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Bound states and fermiophobic unparticle oblique corrections to the photon

Arun M. Thalapillil*

Enrico Fermi Institute and Department of Physics, University of Chicago, 5640 South Ellis Avenue, Chicago, Illinois 60637, USA

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We study the effects of fermiophobic scalar/pseudoscalar oblique corrections on bound state energy levels in muonic atoms. To make the treatment sufficiently general, while including ordinary scalar and axionlike pseudoscalar fields as special cases, we consider unparticle scalar/pseudoscalar operators with couplings predominantly to photons. We derive the relevant vacuum polarization functions and comment on the functional forms of the unparticle Uehling potentials for various scaling dimensions in the point nucleus and finite nucleus approximations. It is estimated that for an infrared fixed point near the scale of electroweak-symmetry breaking, in the low TeV range, and natural values for the model parameters, the energy shifts in the low-lying muonic-lead transitions are typically of the order of a few times 0.1 eV to a few times 0.01 eV. The energy level structures of the unparticle Uehling shifts are inferred using general methods for the scalar and pseudoscalar cases and it is shown that the two cases contribute to the energy shifts with the same sign. It is shown that this conclusion is not changed even when scale invariance is broken and is in fact relatively insensitive to the scale at which it is broken. It is pointed out nevertheless that the estimated magnitude of the unparticle Uehling shift (based on some *natural* values for the model parameters) is a factor of 1000–10000 below the discrepancy in QED/nuclear theory and precision muonic-lead spectroscopy from about two decades ago. We briefly comment on scenarios where the unparticle induced energy shift, if it exists, may be experimentally measurable. One possibility in this direction is if the UV sector, from which the unparticle sector arises, has a large number of fermions. Comments are also made on the possibility of further studying muonic atoms, as a probe for beyond-standard-model physics, in the context of forthcoming experiments, such as those probing lepton flavor violation through coherent muon-electron conversions. For completeness we explore some of the astrophysical and cosmological consequences of a fermiophobic scalar/pseudoscalar unparticle sector. In the fermiophobic context we also estimate a minimum value for the conformal invariance breaking scale.

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I. INTRODUCTION

Recently there has been much interest in the possibility of a scale-invariant hidden sector that couples to the standard model (SM) [1,2]. The operators in such a theory have been referred to as "unparticles" to emphasize their generally nonintegral scaling dimensions. In addition to an n-body phase space resembling that of a nonintegral number of massless particles, the correlation functions of the unparticle operators have interesting nontrivial phases [1]. The above properties make this sector distinct from other beyond-SM extensions.

A model with the above properties was proposed by Banks and Zaks (BZ) [2]. Their theory was a vectorlike $SU(3)$ gauge theory with N_f fermions in the fundamental representation. It was found that for a particular range of N_f, the β function at lowest order is negative [$\beta_1(N_f) < 0$] while the contribution to the β function at the next order is positive [$\beta_2(N_f) > 0$] in

$$\beta(g) \simeq \beta_1(N_f)\frac{g^3}{16\pi^2} + \beta_2(N_f)\frac{g^5}{(16\pi^2)^2} + \cdots.$$

This would mean that as the theory flows to lower energies the small coupling constant g would grow until it hits an infrared fixed point where

$$\beta(g^*) \simeq 0.$$

The theory is scale invariant below this scale and the description in terms of the Banks-Zaks fields at high energies is replaced by one in terms of composite particles of a strongly coupled scale-invariant theory. These composite particles may be identified with the unparticles [1]. We are interested in exploring the case of a scale-invariant fermiophobic unparticle sector that couples only with the massless SM gauge bosons, specifically with a substantial coupling only to the photon. As we shall see a fermiophobic sector might be able to avoid certain constraints compared to a fermiophilic sector. Also, independent of considerations in our study there is great interest in a fermiophobic scalar sector in the context of electroweak-symmetry breaking and the Higgs mechanism [3]. Our main focus will be on a fermiophobic sector that couples predominantly to photons and the effects of the induced oblique corrections on atomic energy levels in muonic atoms.

*madhav@uchicago.edu

If one assumes that the scale-invariant unparticle sector is also conformally invariant, then the scaling dimensions (j) of the gauge invariant primary unparticle operators are tightly constrained by requirements of conformal invariance. For an operator in the (l_1, l_2) Lorentz spin representation, the constraints are [4–6]

$$j \geq l_1 + l_2 + 2 - \delta_{l_1 \times l_2, 0}.$$

These bounds translate to

$$j_{\mathcal{U}} \geq 1, \qquad j_{\mathcal{U}_f} \geq 3/2, \qquad j_{\mathcal{U}_V} \geq 3,$$

for the scalar, fermion, and vector unparticle operators, respectively. These conditions are referred to as Mack's unitarity criteria [4]. We will impose these constraints on the unparticle operators we work with in the present study. Although it is technically possible for a quantum field theory to be scale invariant but not conformally invariant, examples are rare [7].

The unparticle operators in the limit of exact scale invariance may be considered as a sum over resonances having a continuous mass distribution with no mass gap [8]. The Källén-Lehmann spectral density of the unparticle operator takes the form [1]

$$\rho(s^2) \propto (s^2)^{j-2}.$$

This implies that for $j \geq 2$ the theory becomes very UV sensitive and may cause singular behavior [9]. This suggests that we only consider operators with $j < 2$. This is in conflict with the requirements of Mack's unitarity for primary, vector unparticle operators which require $j \geq 3$ [9]. Usually in any unparticle model with couplings to SM fields there must also be additional contact terms between SM operators as first pointed out in [6]. They are generated when we integrate out the ultraheavy UV fields (BZ fields, for example). Without some fine-tuning these terms generally dominate over the SM-unparticle couplings and render unparticle phenomenology to be of lesser interest. In the case of scalar/pseudoscalar unparticles where the Mack's unitarity constraint is $j \geq 1$ and the scaling dimension may be very close to unity, these contact interactions may nevertheless be of lesser importance [6].

Apart from simplicity the arguments above will be our motivation for considering scalar/pseudoscalar unparticles ($\mathcal{U}$) in our study. In the subsequent analysis we will use $\mathcal{U}$ to label a generic scalar/pseudoscalar unparticle, $\mathcal{O}$ for an unparticle scalar and $\tilde{\mathcal{O}}$ for an unparticle pseudoscalar. Also, we label an ordinary scalar by ϕ, pseudoscalar by $\tilde{\phi}$, and a generic scalar/pseudoscalar operator by Φ. The energy scale of the infrared fixed point (where the theory becomes scale invariant) is usually denoted by $\Lambda_{\mathcal{U}}$ and that at which scale invariance is broken by the parameter μ. We will assume that the unparticle scale is in the low TeV range near the scale of electroweak-symmetry breaking, i.e., $\Lambda_{\mathcal{U}} \sim v$. Specifically we will interpret this to mean $\Lambda_{\mathcal{U}} \in [\sim 246\ \text{GeV}, \mathcal{O}(1)\ \text{TeV}]$. There are stringent constraints on

$$p\bar{p} \to h_f \to \gamma\gamma + X,$$

where h_f is a fermiophobic Higgs, from the CDF and D0 Collaborations at the Tevatron [10]. In our case the coupling of the fermiophobic scalar/pseudoscalar sector to the fermions and the heavy gauge bosons ($Z^0, W^{\pm}$) are assumed to be very small or close to zero. Also, the coupling to the $SU(3)_c$ colored massless gluons is assumed to be much smaller than the coupling to the photon. In this case the interaction scale may be in the $\Lambda_{\mathcal{U}} \sim v$ range we consider above, in the low TeV regime, and still be consistent with the collider constraints [11]. We will estimate a lower bound for the scale of μ in the fermiophobic unparticle case later in Sec. III.

Based on all the above arguments we will therefore consider unparticle scalars and pseudoscalars coupling to photons with an interaction scale $\Lambda_{\mathcal{U}} \sim v$ and satisfying

$$1 \leq j_{\mathcal{U}} < 2. \tag{1}$$

Now if one considers two representative fermiophobic and fermiophilic effective interaction terms of the form

$$\Lambda_{\gamma}^{-j} \mathcal{U} F_{\alpha\beta} F^{\alpha\beta}, \qquad \Lambda_{\psi}^{1-j} \mathcal{U}_V^{\alpha} \bar{\Psi} \gamma_{\alpha} \Psi,$$

it may be argued based on scaling arguments and dimensional analysis [12] that in general

$$\Lambda_{\psi} > \Lambda_{\gamma}.$$

This seems to imply, for the relevant values of j, that the coupling to the gauge bosons may lead to larger effects than the corresponding fermiophilic case [12]. Although the true niche for probing such effects may be in high energy colliders (see, for example, [9,11–17], and references therein), especially when the interaction scale is much higher than $\mathcal{O}(1)$ TeV, it is still interesting to explore the effects of a fermiophobic coupling in low-energy experiments. In the fermiophilic case for instance one can put interesting bounds on the unparticle sector from atomic parity violation [18].

We are specifically interested in fermiophobic unparticle contributions to muonic-atom transitions. Although in this case, unlike atomic parity violation in the fermiophilic case [18], no symmetry is violated there could be atomic systems or regions in parameter space where the effect may be measurable. Also, if the fermiophobic unparticle sector is such that it has substantial couplings only to photons and no other gauge bosons, the muonic-atom transitions can play a unique role in constraining it.

We chose to study fermiophobic unparticle contributions to muonic-lead transitions. It will be seen in this specific case that for natural values of the model parameters the estimated unparticle induced energy shifts in low-lying muonic-lead atomic transitions may be of the order of a few 0.1–0.01 eV which is comparable to the bound state QED corrections from light-by-light scattering and the fourth-order Lamb shift to higher orbital angular momentum transitions, respectively. We will comment on cases where this value may be enhanced or measurable, for instance, when there is a very large fermion multiplicity in the UV sector.

Another point is that since oblique corrections may potentially involve new "heavy" particles running in the loops we may expect them to be a probe for unparticles within a wide range of μ values. We will discuss this aspect, in the context of the unparticle Uehling energy shifts, in more detail in Sec. III. There is also the motivation that many of the constraints on the couplings of an unparticle to SM fields [19] are more relaxed in the case of a fermiophobic sector. We will discuss the case of broken scale invariance and the requirements for evading astrophysical and cosmological constraints later in our study.

The scalar/pseudoscalar unparticle propagator in the limit of exact scale invariance is [1]

$$\int d^4x e^{iqx}\langle 0|T(\mathcal{O}_{\mathcal{U}}(x)\mathcal{O}_{\mathcal{U}}(0))|0\rangle = \frac{iA_j}{2\sin(j\pi)}[-(q^2+i\epsilon)]^{j-2}, \tag{2}$$

where

$$A_j = \frac{16\pi^{5/2}\Gamma(j+1/2)}{(2\pi)^{2j}\Gamma(j-1)\Gamma(2j)}$$

with j being the scaling dimension of the unparticle operator. Note that since j is in general a noninteger there is a nontrivial phase $e^{i(2-j)\pi}$ associated with the unparticle propagator. Also note that in the limit of $j \to 1^+$ there is no singular behavior and we recover the ordinary scalar/pseudoscalar propagator. The propagator coefficient $A_j/(2\sin j\pi)$ has unit magnitude at $j=1$ and diverges very close to $j=2$ where the theory is very UV sensitive. In the rest of the region its magnitude is below 1.

The case of broken scale invariance may be parametrized by introducing an effective mass gap μ in the spectral decomposition leading to the scalar/pseudoscalar unparticle propagator [15,16]

$$\int d^4x e^{iqx}\langle 0|T(\mathcal{O}_{\mathcal{U}}(x)\mathcal{O}_{\mathcal{U}}(0))|0\rangle = \frac{iA_j}{2\sin(j\pi)}[-(q^2-\mu^2+i\epsilon)]^{j-2}. \tag{3}$$

μ may be thought of as the scale at which conformal invariance is effectively broken.

Let the fermiophobic unparticle fields be gauge singlets under the SM gauge group. Then the coupling of the scalar unparticle to two photons may be incorporated [1] by a term in the effective low-energy action

$$\mathcal{S}_S^{\rm eff} = \int d^4x \frac{c}{4\Lambda_\gamma^j}\mathcal{O}F_{\mu\nu}F^{\mu\nu} + \cdots. \tag{4}$$

Here ... denotes terms that are suppressed by higher powers of the relevant energy scale and which have been ignored. Λ_γ is a scale relevant to the $\gamma-\mathcal{O}-\gamma$ coupling derived from the fundamental unparticle scale $\Lambda_{\mathcal{U}}$. The coefficient c is assumed to be an $\mathcal{O}(1)$ constant factor. Note that when the unparticle sector is being generated from a UV theory (such as the BZ theory) the coefficient of the above operator goes like [1]

$$\sim \mathcal{O}(1)\left(\frac{\Lambda_{\mathcal{U}}}{M_{\rm UV}}\right)^{d_{\rm UV}}$$

and is in general not of $\mathcal{O}(1)$. Here $M_{\rm UV}$ is the scale of the UV physics and $d_{\rm UV}$ is the dimension of the UV operator coupling to $F_{\mu\nu}F^{\mu\nu}$. But if one assumes, for example, that the scalar unparticle operators $\mathcal{O}_i$ are being generated by the confinement ($\langle\bar{\Psi}_i\Psi_i\rangle \to \mathcal{O}_i$) of fermions ($\Psi_i$) from the UV sector (in this case $d_{\rm UV}=3$), conservatively, the "effective" coefficient

$$c \simeq \sum_{i=1}^{N_f} c_i \sim N_f\mathcal{O}(1)\left(\frac{\Lambda_{\mathcal{U}}}{M_{\rm UV}}\right)^{d_{\rm UV}}$$

could naturally be of $\mathcal{O}(1)$. The reason is that theoretically the number of fermions N_f in the UV theory is permitted to be large and is constrained only by the requirement of conformal invariance at g^*. For example,

$$\frac{33}{2} \gtrsim N_f \gtrsim \frac{306}{38}$$

in the BZ theory [2] and in a recent technicolor inspired $SU(N_T)\times SU(N_U)$ model by Sannino and Zwicky [20]

$$\frac{11}{\gamma^*+2}N_T \gtrsim N_f \gtrsim \frac{11}{\gamma^*+2}N_U + 2,$$

where the critical anomalous dimension satisfies the unitarity bound $\gamma^* \le 2$. Thus we will interpret the effective interaction in Eq. (4) as modeling the effects of various possible unparticle scalar operators ($\mathcal{O} \approx \sum_{i=1}^{N_f}\mathcal{O}_i$), in a semirealistic model, and take the coefficient $c \sim \mathcal{O}(1)$ without loss of generality. But it is nevertheless important to keep in mind that apart from notions of naturalness, nothing excludes a larger value for the coupling, for instance, if there is very large fermion multiplicity (i.e. very large N_f) in the hidden sector. QCD-like models with a possibly large number of colors (N_C) and fermion flavors (N_f) is not uncommon, for example, in many string-inspired models [21].

Similarly the coupling of a pseudoscalar unparticle to two photons may be modeled by

$$\mathcal{S}_{\rm PS}^{\rm eff} = \int d^4x \frac{b}{4\tilde{\Lambda}_\gamma^j}\tilde{\mathcal{O}}F^{\mu\nu}\tilde{F}_{\mu\nu} + \cdots, \tag{5}$$

where

$$\tilde{F}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}.$$

$\tilde{\Lambda}_\gamma$ is a scale relevant to the $\gamma - \tilde{\mathcal{O}} - \gamma$ coupling and the constant b is assumed to be of $\mathcal{O}(1)$. All the assumptions in the scalar case again hold here. The above may be compared to the chiral anomaly induced $\gamma - \pi^0 - \gamma$ Wess-Zumino-Witten coupling

$$\mathcal{S}_{\pi\gamma\gamma} = \int d^4x \frac{-N_c e^2}{48\pi^2 f_\pi} \pi^0 F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{6}$$

leading to

$$\Gamma_\mu \overset{q_\pi \to 0}{\longrightarrow} - \frac{ie^2}{4\pi^2 f_\pi} \epsilon_{\alpha\beta\rho\sigma} k^\rho l^\sigma,$$

valid in the soft pion limit (here N_c is the number of "colors," k^ρ and l^σ are the four-momenta of the two photons) and the $\gamma^* - \gamma - \tilde{\phi}$ vertex (here $\tilde{\phi}$ is a pseudoscalar meson like π^0, η or η')

$$\Gamma_\mu \overset{Q^2 \to \pm\infty}{\longrightarrow} - ie^2\left(\frac{2f_\pi}{Q^2}\right)\epsilon_{\mu\nu\rho\sigma} p^\nu \epsilon^\rho q^\sigma$$

proposed in [22] for $Q^2 = -q^2 \to \pm\infty$ and recently considered in [23] in the context of meson-photon transition form factors in the charmonium energy range. In the above expression q^σ is the four-momentum of the off-shell photon γ^*, p^ν is the four-momentum of the pseudoscalar meson $\tilde{\phi}$, ϵ^ρ is the polarization vector of the outgoing on-shell photon γ, and $f_\pi \simeq 93$ MeV is the pion constant.

The corrections to muonic-atom transitions due to photon vacuum polarization are induced by diagrams like those in Fig. 1. It is our aim to estimate the magnitude of such contributions to low-lying muonic-atom levels. We mention, as an aside, that processes such as those in Fig. 1 for the case of low-energy QCD [where $\mathcal{U}(k)$ is now again a pseudoscalar meson like π^0 or η] contribute, for instance, to muon $(g-2)$. Using chiral perturbation theory based on Eq. (6) and ω vector-meson dominance, in the relevant range $\sqrt{s} < 0.6$ GeV, a diagram such as that in Fig. 1 for the π^0 is expected to contribute [24]

$$a_\mu(\pi^0\gamma, \sqrt{s} < 0.6\ \text{GeV}) = (0.13 \pm 0.01) \times 10^{-10}.$$

A similar contribution from the fermiophobic unparticle sector, for typical model parameter values, is expected to be very small and within experimental limits, since the QED kernel $K(s)$ [25] has a steep cutoff around $\mathcal{O}(1)$ GeV.

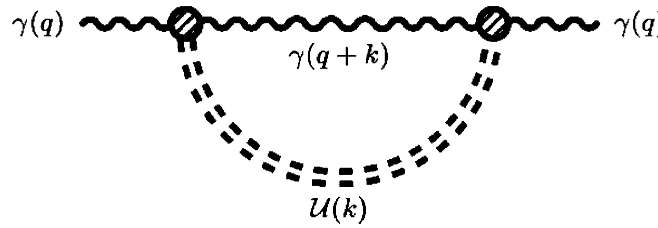

FIG. 1. Photon vacuum polarization by an unparticle scalar or pseudoscalar with an arbitrary scaling dimension j. The vertices are shown with blobs to indicate that they are effective couplings coming from an effective low-energy action of the form (4) or (5).

Let us now proceed to calculate the vacuum polarization functions and estimate the induced effective Uehling potentials for various cases.

II. OBLIQUE CORRECTIONS AND THE UNPARTICLE UEHLING POTENTIAL

We are primarily interested in possible oblique corrections to the photon propagator due to scalar/pseudoscalar unparticles as shown in Fig. 1. If they exist we expect such vacuum polarizations to modify the photon propagator over the usual SM corrections. These scalar/pseudoscalar oblique corrections can show up potentially as very tiny energy shifts in muonic-atom energy levels or in the anomalous magnetic moment of the muon. As we have mentioned previously our main focus will be on possible corrections to the atomic transitions in muonic atoms. We will first consider the case of perfect scale invariance $\mu \to 0$. The case when $\mu \neq 0$ will be discussed in Sec. III after we have determined a minimum value for μ in the fermiophobic case as dictated by astrophysical and cosmological constraints.

Using the Feynman rule for Eq. (4) (see, for example, [26]), the scalar unparticle contribution to the photon polarization tensor is

$$i\Pi^{\mu\nu}_{\mathcal{O}} = -\frac{c^2 A_j}{2\Lambda_\gamma^{2j}\sin(j\pi)} \int \frac{d^4k}{(2\pi)^4} \frac{[q\cdot(q+k)g^{\alpha\mu} - q^\alpha(q+k)^\mu][q\cdot(q+k)g^\nu_\alpha - q_\alpha(q+k)^\nu]}{[(q+k)^2 + i\epsilon][-(k^2-\mu^2)+i\epsilon]^{2-j}}.$$

This simplifies to the expression

$$i\Pi^{\mu\nu}_{\mathcal{O}} = i\Pi_{\mathcal{O}}(q^2, \mu^2, j)(q^2 g^{\mu\nu} - q^\mu q^\nu) \tag{7}$$

with

$$i\Pi_{\mathcal{O}}(q^2,\mu^2,j) = -\frac{c^2 A_j e^{i(2-j)\pi}\Gamma(3-j)}{2\Lambda_\gamma^{2j}\sin(j\pi)\Gamma(2-j)} \int dxdy\delta(x+y-1)y^{1-j} \int \frac{d^4l}{(2\pi)^4} \frac{q^2(x-1)^2 + (l^2/2)}{[l^2 - \Delta(q^2,\mu^2) + i\epsilon]^{3-j}}, \tag{8}$$

where

$$\Delta(q^2, \mu^2) = x(x-1)q^2 + y\mu^2.$$

We note that Eq. (7) has the expected gauge invariant structure and satisfies the Ward identities.

For the pseudoscalar unparticle case the polarization tensor is

$$i\Pi^{\mu\nu}_{\tilde{\mathcal{O}}} = -\frac{b^2 A_j}{2\tilde{\Lambda}_\gamma^{2j}\sin(j\pi)}\int\frac{d^4k}{(2\pi)^4} \times \frac{[\epsilon^{\xi\mu\rho\alpha}\epsilon_{\lambda\alpha\kappa\nu}q_\rho(q+k)_\xi q^\lambda(q+k)^\kappa]}{[(q+k)^2+i\epsilon][-(k^2-\mu^2)+i\epsilon]^{2-j}}.$$

This may be simplified to give

$$i\Pi^{\mu\nu}_{\tilde{\mathcal{O}}} = i\Pi_{\tilde{\mathcal{O}}}(q^2, \mu^2, j)(q^2 g^{\mu\nu} - q^\mu q^\nu), \tag{9}$$

where

$$i\Pi_{\tilde{\mathcal{O}}}(q^2, \mu^2, j) = +\frac{b^2 A_j e^{i(2-j)\pi}\Gamma(3-j)}{2\tilde{\Lambda}_\gamma^{2j}\sin(j\pi)\Gamma(2-j)} \times \int dxdy\delta(x+y-1)y^{1-j} \times \int\frac{d^4 l}{(2\pi)^4}\frac{(l^2/2)}{[l^2-\tilde{\Delta}(q^2,\mu^2)+i\epsilon]^{3-j}}, \tag{10}$$

and again

$$\tilde{\Delta}(q^2, \mu^2) = x(x-1)q^2 + y\tilde{\mu}^2.$$

Regularizing the momentum integral in Eq. (8) using dimensional continuation yields

$$\Pi_{\mathcal{O}}(q^2, \mu^2, j) = -\frac{c^2 A_j}{32\pi^2\Lambda_\gamma^{2j}\sin(j\pi)}\int dxdy\delta(x+y-1) \times y^{1-j}\bigg[\frac{\Delta(q^2,\mu^2)}{j(j-1)}\Delta^{j-1}(q^2,\mu^2) + q^2(x-1)^2\frac{\Delta^{j-1}(q^2,\mu^2)}{(j-1)}\bigg]. \tag{11}$$

For the pseudoscalar case, again regularizing via dimensional continuation

$$\Pi_{\tilde{\mathcal{O}}}(q^2, \mu^2, j) = +\frac{b^2 A_j}{32\pi^2\tilde{\Lambda}_\gamma^{2j}\sin(j\pi)}\int dxdy\delta(x+y-1) \times y^{1-j}\bigg[\frac{\tilde{\Delta}(q^2,\mu^2)}{j(j-1)}\tilde{\Delta}^{j-1}(q^2,\mu^2)\bigg]. \tag{12}$$

We will address the $j \to 1^+$ limit of the expressions in Eqs. (11) and (12) later.

The unparticle polarization functions may be renormalized as

$$\hat{\Pi}_{\mathcal{O}}(q^2, \mu^2, j) = \Pi_{\mathcal{O}}(q^2, \mu^2, j) - \Pi_{\mathcal{O}}(0, \mu^2, j),$$
$$\hat{\Pi}_{\tilde{\mathcal{O}}}(q^2, \mu^2, j) = \Pi_{\tilde{\mathcal{O}}}(q^2, \mu^2, j) - \Pi_{\tilde{\mathcal{O}}}(0, \mu^2, j), \tag{13}$$

so that as $q \to 0$ the residue of the photon propagator tends to unity.

Note that the nontrivial unparticle propagator phase $e^{i(2-j)\pi}$ does not make an appearance in Eqs. (11) and (12). It is found from explicit calculation that the phase is removed during the evaluation of the Euclidean loop integrals. Therefore the unparticle vacuum polarization tensor does not have a complex phase from the propagator contribution and any imaginary part that $\hat{\Pi}_{\mathcal{U}}(q^2, \mu^2, j)$ picks up subsequently, if at all, should come from the kinematic region that q^2 occupies.

We may likewise calculate the polarization functions for ordinary scalars (ϕ) and pseudoscalars ($\tilde{\phi}$) with a two-photon coupling. The ordinary pseudoscalar in this case may be compared to an axionlike particle (a) coupling to photons

$$\mathcal{S}_{a\gamma\gamma} = \int d^4x\frac{g_{a\gamma}}{4f_a}aF_{\mu\nu}\tilde{F}^{\mu\nu},$$

where the scale f_a and the pseudoscalar mass m_a are independent of each other. Substituting $j = 1$ in Eqs. (8) and (10) and integrating over the loop momenta using dimensional regularization we get after an $\overline{MS}$ subtraction

$$\Pi_\phi(q^2, m_\phi^2) = \frac{a^2}{16\pi^2\Lambda_\phi^2}\int dxdy\delta(x+y-1) \times \bigg[\Delta'(q^2,m_\phi^2)\log\frac{\Delta'(q^2,m_\phi^2)}{M^2} + q^2(x-1)^2\log\frac{\Delta'(q^2,m_\phi^2)}{M^2}\bigg],$$
$$\Pi_{\tilde{\phi}}(q^2, m_{\tilde{\phi}}^2) = -\frac{\tilde{a}^2}{16\pi^2\tilde{\Lambda}_\phi^2}\int dxdy\delta(x+y-1) \times \bigg[\Delta''(q^2,m_{\tilde{\phi}}^2)\log\frac{\Delta''(q^2,m_{\tilde{\phi}}^2)}{M^2}\bigg]. \tag{14}$$

Here

$$\Delta'(q^2, m_\phi^2) = x(x-1)q^2 + ym_\phi^2,$$
$$\Delta''(q^2, m_{\tilde{\phi}}^2) = x(x-1)q^2 + ym_{\tilde{\phi}}^2,$$

and we have used the fact, from Eq. (2), that

$$\frac{A_j}{2\sin(j\pi)} \xrightarrow{j\to 1^+} -1, \qquad e^{i(2-j)\pi} \xrightarrow{j\to 1^+} -1.$$

M is an arbitrary subtraction scale. It must be mentioned that this dependence on M will be removed when we explicitly introduce other higher-order interactions in the effective Lagrangian. A natural choice is to take $M \simeq (\Lambda_\phi, \Lambda_{\tilde{\phi}}) \sim v$, the energy scale of the relevant interaction.

The choice would also ensure that in the $m_\Phi \to 0$ limit we can retain to good approximation, as far as numerical computations are concerned, just the lowest order terms in the effective Lagrangian. This is because with this choice, for the range of q we are interested in (specifically $q \lesssim m_{\mu^-}$), we have $q^2 \log(q^2/M^2) \gg q^2$. For instance, in the analogous case in QCD chiral perturbation theory a choice of $M \simeq \Lambda_{\rm QCD} \sim 1$ GeV is appropriate and one can keep to lowest order only the so-called chiral logarithm term to estimate some of the low-energy QCD effects (see, for example, [27], and references cited therein). With this tacitly assumed we proceed to analyze the functional forms of the Uehling potentials in the cases of interest.

If one is interested in bound states, the modified electromagnetic four-potential A'_μ, due to the vacuum polarizations, is given by

$$A'_\mu(x) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot x}[1 - \hat{\Pi}(q^2)]^{-1} \mathcal{G}^{\rm photon}_{\mu\nu}(q^2)\mathcal{J}^\nu_{\rm source}(q)$$
$$\simeq \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot x}[1 + \hat{\Pi}(q^2)]A^{(0)}_\mu(q), \tag{15}$$

where $i\mathcal{G}^{\rm photon}_{\mu\nu}(q^2)$ is the photon propagator, $\mathcal{J}^\nu_{\rm source}(q)$ is the source four-current in momentum space, and $A^{(0)}_\mu(q)$ is the four-vector potential without any vacuum polarization corrections. As is well known $1 - \hat{\Pi}(q^2)$ acts like a dielectric constant for vacuum

$$1 - \hat{\Pi}(q^2) \sim \frac{\epsilon(\omega)}{\epsilon_0}.$$

Thus, the imaginary part of the polarization tensor corresponds to the vacuum becoming absorptive. In general,

$$\hat{\Pi}(q^2) = \hat{\Pi}_{\rm SM}(q^2) + \hat{\Pi}_{\mathcal{U}}(q^2, \mu^2, j) + \hat{\Pi}_{\rm other}(q^2).$$

In our analysis we are interested in the corrections solely due to the unparticle contribution $\Pi_{\mathcal{U}}(q^2, \mu^2, j)$. We noted previously that the nontrivial unparticle propagator phase $e^{i(2-j)\pi}$ is effectively canceled during the evaluation of the loop integrals. From Eqs. (11), (12), and (14), we observe that for

$$x(x-1)q^2 + y\mu^2 < 0 \mapsto {\rm Im}[\Pi_{\mathcal{U}}(q^2, \mu^2, j)] \neq 0,$$
$$x(x-1)q^2 + ym^2_\Phi < 0 \mapsto {\rm Im}[\Pi_\Phi(q^2, m^2_\Phi, j)] \neq 0,$$

and there is a branch cut starting at $q^2 = \mu^2$ and $q^2 = m^2_\Phi$ in analogy with $e^+ - e^-$ vacuum polarization in QED. We will approximate an atomic nucleus of charge Z initially as a static, point source. For a static source of the electromagnetic field the momentum transfer is spacelike ($q^2 < 0$) and the polarization tensor $\hat{\Pi}_{\mathcal{U}}(-\vec{q}^2, \mu^2, j)$ is real which would imply that there is no absorption in vacuum under these circumstances.

Approximating an atomic nucleus of charge Z as a static, point source the electromagnetic 4-current may be calculated in momentum space to be

$$\mathcal{J}^\nu_{\rm source}(q) \simeq -Ze\delta^{\nu 0}\delta(q^0).$$

In our convention the charge e is intrinsically negative. Then the expression for the modified potential becomes

$$A'^{\rm point}_0(\vec{r}) \simeq -Ze\int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}}[1 + \hat{\Pi}(-\vec{q}^2, \mu^2, j)]\mathcal{G}^{\rm photon}_{00}(-\vec{q}^2). \tag{16}$$

The $\Pi_{\mathcal{U}}(q^2, \mu^2, j)$ contributes to a Uehling potential (in a way similar to $e^+ - e^-$ vacuum polarization in QED) and gives

$$V^{\rm point}_{\mathcal{U}}(r, j) \simeq -Ze\int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}}\hat{\Pi}_{\mathcal{U}}(-\vec{q}^2, \mu^2, j) \times \mathcal{G}^{\rm photon}_{00}(-\vec{q}^2). \tag{17}$$

This is the correction to the electromagnetic potential due to fermiophobic, scalar/pseudoscalar unparticle vacuum polarizations.

By simple dimensional analysis of our expressions in Eqs. (11) and (12), using Eq. (17), we may expect the unparticle corrections to the electromagnetic potential in the limit $\mu \to 0$ to be of a functional form

$$V^{\rm point}_{\mathcal{U}}(r, j) \sim \frac{1}{r^{2j+1}} \tag{18}$$

for $j \neq 1$. From Eq. (14) for the ordinary scalar/pseudoscalar case, in the limit $m_\Phi \to 0$, we infer similarly that the potential may go dimensionally as $1/r^3$. This seems to suggest that, due to the generally nonintegral values of the scaling dimension j, the fermiophobic unparticle induced potential may have rich functional dependences. We will have more to say on this issue when we discuss the $j \to 1$ limit for unparticle operators in the context of atomic energy shifts.

We can compare this fermiophobic scenario (where the correction to the potential is through photon vacuum polarizations) to the case of direct unparticle scalar exchange between charged/uncharged fermions (such a coupling could be generated radiatively at a higher order from fermiophobic couplings in the case of the charged particles) as shown in Fig. 2. In the direct-exchange case (fermiophilic)

$$U^{\rm point}_{\mathcal{U}}(r, j) \sim \int \frac{d^3\vec{q}}{(2\pi)^4} e^{i\vec{q}\cdot\vec{r}}(q^2)^{j-2}.$$

Therefore in this scenario we expect the correction to the potential to go like

$$U^{\rm point}_{\mathcal{U}}(r, j) \sim \frac{1}{r^{2j-1}}. \tag{19}$$

Of course in the case of broken scale invariance the direct exchange will be Yukawa suppressed (by the scale breaking parameter μ) leading to a finite range for the force. We are merely interested in the behavior of the

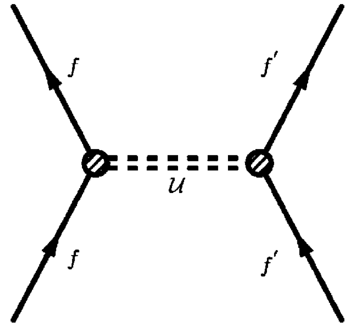

FIG. 2. Direct scalar/pseudoscalar unparticle exchange in a fermiophilic scenario where the unparticles have substantial fermion couplings. These possibilities are heavily constrained, especially when $\mu \to 0$, by searches for a 5th force, atomic parity violation, and other experiments [18,19,28].

direct-exchange potential as a comparison to the vacuum polarization case. As discussed in Sec. I we will require $1 \leq j < 2$ to be consistent with the Mack's unitarity condition and the requirement of UV insensitivity. This immediately implies that for the fermiophobic case under the point nucleus approximation the potential is always more singular than $1/r^2$. This has some implications. We will look more closely at the sign of the unparticle Uehling corrections later, but for now let us assume that as $r \to 0$ the potential energy is negative. Consider an ϵ ball, near $r = 0$, of radius r_ϵ. Then from the uncertainty principle the mean energy at r_ϵ is

$$\langle E\rangle \sim \frac{\hbar^2}{m r_\epsilon^2} - \frac{\xi^2}{r_\epsilon^{2j+1}},$$

where ξ^2 is a positive constant that is determined from some explicit computation. We have $(2j+1) > 2 \ \forall\, j \geq 1$ and this implies that for arbitrarily small r_ϵ, the mean energy $\langle E\rangle$ is arbitrarily negative for a negative potential energy. Therefore the particle must fall to $r = 0$ signifying an instability (see, for example, [29]). This may also be reasoned semiclassically from the criterion that the negative potential energy must be less singular than the centrifugal term, $l(l+1)/r^2$ in the Hamiltonian, to have a stable bound state. Note that for a positive potential energy ($\xi^2 < 0$) this constraint does not apply. Also note that if we had not assumed the scale-invariant unparticle sector to be also conformally invariant, then $j \geq 1$ is no longer necessary; nevertheless for a negative potential energy the scenario is still problematic for $\forall\ j > 1/2$. In this case even for $j = 1/2$ we have to require the condition

$$\xi^2 < \frac{\hbar^2}{8m}$$

to get a stable bound state as is familiar from quantum mechanics [29].

Thus it seems, at least preliminarily, that the correction to the electromagnetic potential in the direct-exchange case is in general less singular than the fermiophobic case for any given value of the scaling dimension (j) under the point nucleus approximation. These expectations are modified to a large extent by some physical considerations. The first obvious modification comes from the fact that near $r = 0$ the finite size of the atomic nucleus becomes important and we would have to calculate the correction to the electromagnetic potential with an appropriate nuclear density profile. This would soften the singular nature of the potential near $r = 0$. One may consider the expression in Eq. (18) as being roughly valid far away from the atomic nucleus in which case it may be approximated by a point source.

More importantly one needs to be careful in the evaluation to get consistent energy shifts as $j \to 1^+$. From Eqs. (11) and (12) we see that the $j \to 1^+$ limit is not continuous. It seems to imply that if one calculates a potential and an energy shift using the quoted unparticle polarization functions, the energy shift increases without bound as one approaches $j = 1$. This does not seem physically sensible. In the $j \to 1^+$ limit, maintaining continuity throughout, one must expect to recover the ordinary scalar (ϕ) and pseudoscalar ($\tilde{\phi}$) situations. This feature seems to be peculiar to the unparticle vacuum polarization diagram we are considering and is related to the fact that the momentum integrals diverge as Λ^{2j}, where Λ is some momentum cutoff, and are only quadratically divergent when $j = 1$. In a similar calculation for the fermiophilic vector unparticle contributing to muon $(g-2)$ (where the photon in the loop is now replaced with a vector unparticle) the relevant momentum integral, for $\mu \to 0$, is [13]

$$\frac{m_{\mu^-}^2}{\Lambda_\psi^{2j-2}} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{[l^2 - \Delta_V(q^2, m_{\mu^-}^2) + i\epsilon]^{4-j}} \sim \frac{m_{\mu^-}^2 \Delta_V^{j-2}(q^2, m_{\mu^-}^2)}{16\pi^2 \Lambda_\psi^{2j-2}}.$$

m_{μ^-} is the muon mass and Λ_ψ is the interaction scale for the fermiophilic coupling. The limit $j \to 1^+$ is now continuous and one obtains

$$\frac{m_{\mu^-}^2 \Delta_V^{j-2}(q^2, m_{\mu^-}^2)}{\Lambda_\psi^{2j-2}} \xrightarrow{j \to 1^+} \frac{m_{\mu^-}^2}{16\pi^2 \Delta_V(q^2, m_{\mu^-}^2)}$$

as expected for an ordinary vector particle in the loop (like the photon).

So to make the expressions (11) and (12) consistent with the $j \to 1^+$ limit and maintain continuity we will consider

$$\frac{\Delta^{j-1}(q^2, \mu^2)}{j-1} \xrightarrow{\forall\, j} (M^2)^{j-1} \log\left[\frac{\Delta(q^2, \mu^2)}{M^2}\right], \qquad (20)$$

where a term that would lead to an infinite energy shift in the $j \to 1^+$ limit has been subtracted out. Here M is again an arbitrary scale. As we mentioned previously for the ordinary scalar/pseudoscalar case a natural and appropriate choice is to take $M \sim \Lambda_{\mathcal{U}}$, the scale of the infrared fixed point, where the theory becomes scale invariant and the description is in terms of unparticles. The crude prescription in Eq. (20) is the one that we will follow in the present study to estimate unparticle Uehling shifts for values of j away from 1 and make the correspondence with the $j = 1$ case. Note that for $j = 1$ the prescription gives the ordinary scalar/pseudoscalar expression. We will be interested mostly in the case when j is close to 1 where, as we shall see in the next section, the unparticle induced atomic energy shifts have their maximum values. Also, as mentioned in Sec. I, very close to $j = 1$ the SM-SM contact terms are relatively less important numerically compared to the case when j is very far from unity [6]. We may therefore be optimistic that the above prescription will certainly capture the essential features of the unparticle Uehling shifts in the interesting region near $j = 1$.

In the $\mu \to 0$ limit we may factor out the q dependent terms in Eqs. (11) and (12). Performing the integration over the Feynman variables we may show that the polarization functions for the scalar and pseudoscalar unparticles are exactly equal $\forall\ j$:

$$\Pi_{\mathcal{O}}(-\vec{q}^2, 0, j) = \Pi_{\tilde{\mathcal{O}}}(-\vec{q}^2, 0, j)$$

assuming $b = c$ and $\Lambda_\gamma = \tilde{\Lambda}_\gamma$. For the ordinary scalar/pseudoscalar cases this equality may be understood readily in terms of the optical theorem. One consequence of our approximation in Eq. (20) is that this relation is no longer exact when j is sufficiently far from unity. We will therefore retain both the scalar and pseudoscalar expressions in all our analyses.

Let us now perform the analysis incorporating the finite extent of the nucleus. Assuming a static situation we have

$$\delta A_0(\vec{r}) = \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \hat{\Pi}_{\mathcal{U}}(-\vec{q}^2, \mu^2, j) \mathcal{G}^{\text{photon}}_{0\mu}(-\vec{q}^2) \times \mathcal{J}^{\mu}_{\text{source}}(\vec{q}).$$

Let

$$\mathcal{J}^{0}_{\text{source}}(\vec{r}) = \rho(\vec{r}) = -Zef(r),$$

where $f(r)$ is a suitable function, which we assume to be spherically symmetric for simplicity, describing the nuclear charge density profile. It must be suitably normalized such that

$$\int d^3\vec{r} f(r) = 1.$$

With this choice one may now perform the integration over the angular variables to obtain

$$\delta A_0(\vec{r}) = V_{\mathcal{U}}(r) \simeq -Ze \int \frac{d|\vec{q}|}{2\pi^2} \frac{\sin(|\vec{q}|r)}{(|\vec{q}|r)} \hat{\Pi}_{\mathcal{U}}(-\vec{q}^2, \mu^2, j) \tilde{f}(|\vec{q}|), \tag{21}$$

where

$$\tilde{f}(|\vec{q}|) = \int d^3\vec{r} e^{i\vec{q}\cdot\vec{r}} f(r). \tag{22}$$

We will choose a simple Gaussian nuclear charge density profile

$$f(r) = f_0 e^{-r^2/(2\zeta)} \tag{23}$$

with

$$f_0 = \frac{1}{(2\pi\zeta)^{3/2}}$$

to give the correct normalization. Note that at $r = \sqrt{2\zeta}$ the nuclear charge density is $1/e$ times its value at the origin. The Gaussian nuclear profile may be considered to be an approximation to a more realistic two-parameter Fermi distribution. Usually the Gaussian density profile is considered more appropriate for light nuclei [30]. In the next section we will see that we are interested in studying muonic lead, which is not a light nucleus. Nevertheless even with the simplistic choice of a Gaussian profile we may expect the analysis to capture all the main features of the finite lead nucleus. With this ansatz, for the nuclear profile, we expect any deviation from the actual case to be of $\mathcal{O}(1)$ and hence comparable to our ignorance about the factors c, b in the effective Lagrangian. The main reason for choosing a Gaussian profile is to simplify the subsequent analysis. The choice would also help us glean simpler analytic results which otherwise would have been more intractable and subject to numerical methods solely.

As previously mentioned for now we will assume exact scale invariance and set $\mu \to 0$. We will comment on this aspect later in Sec. III and will incorporate a nonzero μ at that time. It will be seen that the main conclusions are not drastically changed for the case of broken scale invariance. Defining the dimensionless variable $z = |\vec{q}|r$ we may write Eq. (21) for the two cases, using Eqs. (11) and (12), as

$$V_{\mathcal{O}}(r, j) = \frac{Zec^2 A_j}{64\pi^4 \Lambda_\gamma^{2j} \sin(j\pi)} \int_0^1 dx dy \delta(x + y - 1) y^{1-j} (1 - x)[(1 + j)x - j] \frac{(M^2)^{j-1}}{jr^3} \times \int_0^\infty dz z \sin z e^{-\zeta z^2/(2r^2)} \left[\log(x(1 - x)) + \log\frac{z^2}{M^2 r^2}\right], \tag{24}$$

$$V_{\tilde{\mathcal{O}}}(r,j) = -\frac{Zeb^2 A_j}{64\pi^4 \tilde{\Lambda}_\gamma^{2j}\sin(j\pi)}\int_0^1 dxdy\delta(x+y-1)[x(1-x)y^{1-j}]\frac{(M^2)^{j-1}}{jr^3} \times \int_0^\infty dzz\sin z e^{-\zeta z^2/(2r^2)}\left[\log(x(1-x)) + \log\frac{z^2}{M^2r^2}\right]. \quad (25)$$

One may now perform the integration over the dimensionless parameter z and the Feynman variables. The result may be expressed in terms of the exponential function and the generalized hypergeometric function $F_q^p[(a_1,\ldots,a_p),(b_1,\ldots,b_q);w]$ in a compact form

$$V_{\mathcal{O}}(r,j) = -\frac{Zec^2}{64\pi^4}\left[B_1(j)\frac{e^{-r^2/2\zeta}}{\zeta^{3/2}} + B_2(j)\frac{r^2e^{-r^2/2\zeta}}{\zeta^{5/2}}F_2^2[(1,1),(2,5/2);r^2/(2\zeta)]\right],$$
$$V_{\tilde{\mathcal{O}}}(r,j) = -\frac{Zeb^2}{64\pi^4}\left[\tilde{B}_1(j)\frac{e^{-r^2/2\zeta}}{\zeta^{3/2}} + \tilde{B}_2(j)\frac{r^2e^{-r^2/2\zeta}}{\zeta^{5/2}}F_2^2[(1,1),(2,5/2);r^2/(2\zeta)]\right]. \quad (26)$$

$B_1(j)$, $B_2(j)$, $\tilde{B}_1(j)$, and $\tilde{B}_2(j)$ are functions of the scaling dimension j and are given by

$$B_1(j) = -\frac{\sqrt{\pi}A_j(M^2)^{j-1}}{\sqrt{2}\Lambda_\gamma^{2j}\sin j\pi}\left[\frac{2+j(j-2)(2j-7)-(j-4)(j-3)(1+j(j-3))\mathrm{H}[3-j]}{j(j-4)^2(j-3)^2} - \left\{\frac{j^2-3j+1}{j(j-4)(j-3)}\right\}\log(2M^2\zeta e^{\gamma_E-2})\right],$$
$$B_2(j) = -\frac{\sqrt{\pi}A_j(M^2)^{j-1}}{3\sqrt{2}\Lambda_\gamma^{2j}\sin j\pi}\left[\frac{-j^2+3j-1}{j(j-4)(j-3)}\right], \quad (27)$$

$$\tilde{B}_1(j) = -\frac{\sqrt{\pi}A_j(M^2)^{j-1}}{\sqrt{2}\tilde{\Lambda}_\gamma^{2j}\sin j\pi}\left[\frac{-5+j(5-j)+(j-4)(j-3)\mathrm{H}[4-j]}{j(j-4)^2(j-3)^2} + \left\{\frac{1}{j(j-4)(j-3)}\right\}\log(2M^2\zeta e^{\gamma_E-2})\right],$$
$$\tilde{B}_2(j) = -\frac{\sqrt{\pi}A_j(M^2)^{j-1}}{3\sqrt{2}\tilde{\Lambda}_\gamma^{2j}\sin j\pi}\left[\frac{1}{j(j-4)(j-3)}\right], \quad (28)$$

where γ_E is the Euler-Mascheroni constant and $\mathrm{H}[n]$ is the nth harmonic number given by $\sum_{k=1}^n(1/k)$ for integral values of n and by the Euler integral representation

$$\mathrm{H}[n] = \int_0^1 dx\frac{1-x^n}{1-x}$$

for $n \notin +\mathbb{Z}$. Note that $\sqrt{\zeta}$ has dimensions of length and hence the above expressions for the potentials have the expected dimensions.

The functional forms of the potential energies in the pseudoscalar unparticle case, $eV_{\tilde{\mathcal{O}}}(r,j)$, for two different values of the scaling dimension j are shown in Fig. 3. The profile indicates that for the pseudoscalar unparticle the potential energy is negative. It is seen that the singular nature of the unparticle Uehling correction, near $r=0$, has been eliminated by taking into account the finite size of the nucleus. We will explore in more detail the sign of the unparticle potential energies shortly.

The unparticle Uehling potential would cause a small energy shift in the atomic energy levels. The energy level structure in a bound system may be deduced (without explicitly calculating all the energies) by studying the

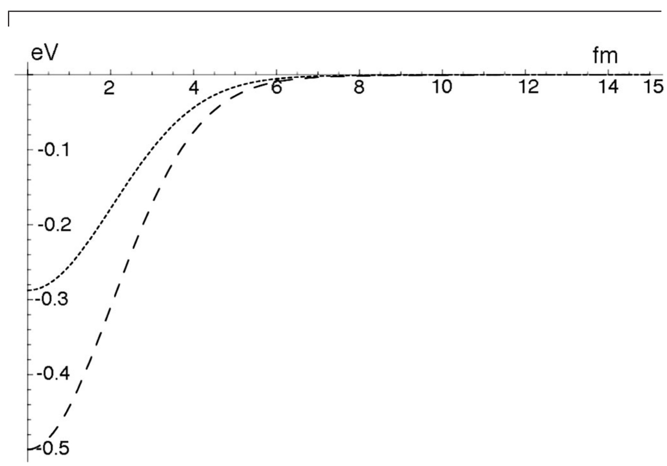

FIG. 3. Potential energies in muonic lead for the pseudoscalar unparticle case based on Eq. (25), assuming perfect scale invariance $\mu \to 0$, for $j = 1.01$ (dashed line) and 1.15 (short-dashed line). We have taken $b \simeq \mathcal{O}(1)$, $\zeta \simeq 4$ fm^2 (to model the case of a $\mu^- - \mathrm{Pb}_{82}^{208}$ nucleus) and adopted a reference value for the interaction scale $\tilde{\Lambda}_\gamma \sim 246$ GeV. Note that the potential goes to zero very rapidly away from $r=0$ and has a constant value at the origin implying that there is no singular behavior.

relevant potential [31]. We will apply some of these methods to study the level structure of the energy shifts ($\delta E^{\mathcal{U}}$) due to unparticle Uehling potentials (24) and (25).

A general result [32] is that for a potential $V(r)$ depending on

$$e\nabla^2 V(r) \equiv \frac{e}{r^2}\frac{d}{dr}\left(r^2\frac{dV(r)}{dr}\right) \gtreqless 0,$$

the energy levels are ordered as

$$E(n, l) \gtreqless E(n, l+1).$$

Here n is the principal quantum number and l is the angular momentum quantum number. Note that for the simple Coulomb potential $\nabla^2 V(r) = 0$ for nonzero r and gives the familiar result that energy levels depend only on the principal quantum number n. For the unparticle Uehling potentials being considered we see that the situation may be complicated somewhat by the fact that for the finite nucleus the Laplacian of the potential energy may change sign. Thus strictly speaking we must consider if

$$\langle e\nabla^2 V_{\mathcal{U}}(r, j, \mu^2)\rangle_{n,l+1} \gtreqless 0. \tag{29}$$

Here the averaging $\langle\cdots\rangle_{n,l+1}$ is over the relevant $(n, l+1)$ wave function [32]. Depending on the sign of the above expression we would have a relation between the unparticle induced energy shifts

$$\delta E^{\mathcal{U}}(n, l, j, \mu^2) \gtreqless \delta E^{\mathcal{U}}(n, l+1, j, \mu^2).$$

Let us proceed to study the Laplacian of the potentials in Eqs. (24) and (25). As mentioned before we will consider the case of $\mu \simeq 0$ presently and concentrate on the low-lying states, i.e. states near $r = 0$. It is noted, from Eqs. (27) and (28), that for both the scalar and pseudoscalar unparticle potentials

$$B_1(j), \quad B_2(j) > 0, \qquad \tilde{B}_1(j), \quad \tilde{B}_2(j) > 0$$

for $\forall\ j \in [1, 2)$. It is also seen that

$$B_1(j) \gg B_2(j), \qquad \tilde{B}_1(j) \gg \tilde{B}_2(j).$$

Thus we infer that both pseudoscalar and scalar unparticle Uehling potential energies are negative $\forall\ j \in [1, 2)$. This reaffirms the indications from Fig. 3. It also means, as indicated by the $B_1(j)$ and $\tilde{B}_1(j)$ coefficients, that the first term in the unparticle Uehling potential dominates over the second term. The Laplacians of the Uehling potentials in Eq. (26) are given by

$$\begin{aligned} e\nabla^2 V_{\mathcal{O}}(r, j, 0) = &-\frac{Ze^2c^2}{64\pi^4}\Bigg[B_1(j)\frac{e^{-(r^2/2\zeta)}}{\zeta^{7/2}}(r^2 - 3\zeta) + B_2(j)\frac{e^{-(r^2/2\zeta)}}{r\zeta^{9/2}}\Bigg\{12r\zeta^2 - 3e^{r^2/2\zeta}\sqrt{2\pi}\zeta^{5/2}\mathrm{Erf}\left[\frac{r}{\sqrt{2\zeta}}\right] \\ &+ r^3(r^2 - 3\zeta)F_2^2[\{1, 1\}, (2, 5/2); r^2/(2\zeta)]\Bigg\}\Bigg], \\ e\nabla^2 V_{\tilde{\mathcal{O}}}(r, j, 0) = &-\frac{Ze^2b^2}{64\pi^4}\Bigg[\tilde{B}_1(j)\frac{e^{-(r^2/2\zeta)}}{\zeta^{7/2}}(r^2 - 3\zeta) + \tilde{B}_2(j)\frac{e^{-(r^2/2\zeta)}}{r\zeta^{9/2}}\Bigg\{12r\zeta^2 - 3e^{r^2/2\zeta}\sqrt{2\pi}\zeta^{5/2}\mathrm{Erf}\left[\frac{r}{\sqrt{2\zeta}}\right] \\ &+ r^3(r^2 - 3\zeta)F_2^2[\{1, 1\}, (2, 5/2); r^2/(2\zeta)]\Bigg\}\Bigg], \end{aligned} \tag{30}$$

where $\mathrm{Erf}[z]$ is the error function defined as

$$\mathrm{Erf}[z] = \frac{2}{\sqrt{\pi}}\int_0^z dt e^{-t^2}.$$

The Laplacians of the unparticle Uehling potentials correspond to vacuum charge densities created by the unparticles

$$\rho_{\mathcal{O}}^{\mathrm{VP}}(r, j) = -\epsilon_0\nabla^2 V_{\mathcal{O}}(r, j, 0),$$
$$\rho_{\tilde{\mathcal{O}}}^{\mathrm{VP}}(r, j) = -\epsilon_0\nabla^2 V_{\tilde{\mathcal{O}}}(r, j, 0).$$

Since the potentials in Eq. (26) fall off faster than $1/r$ (as $r \to \infty$) the net, unparticle induced, vacuum charge vanishes trivially

$$\begin{aligned} Q_{\mathcal{U}}^{\mathrm{VP}} &= \int d^3r\rho_{\mathcal{U}}^{\mathrm{VP}}(r, j) = \int d^3r[-\epsilon_0\nabla^2 V_{\mathcal{U}}(r, j, 0)] \\ &= -\epsilon_0\lim_{R\to\infty}\oint_R d\vec{\sigma}\cdot\vec{\nabla}V_{\mathcal{U}}(r, j, 0) = 0 \end{aligned}$$

as we should expect.

Now from our analytical expressions it is observed, as mentioned above, that for typical values of the scaling dimension j and radial coordinate r, the first term in the unparticle Uehling potential dominates. It is seen from Eq. (30) that the first term changes sign at $r = \sqrt{3\zeta}$. Thus we may suspect that the sign of the Laplacian for the complete Uehling potential would also be dominated by the sign of the first term. Therefore we have for both the potentials

$$\begin{aligned} e\nabla^2 V_{\mathcal{O}}(r, j, 0) &\gtrless 0 \quad \forall\ r \lessgtr \sqrt{3\zeta}, \\ e\nabla^2 V_{\tilde{\mathcal{O}}}(r, j, 0) &\gtrless 0 \quad \forall\ r \lessgtr \sqrt{3\zeta}. \end{aligned} \tag{31}$$

These expectations are confirmed by explicit numerical computations where it is found that the Laplacian of the unparticle Uehling potential indeed changes sign at $\sqrt{3\zeta}$.

We are interested in the low-lying atomic states which will have relatively large energy shifts due to the unparticle Uehling corrections. The higher angular momentum $(p, d, \ldots)$ wave functions are spread out from the origin. Specifically, it is seen that they are mainly nonzero for $r > \sqrt{3\zeta}$ for semirealistic values of ζ that will model an atomic nucleus (for example, $\zeta \simeq 4\ \mathrm{fm}^2$ for the lead nucleus). From Eq. (29) we therefore note that due to this, the sign of $\langle \nabla^2 V_{\mathcal{U}}(r, j, \mu^2)\rangle_{n,l+1}$ will be dominated by the sign of the Laplacian in the region $r > \sqrt{3\zeta}$. Thus for the unparticle scalar and pseudoscalar Uehling potentials

$$\begin{aligned} \langle e\nabla^2 V_{\mathcal{O}}(r, j, \mu^2)\rangle_{n,l+1} &< 0, \\ \langle e\nabla^2 V_{\tilde{\mathcal{O}}}(r, j, \mu^2)\rangle_{n,l+1} &< 0. \end{aligned} \tag{32}$$

This implies that in the case of perfect scale invariance we have

$$\begin{aligned} \delta E^{\mathcal{O}}(n, l, j, 0) &< \delta E^{\mathcal{O}}(n, l+1, j, 0), \\ \delta E^{\tilde{\mathcal{O}}}(n, l, j, 0) &< \delta E^{\tilde{\mathcal{O}}}(n, l+1, j, 0). \end{aligned} \tag{33}$$

This would mean that if we were looking specifically at $l = 0$, 1, and 2 states near $r = 0$, the unparticle Uehling corrections would be ordered as

$$\begin{aligned} \delta E^{\mathcal{O}}_{n,s}(j) &< \delta E^{\mathcal{O}}_{n,p}(j) < \delta E^{\mathcal{O}}_{n,d}(j), \\ \delta E^{\tilde{\mathcal{O}}}_{n,s}(j) &< \delta E^{\tilde{\mathcal{O}}}_{n,p}(j) < \delta E^{\mathcal{O}}_{n,d}(j). \end{aligned}$$

We will explore the level structure of the unparticle Uehling shifts in the next section and confirm explicitly the general results above.

III. MUONIC ATOMS AND ENERGY SHIFTS

Muonic atoms are created when muons are stopped in matter [33]. The muon initially undergoes scattering and as it loses energy it may be captured by one of the atoms in a higher orbital. From here it cascades down, via various processes, to the innermost orbits where it persists until decay. The whole cascade process is expected to occur within $\sim 10^{-9}$–10^{-12} s and the muon spends $\sim 10^{-7}$–10^{-6} s in the inner orbits until decay. During cascade and its time in the innermost orbits we may probe the energy levels of the system (see [34] and references therein, for instance).

Muonic atoms are especially suited for probing the effects of oblique corrections to the photon propagator [33,34]. In the QED induced hydrogenic Lamb shift [35] the $2s_{1/2}$ state is found to lie above the $2p_{1/2}$ state by

$$\delta E^{\mathrm{total}}_{\mathrm{Lamb}} = E_{2s_{1/2}} - E_{2p_{1/2}} \simeq 1058\ \mathrm{MHz},$$

which corresponds to about 4.4×10^{-6} eV. The QED vacuum polarization contribution to the Lamb shift [36] on the other hand is found to be

$$\delta E^{\mathrm{VP}}_{\mathrm{Lamb}} = E_{2s_{1/2}} - E_{2p_{1/2}} \simeq -27.1\ \mathrm{MHz}.$$

For later comparison to the unparticle case we note that the above $e^+ - e^-$ vacuum polarization contribution is about 1×10^{-7} eV. Thus in ordinary atoms the vacuum polarization effects (oblique corrections) contribute less significantly than the other QED corrections. Things are dramatically different in the case of muonic atoms owing to the much higher mass of the muon compared to the electron. Now, the energy shift due to vacuum polarization may be related to the Uehling correction $\delta V(\vec{r})$ as

$$\delta E^{\mathrm{VP}}_{nl} = \int d^3 r \Psi^*_{nl}(\vec{r}) e \delta V(\vec{r}) \Psi_{nl}(\vec{r}). \tag{34}$$

We are specifically interested in the low-lying atomic states, and the S and P states are particularly attractive since the modification to the potential, as we saw previously, mostly occurs near $r = 0$ and falls off sharply as one goes away from the center. Taking the wave function $\Psi_{nl}(\vec{r})$, to lowest order, to be the Schrödinger wave function we note that near $r = 0$

$$|\Psi^{e^-}_{nl}|^2 \sim m^3_{e^-}.$$

Thus we expect that in muonic atoms, where an electron in the inner orbital is replaced by a muon, the energy shift due to oblique corrections would be enhanced as

$$\delta E^{\mu^-}_{nl} \simeq \left(\frac{m_{\mu^-}}{m_{e^-}}\right)^3 \delta E^{e^-}_{nl}.$$

Putting the masses in, $(m_{\mu^-}/m_{e^-})^3 \simeq 10^7$. This is a huge enhancement to the vacuum polarization contribution. In fact, it is found that in muonic hydrogen the $2s_{1/2}$ state lies below the $2p_{1/2}$ level with QED vacuum polarization contributing about 206 meV compared to about 0.6 meV from other QED corrections [34]. Another way to understand this enhancement is by looking at the wave function profiles in muonic atoms and noting that the wave function penetrates into the nucleus much more than in the electron case. In all our numerical computations we use the hydrogenic Schrödinger wave functions to lowest order, with m_{e^-} replaced everywhere with m_{μ^-} and taking the appropriate value for the atomic number Z. One must in principle use the bound state solutions to the Dirac equation but for the purposes of our estimations the Schrödinger wave functions are sufficient. One aspect that we would like to point out though is that the bound state solutions to the Dirac equation are more localized near $r = 0$ compared to the Schrödinger wave functions and hence will in general yield a slightly higher estimate for the corrections to the energy levels. Any error that comes from this simplification will be at most an $\mathcal{O}(1)$ factor and similar to our ignorance regarding the coefficients in the Lagrangian or the interaction scales. Thus all calculated energy shifts in our study must be interpreted as accurate only up to undetermined $\mathcal{O}(1)$ factors to accommodate our ignorance of the interaction scales, coefficients, and other approximations we have made.

The typical binding energy of an atom with atomic number Z goes as

$$E_{\rm bound} \sim Z^2.$$

But from Eq. (34) we see that

$$\delta E_{nl}^{\rm VP} \sim Z^4,$$

since

$$|\Psi_{nl}|^2 \sim Z^3, \qquad \delta V(\vec{r}) \sim Z.$$

Based on the above expressions we expect

$$\frac{\delta E_{nl}^{\rm VP}}{E_{\rm bound}} \sim Z^2.$$

This motivates the reason for choosing a high Z system while probing for the existence of tiny unparticle induced energy shifts.

Based on the above considerations two very favorable systems for probing unparticle vacuum polarization induced energy shifts are muonic mercury ($\mu^- - \mathrm{Hg}_{80}^{200}$) and muonic lead ($\mu^- - \mathrm{Pb}_{82}^{208}$). We will choose as our reference system muonic lead. With $Z_{\rm Pb} = 82$ and $A_{\rm Pb} = 208$, some of the relevant scales in the muonic-lead system are

$$m_{\mu^-} \simeq 0.1 \text{ GeV}, \qquad \lambda_{\rm Compton}^{\mu^-} \simeq 12 \text{ fm},$$
$$r_{\rm Bohr}^{\mu^-} \simeq 3 \text{ fm}, \qquad r_{\rm nucleus}^{\rm Pb} \approx R_0 A_{\rm Pb}^{1/3} \simeq 7 \text{ fm},$$

where $R_0 \simeq 1.2$ fm. From these crude estimates we see that the muonic Bohr radius in lead is much smaller than the extent of the lead nucleus or the muonic Compton wavelength and hence the muonic wave function will penetrate into the nucleus to a large extent compared to electronic wave functions. This results in an enhancement of the Uehling energy shift as we argued previously. It also implies that finite nuclear effects would be much more important in the low-lying states of muonic atoms, especially the muonic-lead system we are considering [37,38].

Another point to note is that for lead, with $Z = 82$, the parameter αZ used in perturbative QED calculations is no longer small. This makes the QED calculations more complicated and leads to larger theoretical uncertainties. Table I shows some of the QED corrections to the low-lying states of muonic lead. The interested reader is referred to theoretical details in [34] and references therein.

Similarly the various finite nuclear effects can be quantified to a large extent in muonic Pb_{82}^{208} (see [37–39], for example). In Table II we give the current estimates of nuclear polarization (NP) effects in $\mu^- - \mathrm{Pb}_{82}^{208}$ for three different models of transition densities as quoted in [39].

It was noted in the past that there is a discrepancy in the $\Delta 2p$ and $\Delta 3p$ NP calculations with results from muonic-lead spectroscopy [40,41]. This discrepancy seems to have been partially tackled in the work of A. Haga *et al.* [39] and the current discrepancy for $\Delta 2p$ is in the ballpark of about 50 eV as shown in Table III.

Historically the spectroscopic information from low-lying muonic-lead transitions was used to constrain parameters in the theoretical nuclear calculations. In Table IV the results of precision measurements on some of the low-lying transitions in muonic lead are shown. It is noted that the experimental uncertainties are typically of the order of a few tens of eV.

As suggested by Fig. 3 the states $1S$ and $2S$ are most affected by the unparticle Uehling potential. A transition from $1S$ to $2S$ is nevertheless forbidden due to the electric dipole selection rule $\Delta l = \pm 1$. That leaves the possibility of a transition to an $l = 1$ state that might still be sensitive

TABLE I. QED corrections, in eV, for the low-lying states in $\mu^- - \mathrm{Pb}_{82}^{208}$ (see [34,39] and references therein).

QED corrections	$1s_{1/2}$	$2s_{1/2}$	$2p_{1/2}$	$2p_{3/2}$	$3p_{1/2}$	$3p_{3/2}$	$3d_{3/2}$	$3d_{5/2}$
Electronic Uehling and Källén-Sabry	−67 864	−19 537	−32 648	−30 082	−10 871	−10 334	−10 605	−9941
Electronic Wichman-Kroll	492	244	348	335	160	160	186	180
Muonic Uehling corrections	−248	−43	−45	−34	−14	−11	−1	−1
Leading self-energy corrections	3220	696	348	649	149	224	−44	51
Higher-order self-energy corrections	153	25	65	58	21	20	8	6
Electron screening	−5	−25	−13	−13	−52	−54	−37	−39
Recoil correction	−382	−87	−111	−95	−30	−26	−15	−14

TABLE II. Estimates of the NP effects (eV) in $\mu^- - \mathrm{Pb}_{82}^{208}$ taken from [39]. The gauge dependences are shown in brackets.

Nuclear polarization corrections	$1s_{1/2}$	$2s_{1/2}$	$2p_{1/2}$	$2p_{3/2}$	$3p_{1/2}$	$3p_{3/2}$	$3d_{3/2}$	$3d_{5/2}$
TGT model	−2727(4)	−463(1)	−1357(7)	−1425(9)	−561(4)	−749(1)	−226(0)	−43(0)
RIN model	−3599(10)	−611(4)	−1590(10)	−1656(10)	−690(3)	−914(1)	−239(0)	−42(0)
JS model	−5721(28)	−930(8)	−2178(13)	−2214(7)	−929(3)	−1179(2)	−280(0)	−38(0)

TABLE III. Comparison of Δp splittings (keV) in $\mu^- - \mathrm{Pb}_{82}^{208}$ [39] based on the QED results in Table I and NP calculations in Table II.

$\mu^- - \mathrm{Pb}_{82}^{208}$ level	TGT	RIN	JS	Exp.
$2p_{3/2} - 2p_{1/2}$	184.858	184.846	184.829	184.788(27)
$3p_{3/2} - 3p_{1/2}$	47.231	47.208	47.225	47.197(45)

TABLE IV. Precision measurements on some of the low-lying $\mu^- - \mathrm{Pb}_{82}^{208}$ transitions [40,41].

Muonic transition	Energy[a] (keV)	Energy[b] (keV)
$2p_{3/2} \leftrightarrow 1s_{1/2}$	5962.770(420)	5962.854(90)
$2p_{1/2} \leftrightarrow 1s_{1/2}$	5777.910(400)	5778.058(100)
$3d_{3/2} \leftrightarrow 2p_{1/2}$	2642.110(60)	2642.332(30)
$3d_{5/2} \leftrightarrow 2p_{3/2}$	2500.330(60)	2500.590(60)
$3d_{3/2} \leftrightarrow 2p_{3/2}$	2457.200(200)	2457.569(70)
$3p_{3/2} \leftrightarrow 2s_{1/2}$	1507.480(260)	1507.754(70)
$3p_{1/2} \leftrightarrow 2s_{1/2}$	···	1460.558(32)
$2s_{1/2} \leftrightarrow 2p_{1/2}$	1215.430(260)	1215.330(30)
$2s_{1/2} \leftrightarrow 2p_{3/2}$	1030.440(170)	1030.543(27)

[a]D. Kessler *et al.* (1975).
[b]P. Bergam *et al.* (1988).

to the unparticle Uehling effect. We will consider, as the prototypical low-lying muonic transition, excitations between $1S$ and $2P$. Based on a direct fit to muonic-lead transition energy data (including higher level transitions), keeping the variables in the Fermi distribution as free parameters, one may obtain an estimate for the discrepancy between theory and experiment in the $1S - 2P$ transition in $\mu^- - \mathrm{Pb}_{82}^{208}$ [41] as

$$\begin{aligned} &\Delta E_{\mathrm{Exp}}[1s_{1/2} - 2p_{1/2}] - \Delta E_{\mathrm{Calc/Fit}}[1s_{1/2} - 2p_{1/2}] \\ &\quad \simeq 227 \text{ eV}, \\ &\Delta E_{\mathrm{Exp}}[1s_{1/2} - 2p_{3/2}] - \Delta E_{\mathrm{Calc/Fit}}[1s_{1/2} - 2p_{3/2}] \\ &\quad \simeq -89 \text{ eV}. \end{aligned} \tag{35}$$

This estimate is a very conservative one based on a direct fit [41] with a very poor $\chi^2/\mathrm{d.o.f} = 187$. We will take the above estimate as a crude measure of the discrepancy between measurement and calculation. It should be noted also that the NP values used in Eq. (35) are obtained keeping the variables in the Fermi distribution as free parameters and are therefore not completely theoretical.

We will see shortly that, for typical values of the model parameters, this makes the identification of any possible unparticle Uehling corrections in $\mu^- - \mathrm{Pb}_{82}^{208}$ extremely difficult since the magnitude of any such correction will be found to be well below the NP uncertainties. This would imply that any unparticle Uehling correction if present may not probably be unambiguously seen due to the data-fitting procedure required to extract information for the theoretical nuclear calculation in $\mu^- - \mathrm{Pb}_{82}^{208}$, unless we can obtain such information independently.

There are two scenarios where these conclusions could get modified. The first, as we commented on earlier in Sec. I, is if the UV sector has a very large fermion multiplicity (large N_f) which increases the total contribution from the fermiophobic unparticle sector. The second is muonic atoms with intermediate Z which may be more amenable to nuclear polarization calculations.

We now proceed to analyze the $1S - 2P$ level corrections in various cases of interest.

Let us first consider the case of ordinary scalars (ϕ) and pseudoscalars ($\tilde{\phi}$). The relevant corrections to the electromagnetic potential are obtained from expressions in Eq. (14). In this case we find that the energy shift due to scalar oblique corrections is

$$\begin{aligned} \delta E_{nl}^{\phi} &= \int d^3r |\Psi_{nl}^{\mu^-}(\vec{r})|^2 e\delta V_{\phi}(\vec{r}) \\ &= -\frac{Z_{\mathrm{Pb}} e^2 a^2}{16\pi^2 \Lambda_{\phi}^2} \int d^3r |\Psi_{nl}^{\mu^-}(\vec{r})|^2 \\ &\quad \times \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} (\vec{q}^2)^{-1} \tilde{f}(\vec{q}) \\ &\quad \times \int_0^1 dx \{\Delta'(q^2, m_{\phi}^2) - \vec{q}^2(x-1)^2\} \log \frac{\Delta'(q^2, m_{\phi}^2)}{M^2}, \end{aligned} \tag{36}$$

where

$$\Delta'(q^2, m_{\phi}^2) = x(x-1)q^2 + (1-x)m_{\phi}^2$$

and m_{ϕ} is the mass of the scalar as before.

For the pseudoscalar case similarly

$$\begin{aligned} \delta E_{nl}^{\tilde{\phi}} &= +\frac{Z_{\mathrm{Pb}} e^2 \tilde{a}^2}{16\pi^2 \Lambda_{\tilde{\phi}}^2} \int d^3r |\Psi_{nl}^{\mu^-}(\vec{r})|^2 \\ &\quad \times \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} (\vec{q}^2)^{-1} \tilde{f}(\vec{q}) \\ &\quad \times \int_0^1 dx \Delta''(q^2, m_{\tilde{\phi}}^2) \log \frac{\Delta''(q^2, m_{\tilde{\phi}}^2)}{M^2} \end{aligned} \tag{37}$$

and

$$\Delta''(q^2, m_{\tilde{\phi}}^2) = x(x-1)q^2 + (1-x)m_{\tilde{\phi}}^2.$$

$m_{\tilde{\phi}}$ denotes the mass of the pseudoscalar. Let us consider the case when $m_{\phi} = 0$ and $m_{\tilde{\phi}} = 0$. With this choice the above expressions may be evaluated numerically to calculate the energy shift in the muonic-lead $1S - 2P$ transition. We take a, $\tilde{a} \simeq 2$ and adopt the reference value $\Lambda_{\Phi} \sim 246$ GeV with a renormalization scale $M \simeq \Lambda_{\Phi}$. With these parameters the magnitude of the energy shift in the $1S - 2P$ transition for $m_{\Phi} = 0$ is estimated to be

$$\Delta E^{\Phi}_{2p-1s} = |\delta E^{\phi}_{2p} - \delta E^{\phi}_{1s}| \simeq |\delta E^{\tilde{\phi}}_{2p} - \delta E^{\tilde{\phi}}_{1s}| \simeq 0.140 \text{ eV} \sim \mathcal{O}(0.1) \text{ eV}. \quad (38)$$

It is found that the correction to the transition, from the SM value, is positive for both scalars and pseudoscalars. The above estimate may be enhanced due to a large multiplicity in the UV fermion sector and is not uncommon in many string-inspired QCD-like models [21] as we commented earlier.

Let us consider the same calculation for an unparticle scalar/pseudoscalar of scaling dimension j. We parametrize j as a deviation from the ordinary scalar/pseudoscalar case by putting

$$j = 1 + \eta.$$

As argued in Sec. I we require $0 \leq \eta < 1$. Then the energy shifts due to the scalar and pseudoscalar unparticle oblique corrections are, respectively,

$$\delta E^{\mathcal{O}}_{nl}(j) = \frac{Z_{\rm Pb} e^2 c^2 A_j}{2\Lambda_\gamma^{2j}\sin(j\pi)} \int d^3r |\Psi^{\mu^-}_{nl}(\vec{r})|^2 \times \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} (\vec{q}^2)^{-1} \tilde{f}(\vec{q}) \times \int dx dy \delta(x+y-1) y^{1-j} \frac{(M^2)^{j-1}}{(4\pi)^2} \times \left[-\vec{q}^2 (x-1)^2 \log\Delta(q^2,\mu^2)/M^2 + \Delta(q^2,\mu^2) \times \frac{\log\Delta(q^2,\mu^2)/M^2}{(1+\eta)}\right], \quad (39)$$

$$\delta E^{\tilde{\mathcal{O}}}_{nl}(j) = -\frac{Z_{\rm Pb} e^2 b^2 A_j}{2\tilde{\Lambda}_\gamma^{2j}\sin(j\pi)} \int d^3r |\Psi^{\mu^-}_{nl}(\vec{r})|^2 \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} (\vec{q}^2)^{-1} \tilde{f}(\vec{q}) \times \int dx dy \delta(x+y-1) y^{1-j} \frac{(M^2)^{j-1}}{(4\pi)^2} \times \left[\tilde{\Delta}(q^2,\mu^2) \frac{\log\tilde{\Delta}(q^2,\mu^2)/M^2}{(1+\eta)}\right], \quad (40)$$

where $\Delta(q^2, \mu^2)$ and $\tilde{\Delta}(q^2, \mu^2)$ are as defined earlier in Sec. II.

The energy shift in the $1S - 2P$ transition in muonic lead may now be calculated as

$$\Delta E^{\mathcal{O}}_{2p-1s} = \delta E^{\mathcal{O}}_{2p} - \delta E^{\mathcal{O}}_{1s}, \qquad \Delta E^{\tilde{\mathcal{O}}}_{2p-1s} = \delta E^{\tilde{\mathcal{O}}}_{2p} - \delta E^{\tilde{\mathcal{O}}}_{1s},$$

using Eqs. (39) and (40) with the nuclear profile of Eq. (23). The results of the numerical computation, for $\mu \longrightarrow 0$, are shown in Fig. 4. We observe from the plot that the energy shift due to pseudoscalar unparticle vacuum polarization is generally smaller in magnitude than the corresponding scalar case assuming other parameters are identical. This is probably an artifact of our approximation in Eq. (20) as we commented previously. As one approaches $j \longrightarrow 1^+$ though the energy shifts due to scalar and pseudoscalar cases start to approach the same value of $\Delta E^{\mathcal{U}}_{2p-1s} \simeq 0.14$ eV as expected. Note also that in the $j \longrightarrow 1^+$ limit the ordinary scalar/pseudoscalar case is recovered in a continuous manner.

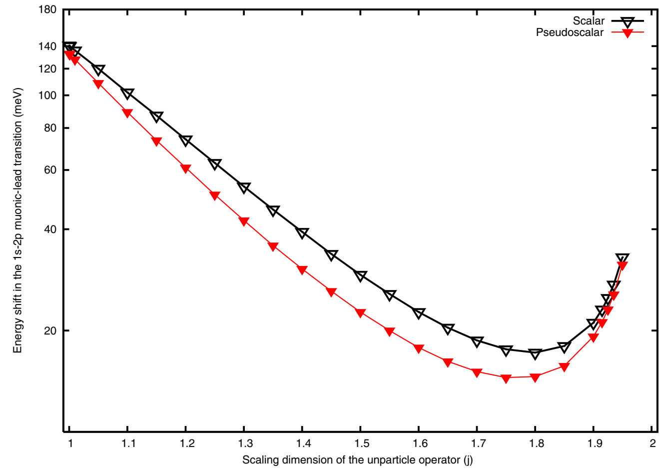

FIG. 4 (color online). Estimates on the magnitude of the energy shift $|\delta E^{\mathcal{U}}_{2p} - \delta E^{\mathcal{U}}_{1s}|$ due to possible scalar/pseudoscalar unparticle vacuum polarizations. We have taken $\mu \simeq 0$, $b = c \sim \mathcal{O}(1)$ and Λ_γ, $\tilde{\Lambda}_\gamma \sim 246$ GeV as the reference value at the renormalization scale $M \simeq \Lambda_{\mathcal{U}} \simeq \Lambda_\gamma$, $\tilde{\Lambda}_\gamma$. As we have mentioned previously the quoted values for the Uehling shifts should be interpreted as being accurate only up to $\mathcal{O}(1)$ factors to accommodate suitable ranges for the interaction scales and other approximations. It is clear from the plot that the Uehling shifts from Eqs. (39) and (40) can under very general scenarios be in the ballpark of a few times 0.1 eV.

Therefore, it may be claimed that for $\mu \longrightarrow 0$ the typical shift in the $1S - 2P$ muonic-lead transition due to the unparticle Uehling potential is in the $\mathcal{O}(0.1)$ eV range with reasonable assumptions about the parameters in the theory. This energy shift, if it exists, is similar in magnitude to QED corrections from the virtual Delbrück effect (light-by-light scattering), to higher angular momentum transitions, in $\mu^- - \mathrm{Pb}^{208}_{82}$ [34]. For example, the virtual Delbrück effect contributes

$$\Delta E^{\rm Delb}_{6h-5g} \simeq 0.4 \text{ eV}$$

in $\mu^- - \mathrm{Pb}^{208}_{82}$. The unparticle induced energy shift may also be compared to the vacuum polarization contribution in the hydrogenic Lamb shift which is in the 10^{-7} eV range (ordinary hydrogen) or eV range (muonic hydrogen). From Eq. (35) we see nevertheless that the discrepancy between measurement and theory of the $2p_{1/2} - 1s_{1/2}$ and $2p_{3/2} - 1s_{1/2}$ muonic-lead transitions is in the range of many eV [40,41]. This means that the estimated energy shift is about 10^2–10^3 times smaller than the discrepancies quoted in Eq. (35). We will analyze a more realistic scenario with

$\mu \neq 0$ later and will note then that the main features are not drastically modified from the simplistic $\mu \to 0$ case.

It is also seen from the plot that the energy shift is very nearly linear, with respect to the scaling dimension j, for values of j close to 1. In fact it may be verified analytically from Eqs. (36), (37), (39), and (40) that for $j = 1 + \eta$ with $\eta \ll 1$ we may expand

$$\delta E^{\Phi}_{nl} - \delta E^{\mathcal{U}}_{nl}(j) \sim C_1\eta + C_2\eta^2 + C_3\eta^3 + \cdots,$$

where dimensionful factors have been omitted and C_n are numerical coefficients. This implies that very close to $j = 1$ we have to leading order

$$\delta E^{\phi}_{nl} - \delta E^{\mathcal{U}}_{nl}(j) \propto \eta$$

as suggested by the plot and Eq. (20). The above expression indicates that close to $j = 1$ the additional energy shift in the $1S - 2P$ transition compared to the ordinary scalar/pseudoscalar case comes from the fractional scaling dimension part (η) of the unparticle. This seems very satisfactory in the sense that any correction to the ordinary scalar/pseudoscalar case is just proportional to the "unscalarness" of the unparticle. This seems to be in the spirit of deconstruction [8] and is similar to the observation in [42]

$$\sigma_{\mathcal{U}}(j) = (2 - j)\sigma_{\phi}(j \to 1)$$

for the unparticle production cross section when the unparticle sector is gauged under the SM.

One also observes from Fig. 4 the onset of singular behavior as $j \to 2$ as discussed in Sec. I. As pointed out this is a pathology arising from the fact that close to $j = 2$ the model becomes more and more UV sensitive. One may try to mitigate this singular behavior near $j = 2$ by adding local contact terms [6].

Concerning the sign of the unparticle energy shifts it is inferred that there are no differences between the scalar and pseudoscalar unparticles. It found from our expressions in (39) and (40) that the scalar as well as the pseudoscalar unparticle corrections to the $1S - 2P$ transitions are positive

$$\begin{aligned} \Delta E^{\text{total}}_{2p-1s} &\simeq \Delta E^{\text{SM}}_{2p-1s} + |\Delta E^{\mathcal{O}}_{2p-1s}|, \\ \Delta E'^{\text{total}}_{2p-1s} &\simeq \Delta E^{\text{SM}}_{2p-1s} + |\Delta E^{\tilde{\mathcal{O}}}_{2p-1s}|. \end{aligned} \tag{41}$$

In the present case, for $\mu \to 0$, from Eqs. (32) and (33), we would expect the level structure of the Uehling shifts to be ordered as

$$\begin{aligned} &\Delta E^{\mathcal{U}}_{2s-1s} < \Delta E^{\mathcal{U}}_{2p-1s}, \\ &\Delta E^{\mathcal{U}}_{3s-1s} < \Delta E^{\mathcal{U}}_{3p-1s} < \Delta E^{\mathcal{U}}_{3d-1s}. \end{aligned}$$

These expectations from Eqs. (32) and (33) are confirmed by numerical calculations. The variations between the $l = 0$, 1, and 2 states are generally found to be of the order of a few 0.01 meV for $n = 3$ and of the order of a few meV for $n = 2$.

Let us now turn our attention to the case of broken scale invariance where $\mu \neq 0$. Even in the case of a fermiophobic unparticle an effective interaction term with Higgs fields of the form

$$\Lambda_h^{2-j}\mathcal{U}\mathcal{H}^2$$

is expected to be present. Here $\mathcal{H}$ stands for a generic Higgs field. As pointed out in the literature this interaction term is unique [15,43] in the sense that it is a superrenormalizable term (since $1 \leq j < 2$ from Mack's unitarity). Even if one starts with a near zero coupling of this term, at a high energy, it will be generated through renormalization group flow and will break the conformal invariance of the unparticle sector, at a scale μ, when the Higgs develops a vacuum expectation value. Thus there is a very strong theoretical reason to expect scale invariance, in the fermiophobic unparticle sector, to be broken at some scale after electroweak-symmetry breaking.

Moreover, from the observational viewpoint, it is expected that there should be very stringent constraints on massless/light scalar degrees of freedom from cosmology and astrophysics. For a scalar/pseudoscalar unparticle coupling to two photons with $\mu \to 0$ (i.e. perfect scale invariance) there are very strong bounds on the couplings from supernovae cooling [19,44,45], for example. Also, in the case of perfect scale invariance, there are very tight bounds on the couplings from big-bang nucleosynthesis (BBN) [45,46]. These constraints effectively render any collider or low-energy experiments ineffective in probing the unparticle sector.

We would therefore like to estimate a lower bound on the scale invariance breaking effective mass (μ), for the fermiophobic unparticle, that would let us evade some of these constraints. So, assuming $\mu \neq 0$, let us first look at the typical Primakoff process with a fermiophobic unparticle that could potentially contribute to supernovae (SN) cooling

$$\gamma(k_1) + \gamma(k_2) \overset{\mu \neq 0}{\longrightarrow} \mathcal{O}(p).$$

Here $\mathcal{O}$ is a fermiophobic scalar unparticle and one of the photons is sometimes assumed to be off shell. We are only interested in estimating a lower bound for the scale breaking parameter μ that would not violate supernovae constraints. Thus for simplicity we assume that the photons are transverse and ignore any plasmon effects (where the photon gets a longitudinal polarization in the plasma). Using

$$|\langle 0|\mathcal{O}_{\mathcal{U}}(0)|P\rangle|^2\rho(P^2) = A_j\Theta(P^0)\Theta(P^2 - \mu^2)(P^2 - \mu^2)^{j-2}$$

and the Feynman rule from Eq. (4) we may compute the invariant amplitude to get

$$|\mathcal{M}_{\mu\neq 0}(\gamma+\gamma\to\mathcal{O})|^2 = \frac{c^2 A_j}{4\Lambda_\gamma^{2j}}\Theta(p^0)\Theta(s-\mu^2) \times s^2(s-\mu^2)^{j-2},$$

where $s = (k_1 + k_2)^2 = p^2$ and Θ is the Heaviside function. From the above amplitude we may estimate the unparticle Primakoff cross section to be

$$\sigma_{\mu\neq 0}(\gamma+\gamma\to\mathcal{O}) \simeq \frac{c^2 A_j}{8\Lambda_\gamma^{2j}} s(s-\mu^2)^{j-2}.$$

Using the above expression, the emissivity may be calculated from

$$\dot{\epsilon}_{\mu\neq 0}(\gamma+\gamma\to\mathcal{O}) \simeq \frac{\langle n_\gamma^1 n_\gamma^2 \sigma_{\mu\neq 0}(\gamma+\gamma\to\mathcal{U}) v_{\rm rel} E_{\rm CM}\rangle}{\rho_{\rm SN}}$$

and gives

$$\dot{\epsilon}_{\mu\neq 0}(\gamma+\gamma\to\mathcal{O}) \simeq \frac{1}{\rho_{\rm SN}}\int\frac{d^3\vec{k}_1}{(2\pi)^3}\frac{2}{e^{E_1/kT_{\rm SN}}-1}\int\frac{d^3\vec{k}_2}{(2\pi)^3} \times \frac{2}{e^{E_2/kT_{\rm SN}}-1}\frac{s(E_1+E_2)}{2E_1E_2}\frac{c^2A_j}{8\Lambda_\gamma^{2j}} \times s(s-\mu^2)^{j-2}, \quad (42)$$

where $\rho_{\rm SN}$ is the average supernovae core density and $T_{\rm SN}$ is the mean core temperature in the supernovae. We define in the usual way the dimensionless variables $x_1 = E_1/kT_{\rm SN}$ and $x_2 = E_2/kT_{\rm SN}$ to write the emissivity as

$$\dot{\epsilon}_{\mu\neq 0}(\gamma+\gamma\to\mathcal{O}) \simeq \frac{c^2 A_j T_{\rm SN}^{2j+5}}{16\pi^4\Lambda_\gamma^{2j}\rho_{\rm SN}}\int_0^\infty dx_1 \times \int_{\mu^2/4T_{\rm SN}^2 x_1}^\infty dx_2 \frac{x_1x_2(x_1+x_2)^{2j+1}}{(e^{x_1}-1)(e^{x_2}-1)} \times\left(1-\frac{\mu^2}{T_{\rm SN}^2(x_1+x_2)^2}\right)^{j-2}. \quad (43)$$

In the limit of perfect scale invariance, $\mu\to 0$, we recover from the above the expressions in [19,44]. Equation (43) may be compared with the fermiophilic case of a vector unparticle ($\mathcal{U}_V$), with $\mu\neq 0$, coupling to fermions. In this case a process such as $\nu\nu\to\mathcal{U}_V$ in the supernovae core can lead to cooling and the emissivity is given by [16]

$$\dot{\epsilon}_{\mu\neq 0}(\nu+\nu\to\mathcal{U}_V) \simeq \frac{g_\nu^2 A_j T_{\rm SN}^{2j+3}}{16\pi^4\Lambda_\psi^{2j-2}\rho_{\rm SN}}\int_0^\infty dx_1 \times\int_{\mu_V^2/4T_{\rm SN}^2x_1}^\infty dx_2\frac{(4x_1x_2)^j(x_1+x_2)}{(e^{x_1}+1)(e^{x_2}+1)} \times\left(1-\frac{\mu_V^2}{4T_{\rm SN}^2x_1x_2}\right)^{j-2}.$$

Note that the power law dependence of $T_{\rm SN}$ on the scaling dimension j is very different, between the fermiophobic and fermiophilic cases, even if one assumes that the other parameters $\mu\simeq\mu_V$, $\Lambda_\gamma\simeq\Lambda_\psi$ and $\mathcal{O}(1)$ coefficients $c\simeq g_\nu$. But we shall see below that the bound on the scale breaking parameter comes out to be of the same order of magnitude in both the cases due to the dominance of the Boltzmann suppression factor.

To be consistent with supernovae models and constraints from SN1987A we require [47]

$$\dot{\epsilon}_{\rm SN} \lesssim 10^{15}\ {\rm J/kg\,s}.$$

Equation (43) may be integrated and expressed in a compact form in the limit of $\mu\gg T_{\rm SN}$ with the Boltzmann suppression term factored out. An explicit numerical computation with the standard values

$$T_{\rm SN}\simeq 30\ {\rm MeV}, \qquad \rho_{\rm SN}\simeq 10^{18}\ {\rm kg/m^3},$$

and the above emissivity criterion gives

$$\mu \gtrsim 1.25\ {\rm GeV}. \quad (44)$$

This lower bound for the effective mass, of a fermiophobic scalar/pseudoscalar unparticle coupling only to photons, is found to be very close to the result in the fermiophilic case $\nu+\nu\to\mathcal{U}_V$, where it was found that [16]

$$\mu_V \gtrsim 1\ {\rm GeV}.$$

Based on the bound in Eq. (44) we may take a minimal value for the effective unparticle mass to be $\mu\simeq 2$ GeV. This choice already leads to an emissivity well below the allowed limit. A crude upper bound on the scale breaking parameter is given by the requirement that $\mu\ll\Lambda_{\mathcal{U}}$. Since we have adopted the prejudice that the unparticle scale is probably close to the electroweak-symmetry breaking scale this implies that in our model we are assuming $\mu\ll\mathcal{O}(1)$ TeV.

Let us now turn our attention to some of the constraints from cosmology. For the fermiophobic unparticle sector not to interfere with BBN, we must require that the unparticles stay decoupled during that epoch [45]. In the case of a fermiophobic unparticle with $\mu\to 0$ the condition $\Gamma_{\rm SM\to\mathcal{O}} < H$ (during the radiation dominated era), by simple dimensional analysis, becomes

$$\Gamma_{\gamma\gamma\mathcal{O}} \sim \frac{c^2}{\Lambda_\gamma^{2j}}T^{2j+1} < \left(\frac{T^2}{10^{18}\ {\rm GeV}}\right).$$

This implies that in the range $j\in[1,2)$ we are considering, the rate $\Gamma_{\gamma\gamma\mathcal{O}}$ redshifts faster than the Hubble parameter (H). It is required that the fermiophobic unparticles decouple before BBN and not get reheated during SM phase transitions to satisfy $\rho_{\mathcal{U}}\ll\rho_{\rm SM}$. This may be achieved by requiring that decoupling happen before the QCD phase transition at an energy $T\gtrsim 1$ GeV [45]. Note that a fermiophobic unparticle sector, coupling to photons, does not generally recouple after BBN with SM fields because $\Gamma_{\gamma\gamma\mathcal{O}}$ redshifts faster than the Hubble parameter. This may again be contrasted with the fermiophilic case of a vector unparticle coupling to fermions through the effective operator

$$\frac{g_f}{\Lambda_\psi^{j-1}}\bar{\psi}\gamma_\mu\psi\mathcal{O}_V^\mu,$$

where recoupling is possible when $1 \leq j \leq 3/2$ [45].

When scale invariance is broken ($\mu \neq 0$), since $\Gamma_{\mathcal{O}} \sim n_{\rm EQ}\langle\sigma|v|\rangle$, the relevant processes are Boltzmann suppressed by factors of $e^{-\mu/T}$ when $\mu > T$. Thus the BBN constraints can be evaded as long as μ is above $\Lambda_{\rm QCD}$ [16]. We must point out that we have tacitly assumed that the fermiophobic unparticles can decay into SM fields [48] when μ is sufficiently nonzero and are therefore not stable. There has been some discussion in the literature on this particular issue [8,46,48]. All the arguments above suggest that while calculating the Uehling shifts with $\mu \neq 0$ we must also consider the possibility that μ is higher than the minimal value of 2 GeV. Thus without loss of generality we will consider a range

$$2\ {\rm GeV} \lesssim \mu \ll M_Z \simeq 91\ {\rm GeV}$$

for both the scalar and pseudoscalar unparticles such that there is still a substantial conformal window $(\mu, \Lambda_{\mathcal{U}}]$. The above choice would also ensure that any modification to the gauge kinetic term ($\Delta\alpha^{-1}$) near the scale μ is within experimental limits [12] and that the effects due to SM Higgs-unparticle mixing, if present, are suppressed [16]. We will demonstrate in a short while that the actual energy shifts are relatively insensitive to small shifts in the μ parameter.

The pseudoscalar unparticle potential energies for two choices of the scale breaking parameter μ are shown in Fig. 5. This figure is to be compared with that in the case of perfect scale invariance illustrated in Fig. 3. It is noted that the potential in the $\mu \neq 0$ case has been suppressed by $\mathcal{O}(1)$ factors compared to the $\mu \to 0$ case. The general features of the unparticle Uehling potential nevertheless remain unchanged between the $\mu \to 0$ and $\mu \neq 0$ scenarios.

Let us now calculate the energy shifts in the muonic-lead transitions when $\mu \neq 0$. Consider the pseudoscalar unparticle with scale invariance broken. We may rewrite Eq. (40) again by defining the dimensionless variable $z = |\vec{q}|r$. The expression becomes after simplification

$$\delta E_{nl}^{\tilde{\mathcal{O}}}(j) = -\frac{Z_{\rm Pb}e^2b^2A_j\mu^2}{64\pi^4\tilde{\Lambda}_\gamma^{2j}\sin(j\pi)}\int d^3r|\Psi_{nl}^{\mu^-}(\vec{r})|^2 \times \int_0^1 dx(1-x)^{2-j}\frac{(M^2)^{j-1}}{jr}\int_0^\infty dz\frac{\sin z}{z} \times e^{-\zeta z^2/(2r^2)}\Big[\Big(1+\frac{xz^2}{\mu^2r^2}\Big)\log\Big\{1+\frac{xz^2}{\mu^2r^2}\Big\} + \frac{xz^2}{\mu^2r^2}\log\Big\{\frac{(1-x)\mu^2}{M^2}\Big\}\Big], \tag{45}$$

where we have made an oversubtraction at $q = 0$ to get a consistent $q \to 0$ limit. From the above expression we expect that for any fixed value of the radial coordinate (r) the dominant contribution to the z integral should come from the region of integration with

$$z^2 \lesssim \frac{2r^2}{\zeta} \tag{46}$$

or its vicinity. For the muonic-lead system we are primarily interested in, a suitable value of the nuclear charge density parameter ζ, in Eq. (23), was found to be $\zeta \simeq 4\ {\rm fm}^2$. We found that a minimal value of μ satisfying astrophysical constraints is $\mu \simeq 2\ {\rm GeV} \equiv 10.14\ {\rm fm}^{-1}$. From these observations we note therefore that for a fixed value of r

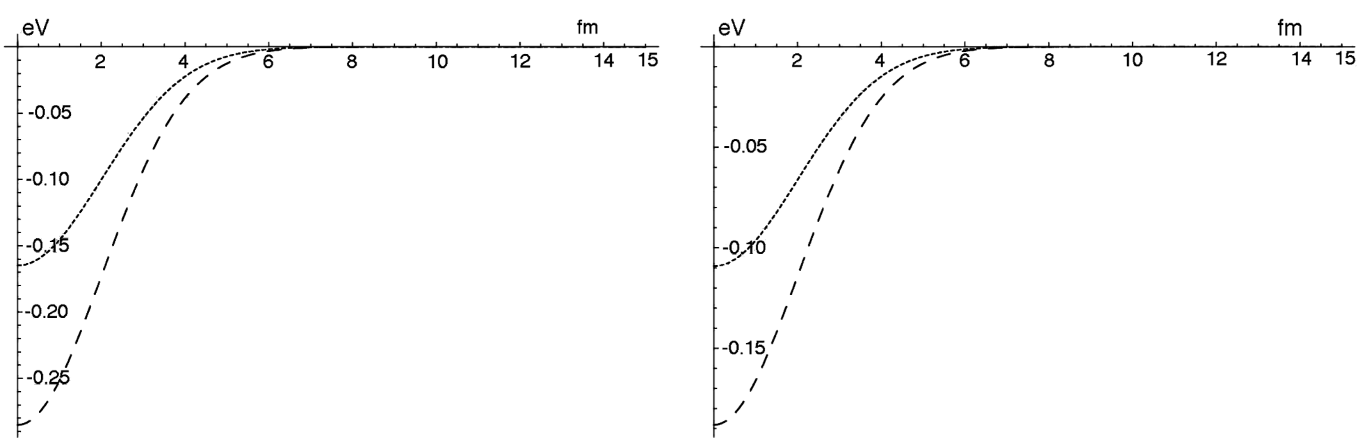

FIG. 5. The unparticle Uehling potential energies for the pseudoscalar case with the same parameters as Fig. 3 except assuming μ is 2 GeV (left panel) and 10 GeV (right panel). The $j = 1.01$ (dashed lines) and 1.15 (short-dashed lines) potential energies are again illustrated as functions of r. Note that the magnitude of the potential at any specific value of r has been reduced in both cases compared to the case of perfect scale invariance $\mu \to 0$.

$$xz^2 \lesssim \frac{2r^2}{\zeta} \ll \mu^2 r^2.$$

This means that in the relevant region of the parameter space we may expand the logarithm in Eq. (45) as

$$\log\left\{1 + \frac{xz^2}{\mu^2 r^2}\right\} \simeq \frac{xz^2}{\mu^2 r^2} - \frac{1}{2}\left(\frac{xz^2}{\mu^2 r^2}\right)^2 + \cdots$$

to get an integral that is separable in x and z variables. Note that this approximation becomes more and more accurate as we raise the value of μ. The variable separable integral may now be evaluated analytically or numerically to calculate the expression in Eq. (45). The calculated energy shifts to the $1S - 2P$ transition for various values of the scaling dimension j and μ are shown in the left panel of Fig. 6. It is noted that compared to the $\mu \to 0$ case the energy shifts are smaller, assuming all other parameters remain the same, by $\mathcal{O}(1)$ factors. Thus we find that incorporating broken scale invariance with a nonzero value of μ does not seem to alter the energy shifts drastically from their $\mu \to 0$ values and the changes are only by whole number factors. Once again, maintaining continuity, the ordinary pseudoscalar case is recovered as j approaches 1 due to Eq. (20). If μ is very large and the corresponding energy shifts very small, then the finite contributions from the higher-order counterterms in the effective Lagrangian may become important and the numerical approximation we adopt, of keeping only the lowest order terms from (4) and (5), may break down.

In the scalar unparticle case the expression for the energy shift becomes

$$\begin{aligned}\delta E^{\mathcal{O}}_{nl}(j) = {} & \frac{Z_{\rm Pb} e^2 c^2 A_j \mu^2}{64\pi^4 \Lambda_\gamma^{2j} \sin(j\pi)} \int d^3r |\Psi^{\mu^-}_{nl}(\vec{r})|^2 \\ & \times \int_0^1 dx (1-x)^{2-j} \frac{(M^2)^{j-1}}{jr} \\ & \times \int_0^\infty dz \frac{\sin z}{z} e^{-\zeta z^2/(2r^2)} \\ & \times \left[\left(1 + ((1+j)x - j)\frac{z^2}{\mu^2 r^2}\right)\log\left\{1 + \frac{xz^2}{\mu^2 r^2}\right\}\right. \\ & \left. + ((1+j)x - j)\frac{z^2}{\mu^2 r^2}\log\left\{\frac{(1-x)\mu^2}{M^2}\right\}\right] \qquad (47)\end{aligned}$$

after an oversubtraction at $q = 0$. In the relevant region where $xz^2 \lesssim 2r^2/\zeta \ll \mu^2 r^2$ we may again simplify the integral and compute it to obtain the energy shifts. The calculated values are shown in Fig. 6, right panel, for various values of the μ parameter. The limit of $j \to 1^+$, as before, corresponds to the ordinary scalar case. Again, the correction from the scalar unparticle vacuum polarization is found to be larger in magnitude than the corresponding pseudoscalar unparticle case. But this may, as before, be an artifact of our approximation in Eq. (20). The corrections are both positive as in the $\mu \to 0$ case.

Once again we observe that the variation of the scalar unparticle Uehling shift with μ is only by $\mathcal{O}(1)$ factors. This implies that the energy shift is relatively insensitive to changes of the scale breaking parameter, across a wide range, in both the scalar and pseudoscalar cases. More specifically the Uehling shifts are relatively unchanged even for $\mu \gg m_{\mu^-} \simeq 0.1$ GeV. In this sense the unparticle

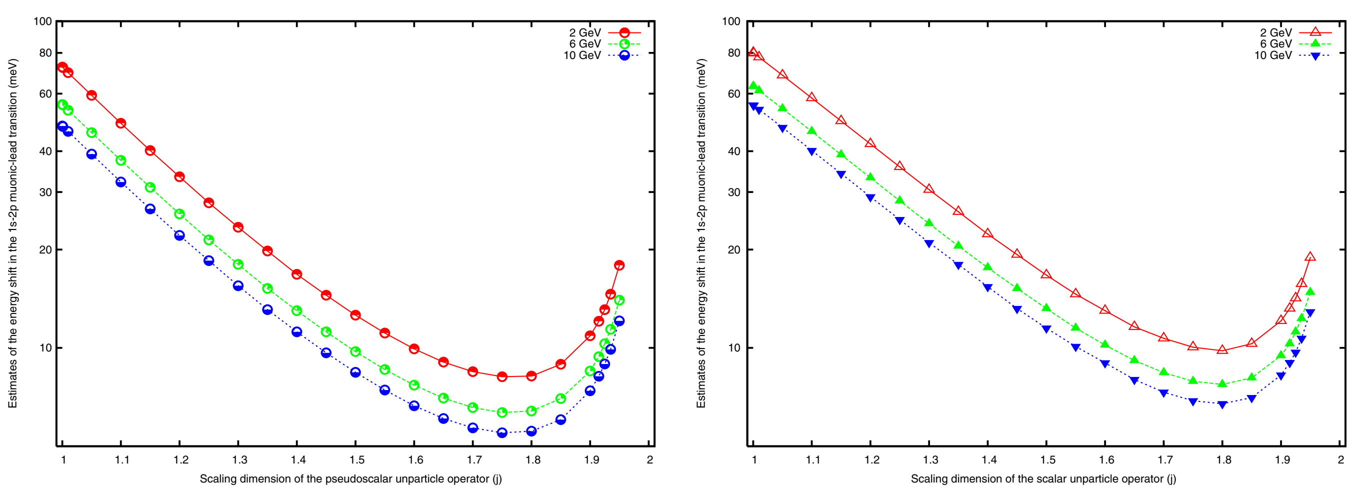

FIG. 6 (color online). Magnitude estimates of the energy shift in the $1S - 2P$ muonic-lead transition due to pseudoscalar (left panel) and scalar (right panel) unparticle vacuum polarizations. The plots are for $\mu \simeq 2$, 6, and 10 GeV. With these choices of μ there is still a substantial conformal window between μ and $\Lambda_{\mathcal{U}}$. The other parameters are taken to be the same as in Fig. 4. Note that the energy shift is lower in both cases compared to the $\mu \to 0$ case, but not drastically. In fact it is observed that the energy shifts are relatively substantial even for $\mu \gg m_{\mu^-} \simeq 0.1$ GeV compared to $\mu \to 0$. We will explore the reasons for this insensitivity to μ shortly. In general it is observed that as we increase the effective mass the energy shift decreases. As before the energy shifts should be interpreted as accurate only up to undetermined $\mathcal{O}(1)$ factors.

oblique correction is sensitive to a wide range of μ values within our approximations. Let us try to understand this a little better.

Let us look at the expression

$$\int_0^\infty dz \frac{\sin z}{z} e^{-\zeta z^2/(2r^2)}\bigg[\frac{\mu^2}{r}\log\bigg\{1+\frac{xz^2}{\mu^2 r^2}\bigg\} + \frac{xz^2}{r^3}\log\bigg\{(1-x)\bigg(\frac{\mu^2}{M^2}+\frac{xz^2}{M^2r^2}\bigg)\bigg\}\bigg]$$

from Eq. (45) for the pseudoscalar unparticle that we have rewritten to include all the dependences on μ. The first term in the above expression tends to zero in the $\mu \to 0$ limit and is absent in the case of perfect scale invariance. Even in the $\mu \neq 0$ limit we note that since $xz^2 \lesssim 2r^2/\zeta \ll \mu^2 r^2$ in the relevant region the first term is relatively suppressed. Thus the correction from the first term when $\mu \neq 0$ is small, even though the corresponding term in the case of $\mu \to 0$ is completely absent. Now let us look at the second term in the above expression, specifically the μ^2/M^2 factor inside the logarithm. Again that factor inside the logarithm is obviously absent when $\mu \to 0$. Since we have adopted the ansatz that the unparticle scale is in the vicinity of the electroweak scale we have up to $\mathcal{O}(1)$ factors $M \simeq \Lambda_{\mathcal{U}} \sim v$. Thus at the level of our approximation for typical values of allowed μ we have $\mu^2/M^2 \ll 1$. This again implies that the correction in the $\mu \neq 0$ case, with respect to the $\mu \to 0$ case, from the second term is not very drastic.

For the scalar unparticle case the arguments proceed exactly as above for the relevant expression

$$\int_0^\infty dz \frac{\sin z}{z} e^{-\zeta z^2/(2r^2)}\bigg[\frac{\mu^2}{r}\log\bigg\{1+\frac{xz^2}{\mu^2 r^2}\bigg\} + \frac{((1+j)x-j)z^2}{r^3}\log\bigg\{(1-x)\bigg(\frac{\mu^2}{M^2}+\frac{xz^2}{M^2r^2}\bigg)\bigg\}\bigg],$$

and once again we conclude that the corrections to the $\mu \to 0$ case from the additional factors are not very large. Thus the relative stability of the Uehling energy shifts to variations in the scale breaking parameter μ may be traced to the momentum cutoff imposed by $e^{-\zeta z^2/(2r^2)}$ leading to the observation in Eq. (46) and the fact that $\mu \lesssim M \simeq \Lambda_{\mathcal{U}} \sim v$ for typical values.

The typical values for the Uehling energy shift in Fig. 6 are in the range $\mathcal{O}(0.1)$–$\mathcal{O}(0.01)$ eV. The $\mathcal{O}(0.1)$ eV as we commented previously is comparable to the contribution from light-by-light scattering in QED. One may get a feel for the lower value $\mathcal{O}(0.01)$ eV in the range by noting that it is of the same order of magnitude as the corrections from the QED fourth-order Lamb shift, to higher angular momentum states in $\mu^- - \mathrm{Pb}^{208}_{82}$ [34]. For higher angular momentum states the muon's anomalous magnetic moment induces an additional spin-orbit interaction in the muonic atom and this contributes to an energy shift. For instance in the $\mu^- - \mathrm{Pb}^{208}_{82}$ $5g-4f$ transition the leading contribution to the fourth-order Lamb shift at order $\alpha^2(Z\alpha)$ was estimated to be [34]

$$\Delta E^{4-\mathrm{LS}}_{5g-4f} \simeq 0.025 \text{ eV}.$$

The values of the Uehling shifts in Fig. 6 may again be compared to the uncertainties in the precision measurements of $2p_{1/2}-1s_{1/2}$ and $2p_{3/2}-1s_{1/2}$ transitions in Table IV and the discrepancy between theory and measurement for these transitions in Eq. (35). It is noted that for typical values of the model parameters (j, Λ_γ, c, and μ) the Uehling shift is again about a factor of 10^3–10^4 below the values in (35).

We may also calculate the Uehling shifts for the low-lying $l=1$ and $l=2$ states with respect to $1S$. It is observed that for $\mu \neq 0$ the unparticle scalar and pseudoscalar corrections again follow a hierarchy

$$\Delta E^{\mathcal{U}}_{2s-1s} < \Delta E^{\mathcal{U}}_{2p-1s},$$
$$\Delta E^{\mathcal{U}}_{3s-1s} < \Delta E^{\mathcal{U}}_{3p-1s} < \Delta E^{\mathcal{U}}_{3d-1s},$$

consistent with our expectations in Eq. (33), for the case of perfect scale invariance, and also with our computations for the $\mu \to 0$ case before. We conclude that the choice of $\mu \neq 0$ does not change the level structure of the Uehling shifts. Moreover the variations are generally of the same order of magnitude as in the $\mu \to 0$ case. For example, assuming $\mu \simeq 2$ GeV and $\Lambda_{\mathcal{U}} \sim v$, the variations between the $l=0$ and 1 states for $n=2$ are again of the order of a few meV and for $n=3$ of the order of a few 0.01 meV.

A speculation is that one may do precision spectroscopy of the low-lying atomic states in parallel with proposed efforts to observe coherent muon-electron conversion in muonic atoms (see [49] and references therein). The relevant process in coherent muon-electron conversion is

$$\mu^- + N \to e^- + N,$$

where N is a nucleon. It is believed that probes of resonant muon-electron conversion near a nucleus may be able to achieve a higher sensitivity to lepton-flavor violation (LFV) compared to direct conversions [50]

$$\mu^- \to e^- + \gamma.$$

This opens the possibility that one may also perform precision spectroscopy on low-lying muonic-atom states in these forthcoming experiments and reduce some of the discrepancies in Eq. (35). But reducing the discrepancy to a level of $\mathcal{O}(0.01)$ eV looks very improbable to us.

Since muon conversion is a coherent process one might expect that the probability of muon conversion in an atom X^A_Z would go like $\sim Z^2$ (or $\sim A^2$). So when normalized to the muon-capture cross section we have heuristically

$$R_{\mu e} = \frac{\Gamma[\mu^- + (A,Z) \to e^- + (A,Z)]}{\Gamma[\mu^- + (A,Z) \to \nu_\mu + (A, Z-1)]} \sim \frac{Z^2}{Z} \approx Z,$$

while for the Uehling shift we are interested in, as we noted previously, the dependence on the atomic number Z goes like

$$\frac{\delta E_{nl}^{\mathrm{VP}}}{E_{\mathrm{bound}}} \sim Z^2.$$

It is not clear that the muonic-lead system, with $Z = 82$, that we are considering is suitable for the LFV measurements since it has been shown through detailed calculations [51] that for coherent muon-electron conversion the most ideal range is $Z \in [30, 60]$. Thus it would be interesting, as we have previously mentioned, to explore the possibility of measuring the scalar/pseudoscalar Uehling shifts in other atomic systems, with an intermediate Z value, where the energy shifts may still be substantial while the system is also of interest to coherent muon-electron conversion experiments. It would also be interesting to estimate the unparticle vacuum polarization effects in muonic atoms where high-precision LASER spectroscopy might be possible [52]. We hope that this work will be a modest pointer in this direction.

It is of course an open possibility that if for some reason the interaction energy scales $(\Lambda_\gamma, \tilde{\Lambda}_\gamma)$ are too large $[\gg \mathcal{O}(1)\ \mathrm{TeV}]$ or the coefficients (b, c) in Eqs. (4) and (5) are very small the energy shifts will be even more suppressed and the unparticle Uehling shift, even if it exists, will be nearly impossible to detect. Even under optimistic assumptions about the model parameters it is possible that the theoretical difficulties in calculating the required higher-order QED/nuclear effects in muonic atoms may be insurmountable or that obtaining precision spectra of the low-lying states is very difficult. In such a scenario the only hope would be to look for fermiophobic unparticles in very high energy colliders or other systems that are more sensitive to fermiophobic unparticles.

IV. SUMMARY

In this work we tried to study some of the probable effects of a fermiophobic scalar/pseudoscalar sector on bound state energy levels, specifically low-lying muonic-atom levels, as a consequence of oblique corrections to the photon propagator.

Considering the scalar and pseudoscalar fields to be unparticle operators, without loss of generality, we examined the functional forms of the vacuum polarization functions and the induced Uehling potentials. Some interesting theoretical observations were made on the singular nature of the unparticle induced Uehling potential and the behavior of the energy shifts in the limit of the scaling dimension approaching unity.

It was estimated that for an unparticle scale near the scale of electroweak-symmetry breaking, in the low TeV range, the energy shifts in the low-lying muonic-lead transitions could typically be of the order of a few 0.1 eV to a few 0.01 eV for some natural values of the model parameters. It was also pointed out that these magnitudes are comparable to bound state QED corrections, to the higher orbital angular momentum transitions in muonic lead, from the virtual Delbrück effect (light-by-light scattering) and the fourth-order Lamb shift [at order $\alpha^2(Z\alpha)$], respectively. These conclusions are relatively unchanged even when one incorporates a breaking of the scale invariance by introducing an effective unparticle mass μ.

But the current discrepancy between muonic-lead spectroscopy and theory, especially nuclear theory, makes an interpretation of the unparticle Uehling shift, if it really exists, extremely challenging. A conservative estimate is that such an interpretation would require an improvement in the discrepancy between theory and experiment, from about 20 years back, by a factor of 1000–10 000. The recent, partial resolution of the long-standing discrepancy in the $\Delta 2p$ and $\Delta 3p$ NP calculations with results from muonic-lead spectroscopy [40,41] by Haga and co-workers [39] is a promising step in this direction.

We also mentioned that in cases where the UV sector has a very large fermion multiplicity, the above contribution may be greatly enhanced, but appealing to arguments of naturalness we do not think this is very plausible. But many interesting models being considered today (for example, string-inspired QCD-like models [21]) allow for the possibility of a large fermionic sector and this perhaps beseeches us not to discard the possibility of fermiophobic unparticle oblique corrections prematurely.

The other interesting direction is to consider intermediate-Z muonic atoms where the nuclear uncertainties may be much better controlled while at the same time have sufficient fermiophobic unparticle contributions to muonic-atom transitions, by virtue of the $\delta E^{\mathrm{VP}_{nl}}/E_{\mathrm{bound}} \sim Z^2$ enhancement. This is left for future work.

In the context of the present study, of a possible fermiophobic unparticle scalar/pseudoscalar sector, we also briefly considered constraints from astrophysics and cosmology and put bounds on the fermiophobic unparticle effective masses. Finally we speculated on improving muonic-lead spectroscopy and theory in the context of forthcoming experiments that will study coherent muon-electron conversion.

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