Grand unification through gravitational effects

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We systematically study the unification of gauge couplings in the presence of (one or more) effective dimension-5 operators $cHG_{\mu\nu}G^{\mu\nu}/4M_{\rm pl}$, induced into the grand unified theory by gravitational interactions at the Planck scale. These operators alter the usual condition for gauge-coupling unification, which can, depending on the Higgs content H and vacuum expectation value, result in unification at scales M_X significantly different than naively expected. We find nonsupersymmetric models of $SU(5)$ and $SO(10)$ unification, with natural Wilson coefficients c, that easily satisfy the constraints from proton decay. Furthermore, gauge-coupling unification at scales as high as the Planck scale seems feasible, possibly hinting at simultaneous unification of gauge and gravitational interactions. In the Appendix we work out the group theoretical aspects of this scenario for $SU(5)$ and $SO(10)$ unified groups in detail; this material is also relevant in the analysis of nonuniversal gaugino masses obtained from supergravity.

I. INTRODUCTION AND OUTLINE

What are the boundary conditions for grand unification? One typically assumes that the gauge couplings of the broken subgroups must become numerically equal at the unification scale M_X [1]. However, effects from physics above the unification scale can alter the gauge-coupling unification condition. In an effective field theory approach, such effects can be caused by dimension-5 operators of the form $cHG_{\mu\nu}G^{\mu\nu}/4M_{\rm pl}$, which shift the coefficients of the gauge kinetic terms in the low-energy theory after the Higgs H acquires a vacuum expectation value in grand unified symmetry breaking [2,3]; one obvious source of such operators is quantum gravitational effects. Indeed, it would be *unnatural* (or require some special explanation) to assume that the Wilson coefficients c above be zero or especially small [4]; the default assumption should be that these coefficients are of order unity in grand unified models, with consequent unification conditions.

In conventional unification models, one might expect $\langle H \rangle \sim 10^{16}$ GeV, plausibly leading to effects from quantum gravity of order a fraction of a percent, $\langle H \rangle / M_{\rm pl} \sim 10^{-3}$, on the gauge-coupling unification condition. In [5] we showed that these dimension-5 operators can be even more relevant than previously suspected since the Planck mass $M_{\rm pl}$ tends to be smaller than naively assumed due to its renormalization group evolution [6,7] under the influence of the large number of fields in supersymmetric grand unified theories. It was noted [5] that these dimension-5 operators introduce in supersymmetric unification models an uncertainty that can be bigger than the two-loop effects which are considered to be necessary to obtain good numerical unification of the gauge couplings.

The aim of this paper is different. We study whether the dimension-5 operators discussed above can lead to perfect gauge-coupling unification *without* supersymmetry by their modifying of the gauge-coupling unification condition. This unification scheme has been studied previously in the literature for models with and without supersymmetry, e.g., in [2,3,5,8–15], but in less detail and generality and mostly only the effect from a single gravitational operator has been considered.

In particular, in this paper we examine, in a systematic way, the effects when two or more dimension-5 operators are present in a theory. Unification under multiple dimension-5 operators has been studied before [13] for the supersymmetric case [16], for which, however, viable gauge-coupling unification is well known [17–19]. The main result of the present paper is that the measurement of the gauge couplings at the Z mass [20] is compatible with nonsupersymmetric grand unification based on $SU(5)$ or bigger groups like $SO(10)$. That is, grand unification does not require a supersymmetric extension of the standard model for a range of natural values of the Wilson coefficients c.

As opposed to models with one dimension-5 operator, in unified models with two or more such operators, the unification scale M_X can be varied in a continuous and controlled manner with the Wilson coefficients c and Higgs value expectation values (VEVs), and we examine this quantitatively in $SU(5)$ and $SO(10)$ theories, primarily for the nonsupersymmetric case (cf. [13] for the supersymmetric case). We find that unification under this scheme can naturally raise the unification scale M_X much above the conventional 10^{15} GeV to 10^{16} GeV, and even allows one to obtain gauge-coupling unification at or near the Planck mass which is suggestive of unification of all gauge and gravitational forces at a common scale (see also [21,22]). Any such unification is safe from the proton decay constraint. This, and the fact that exact gauge-

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coupling unification in the first place can be accomplished, may be a useful tool for model building.

Note, due to the gravitational origin of these dimension-5 operators, the unification scheme considered here is distinct from the Lavoura-Wolfenstein result [23] or similar effects [24], according to which nonsupersymmetric unification can result from particle thresholds. In particular, as can be seen from the form of these effective operators, their effect increases as the unification scale gets closer to the Planck scale, i.e., as the Higgs VEVs get bigger, which in turn allows one to self-consistently shift M_X to values much larger than conventionally assumed.

Realistic supersymmetric unification theories have issues linked to their large particle content. One is the presence of Landau poles between the unification and the Planck scale, another is possible lack of calculability in these theories due to potentially large running of the Planck mass [5]. Both issues are avoided in their nonsupersymmetric counterparts. On the other hand, if supersymmetry is abandoned, the unification scale needs to be stabilized with respect to the gravitational scale (as, of course, does the weak scale). This "little hierarchy problem" can be avoided by one of the scenarios mentioned above where grand unification happens close to the Planck mass, thereby reducing or eliminating the hierarchy between the two scales.

Outline of the paper: For a physical description of the scenario and effects and for exemplary numerical results, see the Introduction and Summary, Secs. I and VIII. More detail is given in the main text, and the group theoretical formalism in the Appendix.

In Sec. II, we introduce the dimension-5 operators $cHG_{\mu\nu}G^{\mu\nu}/4M_{\rm pl}$ under investigation, first specifically for the case of an $SU(5)$ grand unified theory, and also discuss possible sensible choices for the Planck scale $M_{\rm pl}$ suppressing these operators. We then describe how these operators modify the usual gauge-coupling unification condition. This section sets up the notation necessary to understand the figures, tables, and most of the details in the main text.

In Sec. III, we look at numerical results for possible gauge-coupling unification under the modified gauge-coupling unification condition in nonsupersymmetric $SU(5)$ models. First (Sec. III A) we review the known effects in the presence of a single such operator [2,3,9,15], also addressing uncertainties in low-energy measurements of the gauge couplings and modifications due to two-loop running. Then, in Sec. III B, it is numerically shown that (and how) the unification scale may be *continuously* varied in models with two different dimension-5 operators. This, being one of the main results of this paper, is also true in models with more such operators, and we give the general treatment in Sec. III C, also deriving an estimate to assess which unification scales M_X are achievable naturally.

With these numerical results and estimates, in Sec. IV we see that even the lowest unification scales M_X that are achievable in a natural way through our effect, are not in conflict with the current lower bounds on the proton lifetime, coming from the nonobservation of proton decay so far. Thus, the dimension-5 operators can facilitate exact gauge-coupling unification without supersymmetry and also allow the proton decay limit to be evaded [unlike in naive $SU(5)$ unification]. We also investigate how further improvements in the proton lifetime bound constrain our models.

In Sec. V, we note that numerical gauge-coupling unification is possible near or at the Planck scale in a very natural way in models with two or more dimension-5 operators. We speculate that this might hint at unification of the gauge interactions and gravity at a common scale ("gauge-gravity unification," see [21]).

Section VI contains the scenario for an $SO(10)$ grand unified gauge group. The setup and unification results for exemplary cases are described, analogous to the $SU(5)$ case in the preceding sections. The main differences to $SU(5)$ are emphasized, namely, the fact that a continuously varying M_X can be achieved with merely a single dimension-5 operator, and the possible role of these operators in $SO(10)$ breaking with intermediate scales is briefly described.

In Sec. VII we briefly look at the effect of the dimension-5 operators in models of supersymmetric unification [16–18], making contact to the literature (e.g., [5,13,15]). We find that in supersymmetric models the unification scale can also be shifted around, although unification close to the expected 2×10^{16} GeV seems most likely. The main part of the present paper focuses on nonsupersymmetric models as it is those where viable unification is commonly believed to be difficult or impossible (for further reasons, see the last paragraph of Sec. II).

We conclude and summarize the main ideas and mechanisms in Sec. VIII, illustrated with some numerical examples showing the size of the effects.

In an extensive Appendix, we present all normalization conventions and group theoretical details. In particular we carefully derive and give all of the Clebsch-Gordan coefficients $\Phi_{(r)t}^s$ associated with $SU(5)$ and $SO(10)$ breaking (the latter case takes up most of the work) to the standard model, in different bases $\{t\}$; only the coefficients relating to the standard model gauge group factors s have been given before in the literature, see especially [25], and often only their relative sizes, whereas we here follow a uniform absolute normalization scheme across different representations r. These analytical results are also useful for the treatment of nonuniversal gaugino masses obtained from $\mathcal{N} = 1$ supergravity models of $SU(5)$ or $SO(10)$, for which the group theory involved is very similar, see, e.g., [26,27]. The Appendix furthermore describes the modified gauge-coupling unification condition and a systematic

method (which was used for the numerical work in this paper) for its solution.

II. DIMENSION-5 OPERATORS AND UNIFICATION CONDITION—$SU(5)$ CASE

In this section we describe, specifically for the case of nonsupersymmetric $SU(5)$ grand unification models [28], first the operators necessary for our scenario and then their effect on the condition for gauge-coupling unification. The $SO(10)$ case holds additional subtleties and possibilities, to be deferred to Sec. VI. We present actual numerical unification results for the $SU(5)$ case in Sec. III and elaborate on physically interesting scenarios in Secs. IV and V.

At the energies available to us in present-day particle physics experiments, nature is described very well by non-Abelian gauge field theories which are nongravitational, although we know that at some energy scale a more complete description must take gravity into account. In other words, at our energies the effective Lagrangian of the world is a gauge theory with certain additional nonrenormalizable operators of dimension 5 and higher, accounting for the gravitational interactions that have been integrated out. These operators must obey the symmetries (gauge and Lorentz invariance, etc.) of the low-energy theory and are suppressed by powers of the gravitational scale M_{pl}, so they are seemingly negligible at our energies. By this logic, grand unification also appears as an effective theory, valid between the unification scale M_X and well below the Planck scale M_{pl}, and contains higher-dimensional operators induced by gravity and suppressed by M_{pl}; at energies $\sim M_X$, however, such operators are potentially much more significant due to the proximity of the scales M_X and M_{pl}.

One set of such dimension-5 operators, that may have important effects in grand unification, are singlets formed from gauge field strengths $G_{\mu\nu}$ and Higgs multiplets H_i of the grand unified gauge group $G = SU(5)$,

$$\mathcal{L} = \frac{c_i}{4M_{pl}} H_i^{ab} G_{\mu\nu}^a G^{b\mu\nu}, \qquad (1)$$

suppressed by one power of the Planck mass M_{pl} such that c_i are dimensionless (Wilson) coefficients. The index i is summed implicitly and includes the possibility that the effective Lagrangian may contain several such operators involving different Higgs multiplets H_i of the theory, which will be one of our main tools later on. In an $SU(5)$ gauge theory, operators (1) can be formed gauge invariantly only with H_i in the representations $r_i = \mathbf{1}, \mathbf{24}, \mathbf{75}$, or $\mathbf{200}$ (although the theory might contain additional multiplets in other representations); these irreducible representations (irreps) can all uniformly be written in component notation with symmetric adjoint indices a, b of the gauge group, which establishes a *common* normalization for the different operators i in (1). (Here and later, see the Appendix for a careful treatment of the relevant group

theoretical aspects and normalization conventions; see also [29].)

The dimension-5 operators (1) are suppressed by the Planck scale M_{pl}, the energy scale at which quantum gravity sets in, which, as an interaction not accounted for by the renormalizable terms in the Lagrangian, induces these effective operators. There is some ambiguity (or arbitrariness) as to whether the appropriate suppression scale M_{pl} should be taken to be the "naive" Planck scale $G_N^{-1/2} = 1.2 \times 10^{19}$ GeV or, more commonly, the reduced Planck scale $(8\pi G_N)^{-1/2} = 2.4 \times 10^{18}$ GeV, as this is the quantity that controls quantum gravity computations. To leave this choice open, we parametrize

$$M_{pl} \equiv \frac{G_N^{-1/2}}{\xi} = \frac{1.2 \times 10^{19} \text{ GeV}}{\xi}; \qquad (2)$$

$\xi = 1$ corresponds to the choice of the naive Planck scale as the suppression scale, $\xi_{red} = \sqrt{8\pi} \approx 5$ to the reduced Planck scale, which is what we assume (implicitly) in most discussions. The smaller a suppression scale M_{pl} one accepts, i.e., the bigger ξ one chooses, the more pronounced the effects of the operators (1) will be, at fixed Wilson coefficients c_i. Effects of equal size can be achieved for indirectly proportional coefficients $c_i \to c_i/\xi$ when changing ξ. [Conservatively, we put an explicit factor $1/4$ in (1) to avoid overcounting of terms in the contraction of two gauge field strengths, just as in the canonical gauge boson kinetic term.]

Also concerning the choice of an appropriate suppression scale M_{pl}, we have shown previously [5,7] that the fundamental value of Newton's constant (i.e., at high energies) is different from its observed low-energy value G_N used in (2): matter field fluctuations of N_0 real scalar, $N_{1/2}$ Weyl fermion, and N_1 gauge boson fields lead to a running of Newton's constant

$$\frac{1}{G(\mu)} = \frac{1}{G_N} - \frac{\mu^2}{12\pi}(N_0 + N_{1/2} - 4N_1) \qquad (3)$$

at one loop, similar to the running of gauge couplings (see also [6,30,31]). Then determining the fundamental gravitational scale M_{pl} via $G(\mu = M_{pl})^{-1/2} \sim M_{pl}$ yields a value lower by a factor of $\eta \equiv \sqrt{1 + (N_0 + N_{1/2} - 4N_1)/12\pi}$ than without the running. This additional change in the suppression scale M_{pl} is easily incorporated into our parametrization (2) via $\xi \to \xi^{run} = \xi\eta$, and Table I illustrates the size of this running effect in various grand unified models. Furthermore, as illustrated in the last row of the table, appropriate choices of ξ can accommodate suppression scales M_{pl} of other origin as well, e.g., string compactification scales as in [3].

Because they respect the symmetries of the theory and because quantum gravity effects *are* mediated at the scale M_{pl}, operators (1) should be expected in the effective

TABLE I.   For models with various particle contents, the third column shows the effect $M_{\mathrm{pl}} \rightarrow M_{\mathrm{pl}}/\eta$ on the Planck scale entailed by the running (3) of Newton's constant. The fourth column gives numerical values for ξ in (2) if the additional reduction factor $\sqrt{8\pi}$ from perturbative quantum gravity is taken into the suppression scale, as commonly done. Throughout we assume three generations of fermions, and grand unified models are characterized by their gauge group and Higgs content. In almost all cases—and certainly in all models of our main interest, namely, in nonsupersymmetric grand unified models with several different Higgs multiplets at the unification scale—reasonable values for M_{pl} are smaller than the naive value 1.2×10^{19} GeV, in some cases by as much as roughly an order of magnitude, i.e., $1 \leq \xi \lesssim O(10)$ in (2). The last column quantifies roughly how many orders of magnitude above the unification scale M_X the grand unified gauge theories enter nonperturbative strong coupling regime $g_G(M_{\mathrm{np}}) = \sqrt{4\pi}$, center dots ($\cdots$) indicating that the theory is asymptotically free; here, we have assumed $g_G(M_X)^2/4\pi = 1/40$ for the nonsupersymmetric and (conservatively) $g_G(M_X)^2/4\pi = 1/30$ for the supersymmetric cases. When $M_{\mathrm{np}} > M_{\mathrm{pl}}$, the unified field theory has a chance of describing Nature perturbatively up to the onset of quantum gravity at M_{pl}.

Particle physics model	$N = N_0 + N_{1/2} - 4N_1$	$\eta = \sqrt{1 + N/12\pi} = \xi^{\mathrm{run}}$	$\xi^{\mathrm{run}}_{\mathrm{red}} = \sqrt{8\pi}\eta$	$\log_{10} M_{\mathrm{np}}/M_X$
No running of G_N		1	5.0	
Standard model	1	1.0	5.1	
$SU(5)$w/**5, 24**	-17	0.74	3.71	$\cdots$
$SU(5)$w/**5, 200**	159	2.3	11.5	42
$SU(5)$w/**5, 24, 75**	58	1.6	8.0	$\cdots$
$SU(5)$w/**5, 24, 75, 200**	258	2.8	14.0	14
$SO(10)$w/**10, 16, 45**	-35	0.27	1.34	$\cdots$
$SO(10)$w/**10, 16, 210**	130	2.1	10.6	$\cdots$
$SO(10)$w/**10, 16, 770**	690	4.4	22.0	3.9
SUSY-$SU(5)$w/**5, $\bar{5}$, 24**	165	2.3	11.6	$\cdots$
SUSY-$SU(5)$w/**5, $\bar{5}$, 24, 75**	390	3.4	16.9	3.6
SUSY-$SU(5)$w/**5, $\bar{5}$, 200**	693	4.4	22.1	0.85
SUSY-$SO(10)$w/**10, 16, $\overline{16}$, 45, 54**	432	3.5	17.7	11
SUSY-$SO(10)$w/**10, 16, $\overline{16}$, 210**	765	4.6	23.1	1.8
SUSY-$SO(10)$w/**10, 16, $\overline{16}$, 770**	2445	8.1	40.7	0.27
Compactification scale M_c (e.g., [3])		up to $\xi \sim 100$		

Lagrangian of grand unification and, although their sizes cannot be computed in this effective field theory point of view, they are expected *a priori* with natural Wilson coefficients of order $|c_i| \sim O(1)$ (for values much bigger than this, the proposed effective theory is not a good low-energy description and one loses perturbative control, whereas $|c_i| \ll 1$ seemingly constitutes fine-tuning [4]). And even though irrelevant for our purposes here, there are several known ways to generate these operators. For example, they arise from an $\mathcal{N} = 1$ supergravity ultraviolet completion with noncanonical gauge kinetic function $f^{ab}(H_i)$ as the lowest order (nontrivial) terms in an expansion of f^{ab} in M_{pl} [26,27] (in this scenario, supersymmetry needs to be broken in the grand unified theory at M_{pl} to conform to our nonsupersymmetric analysis). Gravitational instantons can induce such effects also [32]. And in descending from higher-dimensional completions to four dimensions, spontaneous compactification generates operators (1) suppressed by the compactification scale $M_c = M_{\mathrm{pl}}$ [33]. At any rate, lacking knowledge of quantum gravity, it seems most reasonable to assume the presence of effective operators (1) with coefficients c_i of order one.

We now describe the effect of the operators (1) on the condition for gauge-coupling unification. At the scale M_X of grand unified symmetry breaking, some Higgs multiplets H_i acquire nonzero VEVs. For simplicity and definite-

ness we assume that all Higgs fields, except for the multiplet containing the standard model Higgs, acquire VEVs at the scale M_X; as will become clear, other multiplets, that get nonzero VEVs only at lower scales, contribute proportionally less, but can in principle be treated equivalently (this assumption also avoids the introduction of further mass hierarchies into the model).

The VEVs $\langle H_i \rangle$, acquired well above the electroweak scale, have to be invariant under the standard model subgroup $G_{321} = SU(3)_C \times SU(2)_L \times U(1)_Y \subset SU(5)$. For each multiplet H_i that can occur in the dimension-5 operators (1), this requirement determines its VEVs $\langle H_i^{ab} \rangle$ up to an overall scale v_i (see the Appendix). Replacing H_i by their VEVs, the operators (1) modify the kinetic terms of the gauge bosons in the Lagrangian at the unification scale M_X by adding to them:

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_i \frac{c_i}{4M_{\mathrm{pl}}} \langle H_i^{ab} \rangle G^a_{\mu\nu} G^{b\mu\nu}$$

$$= -\frac{1}{4}(1 + \epsilon_3) F^a_{\mu\nu} F^{a\mu\nu}_{SU(3)} - \frac{1}{4}(1 + \epsilon_2) F^a_{\mu\nu} F^{a\mu\nu}_{SU(2)}$$

$$- \frac{1}{4}(1 + \epsilon_1) F_{\mu\nu} F^{\mu\nu}_{U(1)} + \ldots, \tag{4}$$

where the ellipses denote the non-standard model gauge bosons of $SU(5)$, which become massive and (quasi) non-

dynamical below M_X. The ϵ_s indicate the modifications to the gauge kinetic terms,

$$\epsilon_s = \sum_i \frac{c_i}{M_{\rm pl}} v_i \delta_s^{(i)} \qquad \text{(for } s = 3, 2, 1); \qquad (5)$$

they generically differ for each factor $s = 3, 2, 1$ of the standard model gauge group G_{321} and depend on the Higgs content H_i and on the sizes $c_i v_i / M_{\rm pl}$ of the VEVs and Wilson coefficients relative to the suppression scale (thus, a Higgs that acquires its VEV $v \sim M_I \ll M_X$ at an intermediate scale contributes much less than a Higgs at M_X). The $\delta_s^{(i)}$ are group theoretical factors (Clebsch-Gordan coefficients) specific to the embedding $G_{321} \subset SU(5)$ and characterize the possible standard model singlet VEVs $\langle H_i \rangle$; they depend only on the representation r_i of H_i and are given in Table II.

In the effective Lagrangian below the unification scale M_X, one would like to have canonically normalized gauge fields as opposed to (4), since it is the coupling constants associated with those that obey the familiar renormalization group (RG) equations (β-functions) and that are measured in low-energy experiments. This can be achieved by a finite (and usually small, see later) redefinition $A_{(s)}^\mu \rightarrow (1 + \epsilon_s)^{1/2} A_{(s)}^\mu$ of the gauge fields associated with each standard model factor $s = 3, 2, 1$, which has to be accompanied by a redefinition $g_s \rightarrow (1 + \epsilon_s)^{-1/2} g_s$, so as not to affect the interaction strength. Gauge-coupling unification requires that at the unification scale M_X the couplings *before* this redefinition meet at a common value $g_G = g_G(M_X)$, the gauge coupling of the unified group G at the unification scale. In terms of the rescaled couplings $\alpha_s \equiv g_s^2/4\pi$, this reads:

$$(1 + \epsilon_1)\alpha_1(M_X) = (1 + \epsilon_2)\alpha_2(M_X) = (1 + \epsilon_3)\alpha_3(M_X)$$

$$= \frac{g_G^2}{4\pi} \equiv \alpha_G. \qquad (6)$$

Under our assumption of one-step breaking at M_X to the standard model, the running coupling functions $\alpha_s(\mu \leq M_X)$ are fixed by the fairly precise low-energy measurements [20] and their RG evolution [17]; the low-energy values and β-functions to one loop are given in the Appendix (A30)–(A32).

TABLE II. The Clebsch-Gordan coefficients δ_s in (5) associated with the embedding $G_{321} \subset SU(5)$, for each irrep r of $SU(5)$ that can occur as a multiplet H_i in (1). Sensible normalization conventions (see the Appendix), which ensure uniform treatment of different dimension-5 operators (1), fix the values of all $\delta_s^{(r)}$ up to an overall sign for each r.

$SU(5)$ irrep r	$\delta_1^{(r)}$	$\delta_2^{(r)}$	$\delta_3^{(r)}$
1	$-1/\sqrt{24}$	$-1/\sqrt{24}$	$-1/\sqrt{24}$
24	$1/\sqrt{63}$	$3/\sqrt{63}$	$-2/\sqrt{63}$
75	$5/\sqrt{72}$	$-3/\sqrt{72}$	$-1/\sqrt{72}$
200	$-10/\sqrt{168}$	$-2/\sqrt{168}$	$-1/\sqrt{168}$

In Sec. III we will be looking for nonsupersymmetric models of $SU(5)$ grand unification (specified by their Higgs content H_i) which feature exact unification of the standard model gauge couplings $\alpha_s(\mu)$ under the unification condition (6). Such gauge-coupling unification can happen naturally at scales $M_X > 10^{16}$ GeV, larger than normally expected, so as to escape the proton decay limit usually encountered in nonsupersymmetric grand unification, and even at scales as large as the Planck scale. Through several criteria we will assess reasonableness of these models, in particular, by checking whether their required Wilson coefficients have natural sizes $|c_i| \sim O(1)$ and also by looking at the modifications to the gauge kinetic terms in (4) (e.g., if $\epsilon_s < -1$ for any s, the gauge kinetic terms in the broken theory would have the wrong sign).

One other criterion for all models, necessary to sensibly claim unification, is the requirement that the masses of all non-standard model gauge bosons ("superheavy gauge bosons") be close to the scale of grand unified symmetry breaking, i.e., close to the unification scale M_X. This is to ensure rightful use of the standard model RG equations with three differently evolving gauge couplings up to the scale M_X, where the gauge couplings are to unify at g_G (6), and of the running of a single coupling of the unified gauge theory thenceforth with all gauge bosons being massless. At least this is necessary in the absence of intermediate scales, which is usually understood for $SU(5)$ unification models (see above). In $SU(5)$, all superheavy gauge bosons get equal masses from grand unified symmetry breaking,

$$M_{\rm gb} = g_G \sqrt{\sum_i \frac{C_2(r_i)}{12} v_i^2}, \qquad (7)$$

where the sum now runs over all Higgs multiplets i in the theory that acquire nonzero VEV v_i at M_X, with the Casimir invariants C_2 of the representations r_i (cf. Table V). For definiteness, and to conform to standard treatment, we take the above requirement then to be

$$M_{\rm gb} = M_X. \qquad (8)$$

Furthermore, since Higgses in any of the representations **24**, **75**, **200** can achieve grand unified symmetry breaking $SU(5) \rightarrow G_{321}$, we will make the simplifying assumption that the theory only contain Higgses able to form the dimension-5 operators (1); the more general case is not harder to treat but increases the particle content of the models and requires larger Wilson coefficients to achieve the desired effects.

A requirement similar to (8) should be put on the masses of the superheavy scalars as well, i.e., the Higgs fields that acquire nonzero VEV at M_X, which we, however, will neglect since generically most superheavy scalars are more massive than the (lightest of the) superheavy gauge bosons and since the superheavy scalars do not influence

the running of the standard model gauge couplings at one loop. Also, we neglect heavy particle thresholds [23,24], to demonstrate unification due specifically to the dimension-5 operators (1) as a proof of principle, cleanly separated from other effects.

In this paper we focus on nonsupersymmetric grand unification. Contrary to the standard lore, and this is one of our main findings (Secs. III and IV), the altered unification condition (6) can yield successful nonsupersymmetric unification models which satisfy the proton lifetime constraint. Furthermore, nonsupersymmetric grand unified models hold the attractive hope of describing physics up to the Planck scale in a perturbative way, whereas supersymmetric models commonly become strongly coupled before reaching the Planck scale. For several nonsupersymmetric as well as supersymmetric grand unified models, the right column in Table I indicates roughly how many orders of magnitude above the unification scale M_X the theories become nonperturbative (i.e., near a Landau pole) and illustrates that generally only nonsupersymmetric models are safe in that regard, requiring roughly 3 orders of magnitude between M_X and the nonperturbative regime M_{np}. Furthermore, as they contain fewer scalars and fermions, the running (3) of Newton's constant and the associated change of the Planck scale M_{pl} from its naive or reduced value tend to be smaller in the nonsupersymmetric models (see Table I), diminishing uncertainties in these models [5].

III. MODELS AND UNIFICATION RESULTS IN THE NONSUPERSYMMETRIC $SU(5)$ CASE

In this section, we will quantitatively examine the effects of multiple gravitationally induced dimension-5 operators (1) and of the subsequently modified condition (6) for gauge-coupling unification in several nonsupersymmetric $SU(5)$ models. All models are presumed to have the minimal fermionic content of three standard model families, but differ in their Higgs content responsible for grand unified symmetry breaking; without loss (see above) we only consider Higgs representations **1**, **24**, **75**, **200**. As the Higgs multiplets under consideration are suitable for breaking $SU(5)$ down to the standard model, we do not consider the Higgs (scalar) potential in these unified theories explicitly; rather, we take as the parameters of the models the Higgs VEVs v_i directly, which are acquired in grand unified symmetry breaking as a consequence of the Higgs potential, and the Wilson coefficients c_i of the dimension-5 operators (1).

Our method is then as follows. After specifying a grand unified model by its Higgs content, we scan its parameter space $\{c_i, v_i\}$ for points that, below the breaking scale M_X, yield the actual running gauge couplings $\alpha_s(\mu)$ of the standard model. Phrased in a bottom-up language, we are looking for points $\{c_i, v_i\}$ in parameter space that result in a

unification condition (6), according to which the actually observed gauge couplings $\alpha_s(\mu)$ of the standard model unify, while simultaneously requirement (8) holds, namely, that the superheavy gauge boson masses (7) be equal to the unification scale. In fact, any such point $\{c_i, v_i\}$ determines M_X and the unified gauge coupling $g_G \equiv g_G(M_X)$ uniquely; see the Appendix below (A33) where we also outline how all points $\{c_i, v_i\}$ may be found.

In the next subsection we look at the known scenario in which the theory contains only one Higgs that can form a dimension-5 operator, as a warm-up and comparison to established results [2,3,15]. For the case of two or more operators (1), we find points in parameter space that yield physically viable unification. These effects are described in detail exemplarily for scenarios with two dimension-5 operators in Sec. III B, before moving on to a more general treatment of models with any number of dimension-5 operators where we also pay special attention to naturalness.

A. Unification in models with one dimension-5 operator

When the modifications ϵ_s to the gauge kinetic terms (4) come from only one Higgs (1), their ratio $\epsilon_1:\epsilon_2:\epsilon_3$ is completely determined; see (5). For the given running gauge-coupling functions $\alpha_s(\mu)$ of the standard model, this ratio uniquely determines, via the unification condition (6), the possible unification scale M_X, the unified coupling $\alpha_G = g_G^2/4\pi$, and the required absolute sizes of all ϵ_s; subsequently, the required Higgs VEV v and Wilson coefficient c necessary for unification may be computed. It then has to be checked whether all these values are physically reasonable.

For numerical results, see Table III which shows these quantities for the $SU(5)$ models with a sole Higgs multiplet **24**, **75**, or **200**. A sole singlet **1** cannot modify condition (6) to make the standard model gauge couplings unify since $\epsilon_1 = \epsilon_2 = \epsilon_3$ in this case, neither can a singlet VEV break the unified symmetry in the first place. Although the required modifications ϵ_s to the gauge kinetic terms are not problematic (in particular, $\epsilon_s > -1$) and even small, the necessary Wilson coefficients c are big in the case of a **75** and especially large for a **24** Higgs, regardless of any reasonable choice of suppression scale; operators (1) with such large coefficients are not expected naturally in an effective theory and might preclude a perturbative treatment. Furthermore, in the case of a **24** the unification scale M_X would be far too low to satisfy current constraints on the proton lifetime, and possibly slightly too low for a **75** (for details, see Sec. IV). For a **200** Higgs the Wilson coefficient c has natural size and, depending on convention (2), the unification scale may still lie below or at the Planck scale M_{pl}, which may be taken as a hint to a scenario of simultaneous gauge-gravity unification (see Sec. V).

These unification results largely agree with [15]. There is, however, some disagreement on a sensible normaliza-

tion among the dimension-5 operators with Higgses in different representations, which directly affects the required sizes of the Wilson coefficients c. We have chosen to write all possible dimension-5 operators in a common form (1) with uniform normalization conventions (see the Appendix), and believe that this allows for a sensible cross comparison between the Wilson coefficients of different operators. Furthermore, within these conventions, $|c| \sim 1$ is believed to be a natural size for the dimension-5 operators from an effective field theory point of view (see Sec. II and the Appendix). Since the low-energy gauge couplings $\alpha_s(m_Z)$ had not been measured well at the time and were therefore partly treated as free parameters, Table III cannot be directly compared to the pioneering work [2,3], that looked only at the case of a sole **24** Higgs, but the results tend in the same direction.

Table III also shows results obtained by using two-loop renormalization group equations [17] (neglecting matter fields) instead of (A30). A full two-loop treatment would further incorporate the effect of thresholds, as partly done in [15], and couplings to the fermion and Higgs fields, increasing arbitrariness and decreasing predictivity [23,24]; there would also be some renormalization scheme dependence. The one-loop values are not significantly altered, so a one-loop analysis can show, with reasonable numerical accuracy, whether the modified condition (6) can make the standard model gauge couplings unify, as a proof of principle.

Furthermore, endowing the measured low-energy couplings (A32) with error bars would put uncertainties on the M_X and c necessary to achieve unification under these initial conditions; for example, in the case of a **75**, the error bars (less than 4% [20]) given in (A32) result in $M_X = (8.1 \pm 6.9) \times 10^{15}$ GeV and $c = (-129 \pm 93)/\xi$, which would not significantly influence whether (natural) unification is considered a reasonable possibility or not. (In a previous paper [5], however, the relative effect from the uncertainty on the necessary Wilson coefficients c was seen to be much bigger, as very small Wilson coefficients are required in supersymmetric unification models in the first place.) Besides being predictive, our analysis shows the effect due solely to the modified unification condition

(6) cleanly separated from other effects like, e.g., threshold corrections [23,24].

B. Unification in models with two dimension-5 operators

In the case of multiple dimension-5 operators (1) the modification to the gauge kinetic terms (4) is a linear combination of the effects from single operators, weighted by the Wilson coefficients and Higgs VEVs; see (5). The idea is that now, as one can vary these contributions with the model parameters in a continuous way, one can find subsets of parameter space that result in perfect gauge-coupling unification at continuously variable scales M_X.

In the case of two dimension-5 operators (not in the same representation) these subsets are two dimensional, by counting the number of parameters (c_i and v_i for each Higgs i) less the two unification constraints (6): $2 \cdot 2 - 2 = 2$. To emphasize the possibility of varying M_X continuously, we choose to parametrize these subsets by the unification scale M_X itself and by the ratio $v_1 : v_2$ of the two Higgs VEVs, which, without fine-tuning the Higgs potential, should probably be within an order of magnitude of each other.

Numerical results for the model with a **24** and a **75** Higgs are shown in Fig. 1. For a given choice of maximally (naturally) acceptable Wilson coefficients $\max\{|c_{24}|, |c_{75}|\}$, the lowest unification scale M_X can be achieved for the (not finely-tuned) VEV ratio of roughly $v_{24} : v_{75} = 1{:}3$ (solid bold curve in the figure). For example, demanding $|c_{24}|, |c_{75}| < 1$ and choosing the suppression scale $M_{\rm pl} = 1.2 \times 10^{19}$ GeV$/\xi_{\rm red} = 2.4 \times 10^{18}$ GeV, then any unification scale $M_X \gtrsim 5 \times 10^{17}$ GeV can be achieved, whereas, if one allows $|c_{24}|, |c_{75}| < 5$ with $M_{\rm pl} = 1.2 \times 10^{19}$ GeV$/\xi_{\rm red}^{\rm run} = 1.5 \times 10^{18}$ GeV, then any $M_X \gtrsim 3 \times 10^{16}$ GeV is possible, getting close to the constraint on proton decay (Sec. IV). The right panel shows the modifications to the gauge kinetic terms (4) necessary to achieve unification at M_X in this model; in no case are big ϵ_s required that would invalidate the analysis (in particular, $\epsilon_s > -1$ always).

Similar results obtain for the two models with **24**, **200**, respectively, **75**, **200** Higgses; see Fig. 2. Within the allow-

TABLE III. For all suitable Higgs irreps, this table shows the parameters a nonsupersymmetric $SU(5)$ model must have if unification of the standard model happens by means of mechanism (6) with only one dimension-5 operator (1). Columns 2–6 are obtained by using the one loop β-functions for the standard model gauge couplings, columns 7–8 by two loop for comparison. The size of the required Wilson coefficients c depends on the chosen suppression scale (2) parametrized by ξ, four plausible values of which are given for each model: naive or reduced Planck scale, either or not taking the running of Newton's constant into account, cf. Table I. For the last column, see below estimate (13).

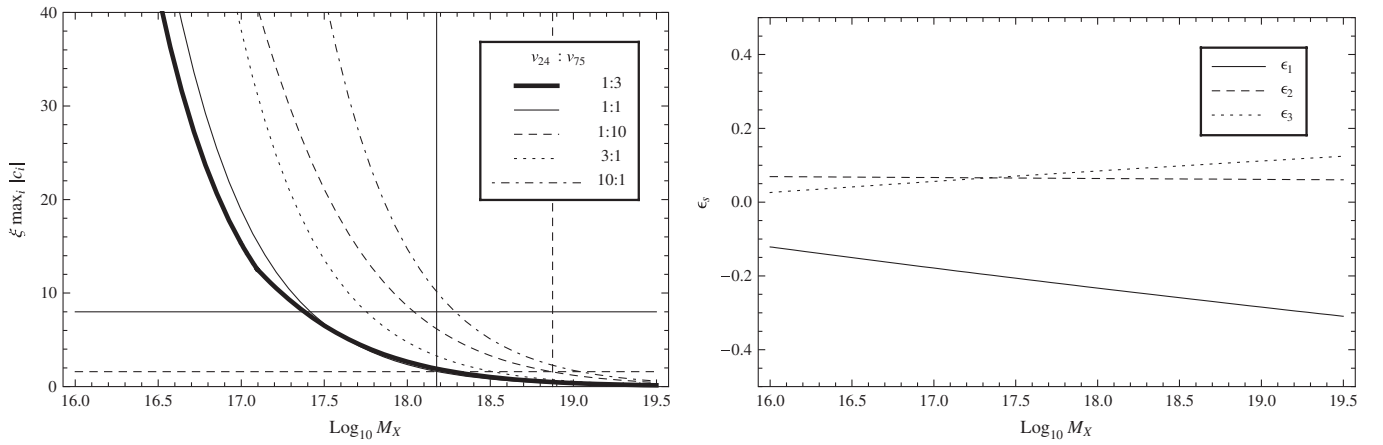| H irrep	M_X/GeV	$1/\alpha_G$	c	v/GeV	$\max_s\|\epsilon_s\|$	$M_X^{(2\mathrm{lp})}$/GeV	$c^{(2\mathrm{lp})}$	$\xi = 1$	$\xi_{\rm red} = \sqrt{8\pi}$	$\xi^{\rm run}$	$\xi_{\rm red}^{\rm run}$	O in (13)
1			Unification by mechanism (6) (and symmetry breaking) impossible									
24	4.6×10^{13}	40.6	$18700/\xi$	1.3×10^{14}	0.076	4.0×10^{13}	$19200/\xi$	1	5	0.74	3.71	0.072
75	8.1×10^{15}	43.3	$-129/\xi$	1.8×10^{16}	0.116	3.7×10^{15}	$-248/\xi$	1	5	1.4	6.9	0.087
200	5.2×10^{18}	53.4	$0.53/\xi$	1.1×10^{19}	0.363	9.8×10^{17}	$2.6/\xi$	1	5	2.3	11.5	0.23

FIG. 1.   Numerical gauge-coupling unification results for the $SU(5)$ model with a **24** and a **75** Higgs. Left panel: For different fixed ratios $v_{24}{:}v_{75}$ of the Higgs VEVs, the curves characterize the size of the Wilson coefficients c_i necessary to achieve unification at M_X. The straight horizontal dashed and solid lines indicate two possible choices for the parameter $\xi^{\mathrm{run}} = 1.6$ and $\xi^{\mathrm{run}}_{\mathrm{red}} = 8.0$ in (2) (cf. also Table I), and the vertical lines the corresponding suppression scales $M_{\mathrm{pl}} = 1.2 \times 10^{19}$ GeV$/\xi$ (it may not make sense to consider scales M_X larger than M_{pl}). Right: The ϵ_s in (6) necessary to achieve unification at a given M_X (not dependent on the ratio $v_{24}{:}v_{75}$ of Higgs VEVs).

ances of the previous paragraph, the widest unification scales achievable are $M_X \gtrsim 1.4 \times 10^{16}$ GeV, namely, for the model with **75**, **200** Higgses, when accepting $\max\{|c_{75}|, |c_{200}|\} < 5$ and assuming $M_{\mathrm{pl}} = 1.2 \times 10^{19}$ GeV$/\xi^{\mathrm{run}}_{\mathrm{red}}$. Requiring $\max_i |c_i| < 1$ and assuming the nonreduced Planck scale, one can achieve unification at any $M_X \gtrsim 8 \times 10^{17}$ GeV (roughly the same for both models).

In models with two dimension-5 operators (1) involving Higgs multiplets in *identical* irreps (e.g., two **24** Higgses), one cannot shift the unification scale continuously since the ratio between the ϵ_s cannot be varied continuously. In fact, only the unification scales M_X given in Table III are possible, with Higgs VEVs and Wilson coefficients modi-

fied by $O(1)$ factors. Furthermore, in models with one singlet and one nonsinglet Higgs (e.g., one **1** and one **24** Higgs) the unification equations do not have physically sensible solutions (formally, the equations yield $\alpha_s(M_X) \approx 0$ at any M_X).

C. Effects in general models

The behavior from Figs. 1 and 2 is generic for models with at least two dimension-5 operators, as we will see now. In particular, there are no points in parameter space which yield unification at some given M_X for significantly smaller Wilson coefficients than shown in these figures. This is true in models with at least two dimension-5 operators (1) and involving Higgses in at least two unequal
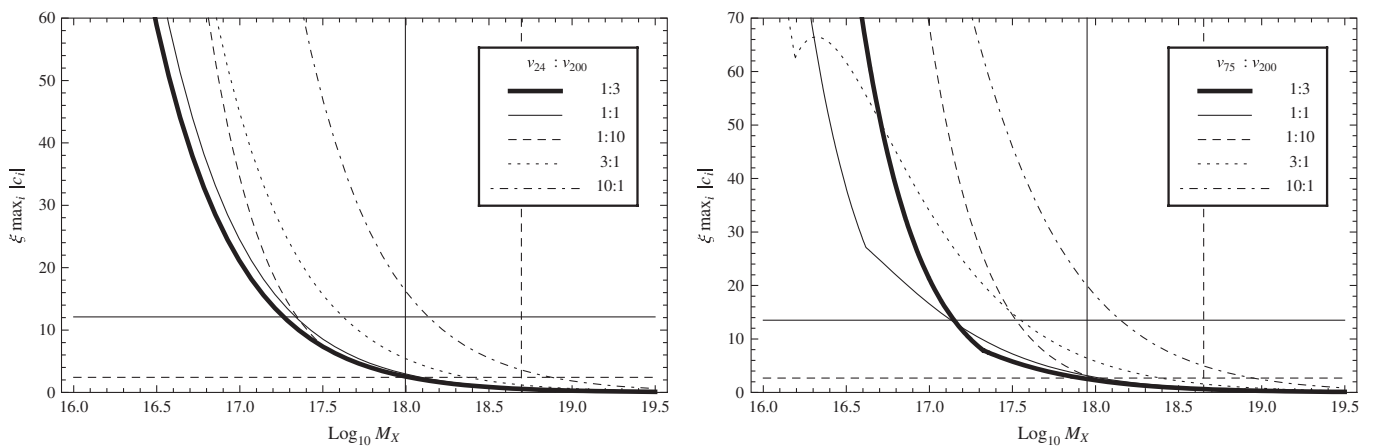

FIG. 2.   Similar to the left panel of Fig. 1, but for the models with **24**, **200** Higgses (left: $\xi^{\mathrm{run}} = 2.42$, $\xi^{\mathrm{run}}_{\mathrm{red}} = 12.1$) and **75**, **200** Higgses (right: $\xi^{\mathrm{run}} = 2.68$, $\xi^{\mathrm{run}}_{\mathrm{red}} = 13.6$). For the **24**, **200** model, the ratio of Higgs VEVs that achieves any chosen M_X with the smallest Wilson coefficients is roughly $v_{24}{:}v_{200} = 1{:}3$ (solid bold curve), and for the **75**, **200** model this ratio depends on M_X and lies between 1:3 and 3:1, requiring no fine-tuning of the Higgs potential. In both models, $|\epsilon_s| < 0.5$ at all unification scales M_X shown here (cf. right panel of Fig. 1).

irreps from the set **24**, **75**, **200**; in all other models, acceptable exact unification via mechanism (6) is impossible for general M_X, as we have just seen, and can at most happen at the discrete scales M_X from Table III.

Writing $c \equiv \sum_{i=1}^{h} c_i$ and $v \equiv \sum_{i=1}^{h} v_i$ with h the number of Higgs multiplets at the grand unification scale, Eqs. (5)–(8), together yield for the Wilson coefficients c_i:

$$c_i = \frac{c_i}{c} \frac{M_{\rm pl}}{M_X} \sqrt{4\pi\alpha_G(M_X)} \sqrt{\sum_{j=1}^{h} \frac{C_2(r_j)}{12} \left(\frac{v_j}{v}\right)^2 \left(\frac{\alpha_G(M_X)}{\alpha_s(M_X)} - 1\right)}$$

$$\times \left(\sum_{k=1}^{h} \frac{c_k}{c} \frac{v_k}{v} \delta_s^{(k)}\right)^{-1} \qquad \text{(for any } s = 1, 2, 3). \qquad (9)$$

(Higgs fields in representations $r_i \notin \{\mathbf{1}, \mathbf{24}, \mathbf{75}, \mathbf{200}\}$ can easily be accommodated by setting $c_i \equiv \delta_s^{(i)} \equiv 0$). Then setting $v_i/v \approx 1/h$ (no large hierarchies between VEVs v_i), $\max_i |c_i/c| \gtrsim O(1-5)/h \gtrsim 1/h$ (the constant of proportionality depends on the hierarchy between and the signs of the c_i), $\alpha_G(M_X) \gtrsim 1/50$ (typical for nonsupersymmetric unification; see also Table III) and $\sum_j C_2(r_j) = h\bar{C}_2 \gtrsim 5h$ (see Table V), and using that (9) is valid for all $s = 1, 2, 3$ gives

$$\max_i |c_i| \gtrsim \frac{1}{h} \frac{M_{\rm pl}}{M_X} \sqrt{\frac{4\pi}{50}} \sqrt{\frac{5h}{12} \frac{1}{h^2}} \left| \frac{\alpha_G(M_X)}{\alpha_s(M_X)} - 1 \right|$$

$$\times \left(\max_{s'} \left| \sum_k \frac{c_k}{c} \frac{1}{h} \delta_{s'}^{(k)} \right| \right)^{-1}$$

$$\gtrsim \frac{1}{\sqrt{3h}} \frac{M_{\rm pl}}{M_X} \left(\frac{1}{3} \sum_{s=1}^{3} \left| \frac{\alpha_G(M_X)}{\alpha_s(M_X)} - 1 \right| \right) (\max_{s',k} |\delta_{s'}^{(k)}|)^{-1}.$$

$$(10)$$

The last factor depends on the group theory constants $\delta_s^{(r)}$ associated with the embedding of the standard model into $SU(5)$ and numerically equals $(10/\sqrt{168})^{-1}$ (see Table II); the other factor largely characterizes how well (or how badly) the actual running couplings $\alpha_s(\mu)$ of the standard model unify without any modification to the unification condition since it can be estimated with (6):

$$\sum_{s=1}^{3} \left| \frac{\alpha_G(M_X)}{\alpha_s(M_X)} - 1 \right| \approx 2 \left| \frac{\alpha_1(M_X) - \alpha_2(M_X)}{\alpha_1(M_X) + \alpha_2(M_X)} \right|$$

$$+ \left\{ \begin{matrix} \alpha_1 \to \alpha_2 \\ \alpha_2 \to \alpha_3 \end{matrix} \right\} + \left\{ \begin{matrix} \alpha_1 \to \alpha_3 \\ \alpha_2 \to \alpha_1 \end{matrix} \right\} \gtrsim 0.25$$

$$(\text{for } 10^{15} \leq M_X/\text{GeV} \leq 10^{19}), \qquad (11)$$

yielding finally:

$$\max_i |c_i| \gtrsim \frac{1}{15\sqrt{h}} \frac{M_{\rm pl}}{M_X} \qquad (\text{for } 10^{15} \leq M_X/\text{GeV} \leq 10^{19}).$$

$$(12)$$

This estimate captures unification according to the modified unification condition (6) pretty well for models with several Higgs multiplets (as specified above), as can be seen from numerical studies: from the numerical behavior, to achieve equality in (12), $1/15$ should be replaced

by some factor of order $O(0.08 \to 4)$, depending on the Higgs content of the model, on the ratio of VEVs and c_i's, and on M_X. The following lower bound can be strict in some models, but has leeway in most situations:

$$\max_i |c_i| \gtrsim \frac{O(0.1)}{\sqrt{h}} \frac{M_{\rm pl}}{M_X} = \frac{O(0.1)}{\xi\sqrt{h}} \frac{1.2 \times 10^{19} \text{ GeV}}{M_X} \qquad (13)$$

$$(\text{for all } 10^{13} \leq M_X/\text{GeV} \leq 10^{20}).$$

Furthermore, unification at any given M_X *can* be achieved in any such model when allowing coefficients c_i of the size (13) with the constant being $O(0.5)$.

For justification of (13) and its $O(0.1)$ factor in a model with three Higgs multiplets, see Fig. 3, especially the right panel where the $O(0.1)$ factor to achieve equality in (13) has been plotted for 30 000 randomly sampled points of the parameter space that yield unification. (One can formally achieve small Wilson coefficients by making the number h of Higgs multiplets large, but these models face issues such as Landau poles close to the unification scale and small Higgs VEVs which lead to scalar masses potentially far below the unification scale.) An estimate like (13) allows one to quickly judge whether, for some given Higgs content, unification via (6) at a desired scale M_X is possible with natural-size Wilson coefficients c_i. Models containing Higgs multiplets in representations $r \notin \{\mathbf{24}, \mathbf{75}, \mathbf{200}\}$ have larger required c_i than models without such multiplets; see (9). Estimate (13) is also quite accurate at the unification point M_X in models with only one dimension-5 operator, see last column of Table III; however, in those models, M_X cannot be continuously shifted to other values as in the models considered in this subsection here.

The ingredients of the unification mechanism discussed in this paper are well illustrated by (13): For models with Higgs multiplets in at least two of the irreps **24**, **75**, **200**, unification at continuously variable scale M_X may be achieved due to the presence of gravitationally induced effective dimension-5 operators (1) in the grand unified theory, since two or more such operators allow for a continuous set of solutions to unification condition (6). To actually achieve unification at a given M_X, however, these operators need to have the right sizes c_i, see (13) for an estimate, and these sizes influence whether one considers unification at M_X possible in a natural way (requiring, very roughly, $|c_i| \sim O(0.1 \to 10)$ or so). The sizes of the c_i are further directly affected by the choice of the Planck scale $M_{\rm pl}$ (2) that suppresses the dimension-5 operators (1). Reasonable choices in (13) are $\xi = 1$ (naive Planck scale) or, more commonly, $\xi_{\rm red} = \sqrt{8\pi} \approx 5$ (reduced Planck scale), possibly further enhanced by factors $O(1.5 \to 5)$ if the running of Newton's constant (3) in such models is taken into account as another, additional gravitational effect (cf. η in Table I). These latter considerations about the appropriate choice of the Planck scale merely influence the required numerical values of the Wilson coefficients c_i by
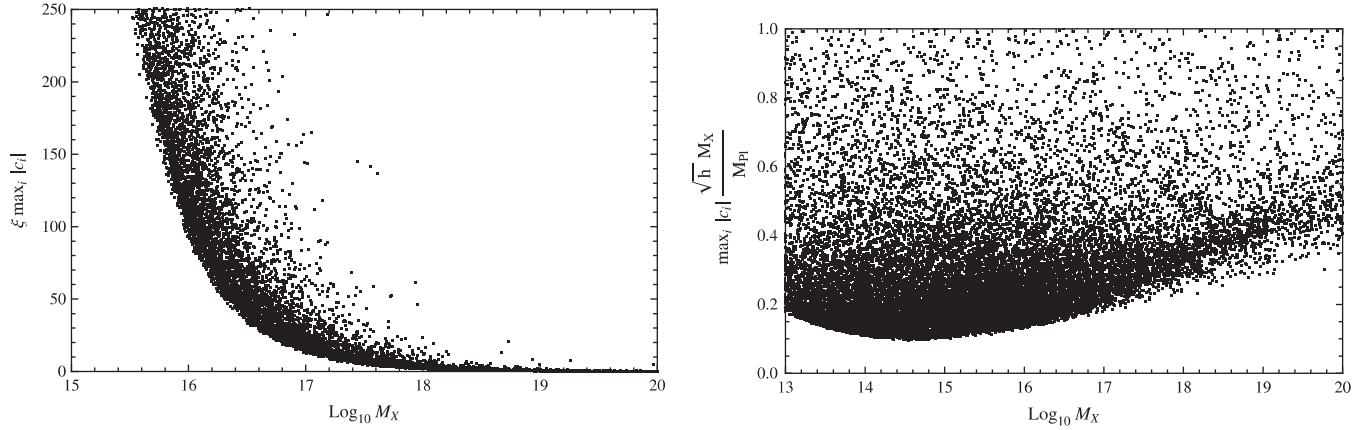
FIG. 3.   For a model with the three ($h = 3$) Higgs multiplets **24**, **75**, and **200**, a random sample is taken of 30 000 points $\{c_i, v_i\}$ of the parameter space which yield gauge-coupling unification. The point plot on the left characterizes the required sizes of the Wilson coefficients c_i to achieve unification, as in Figs. 1 and 2; unification at smaller scales M_X requires proportionally larger Wilson coefficients. To illustrate the validity of (13), the right panel shows, for each sample point, the $O(0.1)$ constant to achieve equality in (13); numerical gauge-coupling unification at any 10^{13} GeV $< M_X < 10^{20}$ GeV can be achieved by choosing 0.5 as this constant, and a lower bound is obtained by the choice 0.1. However, this should not be mistaken: unification at small $M_X \lesssim 10^{16}$ GeV requires large dimensionless Wilson coefficients c_i (see left panel).

factors of order $O(1 \rightarrow 25)$, and thus are only secondary to the possibility of continuous variability of the unification condition (6), which facilitates exact gauge-coupling unification for some values of the c_i in the first place.

In this section we have discussed how models can achieve exact gauge-coupling unification through unification condition (6) and whether this is possible with natural coefficients. The former is a formal numerical requirement which grand unified theories ought to obey. But it is a further question whether those models are physically sensible and viable, e.g., if they are allowed by experimental constraints. In the following two sections (IV and V), we will look at some of the physics of these unification models.

IV. NONSUPERSYMMETRIC UNIFICATION NEAR THE PROTON LIFETIME LIMIT

In this and in the following section we will demonstrate and analyze two possible implications of models which employ the unification mechanism just described.

Here, we display and examine models which allow nonsupersymmetric unification of the standard model into $SU(5)$ without intermediate symmetry breaking scales, while easily avoiding constraints from proton decay, contrary to a widely held belief. This scenario seems attractive as it does not require presently unobserved supersymmetry and as nonsupersymmetric models offer the possibility of describing physics up to the onset of quantum gravity in a perturbative way; see the right column in Table I and the last paragraph of Sec. II. However, the proposed models are not completely minimal in their Higgs content as mechanism (6) requires at least two multiplets for continuously shifting M_X around (the minimal choice being a **24**

and a **75** Higgs), whereas one **24** alone would already be sufficient to break the grand unified symmetry; nevertheless, as described in Sec. II, when this Higgs content is assumed, the presence of the gravitational operators (1) is to be expected and does not further detract from minimality.

In nonsupersymmetric models, proton decay is mediated by gauge $d = 6$ operators (baryon number violating operators after integrating out the superheavy gauge fields) and Higgs $d = 6$ operators (after integrating out superheavy Higgses). The latter are strongly dependent on the Higgs sector and Yukawa couplings; since they are, apart from the potentially dangerous triplet route, generically less important than the gauge $d = 6$ operators (see [34] for a review), we will concentrate on the gauge contribution for the following estimates (also neglecting potential "textures" in flavor space that could partially rotate away the effective gauge $d = 6$ interactions, yielding a slower decay rate and weaker bounds [35]). Under these assumptions, the proton decay rate $1/\tau_{(p \rightarrow e^+ \pi^0)}$ through the dominant (in our models) channel $p \rightarrow e^+ \pi^0$ is related to the superheavy gauge boson masses M_X (8) by

$$\tau_{(p \rightarrow e^+ \pi^0)} = O(1) \frac{M_X^4}{\alpha_G^2 m_p^5}, \tag{14}$$

with the proton mass m_p and we take the $O(1)$ constant of proportionality to be 1 for the following. The current experimental bound on this decay channel is [20,36]

$$\tau_{(p \rightarrow e^+ \pi^0)} > f \times 10^{33} \text{ years with } f = 1.6 \quad \text{or} \quad f = 5.4. \tag{15}$$

This sets a proton lifetime bound on the unification scale

M_X in nonsupersymmetric models:

$$M_X > (40\alpha_G)^{1/2}\left(\frac{f}{1.6}\right)^{1/4} 2.4 \times 10^{15} \text{ GeV}$$
$$\approx (1.9 - 3.8) \times 10^{15} \text{ GeV},$$

(16)

where the latter illustrates the range given by (15) and $\alpha_G = 1/60 - 1/30$ [typical for nonsupersymmetric models using (6), cf. also Table III]. Within the next ten years, if proton decay remains unobserved, improvements in the bound (15) up to $f = 100$ are expected [37], constraining $M_X > 8 \times 10^{15}$ GeV.

One can now see that all of our models that achieve natural gauge-coupling unification, i.e., with Wilson coefficients of order $|c_i|_{\max} \lesssim O(10)$, satisfy the proton decay constraint (16) easily, and are also fairly safe against the expected ten-year improvements in the bound. For example, of the models with one Higgs multiplet (see Table III), only the **200** model (and possibly the **75** model, depending on the choice of M_{pl}) feature natural gauge-coupling unification, with the unification scale 5×10^{18} GeV (respectively, 8×10^{15} GeV) above the bound (16) in either case. This also holds for models with two Higgs multiplets; see Figs. 1 and 2. Under the condition $|c_i|_{\max} < 5$, the lowest unification scale $M_X = 1.4 \times 10^{16}$ GeV is here achieved in the model with a **75** and a **200** Higgs (Fig. 2, right panel) when assuming $\xi = \xi_{\text{red}}^{\text{run}} = 13.6$; the other two models with two multiplets automatically have $M_X > 3 \times 10^{16}$ GeV if gauge-coupling unification with Wilson coefficients of this size is required.

Quite generally, any reasonable nonsupersymmetric unification model featuring natural gauge-coupling unification via (6) satisfies the proton decay constraint (16). This can be seen from the general estimate (13) of the parameters necessary for gauge-coupling unification:

$$M_X \gtrsim \frac{O(0.1)}{\xi\sqrt{h}} \frac{1.2 \times 10^{19} \text{ GeV}}{\max_i |c_i|}$$
$$\gtrsim \frac{0.1}{15\sqrt{4}} \frac{1.2 \times 10^{19} \text{ GeV}}{10}$$
$$= 4 \times 10^{15} \text{ GeV},$$

(17)

so that (16) still holds despite all the very conservative parameter choices in (17) which are by no means necessary or particularly desirable. Models with more natural parameters (i.e., with smaller $|c_i|_{\max}$, reasonable h, and possibly choosing $\xi_{\text{red}} = \sqrt{8\pi}$) satisfy the proton lifetime constraint very easily.

The intuitive reason why our models naturally obey the proton decay constraint so readily is clear: In nonsupersymmetric models, the couplings $\alpha_s(\mu)$ miss each other by quite a bit, so the modifications to the unification condition (6) need to be relatively sizeable $\epsilon_s \sim O(0.1)$ in order to achieve exact gauge-coupling unification; then, since we require naturalness $|c_i|_{\max} \lesssim O(10)$, these sizes $\epsilon_s \sim$

$c_i M_X/M_{\text{pl}}$ of the effective gravitational corrections (4) must be due mainly to the proximity of the Planck scale $M_{\text{pl}} = 1.2 \times 10^{19}$ GeV$/\xi$ to the unification scale M_X and, consequently, to the gauge boson masses (8). Typically, such heavy gauge bosons ensure the proton decay constraint (16).

On the other hand, whereas the natural models easily satisfy current proton decay limits, there are situations conceivable in which the constraints in the not too distant future will come close to testing some of the models, either excluding them or, if the proton is actually seen to decay, strongly restricting their parameter space. For example, a model with Higgs content **24**, **75**, and **200** can achieve unification at $M_X = 8 \times 10^{15}$ GeV, the projected constraint in less than ten years from now [37], if merely the reduced Planck scale ($\xi = \xi_{\text{red}} = \sqrt{8\pi} \approx 5$) and somewhat large coefficients $|c_i|_{\max} \sim 20$ are accepted (see Fig. 3). Alternatively, the smallest model with two Higgs multiplets **24** and **75** can reach this limit when taking $\xi = \xi_{\text{red}}^{\text{run}} = 8$ and allowing $|c_i|_{\max} \sim 15$, and similarly can the other models with two multiplets (in each example, only the product $\xi|c_i|_{\max}$ is fixed whereas the sizes of both factors can be traded back and forth). Regarding the exact numerical values in this discussion, one has to keep in mind that there is some uncertainty in the exact numerical $O(1)$ factor in (14), although it enters the final bound (16) only to the power of $1/4$, and in higher-order and threshold corrections to the gauge coupling running.

We have seen that the natural models with gauge-coupling unification through (6) are physically viable as they automatically evade the proton lifetime constraint that naively excludes nonsupersymmetric unification. However, some of the less natural unification models will come close to the proton limit in the near future, which will exclude some of them, or constrain their parameter space drastically if proton decay is observed.

V. SIMULTANEOUS UNIFICATION OF GAUGE AND GRAVITATIONAL INTERACTIONS

As can be seen from Figs. 1–3 or from estimate (13), gauge-coupling unification at large $M_X \sim M_{\text{pl}}$ is possible through mechanism (6) naturally, i.e., with small Wilson coefficients c_i. Unification of the three standard model gauge interactions with each other on the one hand and with the gravitational interaction on the other hand at the same scale ("gauge-gravity unification"; cf. [21,22]) therefore constitutes another scenario naturally achievable through the modified unification condition (6). However, one should keep in mind that our analysis is based on an effective theory approach and we should therefore consider carefully the expansion we are using. If the operators of dimension 5 and higher that we are considering are of nonperturbative nature, then the expansion is in powers of $1/M_{\text{pl}}$ and one may worry that it could break down once the Higgs VEVs get close to the Planck mass M_{pl}. In any

case, it is an interesting numerical coincidence that the corrections to the unification condition allow one to shift the unification scale close to the Planck scale for a natural set of parameters.

Approximate parameter values c_i of such gauge-gravity unification models can be read off from estimate (13) by setting $M_X = M_{pl}/O(1)$:

$$\max_i |c_i| \approx O(1) \frac{O(0.2 \rightarrow 1)}{\sqrt{h}}, \qquad (18)$$

where the $O(0.2 \rightarrow 1)$ estimate stems from the $M_X \approx M_{pl}$ range of numerical studies like Fig. 3 (right panel) and h denotes the number of Higgs multiplets at the grand unification scale. This suggests the possibility of very natural gauge-coupling unification at or near M_{pl} and is already apparent in the only model with one Higgs multiplet that achieves unification close to the Planck scale: The **200** model has $M_X = 5.2 \times 10^{18}$ GeV, which is related to the Planck scale by $M_X = 0.4\xi M_{pl}$, and the required Wilson coefficient is natural $c = 0.53/\xi$ (see Table III). As another example, gauge-coupling unification can be achieved for appropriate parameter choices in the model with a **24** and a **75** (Fig. 1) at the scale $M_X = M_{pl} = 1.2 \times 10^{19}$ GeV$/\xi$ for any of the reasonable exemplary choices $\xi = 1$, $\xi_{red} = \sqrt{8\pi}$, $\xi^{run} = 1.6$, or $\xi_{red}^{run} = 8.0$; the corresponding model parameters $|c_i|_{max} = 0.23, 0.21, 0.23$, and 0.20 are all natural and of almost equal sizes independent of ξ; see (18). Similar numerical estimates hold for the other two- and three-Higgs models. In these models, any $O(1)$ factor in the relation $M_X = M_{pl}/O(1)$ can be easily accommodated as well by corresponding $O(1)$ changes to the Wilson coefficients, whereas this is not possible for the sole-**200** model as it only permits one discrete unification scale (Table III).

In the effective field theory spirit of Sec. II, operators of dimension higher than 5 are also present, e.g., higher-dimensional generalizations of (1):

$$\mathcal{L} = \frac{c_6}{4M_{pl}^2} H_1 H_2 G_{\mu\nu} G^{\mu\nu} + \frac{c_7}{4M_{pl}^3} H_1 H_2 H_3 G_{\mu\nu} G^{\mu\nu} + \ldots \qquad (19)$$

After the Higgs multiplets acquire VEVs at the scale $M_X = M_{pl}/O(1)$, they contribute to the gauge kinetic terms (4) as well:

$$\mathcal{L} = \sum_{s=1}^{3} -\frac{1}{4}\left(1 + \epsilon_s + \frac{c_6 \delta_s^{(6)}}{g_G^2 O(1)^2} + \frac{c_7 \delta_s^{(7)}}{g_G^3 O(1)^3} + \ldots\right)$$
$$\times F_{(s)\mu\nu}^a F_{(s)}^{a\mu\nu}, \qquad (20)$$

with the corrections $\epsilon_s \sim c_5 \delta_s M_X/g_G M_{pl}$ from the dimension-5 operators (5). Depending on the group theory factors $\delta_s^{(6,7)}$ (analogous to the δ_s in Table II) and on the constant in the relation $M_X = M_{pl}/O(1)$, this expansion might or might not be controlled perturbatively. If it is not, one cannot claim perturbative gauge-gravity unification at

the Planck scale. Nevertheless, the fact that mechanism (6) allows one in principle to naturally adjust the unification scale to a high scale $\sim M_{pl}$ might at least be taken as a hint that gauge-gravity unification is a possible scenario, even if the necessary parameter values or the last piece of the evolution cannot be computed perturbatively.

VI. THE NONSUPERSYMMETRIC $SO(10)$ CASE

In this section we give an account of the effects from the dimension-5 operators (1) in models with grand unified group $G = SO(10)$. We will describe the setup for $SO(10)$ by emphasizing the differences to the $SU(5)$ case described in Sec. II and give a few numerical unification results similar to Sec. III. We find that the effects from Secs. IV (on proton decay) and V (on gauge-gravity unification) can occur with similar sizes for $SO(10)$ as well, although the $SO(10)$ formalism is more general than the $SU(5)$ one; in particular, a continuously variable unification scale M_X can now be achieved with only one single dimension-5 operator. For this reason, and for the beauty of $SO(10)$ unification, we find this treatment worthwhile.

For $G = SO(10)$, there are *two* inequivalent ways to embed the standard model group $G_{321} = SU(3)_C \times SU(2)_L \times U(1)_Y$ into G consistent with the charge assignments of the standard model fermions: the "normal embedding" $G_{321} \subset SU(5) \subset SO(10)$ [38], and the "flipped embedding" $G_{321} \subset SU(5) \times U(1)_X \subset SO(10)$ with $G_{321} \not\subset SU(5)$ [39] (see the Appendix for a careful treatment of the $SO(10)$ group theory). Statements without qualifier in the following apply to either embedding. Note, nowhere are we implying that either $SU(5)$ or $SU(5) \times U(1)_X$ (or any other subgroup) be intermediate unbroken symmetries at any scale.

In the $SO(10)$ grand unified theory, the gravitationally induced effective dimension-5 operators (1) can be formed with Higgs multiplets H_i in any of the four irreps $r_i = \mathbf{1}$, **54**, **210**, **770**. However, contrary to the $SU(5)$ case, the VEV $\langle H_i \rangle$ is not necessarily uniquely specified (up to normalization) by requiring it to be invariant under the standard model group G_{321}; rather, this G_{321} invariance merely restricts to a 3-, respectively, 4-dimensional subspace inside the **210**, respectively, **770** irreps, so that the length as well as the direction of the VEV has to be specified for Higgses in these irreps. This will have the important consequence that with merely *one* dimension-5 operator (1), built with either a **210** or **770** Higgs, one can continuously vary the ratio among the ϵ_s in the unification condition (6) by continuous variation of the VEV direction, ultimately leading to a continuously variable unification scale M_X; the $SU(5)$ case could achieve this continuity only with at least two Higgs multiplets.

To parametrize these VEV directions for calculations, one has to specify a basis in these 3- or 4-dimensional subspaces. Two possible, distinct choices of this basis are given in the Appendix: the vectors in the first, respectively,

second basis have definite transformation properties under the $SU(5) \times U(1)_X$, respectively, $SU(4)_C \times SU(2)_L \times SU(2)_R$ maximal subgroups of $SO(10)$ (see the first two columns of Tables VII, respectively, VIII for these transformation properties). We choose to classify according to these two subgroups since they are the only maximal subgroups that can occur as intermediate symmetries in $SO(10) \to G_{321}$ breaking consistent with the standard model charge assignments. These two choices therefore readily facilitate the analysis of $SO(10)$ breaking with intermediate gauge symmetries; see below (the absence of intermediate symmetries is assumed for now).

With such a chosen basis, the modifications ϵ_s to the three standard model gauge kinetic terms (4) are now, instead of (5), double sums (A28)

$$\epsilon_s = \sum_i \frac{c_i}{M_{\mathrm{pl}}} \sum_j \upsilon_{(i)j} \delta_s^{(i)j} \qquad \text{(for } s = 3, 2, 1), \qquad (21)$$

where the second sum runs over the directions $j = 1, \ldots, 3(4)$ if H_i is in the **210** (**770**) representation (for the **1** and **54** irreps, $j = 1$ only). The Clebsch-Gordan coefficients $\delta_s^{(i)j}$ depend on the embedding $G_{321} \subset SO(10)$ (normal or flipped), on the representation r_i of the Higgs H_i and on the choice of basis vectors j in the standard model singlet subspace of the respective representation. These $\delta_s^{(i)j}$ can all be taken from Table VII [for basis vectors with definite $SU(5) \times U(1)_X$ transformation properties] and Table VIII [definite $SU(4)_C \times SU(2)_L \times SU(2)_R$ transformations] via (A28).

One other difference between the $SO(10)$ and the $SU(5)$ cases is the fact that none of the **54**, **210**, or **770** multiplets can give mass to all of the non-standard model ("superheavy") gauge bosons and that, furthermore, the masses of the gauge bosons which actually do acquire mass from these multiplets are generally unequal. Therefore, both the statement from the $SU(5)$ case that the mass M_{gb} of the superheavy gauge bosons may originate solely from Higgses involved in dimension-5 operators (1) and the requirement (8) that this mass be close (or equal) to the unification scale M_X have to be amended. First, additional Higgs multiplets in other irreps (typically **45**, **16**, or **126**; see also Table I) are necessary to break the $SO(10)$ gauge symmetry down to the standard model G_{321} and they contribute to the masses of the superheavy gauge bosons as well. And for definiteness, we now assume that the Higgs multiplets involved in dimension-5 operators (1) account for half of the average squared superheavy gauge boson masses, and we require that this *averaged* superheavy mass now be equal to the unification scale; see (A25) in the Appendix. The necessity for some such choice causes uncertainty in actual numerical unification calculations greater than in the $SU(5)$ case.

As mentioned above, continuously variable unification scales M_X can be achieved in $SO(10)$ models with a single Higgs in either the **210** or the **770** representation, since the

direction $\{\upsilon_{(i=1)j}\}_{j=1,\ldots,3(4)}$ of their VEV can be varied continuously in (21). Minimizing the Higgs content and the number of dimension-5 operators in this way seems attractive. Further note, that a **770** causes the strong coupling regime of the grand unified theory to be quite close to the unification scale (Table I), although, depending on the gap between the unification and the Planck scale, the theory may still be perturbative up to the onset of quantum gravity.

As a numerical example, Fig. 4 shows gauge-coupling unification results in the $SO(10)$ model with a single **210** Higgs [identical results obtain for both the normal and the flipped embedding due to the orthogonal relation (A18)], similar to Fig. 3 for $SU(5)$ with three Higgses. The lower bound (13) holds here as well, even with the same factor $O(0.1)$, and unification *can* be achieved for any M_X in the displayed range if this factor is allowed to be 0.5 (see right panel). Therefore, unification is naturally safe from the proton lifetime limit in $SO(10)$ as well, but can also come close to it for some points in the parameter space, and unification near or at the Planck scale is achievable in a very natural way, both similar to the corresponding $SU(5)$ scenarios described in detail in Secs. IV and V.

In a scenario with several breaking scales, if the intermediate gauge symmetry is $SU(5) \times U(1)_X$ [or merely $SU(5)$] or the Pati-Salam group $SU(4)_C \times SU(2)_L \times SU(2)_R$, respectively, the directions of both the **210** and **770** VEVs at the unification scale M_X are fixed, as these VEVs have to be singlets under the intermediate group, which allows only the $\delta_s^{(i)j}$ from rows 1, 3, 6 in Table VII, respectively, rows 1, 2, 3, 6 in Table VIII. Therefore, similar to the case of an $SU(5)$ unified group, only linear combinations of the effects from at least two operators (1) can achieve continuously varying M_X. The effect of a dimension-5 operator with a Higgs that acquires its VEV at an intermediate scale $M_I < M_X$ is suppressed relatively by $\sim M_I/M_X$. Note, since there are no singlet VEVs for a **54** Higgs under the $SU(5)$ intermediate group, it can acquire nonzero VEV only at the final breaking scale to the standard model group G_{321} and is therefore almost irrelevant for our effect in the case of an $SU(5)$ intermediate gauge symmetry.

In several previous analyses of the $SO(10)$ case (e.g., [15]), only Higgs VEVs that are invariant under intermediate gauge groups larger than G_{321} have been considered [namely under the $SU(5)$ Georgi-Glashow or the $SU(4)_C \times SU(2)_L \times SU(2)_R$ Pati-Salam subgroups], thereby omitting most of the possibilities discussed above for $SO(10)$ breaking without intermediate symmetries, i.e., the continuous variability of M_X with merely one dimension-5 operator.

Exactly analogous effects obtain for other grand unified gauge groups, like E_6, and they are even of similar size as in $SU(5)$ and $SO(10)$ [15,25]. Furthermore, the formalism necessary for the $SO(10)$ treatment is already the most

general for any unification group, so we leave the analysis of gauge-coupling unification through the mechanism of dimension-5 operators (1) at the $SU(5)$ and $SO(10)$ cases discussed so far.

VII. COMPARISON TO THE SUPERSYMMETRIC CASE

The influence of the dimension-5 operators (1) has been studied [2] in models of supersymmetric grand unification [16–18] as well, although most studies have focused on the effect of the supersymmetrized version of (1) in the creation of nonuniversal gaugino masses in a scenario where the UV completion is a $\mathcal{N} = 1$ supergravity theory; see [25] and references therein. The group theory needed to calculate the resulting gaugino mass ratios is very similar to the formalism for obtaining the modified unification condition (6) [26,27]; see the Appendix for the detailed $SU(5)$ and $SO(10)$ group theory.

The effect of these dimension-5 operators on the unification of gauge couplings in supersymmetric theories and on the unification scale has been noted before [2,5,13,15,26,27]. In some of these works, however, the low-energy inputs $\alpha_s(m_Z)$ or the supersymmetry (SUSY) breaking scale m_{SUSY} [above which the β-function coef-

ficients (A31) are taken to be $(b_1, b_2, b_3) = (33/5, 1, -3)$] have been treated as uncertain parameters, and mostly only the effect from a single dimension-5 operator had been taken into account (except in [13]).

Fixing the low-energy gauge-coupling values (A32) and $m_{SUSY} = 1$ TeV [19], as roughly required to solve the hierarchy problem of the standard model, one obtains for supersymmetric $SU(5)$ models with one dimension-5 operator the unification possibilities in Table IV (at one loop), in analogy with Table III for nonsupersymmetric $SU(5)$ [note, supersymmetric $SU(5)$ with a **200** Higgs enters strong coupling shortly above the unification scale, cf. Table I].

For a general number of Higgs fields, the situation is shown in Fig. 5, in analogy to Fig. 3. As can be seen, exact supersymmetric gauge-coupling unification around 2×10^{16} GeV can be achieved with natural-sized dimension-5 operators (1), which is expected as this is where the standard model gauge couplings come closest to each other in supersymmetric models; but higher unification scales are also possible in a natural way with certain values of the coefficients c_i. Note that the supersymmetric unification model from Fig. 5 has a Landau pole roughly half an order of magnitude above the unification scale M_X, cf. Table I.

TABLE IV. This table shows the parameters that a SUSY-$SU(5)$ model must have if, at one loop, unification happens by means of mechanism (6) with only one dimension-5 operator (1). The sizes of the required Wilson coefficients c depend on the chosen suppression scale (2), parametrized by ξ.

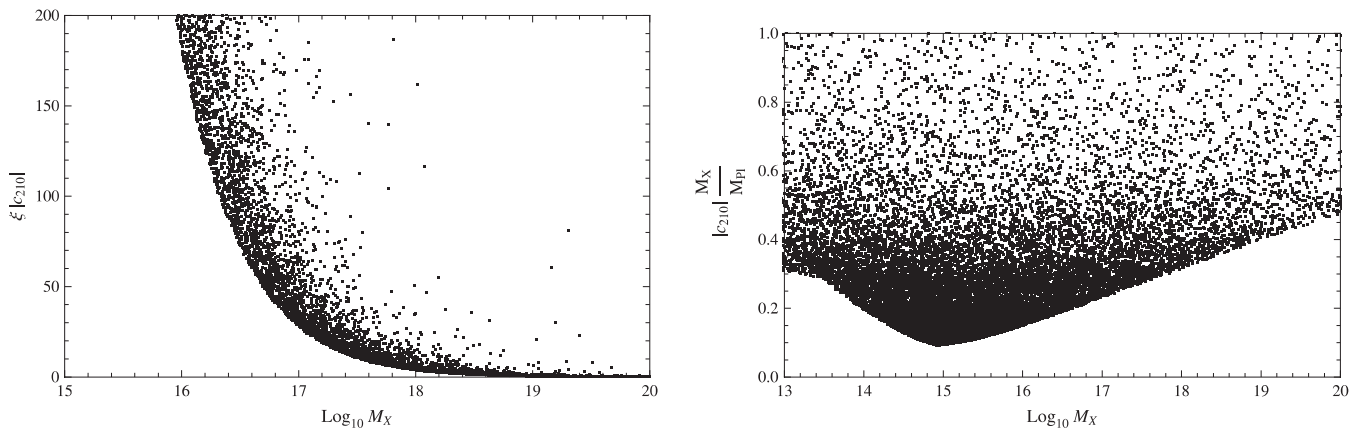| H irrep	M_X/GeV	$1/\alpha_G$	c	ν/GeV	$\max_s\|\epsilon_s\|$	$\xi = 1$	$\xi_{red} = \sqrt{8\pi}$	ξ^{run}	ξ^{run}_{red}
1	Unification by mechanism (6) impossible								
24	1.1×10^{16}	25.9	$31.1/\xi$	2.4×10^{16}	0.024	1	5	2.3	11.6
75	3.2×10^{16}	25.8	$-12.0/\xi$	5.6×10^{16}	0.033	1	5	3.1	15.4
200	8.5×10^{16}	26.8	$13.1/\xi$	1.2×10^{17}	0.105	1	5	4.4	22.1

FIG. 4. For the $SO(10)$ model with a **210** Higgs, a random sample is taken of 30 000 points $\{c_{210}, \nu_{(210)j}\}$ of the parameter space which yield gauge-coupling unification; cf. Figure 3. Estimate (13) is valid here as well (see right panel), and numerical gauge-coupling unification at any 10^{13} GeV $< M_X < 10^{20}$ GeV can be achieved with the $O(0.1)$ constant in (13) being at most 0.5 (the caveat from Fig. 3 applies here as well).
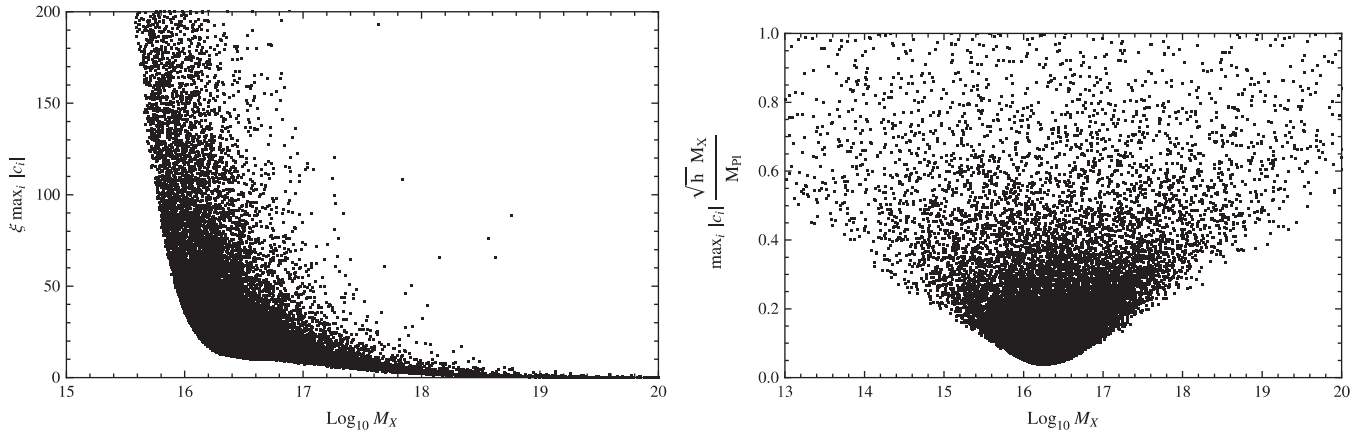
FIG. 5.   Similar to Fig. 3, but for the supersymmetric $SU(5)$ model with the three ($h = 3$) Higgs multiplets **24**, **75**, and **200**. Again, numerical gauge-coupling unification at any 10^{13} GeV $< M_X < 10^{20}$ GeV can be achieved by choosing 0.5 as the constant in (13) (but cf. caveat from Fig. 3). Most of the randomly chosen unification points lead to unification scales $M_X \sim 2 \times 10^{16}$ GeV, where, in supersymmetric models, the standard model gauge couplings almost meet.

VIII. SUMMARY AND CONCLUSION

We have studied the effects from multiple gravitationally induced dimension-5 interactions $c_i H_i G_{\mu\nu} G^{\mu\nu}/4M_{\rm pl}$ (1), naturally present in any grand unified theory, on the unification of gauge couplings. These operators can modify the canonical gauge kinetic terms (4) and the coupling unification condition (6) by up to $\epsilon_s \sim c_i \langle H_i \rangle / M_{\rm pl} \sim c_i M_X / g_G M_{\rm pl} \sim 0.1\%$–30% (5) after the Higgs multiplets acquire VEVs in grand unified symmetry breaking. The size of the effect depends on the unification scale M_X, the Wilson coefficients $0.1 \lesssim |c_i| \lesssim 10$, and the suppression scale 5×10^{17} GeV $\lesssim M_{\rm pl} \lesssim 1.2 \times 10^{19}$ GeV (2).

Modifications ϵ_s of this size can effect perfect gauge-coupling unification, at scales M_X significantly different than naively expected, as the standard model gauge couplings $\alpha_s(\mu)$ ($s = 1, 2, 3$) differ from each other by at most $\lesssim 50\%$ in the wide range 10^{13} GeV $< \mu < 10^{19}$ GeV, both in the case with and without supersymmetry. When two or more dimension-5 operators are present in a theory [or a single **210** or **770** in $SO(10)$, see Sec. VI], then the scale M_X of gauge-coupling unification can be varied in a continuous and controlled way as a function of the Wilson coefficients c_i and Higgs VEVs $\langle H_i \rangle$, which are practically model parameters. We have focused on nonsupersymmetric $SU(5)$ and $SO(10)$ theories, for which grand unification seemed, previously, to be difficult [19], and we find that gauge-coupling unification is possible, in a natural way, at any $M_X \sim 10^{17}$ GeV $- 1.2 \times 10^{19}$ GeV. For illustration we now give a few numerical unification results (all nonsupersymmetric models):

(a) $SU(5)$ with **24** and **75** Higgses, see Fig. 1: Perfect gauge-coupling unification at $M_X = 10^{17}$ GeV happens for some choice of parameter values c_{24}, c_{75} with $|c_i| \lesssim 3$, if $M_{\rm pl} = 1.2 \times 10^{19}$ GeV$/\sqrt{8\pi} = 2.4 \times 10^{18}$ GeV is assumed; the modifications ϵ_s to the gauge kinetic terms are all less than 15%.

(b) $SU(5)$ with **24**, **75**, and **200**, Fig. 3 and estimate (13): Unification at any $M_X > 10^{18}$ GeV is possible for several choices of c_{24}, c_{75}, c_{200} with $|c_i| < 2$, when $M_{\rm pl} = 1.2 \times 10^{19}$ GeV is assumed. In fact, the bigger a unification scale one wants to achieve, the smaller Wilson coefficients are required: $\max_i |c_i| \approx 2 \cdot (10^{18}$ GeV$/M_X)$; analogous statements hold (at fixed $M_{\rm pl}$) in all examples (a)–(c) with continuous variability of M_X, cf. (13).

(c) $SO(10)$ with a single **210**, Fig. 4: Exact unification at $M_X = 3 \times 10^{17}$ GeV is possible with $c_{210} \approx 2.5$ for a certain (continuous) set of directions of the Higgs VEV $\langle H^{ab} \rangle$, if $M_{\rm pl} = 1.2 \times 10^{19}$ GeV$/\sqrt{8\pi} = 2.4 \times 10^{18}$ GeV is assumed.

(d) $SU(5)$ with **24** and **75**, Fig. 1 and Eq. (18): Numerical gauge-coupling unification at the very high scale $M_X = M_{\rm pl}$, i.e., at the Planck scale, is possible for some choice of Wilson coefficients with $|c_{24}|$, $|c_{75}| < 0.25$ (irrespective of the specific choice of $M_{\rm pl}$).

(e) $SU(5)$ with a single **200**, Table III: Exact gauge-coupling unification can happen, but only for a discrete choice $c_{200} = 0.5$ [15] (here, $M_{\rm pl} = 1.2 \times 10^{19}$ GeV is assumed); in this case, the unification scale $M_X = 5.2 \times 10^{18}$ GeV cannot be shifted continuously.

We have shown (Sec. IV) that nonsupersymmetric unification, achieved via these dimension-5 operators, is not in conflict with the current bound on the proton lifetime (16), since, with natural-sized Wilson coefficients c_i, only unification scales $M_X \gtrsim 10^{16}$ GeV are reasonably possible, cf. (17). Consequently, experimental improvements of the proton decay bound will constrain our models only weakly within the foreseeable future. Our main conclusion is that fairly minimal models of nonsupersymmetric $SU(5)$ and $SO(10)$ unification are easily viable through the mecha-

nism described. In particular, supersymmetry does not have to be invoked to save the idea of grand unification.

We also note that, with two or more dimension-5 operators in the theory, gauge-coupling unification at or near the Planck scale $M_X \sim M_{pl}$ can happen for very natural parameter choices $c_i \sim O(0.1 \to 1)$, cf. (18) in Sec. V. Of course, our approach cannot be fully justified very near the Planck scale, due to incalculable corrections from quantum gravity. But the fact that one can, by continuous variation of the Wilson coefficients c_i in a natural domain, push the unification scale M_X close towards the Planck scale, may be suggestive of a scenario of simultaneous gauge-gravity unification.

As we have seen, when unification happens according to the mechanism presented in this paper, then unification scales M_X much larger than usually assumed are favored, in the sense that probably only those are achievable in a natural way. Thus, the mechanism here is distinct from some other known unification scenarios [23,24]. The difference lies in the fact that the importance of the dimension-5 operators (1) increases for unification scales M_X closer to the Planck scale (implying larger Higgs VEVs $\langle H \rangle$), which causes a bigger modification to the unification condition (6) and makes this larger M_X consistent. But, for example, two-loop and threshold effects are not significantly enhanced at larger unification scales, since they originate from within the grand unified theory, rather than from intrinsically shorter distance effects like strong quantum gravity.

APPENDIX: NORMALIZATION CONVENTIONS AND GROUP THEORY

In this appendix, we give a self-contained account of the group theoretical aspects and normalization details that were omitted in the main text.

Our first goal is to establish conventions for the quantities appearing in the dimension-5 operators (1) of interest, aiming towards criteria for assessing the individual size and importance of each operator i based on its Wilson coefficient c_i, and also the relative sizes of different operators (1) appearing in the same Lagrangian. We intend to establish normalization conventions in such a way that operators with Wilson coefficients $|c_i| \sim 1$ are expected in the effective Lagrangian of a grand unified theory after integrating out gravitational interactions, which then helps judge the naturalness of a particular model, i.e., its reasonableness: If $|c_i| \gg 1$ (for any i), the effective Lagrangian might not be a good low-energy description and one might lose perturbative control, whereas $|c_i| \ll 1$ constitutes

fine-tuning, as setting $c = 0$ does not enhance the symmetry [4]. Note, the calculations and numerical gauge-coupling unification results in this paper are valid, at face value, regardless of such criteria.

The effective operators $\sim H G_{\mu\nu} G^{\mu\nu}$ we are considering are formed from two gauge field strengths $G_{\mu\nu}$, transforming in the adjoint representation **G** of the grand unified gauge group G, and one Higgs multiplet H, transforming in an irreducible representation (irrep). The contraction $H G_{\mu\nu} G^{\mu\nu}$ can yield a gauge singlet only if H transforms in an irrep contained in the (conjugate of the) symmetric product $(\mathbf{G} \otimes \mathbf{G})_s$ of two adjoint representations of the gauge group. For $SU(5)$ and $SO(10)$ unified gauge groups G, these possible irreps are the direct summands in the following decompositions (for this and other group theory facts, see [29]):

$$G = SU(5), \qquad \mathbf{G} = \mathbf{24}:$$
$$(\mathbf{24} \otimes \mathbf{24})_s = \mathbf{1} \oplus \mathbf{24} \oplus \mathbf{75} \oplus \mathbf{200};$$
$$G = SO(10), \qquad \mathbf{G} = \mathbf{45}:$$
$$(\mathbf{45} \otimes \mathbf{45})_s = \mathbf{1} \oplus \mathbf{54} \oplus \mathbf{210} \oplus \mathbf{770}. \tag{A1}$$

[In this paper, the **210** of $SO(10)$ will always denote the 210-dimensional irrep in the same congruency class as the **54** and **770**, namely, in the congruency class 0, to which also the adjoint $\mathbf{G} = \mathbf{45}$ belongs.] The irreps on the right-hand sides of (A1) are all real representations, as the adjoint **G** is real itself.

Since they are contained in $(\mathbf{G} \otimes \mathbf{G})_s$, each multiplet H in an irrep (A1) can be written in component form as H^{ab} with symmetric indices a, b of the respective adjoint representation **G**. Under an infinitesimal gauge transformation $(1 + i\alpha^c T^c)$ these components then transform according to

$$H^{ab} \to H^{ab} + i\alpha^c((t_\mathbf{G}^c)^{aa'} \delta^{bb'} + \delta^{aa'}(t_\mathbf{G}^c)^{bb'}) H^{a'b'}$$
$$= H^{ab} + i\alpha^c [t_\mathbf{G}^c, H]^{ab}, \tag{A2}$$

with the representation matrices $(t_\mathbf{G}^c)^{ab} \equiv -if^{abc}$ of the adjoint **G**, where f^{abc} are completely antisymmetric structure constants of the gauge group. We normalize the real fields H^{ab} and the gauge fields $G_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_G f^{abc} A_\mu^b A_\nu^c$, along with the gauge coupling g_G and the structure constants f^{abc}, in such a way that their kinetic terms have the (standard) form

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu H^{ab} - ig_G A_\mu^c [t_\mathbf{G}^c, H]^{ab})(\partial^\mu H^{ab}$$
$$- ig_G A^{d\mu}[t_\mathbf{G}^d, H]^{ab}) - \tfrac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} \tag{A3}$$

with the index of the adjoint representation $C(\mathbf{G})\delta^{ab} = \mathrm{tr}(t_\mathbf{G}^a t_\mathbf{G}^b) = f^{acd}f^{bcd}$ normalized to

$$SU(5): \quad C(\mathbf{G}) = C(\mathbf{24}) = 5, \tag{A4}$$

$$SO(10): \quad C(\mathbf{G}) = C(\mathbf{45}) = 8. \tag{A5}$$

The assignment (A5) follows from (A4) if one demands the structure constants of $SO(10)$, restricted to its "normal"

$SU(5)$ subgroup which we will be interested in later, to coincide with the $SU(5)$ structure constants normalized according to (A4), which itself follows from the standard assignment $C(\mathbf{5}) = 1/2$ for the fundamental representation $\mathbf{5}$ of $SU(5)$. We use this standard choice $C(\mathbf{N}) = 1/2$ for the fundamental representation $\mathbf{N}$ of any unitary group $SU(N)$.

In this notation, the unique gauge (and Lorentz) singlet contraction is

$$HGG \simeq H^{ab} G^a_{\mu\nu} G^{b\mu\nu}. \qquad (A6)$$

Writing the dimension-5 operators of interest in this common form for all of the admissible Higgs irreps (A1), together with the normalization (A3) of the kinetic terms of the gauge and Higgs fields, establishes a convention (up to the sign) for the—*a priori* arbitrary—overall factor of the singlet contraction of one Higgs multiplet with two gauge fields. It moreover asserts, as a convention, that the two operators $H_i GG$ and $H_j GG$, even with Higgses H_i, H_j in different irreps, are of equal size and strength. The overall sign of the gauge singlet contraction is arbitrary, as it contains the Higgs field H to one power and as there is no invariant way to fix the sign of H through the (quadratic) kinetic term (A3). Consequently, this leaves the overall sign of the Wilson coefficients c_i in (1) open, but allows naturalness assessments based on the absolute values $|c_i|$.

In (1) we wrote the dimension-5 operators as

$$\mathcal{L} = \sum_i \frac{c_i}{M_{\mathrm{pl}}} H_i^{ab} \frac{1}{4} G^a_{\mu\nu} G^{b\mu\nu} \qquad (A7)$$

(the H_i do not all have to be in distinct representations). The justification for the factor $1/4$ is the same as in the gauge boson kinetic term (A3), as the squared time derivatives of the spatial gauge fields appear as $2(\partial_0 A_i^a)^2$ in the contraction $-G^a_{\mu\nu} G^{a\mu\nu}$, whereas squared time derivatives of boson fields canonically appear with factors of $1/2$ as kinetic terms in the Lagrangian $\mathcal{L}$. To avoid the same overcounting, we explicitly put $1/4$ into the operator (A7) as well, noting that this might be a conservative choice (i.e., require a larger value of $|c_i|$ in order to achieve the same effect). M_{pl}, which we parametrize as $M_{\mathrm{pl}} = 1.2 \times 10^{19}$ GeV$/\xi$ [cf. (2)], should be the scale of the physics generating the operators (A7), and we imagine this to be gravitational interactions (see the beginning of Sec. II). This completes the setup of normalization conventions that satisfy the goal formulated at the beginning of this appendix.

After a Higgs field H_i has acquired a (nonzero) vacuum expectation value (VEV) $\langle H_i \rangle$, the effective interaction (A7) contributes a term

$$\mathcal{L} = \frac{c_i \langle H_i^{ab} \rangle}{4 M_{\mathrm{pl}}} G^a_{\mu\nu} G^{b\mu\nu}, \qquad (A8)$$

to the Lagrangian, similar to the gauge boson kinetic term in (A3). Our next goal is to look quantitatively at this contribution and its immediate effects.

The contributions (A8) scale roughly like $\langle H_i^{ab} \rangle / M_{\mathrm{pl}} \sim M/M_{\mathrm{pl}}$ if H_i assumes its VEV at energy scale M, and so are completely negligible for M anywhere near or below the electroweak scale. On the other hand, above the electroweak scale the standard model subgroup $G_{321} = SU(3)_C \times SU(2)_L \times U(1)_Y$ of the grand unified group G is unbroken, so the VEVs of all scalar fields in the theory, in particular, all VEVs appearing in (A8), have to be invariant under G_{321} (this includes zero VEV). This requirement on $\langle H^{ab} \rangle$ constrains the contributions (A8) to the gauge kinetic terms. Next we will find, for the cases of $SU(5)$ and $SO(10)$ grand unified gauge groups G and for each of the allowed irreps (A1), all possible $\langle H_i^{ab} \rangle$ that are invariant under the standard model subgroup $G_{321} \subset G$.

The method employed is similar to [25] (Sec. I), but we do not only want to find relative contributions, but rather also *absolute* values within our normalization conventions above, and we furthermore want to establish a relation between the Higgs VEVs and the masses of the superheavy gauge bosons (see later). Let $\phi \equiv (\phi^{ab})$ be a vector, transforming like $\langle H^{ab} \rangle$ as the symmetric product of two adjoint representations $\mathbf{G}$ of the gauge group G; see (A2). Choose[1] an explicit basis $\{t_{\mathbf{G}}^a\}_a$ of generators of the adjoint representation $\mathbf{G}$ such that they satisfy the normalization (A4) and (A5) and that the generators of the standard model subgroup G_{123} correspond to $a = 1, \ldots, 8$ [for the $SU(3)_C$ factor; called "set I" of the generators], $a = 9, 10, 11$ [$SU(2)_L$; set II], and $a = 12$ [$U(1)_Y$; set III or set III', see later]; this allows for easy examination of the transformation properties of ϕ under the standard model subgroup G_{321}. Also, the remaining generators $a = 13, \ldots, d(G)$ may be identified according to their transformation properties under subgroups of the full gauge group G, and we need such a naming scheme later to address the components of ϕ^{ab} of different superheavy gauge bosons. For our purposes we choose one of the following classifications of generators t^a (applicable to any representation):

(a) For $SU(5)$: The adjoint $\mathbf{G} = \mathbf{24}$ of $SU(5)$, under which the generators transform, branches under the standard model subgroup G_{321} into

$$\mathbf{24} \overset{G_{321}}{\to} \underbrace{(\mathbf{8}, \mathbf{1}, 0)}_{I} \oplus \underbrace{(\mathbf{1}, \mathbf{3}, 0)}_{II} \oplus \underbrace{(\mathbf{1}, \mathbf{1}, 0)}_{III}$$
$$\oplus \underbrace{\left(\left(\mathbf{3}, \mathbf{2}, -\frac{5}{6} \right) \oplus \mathrm{H.c.} \right)}_{IV}. \qquad (A9)$$

Besides the irreps I, II, and III, there is only one

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[1]One way to choose such an explicit basis $t_{\mathbf{G}}^a$ of adjoint generators is to choose an explicit basis $t_{\mathbf{F}}^a$ for the fundamental representation $\mathbf{F}$ of G, normalized to $\mathrm{tr}(t_{\mathbf{F}}^a t_{\mathbf{F}}^b) = C(\mathbf{F})\delta^{ab}$, such that the $t_{\mathbf{F}}^a$ already obey the desired classification, which may be more easily accomplished than the same classification for the $t_{\mathbf{G}}^a$. Here, $\mathbf{F} = \mathbf{5}$ or $\mathbf{10}$ for $G = SU(5)$ or $SO(10)$, respectively, and $C(\mathbf{5}_{SU(5)}) = 1/2$ and $C(\mathbf{10}_{SO(10)}) = 1$. Then $(t_{\mathbf{G}}^a)^{bc} \equiv -i f^{abc}$ with $f^{abc} \equiv -i \, \mathrm{tr}([t_{\mathbf{F}}^a, t_{\mathbf{F}}^b] t_{\mathbf{F}}^c)/C(\mathbf{F})$ have the desired properties.

real representation of G_{321} contained in the **24**, and those 12 generators make up set IV of generators.

(b) For $SO(10)$: Two different classifications are possible since G_{321} can be embedded in two different ways (normal [38] or "flipped" [39] embedding) into $SO(10)$ with the correct charge assignments for each standard model family of fermions from a **16** of $SO(10)$. For either embedding, $SU(5)$ in the following denotes the $SU(5) \subset SO(10)$ subgroup which contains the $SU(3)_C$ and $SU(2)_L$ factors of G_{321}, and $U(1)_X$ the Abelian factor such that $SU(5) \times U(1)_X \subset SO(10)$ is a maximal subgroup.

(i) Normal (Georgi-Glashow-like) embedding $G_{321} \subset SU(5) \subset SO(10)$ [38]:

$$\mathbf{45} \overset{SU(5)}{\to} \mathbf{24} \oplus \mathbf{1} \oplus (\mathbf{10} \oplus \text{H.c.})$$

$$\overset{G_{321}}{\to} \underbrace{(\mathbf{8}, \mathbf{1}, 0)}_{I} \oplus \underbrace{(\mathbf{1}, \mathbf{3}, 0)}_{II} \oplus \underbrace{(\mathbf{1}, \mathbf{1}, 0)}_{III}$$

$$\oplus \underbrace{((\mathbf{3}, \mathbf{2}, -\tfrac{5}{6}) \oplus \text{H.c.})}_{IV} \oplus \underbrace{(\mathbf{1}, \mathbf{1}, 0)}_{V}$$

$$\oplus \underbrace{((\mathbf{3}, \mathbf{2}, \tfrac{1}{6}) \oplus \text{H.c.})}_{VI} \oplus \underbrace{((\bar{\mathbf{3}}, \mathbf{1}, -\tfrac{2}{3}) \oplus \text{H.c.})}_{VII}$$

$$\oplus \underbrace{((\mathbf{1}, \mathbf{1}, 1) \oplus \text{H.c.})}_{VIII}. \tag{A10}$$

The number of generators in sets IV, V, VI, VII, $VIII$ is 12, 1, 12, 6, 2. Sets I, II, III, IV correspond to I, II, III, IV in (A9) (this is the Georgi-Glashow embedding). Note, the generators III [generating $U(1)_Y \subset G_{321}$] and V [generating $U(1)_X$] both have the same standard model quantum numbers (both are standard model singlets), whereas all other sets of generators have distinct ones

(ii) Flipped embedding $G_{321} \subset SU(5) \times U(1)_X \subset SO(10)$ with $G_{321} \nsubseteq SU(5)$ [more precisely, the $U(1)_Y$ factor of G_{321} is not contained in $SU(5)$] [39]:

$$\mathbf{45} \overset{SU(5) \times U(1)_X}{\to} \mathbf{24}(0) \oplus \mathbf{1}(0) \oplus (\mathbf{10}(-\sqrt{\tfrac{2}{3}}) \oplus \text{H.c.})$$

$$\overset{G_{321}}{\to} \underbrace{(\mathbf{8}, \mathbf{1}, 0)}_{I} \oplus \underbrace{(\mathbf{1}, \mathbf{3}, 0)}_{II} \oplus \underbrace{(\mathbf{1}, \mathbf{1}, 0)}_{III'} \oplus \underbrace{((\mathbf{3}, \mathbf{2}, \tfrac{1}{6}) \oplus \text{H.c.})}_{IV}$$

$$\oplus \underbrace{(\mathbf{1}, \mathbf{1}, 0)}_{V'} \oplus \underbrace{((\mathbf{3}, \mathbf{2}, -\tfrac{5}{6}) \oplus \text{H.c.})}_{VI}$$

$$\oplus \underbrace{((\bar{\mathbf{3}}, \mathbf{1}, -\tfrac{2}{3}) \oplus \text{H.c.})}_{VII} \oplus \underbrace{((\mathbf{1}, \mathbf{1}, -1) \oplus \text{H.c.})}_{VIII}. \tag{A11}$$

Again, only the generators III' and V' have the same transformation properties under the standard model. The sets I, II, IV, VI, VII, and $VIII$ of generators here can be identified with the respective sets in (A10), whereas the two generators $t^{III'}$ [generating $U(1)_Y \nsubseteq SU(5)$] and $t^{V'}$ here are linear combinations

of the generators t^{III} and t^V from (A10),

$$\begin{pmatrix} t^{III'} \\ t^{V'} \end{pmatrix} = V \begin{pmatrix} t^{III} \\ t^V \end{pmatrix} \quad \text{with}$$

$$V = \begin{pmatrix} -\tfrac{1}{5} s^{III} s^{III'} & \tfrac{2\sqrt{6}}{5} s^V s^{III'} \\ \tfrac{2\sqrt{6}}{5} s^{III'} s^{V'} & \tfrac{1}{5} s^V s^{V'} \end{pmatrix}, \tag{A12}$$

where s^{III}, s^V, $s^{III'}$, $s^{V'} = \pm 1$ are signs that depend on the choice of signs of the generators t^{III}, t^V, $t^{III'}$, and $t^{V'}$, which are not fixed by our normalization conventions above. V is a real orthogonal matrix, $VV^T = \mathbb{1}_2$, and the gauge bosons $F^{III'}_{\mu\nu}$, $F^{V'}_{\mu\nu}$ corresponding to the generators $t^{III'}$, $t^{V'}$ are related to the gauge bosons $F^{III}_{\mu\nu}$, $F^V_{\mu\nu}$ via $(F^{III'}, F^{V'}) = (F^{III}, F^V) V^T$.

In this explicit basis, the requirement that ϕ^{ab} be invariant under G_{321} then, according to (A2), translates into

$$(t^c_{\mathbf{G} \otimes \mathbf{G}})^{(ab)(a'b')} \phi^{a'b'} \equiv [t^c_{\mathbf{G}}, \phi]^{ab} = 0$$

$$\text{for all } c = 1, \dots, 12. \tag{A13}$$

Solving this system of (linear) equations (along with enforcing symmetry $\phi^{ab} = \phi^{ba}$) yields 4 (respectively, 9) linearly independent solutions ϕ^{ab}_t ($t = 1, \dots, 4$, respectively, $t = 1, \dots, 9$) in the case of $G = SU(5)$ [respectively, $G = SO(10)$], their linear combinations exhausting all standard model singlet VEVs contained in $(\mathbf{G} \otimes \mathbf{G})_s$. In order to find all possible standard model singlet VEVs for each of the Higgs irreps (A1), one has to form linear combinations of the solutions ϕ_t that transform in those irreps. To this end, one evaluates the quadratic Casimir operator on each of the linearly independent solutions ϕ_t, expressing the result again as their linear combination,

$$C_2 \phi_t \equiv \sum_{c=1}^{d(G)} [t^c_{\mathbf{G}}, [t^c_{\mathbf{G}}, \phi_t]] = \sum_u c_{ut} \phi_u, \tag{A14}$$

which is possible since the restriction of the action of the Casimir operator to an irrep is proportional to the identity on that irrep; Table V gives the respective constants of proportionality ("quadratic Casimir invariants") for irreps of several groups of interest here and later. The matrix (c_{ut}) is diagonalizable, $\sum_t c_{ut} v_t^{(w)} = C_2^{(w)} v_u^{(w)}$ with linearly independent eigenvectors $v^{(w)}, w = 1, \dots, 4(9)$ for $G = SU(5)$ [$G = SO(10)$]. The 4 (respectively, 9) vectors $\Phi_w \equiv \sum_t v_t^{(w)} \phi_t$ then form a basis of the standard model singlet subspace as

TABLE V. Quadratic Casimir invariants $C_2(r)$ for some irreps r of several groups of interest, in the conventions described above. $C_2(\mathbf{1}) = 0$ for the singlet **1** of any group.

Group	$C_2(r_1)$	$C_2(r_2)$	$C_2(r_3)$
$SU(5)$	$C_2(\mathbf{24}) = 5$	$C_2(\mathbf{75}) = 8$	$C_2(\mathbf{200}) = 12$
$SO(10)$	$C_2(\mathbf{54}) = 10$	$C_2(\mathbf{210}) = 12$	$C_2(\mathbf{770}) = 18$
$SU(4)$	$C_2(\mathbf{15}) = 4$	$C_2(\mathbf{84}) = 20$	
$SU(2)$	$C_2(\mathbf{3}) = 2$	$C_2(\mathbf{5}) = 6$	

well, and each Φ_w transforms in a representation r of G whose quadratic Casimir invariant $C_2(r)$ is equal to the eigenvalue $C_2^{(w)}$. Since, in the $SU(5)$ as well as the $SO(10)$ case, the different irreps in (A1) have distinct quadratic Casimir invariants, and since each irrep only occurs once in the direct sum decomposition (A1) of $(\mathbf{G} \otimes \mathbf{G})_s$, it is ensured that each Φ_w actually is contained in an irrep r, and r can be read off from the corresponding eigenvalue $C_2^{(w)}$ via Table V.

What one finds, in the $G = SU(5)$ case, is that each of the four irreps **1**, **24**, **75**, and **200**, into which $(\mathbf{24} \otimes \mathbf{24})_s$ decomposes (A1), contains exactly one standard model singlet Φ_w (the 4 eigenvalues of c_{ut} are nondegenerate: 0, 5, 8, 12). Therefore, in $SU(5)$, specifying the irrep of H determines its (standard model singlet) VEV and subsequently its contributions to (A8) up to an overall factor; in particular, the relative contributions between terms $G_{\mu\nu}^a G^{b\mu\nu}$ in (A8) are completely determined.

For the $SO(10)$ unified gauge group, however, only the **1** and **54** irreps contain exactly one standard model singlet; the **210** contains three, and the **770** contains four linearly independent standard model singlets (the 9 eigenvalues of c_{ut} are partly degenerate: 0, 10, 3×12, 4×18). These statements hold for both the normal (Georgi-Glashow) and the flipped embedding of G_{321} into $SO(10)$, and, in fact, it is found that the set of standard model singlets is the same in either case.

In order to distinguish these linearly independent G_{321} singlets within the **210** or the **770** of $SO(10)$, one can specify their transformation properties under subgroups of $SO(10)$ that contain the standard model G_{321} (the normal or the flipped G_{321}, respectively). We consider here two such subgroups, namely, the two maximal subgroups of $SO(10)$ consistent with the charge assignments for the standard model fermions; these are (a) the extended Georgi-Glashow subgroup $SU(5) \times U(1)_X \subset SO(10)$, and (b) the Pati-Salam subgroup $SU(4)_C \times SU(2)_L \times SU(2)_R$. To achieve this practically, the method of the Casimir operator (A14) is employed again; however, the sum over c in (A14) is now restricted to the generators of the desired subgroup, or, more precisely, to the generators of each simple factor of the subgroup, in order to find linear

combinations of the Φ_w that transform in irreps of the subgroup (factors). It turns out that either one of the subgroups (a) or (b) can resolve the degeneracy of the vectors Φ_w transforming within the **210** or the **770** of $SO(10)$; see the second column of Tables VII and VIII for cases (a) and (b), respectively.

So, for each of the irreps r on the right-hand sides of (A1) in the $SU(5)$ as well as the $SO(10)$ case, we have found a basis $\{\Phi_{(r)t}\}_t$, such that any standard model singlet ϕ^{ab} in the irrep r can be written as a linear combination $\phi^{ab} = \sum_t c_t \Phi_{(r)t}^{ab}$ with real coefficients c_t; the index t runs over $t = 1$ only, except for $SO(10)$ in the cases of $r = \mathbf{210}$ (where $t = 1, 2, 3$) or $r = \mathbf{770}$ ($t = 1, 2, 3, 4$), for which the $\Phi_{(r)t}$ can be given definite transformation properties under (maximal) subgroups of $SO(10)$, as described above. To be specific, we normalize the overall magnitude of each $\Phi_{(r)t}$ such that $\sum_{ab} \Phi_{(r)t}^{ab} \Phi_{(r)t}^{ab} = 1$, resulting in $\Phi_{(r)t}^{ab} \Phi_{(r')t'}^{ab} = \delta_{rr'} \delta_{tt'}$. Also, it turns out that $\Phi_{(r)t}^{ab} = 0$ whenever a and b belong to two subsets of generators with different transformation properties under G_{321}, see (A9), (A10), or (A11), and $\Phi_{(r)t}^{ab}$ is proportional to the identity matrix when a and b are restricted to one set of generators. Therefore, nonzero off-diagonal elements $\Phi_{(r)t}^{a \neq b} \neq 0$ can possibly occur only in the $SO(10)$ case when a and b correspond to the two degenerate G_{321}-singlet generators III, V or III', V'. In particular, $\Phi_{(r)t}^{ab}$ is diagonal in a, b on the set a, $b = 1, \ldots, 12$ and is proportional to the identity matrix on each simple factor of the standard model a, $b = 1, \ldots, 8$ [$SU(3)_C$], a, $b = 9$, 10, 11 [$SU(2)_L$], and a, $b = 12$ [$U(1)_Y$]; one could therefore arbitrarily, as an invariant convention, for each r and t choose the sign of $\Phi_{(r)t}$ such that, e.g., the nonzero diagonal entry $\Phi_{(r)t}^{a=b}$ with smallest $a = b$ is positive; however, to save some writing, we reverse this convention for the flipped embedding case in rows 5 and 8 of Table VIII.

We are now ready to write down the matrices $\Phi_{(r)t}^{ab}$, obtained in this way, explicitly. Since, as just described, each matrix $\Phi_{(r)t}$ contains only a few independent entries, they can be written down in an economical way ($\mathbb{I}_d$ denotes the identity matrix of size $d \times d$):

$$\text{for } SU(5): \Phi^{ab} = \begin{pmatrix} \Phi^I \mathbb{I}_8 & & & \\ & \Phi^{II} \mathbb{I}_3 & & \\ & & \Phi^{III} & \\ & & & \Phi^{IV} \mathbb{I}_{12} \end{pmatrix}^{ab} ; \tag{A15}$$

$$\text{for } SO(10): \Phi^{ab} = \begin{pmatrix} \Phi^I \mathbb{I}_8 & & & & & & & \\ & \Phi^{II} \mathbb{I}_3 & & & & & & \\ & & \Phi^{III} & 0 & \Phi^{III,V} & & & \\ & & 0 & \Phi^{IV} \mathbb{I}_{12} & 0 & & & \\ & & \Phi^{III,V} & 0 & \Phi^V & & & \\ & & & & & \Phi^{VI} \mathbb{I}_{12} & & \\ & & & & & & \Phi^{VII} \mathbb{I}_6 & \\ & & & & & & & \Phi^{VIII} \mathbb{I}_2 \end{pmatrix}^{ab} . \tag{A16}$$

For the $SU(5)$ case, the numerical entries of each matrix $\Phi_{(r)t} \equiv \Phi_{(r)}$ (A15) are listed in Table VI (in this case, $t = 1$ only, see above). For both the normal and the flipped embedding of $G_{321} \subset SO(10)$, Table VII lists the entries of each $\Phi_{(r)t}$ (A16) (for the flipped embedding, $\Phi_{(r)t}^{III'}$, $\Phi_{(r)t}^{V'}$, and $\Phi_{(r)t}^{III',V'}$ should be used from Table VII instead of the unprimed $\Phi_{(r)t}^{III}$, etc.), where the G_{321} singlets (labeled by t) within each $SO(10)$ irrep r are further classified according to their transformation under the maximal subgroup $SU(5) \times U(1)_X \subset SO(10)$, see above; Table VIII is similar, but with the G_{321} singlets $\Phi_{(r)t}$ classified according to their transformation properties under the maximal subgroup $SU(4)_C \times SU(2)_L \times SU(2)_R \subset SO(10)$ [of course, all G_{321} singlets are $SU(2)_L$ singlets, since $SU(2)_L \subset G_{321}$]. In all cases, the matrices (A16) with specified transformation properties under any subgroup are identical for the normal and flipped embedding except for the Φ^{III}, Φ^V, and $\Phi^{III,V}$ entries (which correspond to generators t^{III}, t^V that have the same quantum numbers under G_{321}), and these entries are related by

$$\begin{pmatrix} \Phi^{III'} & \Phi^{III',V'} \\ \Phi^{III',V'} & \Phi^{V'} \end{pmatrix} = V \begin{pmatrix} \Phi^{III} & \Phi^{III,V} \\ \Phi^{III,V} & \Phi^{V} \end{pmatrix} V^T, \quad \text{(A17)}$$

with the 2×2 matrix V from (A12).

Furthermore, for each $SO(10)$ irrep r separately, the VEVs $\Phi_{(r)t}$, classified according to $SU(5) \times U(1)_X$ (see Table VII), and the VEVs $\Phi'_{(r)t}$, classified according to $SU(4)_C \times SU(2)_R$ (Table VIII), are linear combinations of each other, in fact, they are related by orthogonal transformations; when their overall signs are chosen as shown in the tables, then, for both the normal and the flipped embedding, the relation between the $\Phi_{(r)t}$ and the $\Phi'_{(r)t}$ is

$$\Phi'_{(\mathbf{1})1} = \Phi_{(\mathbf{1})1}, \qquad \Phi'_{(\mathbf{54})1} = \Phi_{(\mathbf{54})1},$$

$$\begin{pmatrix} \Phi'_{(\mathbf{210})1} \\ \Phi'_{(\mathbf{210})2} \\ \Phi'_{(\mathbf{210})3} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & -\sqrt{\frac{2}{5}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{3}{10}} & \sqrt{\frac{8}{15}} & \frac{1}{\sqrt{6}} \\ \sqrt{\frac{3}{5}} & -\frac{1}{\sqrt{15}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \Phi_{(\mathbf{210})1} \\ \Phi_{(\mathbf{210})2} \\ \Phi_{(\mathbf{210})3} \end{pmatrix},$$

$$\begin{pmatrix} \Phi'_{(\mathbf{770})1} \\ \Phi'_{(\mathbf{770})2} \\ \Phi'_{(\mathbf{770})3} \\ \Phi'_{(\mathbf{770})4} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & -\sqrt{\frac{3}{175}} & \sqrt{\frac{3}{5}} & \sqrt{\frac{12}{35}} \\ \sqrt{\frac{3}{20}} & -\frac{4}{\sqrt{35}} & -\frac{1}{2} & \frac{1}{\sqrt{7}} \\ -\frac{3}{2\sqrt{5}} & \frac{2}{\sqrt{105}} & -\frac{1}{2\sqrt{3}} & \sqrt{\frac{3}{7}} \\ \frac{3}{5} & \frac{16}{5\sqrt{21}} & -\frac{1}{\sqrt{15}} & \sqrt{\frac{3}{35}} \end{pmatrix} \quad \text{(A18)}$$

$$\times \begin{pmatrix} \Phi_{(\mathbf{770})1} \\ \Phi_{(\mathbf{770})2} \\ \Phi_{(\mathbf{770})3} \\ \Phi_{(\mathbf{770})4} \end{pmatrix}.$$

TABLE VI. The standard model singlets $\Phi_{(r)}$ in each of the irreps r (A1) of $SU(5)$ in the explicit version (A15) with the conventions described above.

r	$\Phi_{(r)}^{I}$	$\Phi_{(r)}^{II}$	$\Phi_{(r)}^{III}$	$\Phi_{(r)}^{IV}$
1	$1/\sqrt{24}$	$1/\sqrt{24}$	$1/\sqrt{24}$	$1/\sqrt{24}$
24	$2/\sqrt{63}$	$-3/\sqrt{63}$	$-1/\sqrt{63}$	$-1/2\sqrt{63}$
75	$1/\sqrt{72}$	$3/\sqrt{72}$	$-5/\sqrt{72}$	$-1/\sqrt{72}$
200	$1/\sqrt{168}$	$2/\sqrt{168}$	$10/\sqrt{168}$	$-2/\sqrt{168}$

TABLE VII. The standard model singlets $\Phi_{(r)t}$ in each of the irreps r (A1) of $SO(10)$, classified according to their transformation properties under the $SU(5) \times U(1)_X \subset SO(10)$ maximal subgroup [second column; note, all standard model singlets $\Phi_{(r)t}$ also happen to be $U(1)_X$ singlets], in the explicit version (A16) with the conventions described above. The entries of the matrices $\Phi_{(r)t}$ (A16) agree between the normal and the flipped embedding of $G_{321} \subset SO(10)$, except for the $\Phi_{(r)t}^{III}$, $\Phi_{(r)t}^{V}$, and $\Phi_{(r)t}^{III,V}$ entries, where the primed ones ($\Phi_{(r)t}^{III'}$, etc., see last three columns) should be used instead for the flipped embedding; cf. also (A17). [For both the normal and the flipped embedding, the $SU(5)$ here denotes the $SU(5) \subset SO(10)$ containing $SU(3)_C \times SU(2)_L$.]

$SO(10)$	$SU(5) \times U(1)_X$	$\Phi_{(r)t}^{I}$	$\Phi_{(r)t}^{II}$	$\Phi_{(r)t}^{III}$	$\Phi_{(r)t}^{IV}$	$\Phi_{(r)t}^{V}$	$\Phi_{(r)t}^{III,V}$	$\Phi_{(r)t}^{VI}$	$\Phi_{(r)t}^{VII}$	$\Phi_{(r)t}^{VIII}$	$\Phi_{(r)t}^{III'}$	$\Phi_{(r)t}^{V'}$	$\Phi_{(r)t}^{III',V'}$
$r = \mathbf{1}$	$(\mathbf{1}, 0), t = 1$	$\frac{1}{3\sqrt{5}}$	$\frac{1}{3\sqrt{5}}$	$\frac{1}{3\sqrt{5}}$	$\frac{1}{3\sqrt{5}}$	$\frac{1}{3\sqrt{5}}$	0	$\frac{1}{3\sqrt{5}}$	$\frac{1}{3\sqrt{5}}$	$\frac{1}{3\sqrt{5}}$	$\frac{1}{3\sqrt{5}}$	$\frac{1}{3\sqrt{5}}$	0
$r = \mathbf{54}$	$(\mathbf{24}, 0), t = 1$	$\frac{1}{\sqrt{30}}$	$-\sqrt{\frac{3}{40}}$	$-\frac{1}{2\sqrt{30}}$	$-\frac{1}{4\sqrt{30}}$	0	$\frac{1}{2\sqrt{5}}$	$-\frac{1}{4\sqrt{30}}$	$\frac{1}{\sqrt{30}}$	$-\sqrt{\frac{3}{40}}$	$-\frac{1}{2\sqrt{30}}$	0	$\frac{1}{2\sqrt{5}}$
$r = \mathbf{210}$	$(\mathbf{1}, 0), t = 1$	$\frac{1}{2\sqrt{15}}$	$\frac{1}{2\sqrt{15}}$	$\frac{1}{2\sqrt{15}}$	$\frac{1}{2\sqrt{15}}$	$-\frac{2}{\sqrt{15}}$	0	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{19}{10\sqrt{15}}$	$\frac{2}{5\sqrt{15}}$	$-\sqrt{\frac{2}{125}}$
	$(\mathbf{24}, 0), t = 2$	$\frac{2}{3\sqrt{15}}$	$-\frac{1}{\sqrt{15}}$	$-\frac{1}{3\sqrt{15}}$	$-\frac{1}{6\sqrt{15}}$	0	$-\frac{1}{\sqrt{10}}$	$\frac{1}{6\sqrt{15}}$	$-\frac{2}{3\sqrt{15}}$	$\frac{1}{\sqrt{15}}$	$\frac{7}{15\sqrt{15}}$	$-\frac{4}{5\sqrt{15}}$	$-\frac{13}{15\sqrt{10}}$
	$(\mathbf{75}, 0), t = 3$	$\frac{1}{6\sqrt{3}}$	$\frac{1}{2\sqrt{3}}$	$-\frac{5}{6\sqrt{3}}$	$-\frac{1}{6\sqrt{3}}$	0	0	$\frac{1}{6\sqrt{3}}$	$-\frac{1}{6\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{30\sqrt{3}}$	$-\frac{4}{5\sqrt{3}}$	$\frac{\sqrt{2}}{15}$
$r = \mathbf{770}$	$(\mathbf{1}, 0), t = 1$	$\frac{1}{6\sqrt{10}}$	$\frac{1}{6\sqrt{10}}$	$\frac{1}{6\sqrt{10}}$	$\frac{1}{6\sqrt{10}}$	$\sqrt{\frac{32}{45}}$	0	$-\frac{1}{3\sqrt{10}}$	$-\frac{1}{3\sqrt{10}}$	$-\frac{1}{3\sqrt{10}}$	$\frac{77}{30\sqrt{10}}$	$\sqrt{\frac{8}{1125}}$	$\sqrt{\frac{3}{125}}$
	$(\mathbf{24}, 0), t = 2$	$\frac{1}{3\sqrt{210}}$	$-\frac{1}{2\sqrt{210}}$	$-\frac{1}{6\sqrt{210}}$	$-\frac{1}{12\sqrt{210}}$	0	$\sqrt{\frac{7}{20}}$	$\sqrt{\frac{7}{4320}}$	$-\sqrt{\frac{7}{270}}$	$\sqrt{\frac{7}{120}}$	$-\frac{101}{30\sqrt{210}}$	$\sqrt{\frac{128}{2625}}$	$\frac{97}{30\sqrt{35}}$
	$(\mathbf{75}, 0), t = 3$	$\frac{1}{6\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$	$-\frac{5}{6\sqrt{6}}$	$-\frac{1}{6\sqrt{6}}$	0	0	$-\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$-\frac{1}{30\sqrt{6}}$	$-\sqrt{\frac{8}{75}}$	$\frac{1}{15}$
	$(\mathbf{200}, 0), t = 4$	$\frac{1}{2\sqrt{42}}$	$\frac{1}{\sqrt{42}}$	$\frac{5}{\sqrt{42}}$	$-\frac{1}{\sqrt{42}}$	0	0	0	0	0	$\frac{1}{5\sqrt{42}}$	$\sqrt{\frac{96}{175}}$	$-\frac{2}{5\sqrt{7}}$

TABLE VIII.   Same as Table VII, but with the standard model singlets $\Phi'_{(r)t}$ classified according to their transformation properties under the $SU(4)_C \times SU(2)_L \times SU(2)_R \subset SO(10)$ maximal subgroup [second column: all entries are $SU(2)_L$ singlets, since $SU(2)_L \subset G_{321}$]. [The $SU(4)_C$ here denotes the $SU(4)$ factor in $SU(4) \times SU(2) \times SU(2) \subset SO(10)$ with $SU(4) \supset SU(3)_C$, where one of the two $SU(2)$ factors is $SU(2)_L$.]

$SO(10)$	$SU(4)_C \times SU(2)_R$	$\Phi'^{II}_{(r)t}$	$\Phi'^{III}_{(r)t}$	$\Phi'^{IIII}_{(r)t}$	$\Phi'^{IIV}_{(r)t}$	$\Phi'^{V}_{(r)t}$	$\Phi'^{III,V}_{(r)t}$	$\Phi'^{VI}_{(r)t}$	$\Phi'^{VII}_{(r)t}$	$\Phi'^{VIII}_{(r)t}$	$\Phi'^{III'}_{(r)t}$	$\Phi'^{V'}_{(r)t}$	$\Phi'^{III',V'}_{(r)t}$
$r = 1$	$(\mathbf{1},\mathbf{1})$, $t = 1$	$\frac{1}{3\sqrt5}$	$\frac{1}{3\sqrt5}$	$\frac{1}{3\sqrt5}$	$\frac{1}{3\sqrt5}$	$\frac{1}{3\sqrt5}$	0	$\frac{1}{3\sqrt5}$	$\frac{1}{3\sqrt5}$	$\frac{1}{3\sqrt5}$	$\frac{1}{3\sqrt5}$	$\frac{1}{3\sqrt5}$	0
$r = 54$	$(\mathbf{1},\mathbf{1})$, $t = 1$	$\frac{1}{\sqrt{30}}$	$-\sqrt{\frac{3}{40}}$	$-\frac{1}{2\sqrt{30}}$	$-\frac{1}{4\sqrt{30}}$	0	$\frac{1}{2\sqrt5}$	$-\frac{1}{4\sqrt{30}}$	$\frac{1}{\sqrt{30}}$	$-\sqrt{\frac{3}{40}}$	$-\frac{1}{2\sqrt{30}}$	0	$\frac{1}{2\sqrt5}$
$r = 210$	$(\mathbf{1},\mathbf{1})$, $t = 1$	0	$\frac{1}{\sqrt6}$	$-\sqrt{\frac{3}{50}}$	0	$-\sqrt{\frac{2}{75}}$	$\frac{1}{5}$	0	0	$-\frac{1}{\sqrt6}$	$-\sqrt{\frac{3}{50}}$	$-\sqrt{\frac{2}{75}}$	$\frac{1}{5}$
	$(\mathbf{15},\mathbf{1})$, $t = 2$	$\frac{1}{3\sqrt2}$	0	$-\frac{2\sqrt2}{15}$	0	$-\frac{\sqrt2}{5}$	$-\frac{2}{5\sqrt3}$	0	$-\frac{1}{3\sqrt2}$	0	$-\frac{2\sqrt2}{15}$	$-\frac{\sqrt2}{5}$	$-\frac{2}{5\sqrt3}$
	$(\mathbf{15},\mathbf{3})$, $t = 3$	0	0	$\frac{2}{5}$	$\frac{1}{6}$	$-\frac{2}{5}$	$\frac{1}{5\sqrt6}$	$-\frac{1}{6}$	0	0	$-\frac{2}{5}$	$\frac{2}{5}$	$-\frac{1}{5\sqrt6}$
$r = 770$	$(\mathbf{1},\mathbf{1})$, $t = 1$	$\frac{1}{3\sqrt{10}}$	$\sqrt{\frac{5}{72}}$	$\frac{19}{30\sqrt{10}}$	$-\sqrt{\frac{5}{288}}$	$\sqrt{\frac{32}{1125}}$	$-\sqrt{\frac{3}{500}}$	$-\sqrt{\frac{5}{288}}$	$\frac{1}{3\sqrt{10}}$	$\sqrt{\frac{5}{72}}$	$\frac{19}{30\sqrt{10}}$	$\sqrt{\frac{32}{1125}}$	$-\sqrt{\frac{3}{500}}$
	$(\mathbf{1},\mathbf{5})$, $t = 2$	0	0	$\frac{\sqrt6}{5}$	0	$\sqrt{\frac{8}{75}}$	$-\frac{2}{5}$	0	0	$-\frac{1}{\sqrt6}$	$\frac{\sqrt6}{5}$	$\sqrt{\frac{8}{75}}$	$-\frac{2}{5}$
	$(\mathbf{15},\mathbf{3})$, $t = 3$	0	0	$\frac{2\sqrt2}{5}$	$-\frac{1}{6\sqrt2}$	$-\frac{2\sqrt2}{5}$	$\frac{1}{5\sqrt3}$	$\frac{1}{6\sqrt2}$	0	0	$-\frac{2\sqrt2}{5}$	$\frac{2\sqrt2}{5}$	$-\frac{1}{5\sqrt3}$
	$(\mathbf{84},\mathbf{1})$, $t = 4$	$\frac{1}{6\sqrt{10}}$	0	$\sqrt{\frac{128}{1125}}$	0	$\sqrt{\frac{32}{125}}$	$\frac{8}{5\sqrt{15}}$	0	$-\sqrt{\frac{2}{45}}$	0	$\sqrt{\frac{128}{1125}}$	$\sqrt{\frac{32}{125}}$	$\frac{8}{5\sqrt{15}}$

By classifying the standard model singlets $\Phi_{(r)t}$, in the $SO(10)$ case, into irreps under (maximal) subgroups of $SO(10)$ as in Tables VII and VIII, we do not mean to imply a grand unified symmetry breaking scenario where $SO(10)$ is broken to the standard model G_{321} necessarily via some intermediate gauge group, although this classification is well suited for such a scenario; see the end of Sec. VI. Moreover, such a classification, in particular, can serve as a parametrization for standard model singlets $\langle H_i^{ab} \rangle$ (A8), transforming in irreps r_i, in terms the of the basis vectors $\Phi_{(i)t} \equiv \Phi_{(r_i)t}$:

$$\langle H_i^{ab} \rangle \equiv \sum_t \upsilon_{(i)t} \Phi_{(i)t}^{ab} \equiv \upsilon_i \phi_i^{ab} \qquad (A19)$$

with real scalars $\upsilon_{(i)t}$, υ_i, and where, e.g., for $r_i = \mathbf{210}$,

$$\phi_i^{ab} = \Phi_{(\mathbf{210})1}^{ab} \cos\theta_1 + \Phi_{(\mathbf{210})2}^{ab} \sin\theta_1 \cos\theta_2$$
$$+ \Phi_{(\mathbf{210})3}^{ab} \sin\theta_1 \sin\theta_2, \qquad (A20)$$

such that $\phi_i^{ab}\phi_i^{ab} = 1$ and $\sum_t \upsilon_{(i)t}^2 = \upsilon_i^2$ for all i; then the kinetic terms of the υ_i and $\upsilon_{(i)t}$ field degrees of freedom, before assuming VEVs, are canonical (A3) $\mathcal{L} = \frac{1}{2} \times (\partial_\mu H_i^{ab} + \ldots)(\partial^\mu H_i^{ab} + \ldots) = \frac{1}{2}(\partial_\mu \upsilon_i)^2 + \ldots$. [The two parametrizations in (A19) differ only if H_i is a $\mathbf{210}$ or $\mathbf{770}$ of $SO(10)$, since in all other cases the sum merely runs over $t = 1$.]

We assume, for definiteness and simplicity (see Secs. II and VI), one-step breaking of the grand unified gauge group G to the standard model G_{321} at the unification scale M_X. Below M_X, all Higgs multiplets H_i responsible for grand unified symmetry breaking assume nonzero VEVs (A19), and, in particular, give masses to the non-G_{321} gauge bosons [henceforth called superheavy gauge bosons, although not all of them can get mass from the Higgses

(A1), see below], see (A3):

$$\mathcal{L} = \tfrac{1}{2} g_G^2 A_\mu^a A^{b\mu} \sum_i \mathrm{Tr}(-[t_\mathbf{G}^a, \langle H_i \rangle][t_\mathbf{G}^b, \langle H_i \rangle])$$

$$\equiv \tfrac{1}{2} m_{ab}^2 A_\mu^a A^{b\mu}; \qquad (A21)$$

each Higgs H_i contributes independently to the gauge boson squared mass matrix m_{ab}^2. For an $SU(5)$ grand unified group, each of the 12 superheavy gauge bosons acquires equal mass:

$$SU(5): m_{ab}^2 = \sum_i \frac{C_2(r_i)}{12} g_G^2 \upsilon_i^2 \delta_{ab} \equiv M_{\mathrm{gb}}^2 \delta_{ab} \qquad (A22)$$

$$\text{for } a, b = 13, \ldots, 24,$$

where $C_2(r_i)$ is the quadratic Casimir invariant of irrep r_i [cf. Table V for the irreps (A1)], and υ_i the VEV of H_i [cf. (A19) and below]. In $SO(10)$, the situation is more complicated: Higgs multiplets in any of the irreps (A1) of interest for the dimension-5 operators (A7) fail to give mass to all of the superheavy gauge bosons. In particular, the non-G_{321} gauge boson $G_{\mu\nu}^V$ (or $G_{\mu\nu}^{V'}$ for the flipped embedding) belonging to the set V [or V'; see (A10) or (A11)] of generators is always left massless by Higgses in irreps (A1), provided only that their VEVs are G_{321} singlets; cf. also the second column of Table VII. Also, some of the other superheavy gauge bosons might remain massless, and, at any rate, do not receive equal masses. Higgses H_i in representations r_i other than those on the right-hand side of (A1) are needed to give mass to all of the 33 non-G_{321} gauge bosons in the $SO(10)$ case, and they cannot occur in dimension-5 operators (A7). Independent of the individual directions ϕ_i^{ab} (A19) of the VEVs $\langle H_i \rangle = \upsilon_i \phi_i$, the averaged superheavy gauge boson squared mass is

$$SO(10): \bar{m}_{ab}^2 = \sum_i \frac{C_2(r_i)}{33} g_G^2 v_i^2 \delta_{ab} \equiv M_{\text{gb}}^2 \delta_{ab} \quad (A23)$$

for $a, b = 13, \ldots, 45,$

where the sum runs over all Higgs multiplets at the grand unification scale, whether or not they occur in (A7).

Grand unification asserts that the masses of the superheavy gauge bosons are related to the scale M_X of gauge-coupling unification; namely, their masses have to be somewhere around M_X, such that the renormalization group equations of the standard model apply below the scale M_X, at which the gauge couplings unify and above which the unified gauge coupling evolves according to the β-function of the unified gauge theory. For definiteness in obtaining numerical values, we assert for the analysis in the main text, that the superheavy gauge boson masses [or the averaged superheavy gauge boson mass (A23) in the $SO(10)$ case] have to exactly equal the unification scale:

$$M_{\text{gb}} = M_X. \quad (A24)$$

In the $SU(5)$ case, any of the Higgses in (A1) can give mass to all of the superheavy gauge bosons; so we assert, again for definiteness (see Secs. II and III), that no Higgs multiplets in irreps other than (A1) are present in the theory [nor, for that matter, in the sum (A22)]. In the $SO(10)$ analysis, however, we assert that the Higgs multiplets in irreps (A1) account for some fraction $1/f$, e.g., half, of the average gauge boson squared mass:

$$M_X = M_{\text{gb}} = \sqrt{\frac{f}{d(G) - 12} \sum_{\{i|r_i \text{in}(A1)\}} C_2(r_i) g_G^2 v_i^2} \quad (A25)$$

$$= g_G v \sqrt{\frac{f}{d(G) - 12} \sum_{\{i|r_i \text{in}(A1)\}} C_2(r_i) x_i^2}, \quad (A26)$$

with $f = 1$ (respectively, $f = 2$) for $SU(5)$ [respectively, $SO(10)$], and $d(G) - 12 = 12$ (respectively $= 33$) the number of superheavy gauge bosons. The form (A26) arises when the VEVs $v_i = x_i v$ are to obey some given ratio $x_1:x_2:\ldots$, with an open overall scale v, which will be useful later.

Now, plugging the Higgs VEVs (A19) back into the Lagrangian (A8) and adding the gauge boson kinetic term (A3), one obtains:

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} + \sum_i \frac{c_i}{4M_{\text{pl}}} \sum_t v_{(i)t} \Phi_{(i)t}^{ab} G_{\mu\nu}^a G^{b\mu\nu}$$

$$= -\frac{1}{4}\left(1 - \sum_i \frac{c_i}{M_{\text{pl}}} \sum_t v_{(i)t} \Phi_{(i)t}^I \right) F_{\mu\nu}^a F_{SU(3)}^{a\mu\nu}$$

$$- \frac{1}{4}\left(1 - \sum_i \frac{c_i}{M_{\text{pl}}} \sum_t v_{(i)t} \Phi_{(i)t}^{II} \right) F_{\mu\nu}^a F_{SU(2)}^{a\mu\nu}$$

$$- \frac{1}{4}\left(1 - \sum_i \frac{c_i}{M_{\text{pl}}} \sum_t v_{(i)t} \Phi_{(i)t}^{III} \right) F_{\mu\nu} F_{U(1)}^{\mu\nu} + \ldots$$

$$\equiv -\frac{1}{4}(1 + \epsilon_3) F_{\mu\nu}^a F_{SU(3)}^{a\mu\nu} - \frac{1}{4}(1 + \epsilon_2) F_{\mu\nu}^a F_{SU(2)}^{a\mu\nu}$$

$$- \frac{1}{4}(1 + \epsilon_1) F_{\mu\nu} F_{U(1)}^{\mu\nu} + \ldots \quad (A27)$$

Here, we have used the forms (A15) and (A16) of the standard model singlets $\Phi_{(i)t}$ on the subspace I, II, III (or III') of G_{321} generators and have omitted the superheavy gauge bosons, assuming the Lagrangian (A27) to be valid below the unification scale M_X where they have acquired mass and are integrated out; the last line defines what we mean by ϵ_s, $s = 1, 2, 3$, namely, the corrections to the gauge boson kinetic terms originating from the Higgs multiplets in the dimension-5 operators assuming nonzero VEVs (A8):

$$\epsilon_s = \sum_i \frac{c_i}{M_{\text{pl}}} \sum_t v_{(i)t} \delta_s^{(i)t} \quad (\text{for } s = 1, 2, 3),$$

where $\delta_1^{(i)t} \equiv -\Phi_{(i)t}^{III}$, $\quad \delta_2^{(i)t} \equiv -\Phi_{(i)t}^{II}$, $\quad \delta_3^{(i)t} \equiv -\Phi_{(i)t}^I$;

$$(A28)$$

the $\delta_s^{(i)t}$ can be read off from Tables VI, VII, and VIII for the $SU(5)$ and the $SO(10)$ cases [use $\delta_1^{(i)t} \equiv -\Phi_{(i)t}^{III'}$ for the flipped embedding $G_{321} \subset SO(10)$], and Table II in the main text gives them for $SU(5)$ explicitly (in this case, $t = 1$ only).

As shown in Sec. II, after rescaling $F_{(s)}^{\mu\nu} \to (1 + \epsilon_s)^{1/2} F_{(s)}^{\mu\nu}$ and $g_s \to (1 + \epsilon_s)^{-1/2} g_s$ to the observed low-energy gauge field strengths and gauge couplings by amounts which are different for each of the standard model gauge group factors $s = 1, 2, 3$, the condition for gauge-coupling unification at the scale M_X into the unified group reads, in terms of the observed (running) gauge couplings $\alpha_s = \alpha_s(\mu) = g_s(\mu)^2/4\pi$ of the theory below M_X:

$$(1 + \epsilon_1)\alpha_1(M_X) = (1 + \epsilon_2)\alpha_2(M_X) = (1 + \epsilon_3)\alpha_3(M_X)$$

$$= \frac{g_G^2}{4\pi} \equiv \alpha_G, \quad (A29)$$

where $g_G \equiv g_G(M_X)$ is the gauge coupling of the unified theory at the scale $\mu = M_X$. The functions $\alpha_s(\mu)$ are fixed through their low-energy measurements (e.g., at the scale $\mu = m_Z$ of the Z mass) and their renormalization group evolution; in the nonsupersymmetric standard model to one-loop order with one standard model Higgs doublet:

$$\frac{1}{\alpha_s(\mu)} = \frac{1}{\alpha_s(m_Z)} - \frac{b_s}{2\pi}\ln\frac{\mu}{m_Z} \qquad \text{(A30)}$$

with β-function coefficients

$$b_1 = \tfrac{41}{10}, \qquad b_2 = -\tfrac{19}{6}, \qquad b_3 = -7, \qquad \text{(A31)}$$

and initial values [20]

$$\alpha_1(m_Z) = 0.016\,887 \pm 0.000\,40,$$

$$\alpha_2(m_Z) = 0.033\,22 \pm 0.000\,25, \qquad \text{(A32)}$$

$$\alpha_3(m_Z) = 0.1176 \pm 0.005$$

for the couplings at the scale $m_Z = 91.1876$ GeV. For two-loop evolution, which we use in Sec. III A for comparison to one loop, see [17].

Under the perspective of this Appendix, unification is the numerical requirement that the unification scale M_X, the unified gauge coupling g_G, the Wilson coefficients c_i, and the Higgs VEVs $v_i^2 = \sum_t v_{(i)t}^2$ satisfy Eqs. (A25), (A28), and (A29) simultaneously, with the given running coupling functions $\alpha_s(\mu)$ of the standard model, e.g., (A30)–(A32) to one loop. In the main text, we further demand that the values of these quantities are "natural," so that physically sensible unification can be claimed, and we exhibit such models.

We outline here a procedure to solve these equations simultaneously, after having chosen the unified gauge group [$SU(5)$ or $SO(10)$] and the Higgs content H_i [the multiplets that may occur in the dimension-5 operators (A7)]: Fix any ratio $x_1{:}x_2{:}\ldots$ between the Higgs VEVs

$v_i = x_i v$ (without loss of generality, $\sum_i x_i^2 = 1$ with, e.g., spherical coordinates); for Higgs multiplets H_i in a **210** or a **770** of $SO(10)$, also fix the direction of the VEV $\langle H_i \rangle$ by fixing the ratio $y_{(i)1}{:}y_{(i)2}{:}\ldots$ between the $v_{(i)t} = y_{(i)t} v_i$ (with $\sum_t y_{(i)t}^2 = 1$); see (A19). Also fix the ratio $z_1{:}z_2{:}\ldots$ between the Wilson coefficients $c_i = z_i c$ (with $\sum_i z_i^2 = 1$). With these inputs, the *ratio* between the ϵ_s ($s = 1, 2, 3$) in (A28) is completely fixed:

$$\epsilon_s = \frac{c v}{M_{\text{pl}}} \sum_i z_i \sum_t x_i y_{(i)t} \delta_s^{(i)t}. \qquad \text{(A33)}$$

A key observation is now that any such given ratio $\epsilon_1{:}\epsilon_2{:}\epsilon_3$, along with the given functions $\alpha_s(\mu)$, determines M_X and the values ϵ_s uniquely by solving the two left equalities in (A29); this can be seen analytically to one-loop order when (A30) is plugged back into (A29) (the linear system of equations for $\ln M_X$ has nonvanishing determinant for almost any given ratio $\epsilon_1{:}\epsilon_2{:}\epsilon_3$), but also holds at two-loop order. Then, the unified coupling $\alpha_G = g_G^2/4\pi$ is determined by (A29) as well, and so is the overall scale v of the VEVs via (A26). Finally, with a choice (2) of the Planck scale M_{pl}, the required Wilson coefficients $c_i = z_i c$ can be computed from (A33). Note, that this last step is the only place where the choice of the Planck scale $M_{\text{pl}} = 1.2 \times 10^{19}$ GeV$/\xi$ comes in: the smaller a Planck scale one chooses, the smaller the Wilson coefficients c_i have to be in order to achieve unification (at the same x_i, $y_{(i)t}$, z_i); in fact, the necessary Wilson coefficients are inversely proportional to the choice of ξ in (2).

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