 4\Omega_c^2}\sin^2((1/2)\sqrt{\Omega_0^2 + 4\Omega_c^2}t), \tag{70}$$

where $\Omega_c = |m_c|c^2/\hbar$ and $\Omega_0 = (V_+ - V_-)/\hbar$. Equation (70) shows that the particle undergoes Rabi-like oscillations between both branes. $\Omega_0\hbar$ is an effective potential mimicking the interactions of the particle with its environment [21]. $\Omega_0\hbar$ might contain the contribution of atomic nuclei electrostatic fields or the Earth's gravitational field, for instance [21]. An important point is that the oscillations are strongly suppressed when $\Omega_0\hbar$ becomes greater than $|m_c|c^2$, i.e. when the particle is strongly interacting with its environment through $\mathbf{H}_0$. As a consequence, these spontaneous oscillations will probably be hardly observed.

B. Induced matter swapping between branes

$\mathbf{H}_{cm}$ is a geometrical coupling involving electromagnetic fields of the two branes. $\mathbf{H}_{cm}$ vanishes for null magnetic vector potentials. $\mathbf{H}_{cm}$ also implies Rabi-like oscillations between the branes. This effect was previously considered in previous papers in which the reader will find more detailed explanations [21]. It is only important here to remind that Eq. (64) holds resonant solutions for a magnetic vector potential rotating with an angular frequency ω. One may consider, for instance, a neutral particle, endowed with a magnetic moment, initially ($t = 0$) localized in our brane in a region of curlless rotating magnetic vector potential such that $\mathbf{A}_- = A_p\mathbf{e}(t)$ and $\mathbf{A}_+ = 0$, with $\mathbf{e}(t) = (\cos\omega t, \sin\omega t, 0)$. ω is the angular frequency of the field $\mathbf{A}_-$ and can be possibly null (static field case). Let us assume that the conventional part of the Pauli Hamiltonian $\mathbf{H}_\pm$ can be written as $\mathbf{H}_\pm = V_\pm$. Moreover, $\mathbf{H}_c$ is neglected relative to $\mathbf{H}_{cm}$. From Eq. (64), it can be shown that any particle initially in a spin-down state for instance (according to $\mathbf{e}_3 = (0, 0, 1)$) and localized in our brane at $t = 0$ can be detected in the second brane at time t with a probability [21]

$$P(t) = \frac{4\Omega_p^2}{(\Omega_0 - \omega)^2 + 4\Omega_p^2} \sin^2((1/2) \times \sqrt{(\Omega_0 - \omega)^2 + 4\Omega_p^2} t), \quad (71)$$

where $\Omega_p = g g_s \mu A_p/(2\hbar)$ and $\Omega_0 = (V_+ - V_-)/\hbar$. In addition, in the second brane, the particle is then in a spin-up state. $\Omega_0\hbar$ is still an effective potential mimicking the interactions of the particle with its environment [21]. Equation (71) shows how the induced matter swapping between branes occurs through a Rabi-like phenomenon. The resonant exchange occurs whenever the magnetic vector potential rotates with an angular frequency $\omega = \Omega_0$. It is clear that the situation described in Eq. (71) remains rather simplistic. More realistic descriptions to achieve experimental conditions of matter swapping are suggested elsewhere [30]. It can be easily checked that a static field case allows matter swapping to occur as well. However, the amplitude of the vector potential must be huge to overcome the particle confinement induced by the environment [21]. From an experimental point of view, the resonant mechanism appears as a worth studying alternative.

Again, it is necessary to stress that the matter swapping mechanism described here depends on the $\mathbf{H}_{cm}$ term only. Since this term is predicted by distinct mathematical approaches, we expect that induced matter swapping between branes might be a generic phenomenon of any five-dimensional model containing two lower-dimensional (four-dimensional) sheets.

VIII. DISCUSSION AND CONCLUSIONS

(1) Rewriting the five-dimensional continuous Dirac equation in a two-level form presents many advantages:

(i) It allows a dramatic simplification of the equations and allows a better understanding of the quantum behavior of particles in a two-braneworld setup.

(ii) Connes *et al.* have shown the great potential of noncommutative geometries [17], especially of two-sheeted spacetime representations which are suitable to recover the standard model of particles [17,18]. In the present paper, a formal equivalence between domain-walls approaches and certain noncommutative geometries has been shown. Noncommutativity appears here as a consequence of a two-level simplification. Nevertheless, the arena where physical events take place still remains commutative. This unexpected bridge between domain walls and noncommutativity clearly deserves further studies.

(2) In the present model, the kink-domain wall localized at $z = -d/2$ [i.e. the $(-)$ brane] undergoes left-handed neutrinos, while the antikink-domain wall localized at $z = d/2$ [i.e. the $(+)$ brane] undergoes right-handed neutrinos. As a consequence, and due to the doubling of the wave function, this can be seen as a reminiscence of the mirror-matter concept [31]. Nevertheless, while it is true that hidden sector models and present approach share several common points, it is equally true that they differ in many ways:

(i) In the mirror-matter formalism, there is only one four-dimensional manifold that justifies the left/right parity by introducing implicit new internal degrees of freedom to particles. In the present work, it can be noted that the number of particle families remains unchanged but the particles have now access granted to two distinct branes.

(ii) Moreover, in the mirror-matter approach, the mixing between our world and the mirror world occurs through a photon/mirror photon kinetic mixing [29] (gravitation also is assumed to mediate interactions between matter and mirror matter but it is not relevant in the present discussion). Nevertheless, the present two-brane structure demonstrates the existence of oscillations without recourse to a photon/mirror photon kinetic mixing.

(3) It is suggested that the "swapping effect" (see Sec. VII B) involving induced Rabi-like oscillations of matter between adjacent branes might be a common feature of any braneworld theory involving more than one brane in the bulk. Indeed, the method used in Sec. V to obtain the relevant Eq. (48) is quite general and it does not rely on any assumption concerning the domain walls. The Hamiltonian term $\mathbf{H}_{cm}$ [see Eq. (67)] is due to the $ig\gamma^5$ terms in Eq. (63), which are related to the overlap integral over the fifth dimension of the product of the extra-dimensional fermionic wave functions related to each brane. We expect that in more complex domain-walls models (involving warped metric, for instance, like in Randall-Sundrum braneworlds), we would just have obtained different expressions for g, m_r and δm [see Eqs. (44)]. Finally, if one considers two parallel three-branes in a bulk with more than five dimensions (say $3 + N + 1$, with $N > 1$), the situation should be quite similar to that described in this paper. Indeed, considering a bijective relation between the two three-branes, one can build a fiber bundle linking each point of the branes. Each fiber allows to define a preferential fifth dimension connecting both branes. Moreover, since the fermionic wave functions spread over the $N - 1$ other extra dimensions, they must quickly decrease when going away from the branes. The system would therefore reduce from $3 + N + 1$ to $3 + 1 + 1$ dimensions similar to the setup considered in this paper [9].

As a consequence, we conjecture that at low energy, any multidimensional setup containing two branes can be described by a two-sheeted spacetime in the formalism of the noncommutative geometry. As a result, we also conjecture that the so-called "swapping effect" originally predicted in the context of $M_4 \times Z_2$ geometries [21,22] might be a model-independent feature of any multidimensional setup containing at least two branes. Note that, due to the links expected between domain walls and string theories [26], one might also wonder to what extent the "swapping effect" described here could be hidden in string theories as well.

APPENDIX A: DERIVATION OF THE EXPLICIT EXPRESSION OF $\tilde{H}$

In the following, we detail how Eq. (43) can be derived from Eq. (42). In Eq. (42), the two-level Hamiltonian $\tilde{H}$ was written as

$$\tilde{H} = \frac{1}{1-s^2} \otimes \begin{pmatrix} h_{+,+} - sh_{-,+} & h_{+,-} - sh_{-,-} \\ h_{-,+} - sh_{++} & h_{-,-} - sh_{+,-} \end{pmatrix}. \quad \text{(A1)}$$

Using Eqs. (6), (10), (11), (14)–(18), (37), and (38), the terms of (A1) can be expressed as follows:

$$s = a + \gamma^5 b \quad \text{(A2)}$$

with

$$a = \int \{f_+(z)f_-(z) + \kappa_-(z)\kappa_+(z)\}dz, \qquad b = \int \{f_-(z)\kappa_+(z) + \kappa_-(z)f_+(z)\}dz \quad \text{(A3)}$$

and

$$h_{i,j} = \left\{\int \Pi_i^\dagger(z)\Pi_j(z)dz\right\}\{-i\gamma^0\gamma^\eta\partial_\eta\} + \gamma^0\gamma^5\alpha_{i,j} + \gamma^0\beta_{i,j} + \gamma^0\gamma^\mu G_{i,j} + \gamma^0\gamma^5\tilde{G}_{i,j} \quad \text{(A4)}$$

with

$$\alpha_{i,j} = \lambda\int\{\Phi_i(z) + \Delta\Phi\}\{f_i(z)\kappa_j(z) - \kappa_i(z)f_j(z)\}dz - mb \quad \text{if } i \neq j, \quad \text{and} \quad \alpha_{i,i} = 0 \quad \text{(A5)}$$

and

$$\beta_{i,j} = \lambda\int\{\Phi_i(z) + \Delta\Phi\}\{f_i(z)f_j(z) - \kappa_i(z)\kappa_j(z)\}dz + ma \quad \text{if } i \neq j,$$
$$\beta_{i,i} = m + \lambda\int \Phi_j(z)\{f_i^2(z) - \kappa_i^2(z)\}dz \quad \text{where } i \neq j. \quad \text{(A6)}$$

We note that, according to Eqs. (5) and (6), $\Phi_\pm(z) = \mp\Phi(z \mp d/2)$ and $\Delta\Phi = -\eta$. $\alpha_{i,j}$ is related to the projection of the first derivative ∂_z term of H [see Eq. (37)] onto the independent states of each brane. $\beta_{i,j}$ is related to the projection of the scalar field Φ term of H onto the independent states of each brane.

Recalling that $\mathcal{A}(x,z) = \mathcal{A}^+(x,z) + \mathcal{A}^-(x,z)$ (see Sec. IV B) and using $\mathcal{A} = \gamma^\mu \mathcal{A}_\mu$, one also gets

$$G_{i,j} = \mathcal{A}_{i,j}^+ + \mathcal{A}_{i,j}^- - i\mathcal{A}_{z,i,j}^+ - i\mathcal{A}_{z,i,j}^- \quad \text{and} \quad \tilde{G}_{i,j} = \mathcal{B}_{i,j}^+ + \mathcal{B}_{i,j}^- + i\mathcal{B}_{z,i,j}^+ + i\mathcal{B}_{z,i,j}^- \quad \text{(A7)}$$

with

$$\mathcal{A}_{i,j}^\pm = \int\{f_i(z)f_j(z) + \kappa_i(z)\kappa_j(z)\}\mathcal{A}^\pm dz,$$
$$\mathcal{A}_{z,i,j}^\pm = \int\{f_i(z)\kappa_j(z) - \kappa_i(z)f_j(z)\}\mathcal{A}_z^\pm dz,$$
$$\mathcal{B}_{i,j}^\pm = -\int\{f_i(z)\kappa_j(z) + \kappa_i(z)f_j(z)\}\mathcal{A}^\pm dz,$$
$$\mathcal{B}_{z,i,j}^\pm = -\int\{f_i(z)f_j(z) - \kappa_i(z)\kappa_j(z)\}\mathcal{A}_z^\pm dz. \quad \text{(A8)}$$

Hence, the Hamiltonian (A1) can be written as

$$\begin{aligned}\tilde{H} = &\begin{pmatrix} -i\gamma^0\gamma^\eta\partial_\eta & 0 \\ 0 & -i\gamma^0\gamma^\eta\partial_\eta \end{pmatrix} + \frac{1}{1-s^2} \\ &\otimes \begin{pmatrix} \gamma^0(\beta_{+,+} + b\alpha_{-,+} - a\beta_{-,+}) - \gamma^0\gamma^5(a\alpha_{-,+} - b\beta_{-,+}) & \gamma^0\gamma^5(\alpha_{+,-} + b\beta_{-,-}) + \gamma^0(\beta_{+,-} - a\beta_{-,-}) \\ \gamma^0\gamma^5(\alpha_{-,+} + b\beta_{+,+}) + \gamma^0(\beta_{-,+} - a\beta_{+,+}) & \gamma^0(\beta_{-,-} + b\alpha_{+,-} - a\beta_{+,-}) - \gamma^0\gamma^5(a\alpha_{+,-} - b\beta_{+,-}) \end{pmatrix} \\ &+ \frac{1}{1-s^2} \otimes \begin{pmatrix} \gamma^0(G_{+,+} - aG_{-,+} - b\tilde{G}_{-,+}) + \gamma^0\gamma^5(\tilde{G}_{+,+} - a\tilde{G}_{-,+} - bG_{-,+}) & \gamma^0(G_{+,-} - aG_{-,-} - b\tilde{G}_{-,-}) + \gamma^0\gamma^5(\tilde{G}_{+,-} - a\tilde{G}_{-,-} - bG_{-,-}) \\ \gamma^0(G_{-,+} - aG_{+,+} - b\tilde{G}_{+,+}) + \gamma^0\gamma^5(\tilde{G}_{-,+} - a\tilde{G}_{+,+} - bG_{+,+}) & \gamma^0(G_{-,-} - aG_{+,-} - b\tilde{G}_{+,-}) + \gamma^0\gamma^5(\tilde{G}_{-,-} - a\tilde{G}_{+,-} - bG_{+,-}) \end{pmatrix}. \end{aligned} \quad \text{(A9)}$$

The integrals a, b, $\alpha_{i,j}$ (with $i \neq j$) and $\beta_{i,j}$ (with $i \neq j$) involve the overlapping of the fermionic wave functions of each brane. These terms have to be small enough to act as a correction and for a sake of simplicity, we assume that higher-order terms (i.e. a^2, b^2, ab, $a\alpha_{\mp,\pm}$, $b\alpha_{\mp,\pm}$, $a\beta_{\mp,\pm}$, $b\beta_{\mp,\pm}$, ...) can be fairly neglected.

The terms related to the gauge field [see Eqs. (A7) and (A8)] must be considered with caution. Indeed, since the gauge terms are related to quantum corrections of the five-dimensional gauge field, one must take care of undesirable anomalies that could break the gauge invariance or the Hermiticity of the two-level Hamiltonian. Assuming for instance that the Hamiltonian remains hermitic requires these undesirable terms to vanish. In practice, the terms inducing anomalies can be cancelled by adding convenient extra fermion species, or compensated through some quantum number flows in the bulk [32]. In addition, the integrals $G_{i,j}$ and $\tilde{G}_{i,j}$ (with $i \neq j$) also imply an overlapping of the fermionic wave functions of each brane. These terms should act as perturbative terms. For similar reasons, the higher-order terms $aG_{i,j}$, $bG_{i,j}$, $aG_{i,j}$ and $bG_{i,j}$ can be fairly neglected. Keeping the only relevant gauge field terms (see Sec. IV), the Hamiltonian (A9) reduces then to

$$\tilde{H} = \begin{pmatrix} -i\gamma^0\gamma^\eta\partial_\eta + \gamma^0 m + \gamma^0\delta m_+ + \gamma^0\mathcal{A}^+_{+,+} & \gamma^0\gamma^5 g_{+,-} + \gamma^0 m_{+,-} - \gamma^0\gamma^5\Upsilon \\ \gamma^0\gamma^5 g_{-,+} + \gamma^0 m_{-,+} + \gamma^0\gamma^5\bar{\Upsilon} & -i\gamma^0\gamma^\eta\partial_\eta + \gamma^0 m + \gamma^0\delta m_- + \gamma^0\mathcal{A}^-_{-,-} \end{pmatrix}, \tag{A10}$$

where

$$g_{\pm,\mp} = \mp\lambda\int\{\Phi_\pm + \Delta\Phi\}\{f_-(z)\kappa_+(z) - \kappa_-(z)f_+(z)\}dz,$$
$$m_{\pm,\mp} = \lambda\int\{\Phi_\pm + \Delta\Phi\}\{f_-(z)f_+(z) - \kappa_-(z)\kappa_+(z)\}dz,$$
$$\delta m_\pm = \lambda\int\Phi_\mp\{f^2_\pm(z) - \kappa^2_\pm(z)\}dz \tag{A11}$$

and

$$\Upsilon = \varphi + \gamma^5\phi, \qquad \bar{\Upsilon} = \varphi^* - \gamma^5\phi^* \tag{A12}$$

with

$$\phi = i\int\{f_+(z)\kappa_-(z) - \kappa_+(z)f_-(z)\}\{\mathcal{A}^+_z + \mathcal{A}^-_z\}dz,$$
$$\varphi = i\int\{f_+(z)f_-(z) - \kappa_+(z)\kappa_-(z)\}\{\mathcal{A}^+_z + \mathcal{A}^-_z\}dz. \tag{A13}$$

Since $\tilde{H}$ must be hermitic, the properties of g_{ij} and m_{ij} can be easily deduced. We get $g_{+,-} = -g_{-,+} = -g$ and $m_{-,+} = m_{+,-} = m_r$. Because of the symmetry between both branes, we get $\delta m_\pm = \delta m$. Finally, the effective Hamiltonian reads

$$\tilde{H} = \begin{pmatrix} -i\gamma^0\gamma^\eta(\partial_\eta + iqA^+_\eta) + \gamma^0 m + \gamma^0\delta m + qA^+_0 & -\gamma^0\gamma^5 g + \gamma^0 m_r - \gamma^0\gamma^5\Upsilon \\ \gamma^0\gamma^5 g + \gamma^0 m_r + \gamma^0\gamma^5\bar{\Upsilon} & -i\gamma^0\gamma^\eta(\partial_\eta + iqA^-_\eta) + \gamma^0 m + \gamma^0\delta m + qA^-_0 \end{pmatrix}, \tag{A14}$$

where we have assumed that

$$qA^\pm_\mu = \int\{f^2_\pm(z) + \kappa^2_\pm(z)\}\mathcal{A}^\pm_\mu(x,z)dz \tag{A15}$$

with the field redefinition $a^\pm_\mu \to qA^\pm_\mu$. The constant with the dimension of a charge q gives to the effective gauge vector fields the correct usual physical dimensions. $A^\pm_\mu$ correspond to the usual electromagnetic fields onto the brane (+) or (−).

Assuming the first-order approximation, g (respectively, m_r) is only related to $\alpha_{i,j}$ (respectively, $\beta_{i,j}$) [see Eqs. (A5) and (A6)]. In that case, g depends only on the first-order derivative ∂_z along the continuous extra dimension, while m_r is only related to the scalar field Φ.

APPENDIX B: ESTIMATION OF g, m_r AND δm

In the present appendix we consider the behavior of g, m_r and δm from the expressions given by Eqs. (44). Using expressions (6) and (18), the equations (44) can be written as

$$g = (1/2)\lambda\eta\int\{\tanh((z + d/2)/\xi) - 1\} \times\{f_{R,+}(z)f_{L,-}(z) - f_{L,+}(z)f_{R,-}(z)\}dz,$$
$$m_r = (1/2)\lambda\eta\int\{\tanh((z + d/2)/\xi) - 1\} \times\{f_{L,+}(z)f_{R,-}(z) + f_{R,+}(z)f_{L,-}(z)\}dz,$$
$$\delta m = \lambda\eta\int\tanh((z + d/2)/\xi)f_{R,+}(z)f_{L,+}(z)dz. \tag{B1}$$

The expressions of $f_{L/R,\pm}(z)$ are easily deduced from expressions (21) to (23). It can be shown (with the help of a numerical tool like MATHEMATICA,, for instance) that there is no trivial analytical expressions for such integrals (B1) except if $\varepsilon = \lambda\sqrt{2/\chi} = \lambda\eta\xi$ is an integer. In that case, the above integrals can be written as a ratio of two functions. Each function appears then as a sum of exponential terms: $\exp(-nd/\xi)$ and $(d/\xi)\exp(-nd/\xi)$, where n are integers. In the two-brane solutions considered here, we have $d \gg \xi$

and the integrals can then be approximated by a single exponential function. By considering the first massive mode, several cases can be considered related to the mass $m = m_1 = k\sqrt{2\varepsilon - 1}$ of the particle trapped on a brane (we set $k = 1/\xi$):

	Fermion mass m	Coupling constant g	Off-diagonal mass m_r	Mass correction δm
$\varepsilon = 1$	k	0	0	0
$\varepsilon = 2$	$k\sqrt{3}$	$2ke^{-kd}$	$-2ke^{-kd}$	$8\sqrt{3}k(kd-1)e^{-2kd}$
$\varepsilon = 3$	$k\sqrt{5}$	$24ke^{-2kd}$	$6ke^{-2kd}$	$6\sqrt{5}ke^{-2kd}$
$\varepsilon = 4$	$k\sqrt{7}$	$180ke^{-3kd}$	$100ke^{-3kd}$	$4\sqrt{7}ke^{-2kd}$
$\varepsilon = 5$	$3k$	$1120ke^{-4kd}$	$770ke^{-4kd}$	$10ke^{-2kd}$

A first noticeable point is that the coupling constant g decreases when ε increases. Indeed, it should be kept in mind that the fermionic wave functions become sharply localized when ε increases (see Sec. III A), such that the overlap of the fermionic states of each brane also decreases. It is also noticeable that g can be written as $g = f_\varepsilon k \exp(-nkd)$ (where n is an integer related to ε and f_ε a constant which depends of ε). As a consequence, the phenomenological distance δ between the branes becomes $\delta = f_\varepsilon^{-1}\xi \exp(nd/\xi)$ and is related to the real distance d between branes and to the brane thickness ξ.

Let us consider, for instance, the case $\varepsilon = 2$ (for which $m = k\sqrt{3} = \sqrt{3}/\xi$) assuming a mass m equals to the electron mass. We get $g = 10^3$ m^{-1} i.e. $\delta = 1$ mm when $d = 1,46 \cdot 10^{-11}$ m (i.e. about 22 times the brane thickness). For $d = 2,38 \cdot 10^{-11}$ m (i.e. about 36 times the brane thickness) we get $g = 10^{-3}$ m^{-1}, i.e. $\delta = 1$ km. Therefore, the real distance d between branes is in agreement with our $d \gg \xi$ hypothesis. Furthermore, the values of the coupling constant g are also consistent with observable phenomena in the context of present day technology [21,22,30,33]. It can be noticed that slight fluctuations of the actual distance d between branes lead to strong variations of the coupling constant g (g is multiplied by 10^6 when d is divided by 1, 6). Finally, for a range of distances d between branes, the model remains fully consistent with constraints induced by Newton law variation measurements [16].

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