PHYSICAL REVIEW D **81**, 034018 (2010)

Pionic contribution to neutrinoless double beta decay

J. D. Vergados,[1,2,*] Amand Faessler,[3] and H. Toki[4]

[1]*Physics Department, University of Ioannina, Ioannina, GR 451 10, Greece*
[2]*Theory Division, CERN, Geneva, Switzerland*
[3]*Institute für Theoretische Physik, Universität Tübingen, Germany*
[4]*RCNP, Osaka University, Osaka, 567-0047, Japan*

(Received 3 December 2009; published 12 February 2010)

It is well known that neutrinoless double decay is going to play a crucial role in settling the neutrino properties, which cannot be extracted from the neutrino oscillation data. It is, in particular, expected to settle the absolute scale of neutrino mass and determine whether the neutrinos are Majorana particles, i.e. they coincide with their own antiparticles. In order to extract the average neutrino mass from the data, one must be able to estimate the contribution of all possible high mass intermediate particles. The latter, which occur in practically all extensions of the standard model, can, in principle, be differentiated from the usual mass term, if data from various targets are available. One, however, must first be able to reliably calculate the corresponding nuclear matrix elements. Such calculations are extremely difficult since the effective transition operators are very short ranged. For such operators processes like pionic contributions, which are usually negligible, turn out to be dominant. We study such an effect in a nonrelativistic quark model for the pion and the nucleon.

DOI: 10.1103/PhysRevD.81.034018 PACS numbers: 12.39.Jh, 11.30.Fs, 13.15.+g, 14.60.Pq

I. INTRODUCTION

The discovery of neutrino oscillations can be considered as one of the greatest triumphs of modern physics. It began with atmospheric neutrino oscillations [1] interpreted as $\nu_\mu \to \nu_\tau$ oscillations, as well as ν_e disappearance in solar neutrinos [2]. These results have been recently confirmed by the KamLAND experiment [3], which exhibits evidence for reactor antineutrino disappearance. As a result of these experiments we have a pretty good idea of the neutrino mixing matrix and of the two independent quantities Δm^2; e.g., $m_2^2 - m_1^2$ and $m_3^2 - m_2^2$. Fortunately these two Δm^2 values are vastly different:

$$|\Delta m_{21}^2| = |m_2^2 - m_1^2| = (5.0 - 7.5) \times 10^{-5}\ (\mathrm{eV})^2$$

and

$$|\Delta m_{32}^2| = |m_3^2 - m_2^2| = 2.5 \times 10^{-3}\ (\mathrm{eV})^2.$$

This means that the relevant L/E parameters are very different. Thus, for a given energy the experimental results can approximately be described as two generation oscillations. For an accurate description of the data, however, a three generation analysis [4,5] is necessary.

We thus know that the neutrinos are massive, with two nonzero Δm^2, and they are admixed. We do not know, however, whether they are Majorana, i.e. the mass eigenstates coincide with their antiparticles, or of Dirac type, i.e. the mass eigenstates do not coincide with their antiparticles. Furthermore, we do not know the absolute mass scale as well as the sign of Δm_{32}^2. The first question can be settled by neutrinoless double beta decay ($0\nu\beta\beta-$ decay). The second will also, most likely, be settled by this process.

We should stress, of course, the fact that the light neutrino mediated process is not the only mechanism available for $0\nu\beta\beta$ [6]. Among those are some which involve heavy intermediate particles. These lead to very short ranged two body effective transition operators, which must be dealt with care, due to the presence of the nuclear hard core. To this end three treatments have been proposed:

(i) Treat the nucleons as composite particles (two nucleon mode). This can be done in the context of nonrelativistic quark model or simply by assigning to the nucleon a suitable form factor [7].
(ii) Consider the possibility of six quark clusters in the nucleus [8].
(iii) Consider other particles in the nuclear soup.

The most prominent are pions in flight between the two interacting nucleons [6]. In the present study we will examine the last possibility. This was examined a long time ago [9] and it was revived in the context of R-parity violating supersymmetry a decade later [10–12] as well as recently [13]. It was shown that in the context of R-parity violating supersymmetry the pion mode is more important than the two nucleon mechanism. The same conclusion was reached recently in the context of effective field theory [14]. In the above treatments the pions were treated as elementary particles. This approach is reasonable in particle physics, but one knows, of course, that the hadrons involved are not elementary. Furthermore a crucial factorization approximation has to be made, by inserting only the vacuum as intermediate state [see Eqs. (82) and (85) below]. Finally, even though the hadrons are elementary, in the interesting case of the pseudoscalar coupling an as-

*Vergados@uoi.gr

sumption had to be made about the quark mass, taken to be the current quark mass. In this work we are going to adopt a different procedure. The hadrons will be assumed to have a quark substructure in the context of the harmonic oscillator. In the harmonic oscillator approximation the internal degrees of freedom can be separated from the center of mass motion. In this approach, one derives the effective operator at the quark level by a suitable nonrelativistic expansion of the elementary amplitude. In some processes in our formalism one extra $q\bar{q}$ pair must be produced. This can be achieved either through the weak interaction itself or via the strong interaction. The net result is that, in this new approach, one obtains new types of operators, including some that are nonlocal at the nucleon level. One must weigh these advantages, however, against possible shortcomings of the need for a nonrelativistic reduction of the transition operator at the quark level.

II. THE CONTRIBUTION OF PIONS IN FLIGHT BETWEEN NUCLEONS

As we have mentioned in the Introduction when the intermediate fermion, e.g. the Majorana neutrino, is very heavy the transition operator becomes very short ranged. In this case the usual two nucleon mechanism may be suppressed due to the nuclear hard core and the contribution of other particles in the nuclear soup, such as pions, may dominate. These mechanisms at the nucleon level are illustrated in Fig. 1.

The two body double beta decay operator, associated with heavy intermediate particle exchange, will be normalized in a way which is consistent with the light intermediate neutrino. We begin with the intermediate heavy neutrino. Then

$$\eta_\nu \frac{R_0}{r} \Leftrightarrow \eta_N^{L,R} \frac{4\pi R_0}{m_e m_p} \delta(\mathbf{r}_1 - \mathbf{r}_2). \quad (1)$$

L, R stand for left handed and right handed currents, respectively, with

$$\eta_\nu = \frac{\prec m_\nu \succ}{m_e}, \qquad \eta_N^{L,R} = \prec \frac{m_p}{m_N} \succ . \quad (2)$$

We find it convenient to work in momentum space, which even in the case of the standard $0\nu - \beta\beta$ calculations has a number of advantages, see e.g. Ref. [15]. Then the corresponding expression becomes

$$\eta_\nu \frac{R_0}{r} \delta(\mathbf{r}_1 - \mathbf{r}'_1)\delta(\mathbf{r}_2 - \mathbf{r}'_2) \Leftrightarrow \eta_N^{L,R} \frac{4\pi R_0}{m_e m_p} \Omega_{\beta\beta} \quad (3)$$

$$\Omega_{\beta\beta} = \frac{1}{(2\pi)^3} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1 - \mathbf{p}'_2) A(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}'_1, \mathbf{p}'_2). \quad (4)$$

The function $A(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}'_1, \mathbf{p}'_2)$ depends on the assumed mechanism for the neutrinoless double beta decay.

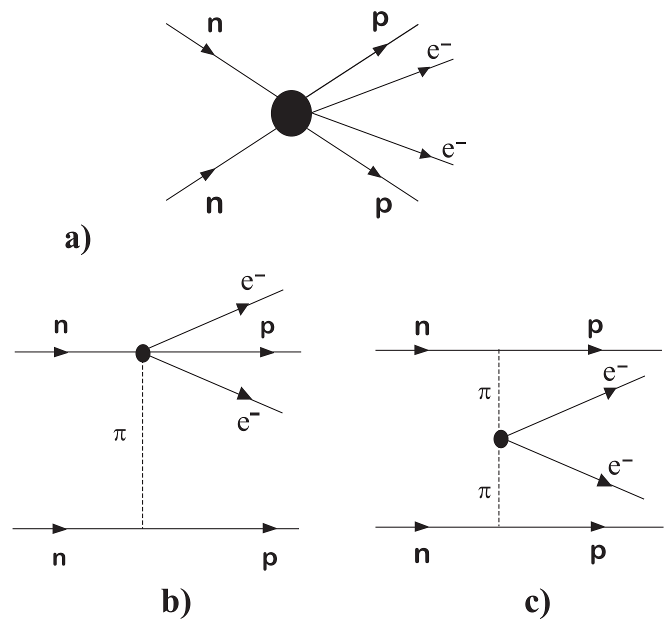

FIG. 1. The double beta decay of two neutrons into two protons at the two nucleon level (a) arising when all the intermediate particles at the quark level are very heavy. The double beta decay of a neutron with the simultaneous production of a π^+, which is then absorbed by another neutron converting it into a proton (b) (one-pion mode). A neutron can also be converted into a proton and a π^-. The π^- then double beta decays into a π^+, which subsequently is absorbed by another neutron converting into a proton (c) (two-pion mode).

The factor $\eta_N^{L,R}$ is not usually included in the nuclear matrix element. The factor $(R_0 m_p)/m_e$ will be absorbed into the effective nuclear operator, while the factor $\frac{4\pi}{m_p^2}$ will eventually be included in the effective coupling, as will be discussed in this work.

With the above expressions the formula for the lifetime due to heavy intermediate neutrinos in left handed V-A theories can be cast in the form

$$[T_{1/2}^{0\nu}]^{-1} = G_{01}\left[\eta_N^L\left(\left(\frac{f_V}{f_A}\right)^2\Omega_F - \Omega_{GT} + \alpha_{1\pi}\Omega_{1\pi} + \alpha_{2\pi}\Omega_{2\pi}\right)\right]^2. \quad (5)$$

The two nucleon contribution $(\frac{f_V}{f_A})^2\Omega_F - \Omega_{GT}$ was inserted in the above equation merely for comparison.

The case of other heavy intermediate particles, as those encountered in the R-parity violating supersymmetry, can be handled in a similar fashion:

$$[T_{1/2}^{0\nu}]^{-1} = G_{01}\left[\frac{3}{8}\left(\eta^T + \frac{5}{3}\eta^{PS}\right)\left(\frac{4}{3}\alpha_{1\pi}\Omega_{1\pi} + \alpha_{2\pi}\Omega_{2\pi}\right)\right]^2 \quad (6)$$

$$\Omega_{k\pi} = \frac{m_p}{m_e}[M_{GT}^{k\pi} + M_T^{k\pi}]. \tag{7}$$

In both cases,

$$M_{GT}^{k\pi} = \sum_{i<j} \tau_+(i)\tau_+(j)\sigma_i \cdot \sigma_j \frac{R_0}{r} F_1^{(k)}(x_\pi) \tag{8}$$

$$M_T^{k\pi} = \sum_{i<j} \tau_+(i)\tau_+(j)[3\sigma_i \cdot \hat{r}_{ij}\sigma_j \cdot \hat{r}_{ij} - \sigma_i \cdot \sigma_j] \times \frac{R_0}{r} F_2^{(k)}(x_\pi), \tag{9}$$

where R_0 is the nuclear radius, $x_\pi = m_\pi r_{ij}$, and

$$F_1^{(1)}(x) = e^{-x}, \qquad F_2^{(1)}(x) = (x^2 + 3x + 3)e^{-x},$$
$$F_1^{(2)}(x) = (x-2)e^{-x}, \qquad F_2^{(2)}(x) = (x+1)e^{-x}. \tag{10}$$

The function $A(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1', \mathbf{p}_2')$ depends on the pion mode under consideration.

III. THE TWO-PION MODE

The spin dependence of the transition operator is in this case trivial. So we will focus on the orbital structure of the operator. The function $A(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1', \mathbf{p}_2')$ is independent of the momenta in the standard V-A theory as well as in the case of the scalar (S-S) theory. It is, however, a model dependent function in the case of pseudoscalar (P-P) interaction encountered, e.g., in R-parity violating supersymmetry (SUSY) mediated double beta decay. In the last case we find

$$A = -\tfrac{1}{3}\mathbf{A}_1 \cdot \mathbf{A}_2, \tag{11}$$

where $\mathbf{A}_i$ is the amplitude resulting from the nonrelativistic reduction of the pseudoscalar involved in the $d \rightarrow u$ coupling, i.e.

$$\bar{u}(p_i')\gamma_5 d(p_i) \rightarrow \mathbf{A}_i \cdot \boldsymbol{\sigma}_i, \tag{12}$$

where $\boldsymbol{\sigma}_i$ is the spin of the quark i and

$$A_i = \frac{1}{2m_d}\mathbf{p}_i - \frac{1}{2m_u}\mathbf{p}_i'. \tag{13}$$

We find it convenient to rewrite them as follows:

$$\mathbf{A}_1 = -\frac{1}{\sqrt{2}}\left(\frac{\boldsymbol{\rho}}{2m_d} - \frac{\boldsymbol{\rho}'}{2m_u}\right) + \left(\frac{1}{2m_d} - \frac{1}{2m_u}\right)\frac{\mathbf{q}}{2} \tag{14}$$

$$\mathbf{A}_2 = \frac{1}{\sqrt{2}}\left(\frac{\boldsymbol{\rho}}{2m_d} - \frac{\boldsymbol{\rho}'}{2m_u}\right) + \left(\frac{1}{2m_d} - \frac{1}{2m_u}\right)\frac{\mathbf{q}}{2}, \tag{15}$$

where $\mathbf{q} = \mathbf{P}_\pi$ is the momentum of the pion in flight between the two nucleons and $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ are the relative internal momenta (see the next subsection). One normally ignores at this level the momentum carried away by the two leptons. The $2\pi\, 0\nu - \beta\beta$ decay contribution in the case of the heavy Majorana neutrino or any other Majorana fermion is explicitly shown in Fig. 2.

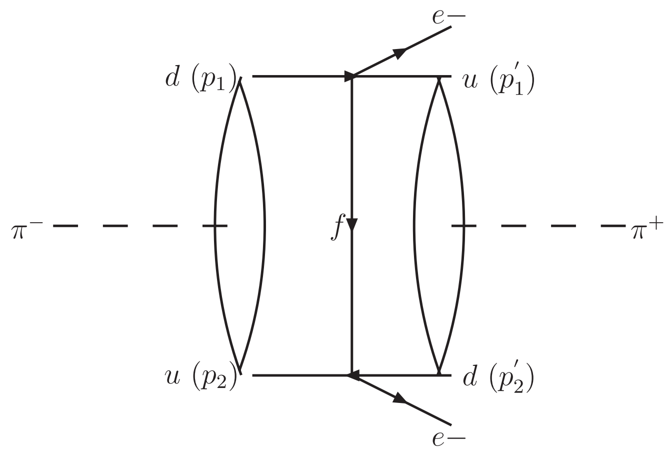

FIG. 2. The $0\nu\beta\beta$ decay of pions in flight (2π mode of Fig. 1) illustrated at the quark level. f stands for an effective exchange of a heavy Majorana fermion (heavy neutrino or, as in R-parity violating supersymmetry, a neutralino, gluino, etc.). The ellipses merely indicate that the pion is a bound state of two quarks.

A. Orbital integrals in the two-pion exchange

The pion wave function is given by

$$\psi_{\mathbf{P}_\pi}(\mathbf{Q}, \rho) = \sqrt{2E_\pi}(2\sqrt{2})^{1/2}(2\pi)^{3/2}\delta(\sqrt{2}\mathbf{Q} - \mathbf{P}_\pi)\phi_\pi(\boldsymbol{\rho}), \tag{16}$$

where $\mathbf{P}_\pi$ is the pion momentum and

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{p}_2 - \mathbf{p}_1), \qquad \mathbf{Q} = \frac{1}{\sqrt{2}}(\mathbf{p}_2 + \mathbf{p}_1) \tag{17}$$

with $\mathbf{p}_1$ and $\mathbf{p}_2$ being the momenta of the quark and antiquark participating in the pion. This wave function is normalized in the usual way:

$$\prec \psi_{\mathbf{P}_\pi} | \psi_{\mathbf{P}_\pi'} \succ = 2E_\pi(2\pi)^3\delta(\mathbf{P}_\pi - \mathbf{P}_\pi'). \tag{18}$$

$\phi_\pi(\boldsymbol{\rho})$ is described by a 1s harmonic oscillator state. In momentum space it takes the form

$$\phi_\pi(\boldsymbol{\rho}) = \phi_\pi(0)e^{-(b_\pi^2\rho^2)/2}, \qquad \phi_\pi(0) = \sqrt{\frac{b_\pi^3}{\pi\sqrt{\pi}}}. \tag{19}$$

Thus the orbital matrix element in this case takes the form

$$\mathrm{ME}_{2\pi} = \mathcal{M}_{2\pi}(2\pi)^3\delta(\mathbf{P}_\pi - \mathbf{P}_\pi'),$$
$$\mathcal{M}_{2\pi} = \frac{1}{2\pi\sqrt{2\pi}}\frac{2m_\pi}{b_N^3}f_{2\pi}^{(1)}(x), \qquad f_{2\pi}^{(1)}(x) = \frac{1}{x^3}, \tag{20}$$

where $x = b_\pi/b_N$. b_π and b_N are the harmonic oscillator (HO) size parameters for the pion and the nucleon, respectively. We have decided to introduce the ratio x as a variable to be adjusted. In V-A theories after incorporating the spin we find

$$\frac{4\pi}{m_p^2}\mathcal{M}_{2\pi} = c_{2\pi} m_\pi^2 \tag{21}$$

with

$$c_{2\pi} = \frac{1}{\sqrt{2\pi}} \frac{4}{b_N^3 m_p^2 m_\pi} \prec |1 - \boldsymbol{\sigma_1} \cdot \boldsymbol{\sigma_2}| \succ f_{2\pi}^{(1)}(x), \tag{22}$$

where $\prec |\boldsymbol{\sigma_1} \cdot \boldsymbol{\sigma_2}| \succ = -3$ is the spin matrix element (ME). One now can construct the effective transition operator in coordinate space at the nuclear level. The effective coupling in V-A theory is given [6] by

$$\alpha_{2\pi} = c_{2\pi} g_r^2 \left(\frac{m_\pi}{2m_N}\right)^2 \frac{1}{4\pi} \frac{1}{6m_\pi^2} \frac{1}{f_A^2} \tag{23}$$

or

$$\alpha_{2\pi} = \frac{2}{3f_A^2} f_{\pi NN}^2 c_{2\pi}. \tag{24}$$

Using $f_{\pi NN}^2 = 0.08$ and $b_N = 1.0$ fm, we find $\alpha_{2\pi} = 0.013$ and 0.11 for $x = 1.0$ and 0.5, respectively. For the scalar interaction, one gets the value $f_S^2/4$ with the value of f_S depending on the specific particle model.

The dependence of the results on the pion size parameter is exhibited in Fig. 3. In the case of the pseudoscalar coupling, since the pion has spin zero, we encounter the combination

$$(\mathbf{A}_1 \cdot \boldsymbol{\sigma_1})(\mathbf{A}_2 \cdot \boldsymbol{\sigma_2}) \Rightarrow -\tfrac{1}{3}(\mathbf{A}_1 \cdot \mathbf{A}_2)(\boldsymbol{\sigma_1} \cdot \boldsymbol{\sigma_2}). \tag{25}$$

In this case, one can show that the orbital amplitude is

$$\mathcal{M}_{2\pi} = \frac{1}{2\pi\sqrt{2\pi}} \frac{2m_\pi}{b_N^3} \left(\frac{1}{4}(\kappa_d^2 + \kappa_u^2) - \frac{1}{6}(\kappa_d - \kappa_u)^2 b_N^2 \mathbf{q}^2\right) f_{2\pi}^{(2)}(x), \tag{26}$$

$$f_{2\pi}^{(2)}(x) = \frac{1}{x^5},$$

where $\mathbf{q}$ is the momentum of the propagating pion and

$$\kappa_d = \frac{1}{2m_d b_N}, \qquad \kappa_u = \frac{1}{2m_u b_N}. \tag{27}$$

The above equation can be rewritten in a way that the pion propagator is manifest:

$$\mathcal{M}_{2\pi} = \frac{1}{2\pi\sqrt{2\pi}} \frac{2m_\pi}{b_N^3} \left(\frac{1}{4}(\kappa_d^2 + \kappa_u^2) + \frac{1}{6}(\kappa_d - \kappa_u)^2 b_N^2 m_\pi^2 - \frac{1}{6}(\kappa_d - \kappa_u)^2 b_N^2 (\mathbf{q}^2 + m_\pi^2)\right) f_{2\pi}^{(2)}(x). \tag{28}$$

In other words there appear two terms $c_{2\pi}^0$ and $c_{2\pi}^q b_N^2(q^2 + m_\pi^2)$ with

$$c_{2\pi}^0 = \frac{1}{\sqrt{2\pi}} \frac{4}{b_N^3 m_p^2 m_\pi} f_{2\pi}^{(2)}(x) \left(\frac{1}{4}(\kappa_d^2 + \kappa_u^2) + \frac{1}{6}(\kappa_d - \kappa_u)^2 b_N^2 m_\pi^2\right) \tag{29}$$

$$c_{2\pi}^q = -\frac{1}{\sqrt{2\pi}} \frac{4}{b_N^3 m_p^2 m_\pi} \frac{1}{6}(\kappa_d - \kappa_u)^2 b_N^2 m_\pi^2 f_{2\pi}^{(2)}(x). \tag{30}$$

The first gives rise to an effective operator similar to that of the V-A theory with a coupling

$$\alpha_{2\pi} = \frac{2}{3f_A^2} \frac{1}{\sqrt{2\pi}} f_{\pi NN}^2 \frac{1}{m_\pi m_p^2 b_N^3} f_{2\pi}^{(2)}(x) \left(\frac{1}{4}(\kappa_d^2 + \kappa_u^2) + \frac{1}{6}(\kappa_d - \kappa_u)^2 b_N^2 m_\pi^2\right) \prec |\boldsymbol{\sigma_1} \cdot \boldsymbol{\sigma_2}| \succ \tag{31}$$

with $\prec |\boldsymbol{\sigma_1} \cdot \boldsymbol{\sigma_2}| \succ = -3$.

The second term, contributing when the u and d quarks are not degenerate, yields a coupling $\alpha_{2\pi}(\Omega_{1\pi})$, where

$$\alpha_{2\pi}(\Omega_{1\pi}) = -\frac{4}{f_A^2} \frac{1}{\sqrt{2\pi}} f_{\pi NN}^2 \frac{m_\pi}{m_p} \frac{1}{m_p b_N} f_{2\pi}^{(2)}(x) \frac{1}{6} \times (\kappa_d - \kappa_u)^2) \prec |\boldsymbol{\sigma_1} \cdot \boldsymbol{\sigma_2}| \succ \tag{32}$$

which is associated with the operator with one pion propagator less, i.e. that encountered in the 1π mode (see

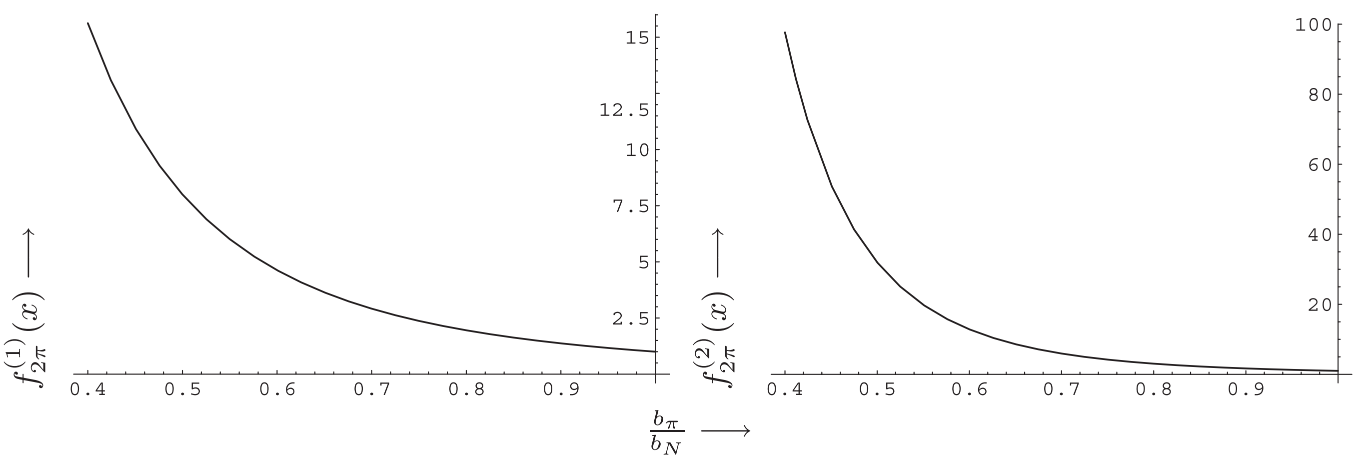

FIG. 3. The function $f_{2\pi}^{(1)}(x)$ on the left and $f_{2\pi}^{(2)}(x)$ on the right as a function of $x = b_\pi/b_N$.

below). Such an operator is absent in the elementary particle treatment, even though the quarks are assumed to be nondegenerate.

IV. THE ONE-PION MODE

In this case a positively charged pion, produced in virtual double beta decay of a neutron into a proton, is absorbed by another neutron converting it into a proton. At the quark level the first of these steps is exhibited in Figs. 4–6. In these figures a $q\bar{q}$ pair is created out of the vacuum. In the first two figures this is achieved as, e.g., in a gluon exchange [16] or a multigluon exchange simulated in the $^3P_0$ model [17–20]. The latter is a fairly old model, which still continues to be successfully applied in the description of meson decays [21]. In Fig. 6 this pair is created by the weak interaction itself.

A. The orbital part at the quark level

Orbital wave functions in momentum space are expressed in terms of Jacobi coordinates:

$$\psi_{\mathbf{P}_\pi} = \sqrt{2E_\pi}(2\sqrt{2})^{1/2}(2\pi)^{3/2}\delta(\sqrt{2}\mathbf{Q}_\pi - \mathbf{P}_\pi)\phi_\pi(\boldsymbol{\rho}) \tag{33}$$

$$\psi_{\mathbf{P}} = (3\sqrt{3})^{1/2}(2\pi)^{3/2}\delta(\sqrt{3}\mathbf{Q}-\mathbf{P})\phi(\boldsymbol{\xi}, \boldsymbol{\eta}) \tag{34}$$

$$\psi_{\mathbf{P}'} = (3\sqrt{3})^{1/2}(2\pi)^{3/2}\delta(\sqrt{3}\mathbf{Q}'-\mathbf{P}')\phi(\boldsymbol{\xi}', \boldsymbol{\eta}'), \tag{35}$$

where $\mathbf{P}_\pi$, $\mathbf{P}$, and $\mathbf{P}'$ are the momenta of the pion and the two nucleons, respectively, and

$$\boldsymbol{\xi} = \frac{1}{\sqrt{2}}(\mathbf{p}_1 - \mathbf{p}_2), \qquad \boldsymbol{\eta} = \frac{1}{\sqrt{6}}(\mathbf{p}_1 + \mathbf{p}_2 - 2\mathbf{p}_3), \qquad Q = \frac{1}{\sqrt{3}}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \tag{36}$$

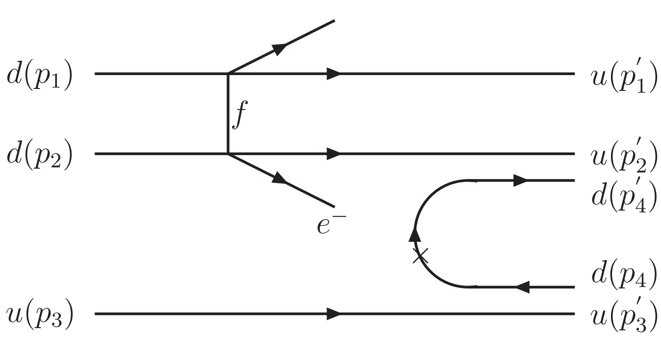

FIG. 4. The pion mediated $0\nu\beta\beta$ decay in the so-called 1π mode. At the top we show the diagram in which the quarks of the pion are spectators, i.e. the heavy intermediate heavy fermion f is exchanged between the other two quarks. $\times$ indicates that a $q\bar{q}$ pair is created out of the vacuum in the context of a multigluon exchange. We will call it direct diagram.

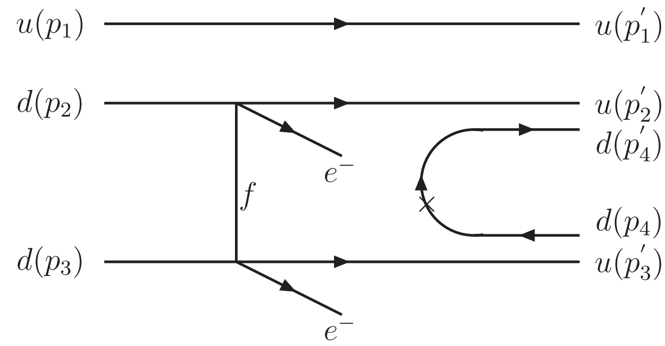

FIG. 5. The same as in Fig. 4 involving the exchange diagram. In this case the quark involved in the pion participates in the exchange of the heavy fermion f, cooperating this way with another quark belonging in the nucleon.

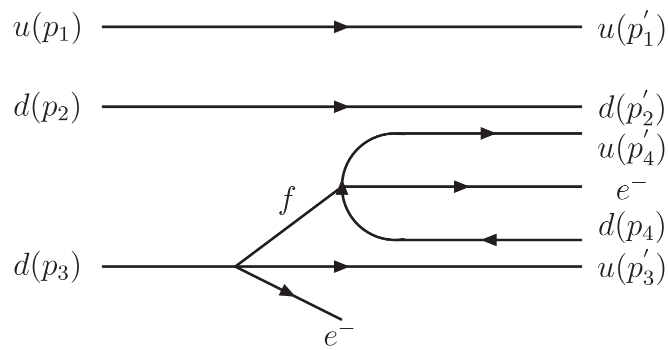

FIG. 6. The same as in Fig. 5 but in a novel mechanism, i.e. one in which the $q\bar{q}$ pair is produced by the weak interaction itself.

$$\boldsymbol{\xi}' = \frac{1}{\sqrt{2}}(\mathbf{p}_1' - \mathbf{p}_2'), \qquad \boldsymbol{\eta}' = \frac{1}{\sqrt{6}}(\mathbf{p}_1' + \mathbf{p}_2' - 2\mathbf{p}_4'), \qquad Q' = \frac{1}{\sqrt{3}}(\mathbf{p}_1' + \mathbf{p}_2' + \mathbf{p}_4') \tag{37}$$

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{p}_3' - \mathbf{p}_4), \qquad Q_\pi = \frac{1}{\sqrt{2}}(\mathbf{p}_3' + \mathbf{p}_4), \tag{38}$$

where $\mathbf{p}_i$, $\mathbf{i} = \mathbf{1}, \mathbf{3}$ are the momenta of the three quarks of one nucleon, $\mathbf{p}_1'$, $\mathbf{p}_2'$, $\mathbf{p}_4'$ the momenta of the three quarks of the other nucleon, and $\mathbf{p}_4$, $\mathbf{p}_3'$ are those of the quarks involved in the pion. This notation was chosen since the interaction preserves the fermion lines $\mathbf{p}_i \leftrightarrow \mathbf{p}_i'$.

The above wave functions were normalized in the usual way:

$$\prec \psi_{\mathbf{P}} | \psi_{\mathbf{P}'} \succ = (2\pi)^3\delta(\mathbf{P} - \mathbf{P}'), \qquad \prec \psi_{\mathbf{P}_\pi} | \psi_{\mathbf{P}_\pi'} \succ = 2E_\pi(2\pi)^3\delta(\mathbf{P}_\pi - \mathbf{P}_\pi'). \tag{39}$$

The internal wave functions are given by

$$\phi(\xi) = \phi(0)e^{-(b_N^2\xi^2)/2}, \qquad \phi(0) = \sqrt{\frac{b_N^3}{\pi\sqrt{\pi}}}\ \text{etc.} \tag{40}$$

The pion wave function has already been defined [see Eq. (19)], except that sometimes we will write

$$\phi_\pi(0) = \phi(0)x^{3/2}, \qquad x = \frac{b_\pi}{b_N}. \tag{41}$$

The integrals over the momentum variables $\mathbf{Q}$, $\mathbf{Q}'$, and $\mathbf{Q}_\pi$ can be trivially performed due to the δ functions. Thus the orbital integral becomes

$$I_{\beta\beta} = (2\pi)^3\delta(\mathbf{P} - \mathbf{P}' - \mathbf{P}_\pi)\mathcal{M} \tag{42}$$

$$\mathcal{M} = \frac{(2\pi)^{3/2}\sqrt{2E_\pi}}{3\sqrt{3}(2\sqrt{2})^{1/2}}\int d^3\boldsymbol{\xi}d^3\boldsymbol{\xi}'d^3\boldsymbol{\eta}d^3\boldsymbol{\eta}'d^3\boldsymbol{\rho}\phi(\boldsymbol{\xi},\boldsymbol{\eta}) \times \phi(\boldsymbol{\xi}',\boldsymbol{\eta}')\phi_\pi(\boldsymbol{\rho})\Omega_{\beta\beta}, \tag{43}$$

where $\Omega_{\beta\beta}$ depends on the mechanism involved as we now discuss.

(1) The $q\bar{q}$ pair is created by the $0\nu\beta\beta$ operator ($0\nu q\bar{q}$ case).
The case in which the $q\bar{q}$ pair is created by the $0\nu\beta\beta$ operator (see Fig. 6). Then up to terms linear in the momentum the effective operator takes the form:

$$\omega_{S(V)} = \sigma_4 \cdot \left(\frac{\mathbf{p}_4}{2m_d} + \frac{\mathbf{p}_4'}{2m_u}\right) \quad \text{(scalar and vector)} \tag{44}$$

$$\omega_P = \sigma_3 \cdot \left(\frac{\mathbf{p}_3'}{2m_u} - \frac{\mathbf{p}_3}{2m_d}\right) \quad \text{(pseudoscalar)} \tag{45}$$

$$\omega_A = i(\sigma_3 \times \sigma_4) \cdot \left(\frac{\mathbf{p}_4'}{2m_u} - \frac{\mathbf{p}_4}{2m_d}\right) \quad \text{(axial current)}. \tag{46}$$

It is, of course, understood that the scalar and pseudoscalar must be multiplied by suitable coupling constants. The full operator takes the form

$$\Omega_{\beta\beta\pi} = \frac{1}{(2\pi)^3}\delta(p_3 - p_3' - p_4 - p_4')\delta(p_1 - p_1') \times \delta(\mathbf{p}_2 - \mathbf{p}_2')\omega_i, \qquad i, S, V, P, A. \tag{47}$$

The product of the three δ functions can be cast in the form

$$\delta(P - P' - P_\pi)\delta(p_1 - p_1')\delta(p_2 - p_2') = \delta(P - P' - P_\pi)\delta(\sqrt{2}(\xi - \xi')) \times \delta\left(\frac{1}{\sqrt{6}}(\eta - \eta') + \frac{\mathbf{q}}{3}\right).$$

By setting $\xi' = \xi$ and $\eta' = \eta + \sqrt{\frac{2}{3}}\mathbf{q}$, we get

$$\omega_{S(V)} = \sigma_4 \cdot \left(\frac{4(3(q - \sqrt{2}\rho)m_d + m_u(-5q - 2\sqrt{6}\eta + 2p_N))}{6m_d m_u}\right) \tag{48}$$

$$\omega_P = \sigma_3 \cdot \left(-\frac{\sigma_3(m_d(q - 2\sqrt{6}\eta + 2p_N) - 3(q + \sqrt{2}\rho)m_u)}{6m_d m_u}\right)\text{(pseudoscalar)} \tag{49}$$

$$\omega_A = i(\sigma_3 \times \sigma_4) \cdot \left(-\frac{\sigma_4(3(q - \sqrt{2}\rho)m_d + m_u(5q + 2\sqrt{6}\eta - 2p_N))}{6m_d m_u}\right)\text{(axial)}. \tag{50}$$

After the integration (see next section) we get

$$\omega_{S(V)} = \sigma_4 \cdot \left(\frac{(3\mathbf{q}m_d + m_u(2\mathbf{p}_N - 3\mathbf{q}))}{6m_d m_u}\right) \Rightarrow \frac{\sigma_4 . \mathbf{p}_N}{m_p} \tag{51}$$

$$\omega_P = \sigma_3 \cdot \left(\frac{(3\mathbf{q}m_u - m_d(3\mathbf{q} + 2\mathbf{p}_N))}{6m_d m_u}\right) \Rightarrow -\frac{\sigma_3 . \mathbf{p}_N}{m_p} \tag{52}$$

$$\omega_A = i(\sigma_3 \times \sigma_4) \cdot \left(\frac{(3\mathbf{q}(m_d + m_u) - 2m_u\mathbf{p}_N)}{6m_d m_u}\right) \Rightarrow i(\sigma_3 \times \sigma_4) \cdot \frac{-3\mathbf{q} + \mathbf{p}_N}{m_p}. \tag{53}$$

The last expressions result in the case of the constituent mass for the quarks, $m_u = m_d = m_p/3$. In the above equations,

$$\mathbf{p}_N = \frac{\mathbf{P} + \mathbf{P}'}{2}, \qquad \mathbf{q} = \mathbf{P} - \mathbf{P}' = \mathbf{P}_\pi. \tag{54}$$

(2) Double beta decay and strong $q\bar{q}$ production ($^3P_0 q\bar{q}$ case).
In this case, one needs the collaborative effect of the $0\nu\beta\beta$ interaction acting between quarks together with the strong interaction, which creates a pion out of the vacuum (*à la* the $^3P_0$ model or multigluon exchange):

$$H = g_r' \boldsymbol{\sigma}_4 \cdot \mathbf{B}\delta(\mathbf{p}_4 + \mathbf{p}_4'), \qquad \mathbf{B} = \mathbf{p}_4 - \mathbf{p}_4', \tag{55}$$

where g_r' is a dimensionless constant proportional to the parameter $g_r = 13.4 \pm 0.1$, which is known from experiment. One finds

$$g_r' = g_r \frac{3\sqrt{3}(2x^2+3)^{3/2}}{80\sqrt[4]{2}\phi_\pi(0)\pi^{3/2} m_p \sqrt{m_\pi}}, \tag{56}$$

where $\phi_\pi(0)$ is the pion wave function at the origin.
(i) The direct term in the one pion contribution.
In this case (see Fig. 4) none of the two interacting quarks participates in the pion as defined above. Thus we get

$$\Omega_{\beta\beta\pi} = \frac{g_r'}{(2\pi)^3}\delta(p_1 + p_2 - p_1' - p_2') \times \delta(p_3 - p_4')\delta(\mathbf{p}_4 + \mathbf{p}_4')\sigma_4 \cdot (\mathbf{p}_4 + \mathbf{p}_4'). \tag{57}$$

The product of the above three δ functions can be cast in the form

$$\delta(P - P' - P_\pi)\delta(p_1 + p_2 - p_1' - p_2') \times \delta(\mathbf{p}_4 + \mathbf{p}_4').$$

The first of these δ functions expresses momentum conservation. Going into the Jacobi variables we find

$$\Omega_{\beta\beta\pi} = \frac{1}{(2\pi)^3}\delta(P - p' - P_\pi) \times \delta\left(\frac{2q + \sqrt{6}(\eta - \eta')}{3}\right) \times \delta\left(\frac{2q - 2\sqrt{6}\eta' - 3\sqrt{2}\rho + 2p_N}{6}\right) \times \sigma_4 \cdot \frac{4q + 2\sqrt{6}\eta' - 3\sqrt{2}\rho - 2p_N}{6}. \tag{58}$$

We find it convenient to use the above δ functions to obtain

$$\eta = -\frac{2q + 3\sqrt{2}\rho - 2p_N}{2\sqrt{6}}, \qquad \eta' = \frac{2q - 3\sqrt{2}\rho + 2p_N}{2\sqrt{6}}. \tag{59}$$

One finds

$$\boldsymbol{\sigma}_4 \cdot \mathbf{B} = \boldsymbol{\sigma}_4 \cdot (\mathbf{q} - \sqrt{2}\boldsymbol{\rho}). \tag{60}$$

Furthermore A terms, appearing in the case of the pseudoscalar contribution, take the form

$$\mathbf{A}_1 = -\frac{(m_u(\sqrt{2}(\rho - 2\xi) - 2p_N) + m_d(\sqrt{2}(2\xi' - \rho) + 2p_N))\sigma_1}{4m_d m_u} \tag{61}$$

$$\mathbf{A}_2 = -\frac{(m_u(\sqrt{2}(2\xi + \rho) - 2p_N) + m_d(2p_N - \sqrt{2}(2\xi' + \rho)))\sigma_2}{4m_d m_u}. \tag{62}$$

Thus using the corresponding δ functions, the η and η' integrations can be done trivially.
(ii) The exchange term in the one-pion contribution.
By this we mean that one of the interacting particles participates in the pion (see Fig. 5). Proceeding as above, we have

$$\Omega_{\beta\beta\pi} = \frac{g_r'}{(2\pi)^3}\delta(p_2 + p_3 - p_2' - p_3')\delta(p_1 - p_1')\delta(\mathbf{p}_4 + \mathbf{p}_4')\sigma_4 \cdot (\mathbf{p}_4 + \mathbf{p}_4'). \tag{63}$$

Going into the Jacobi variables we find

$$\Omega_{\beta\beta\pi} = \frac{1}{(2\pi)^3}\delta(P - P' - P_\pi)\delta\left(\frac{2p_N - \sqrt{2}(\sqrt{3}\eta + \sqrt{3}\eta' + 3(\xi - \xi' + \rho))}{6}\right) \times \delta\left(\frac{2q - 2\sqrt{6}\eta' - 3\sqrt{2}\rho + 2p_N}{6}\right)\sigma_4 \cdot (\mathbf{q} - \sqrt{2}\rho). \tag{64}$$

We find it convenient to use the above δ functions to obtain

$$\xi' = \frac{1}{6}(\sqrt{2}q + 2\sqrt{3}\eta + 6\xi + 3\rho - \sqrt{2}p_N), \qquad \eta' = \frac{2q - 3\sqrt{2}\rho + 2p_N}{2\sqrt{6}}.$$

Thus the ξ' and η' can be done trivially. Furthermore **A** terms, appearing in the case of the pseudoscalar contribution, take the form

$$\mathbf{A}_2 = \frac{\sigma_2(m_d(q + \sqrt{2}(\sqrt{3}\eta + 3(\xi + \rho)) - 4p_N) + m_u(q + \sqrt{6}\eta - 3\sqrt{2}\xi + 2p_N))}{6m_d m_u} \tag{65}$$

$$\mathbf{A}_3 = \frac{\sigma_3(m_u(q - 2\sqrt{6}\eta + 2p_N) - 3(q + \sqrt{2}\rho)m_d)}{6m_d m_u}. \tag{66}$$

B. The $0\nu\beta\beta$ decay amplitude at the nucleon level

Performing the orbital integrals we encountered in the previous section, we must evaluate the spin-flavor ME for the various operators encountered above, classified according to their spin rank. The obtained matrix elements, in units of the nucleon spin ME, are included in Table I). Using these results, one can obtain the needed amplitude at the nucleon level. As expected from the above discussion we will consider three possibilities:

(1) The $0\nu q\bar{q}$ case.
In this case we can write the amplitude as

$$\mathcal{M} = \frac{1}{(2\pi)^{3/2}}\frac{1}{3\sqrt{3}}\frac{\sqrt{2m_\pi}}{\sqrt{2\sqrt{2}}}\boldsymbol{\sigma}N \cdot \mathbf{C}_i \mathrm{ME}(sf)J_{\mathrm{orb}}, \tag{67}$$

where $\boldsymbol{\sigma}N$ is the nucleon spin and ME$(s - f)$ is the spin-flavor matrix element (see Table I) and J_{orb} is the radial integral. One finds

$$J_{\mathrm{orb}} = 6\sqrt{6}\frac{\phi_\pi(0)}{(\phi(0))^2}e^{-(b_N^2 q^2)/6}. \tag{68}$$

The coefficients $\mathbf{C}_i$ can be read off from Eqs. (51)–(53), namely,

$$\mathbf{C}_{S(V)} = \left(\frac{(3\mathbf{q}m_d + m_u(2\mathbf{p}_N - 3\mathbf{q}))}{6m_d m_u}\right) \Rightarrow \frac{\mathbf{p}_N}{m_N} \tag{69}$$

$$\mathbf{C}_P = \left(\frac{(3\mathbf{q}m_u - m_d(3\mathbf{q} + 2\mathbf{p}_N))}{6m_d m_u}\right) \Rightarrow -\frac{\mathbf{p}_N}{m_N} \tag{70}$$

$$\mathbf{C}_A = \left(\frac{(3\mathbf{q}(m_d + m_u) - 2m_u\mathbf{p}_N)}{6m_d m_u}\right) \Rightarrow \frac{3\mathbf{q} - \mathbf{p}_N}{m_N}. \tag{71}$$

The term p_N of the amplitude will lead to a nonlocal effective operator in coordinate space.

(2) The ${}^3P_0 q\bar{q}$ case.
Double beta decay proceeds via two quarks in a state with isospin one, which is color antisymmetric. So the two quarks must be in a spin one state. So there is no contribution in V-A theories, since the vector and the axial vector contributions are identical. For the scalar and pseudoscalar cases, the needed couplings depend on the particle model assumed. In the R-parity violating SUSY the coupling is, e.g. $\frac{3}{8} \times (\eta^T + \frac{5}{3}\eta_{PS})$ found in [11]. In our discussion we will not include such a model dependent coupling. We will distinguish the two possibilities:

(a) The direct term.
In this case we can write the amplitude as

$$\mathcal{M} = \frac{1}{(2\pi)^{3/2}}\frac{1}{3\sqrt{3}}\frac{\sqrt{2m_\pi}}{\sqrt{2\sqrt{2}}}\mathbf{A}_1 \cdot \mathbf{A}_2 J_{\mathrm{orb}}. \tag{72}$$

In the case of the scalar contribution we find from Table I that

$$\mathbf{A}_1 \cdot \mathbf{A}_2 = -\frac{\sqrt{2}}{9}\mathbf{q} \cdot \sigma_N. \tag{73}$$

In the case of the pseudoscalar contribution (see the

TABLE I. The spin-flavor matrix elements of the various spin operators encountered in this work. They are normalized to the matrix element of the nucleon spin.

Ω_s (k indicates the spin ranks)	Process	$\mathrm{ME}sf = \frac{\langle\|\Omega_s\|\rangle}{\langle\|\sigma_N\|\rangle}$
σ_4	Scalar or vector	$-\frac{5\sqrt{2}}{9}$
σ_3	Pseudoscalar	$-\frac{5\sqrt{2}}{9}$
$i\sigma 3 \times \sigma_4$	Axial	$\frac{10\sqrt{2}}{9}$
σ_4	Direct	$-\frac{\sqrt{2}}{9}$
$(\sigma_1 \cdot \sigma_2)\sigma_4$	Direct	$-\frac{\sqrt{2}}{9}$
$[(\sigma_1 \times \sigma_2)k_{12} = 2; \sigma_4]k = 1$	Direct	$\frac{4\sqrt{10}}{9\sqrt{3}}$
$(\sigma_1 \cdot \sigma_4)\sigma_2$	Direct	$-\frac{7\sqrt{2}}{9}$
$(\sigma_2 \cdot \sigma_4)\sigma_1$	Direct	$-\frac{7\sqrt{2}}{9}$
σ_4	Exchange	$\frac{\sqrt{2}}{9}$
$(\sigma_2 \cdot \sigma_3)\sigma_4$	Exchange	$\frac{\sqrt{2}}{9}$
$[(\sigma_2 \times \sigma_3)k_{23} = 2; \sigma_4]k = 1$	Exchange	$\frac{8\sqrt{10}}{9\sqrt{3}}$
$(\sigma_2 \cdot \sigma_4)\sigma_3$	Exchange	$-\frac{13\sqrt{2}}{9}$
$(\sigma_3 \cdot \sigma_4)\sigma_2$	Exchange	$-\frac{13\sqrt{2}}{3}$

Appendix) and in the local approximation $\mathbf{p}_N = 0$, we find

$$\mathbf{A}_1 \cdot \mathbf{A}_2 = \frac{1}{3}\left(\frac{(m_d - m_u)}{\sqrt{4x^2 + 6b_N m_d m_u}}\right)^3 (\sigma_1 \cdot \sigma_2) q \cdot \sigma_4. \tag{74}$$

We expect this to be a good approximation. In any event it makes the operator tractable.

The corresponding orbital integral is

$$J_{\mathrm{orb}} = g'_r \frac{2^3 3^2}{(3 + 2x^2)\sqrt{3 + 2x^2}} \times \frac{\phi_\pi(0)}{(\phi(0))^2} e^{-b_N^2 q^2/6} e^{-b_N^2 p_N^2((2x^2)/(3+2x^2))/6} = g_r \frac{81\sqrt{3}}{10\sqrt[4]{2}\phi^2(0)\pi^{3/2}} e^{-b_N^2 p_N^2((2x^2)/(3+2x^2))/6}. \tag{75}$$

We note with satisfaction that any uncertainties in the pion wave function (w.f.) have dropped out, at least if the nonlocal term in the exponential are ignored.

(b) The exchange term.

The amplitude takes the form

$$\mathcal{M} = \frac{1}{(2\pi)^{3/2}} \frac{1}{3\sqrt{3}} \frac{\sqrt{2m_\pi}}{\sqrt{2\sqrt{2}}} \mathbf{A}_2 \cdot \mathbf{A}_3 J_{\mathrm{orb}}. \tag{76}$$

Again there is no contribution in V-A theories, since the vector and the axial vector contributions are identical. In the case of the scalar contribution we find from Table I that

$$\mathbf{A}_2 \cdot \mathbf{A}_3 = \frac{\sqrt{2}}{9} \mathbf{q} \cdot \sigma_N. \tag{77}$$

In the case of the pseudoscalar contribution for the constituent quark masses we get

$$\mathbf{A}_2 \cdot \mathbf{A}_3 = \left[q^2 \frac{320\sqrt{2}(7x^2 + 1)(56x^2 + 3)^2}{147(28x^2 + 3)^3 m_N^2} + \frac{416\sqrt{2}(588x^4 - 77x^2 + 57)}{63(28x^2 + 3)^2 b_N^2 m_N^2} \right] \sigma_N \cdot q, \tag{78}$$

where $x = b_\pi / b_N$. Note the presence of the q^2 in the first term. This will lead to an operator with a different radial dependence, i.e. $F_i^{(k)}(x) \Rightarrow -\nabla^2 F_i^{(k)}(x)$ [see Eq. (10)]. The corresponding orbital integral for the exchange term is

$$J_{\mathrm{orb}} = g'_r \frac{3^3 2^7 \sqrt{2}}{(3 + 28x^2)\sqrt{3 + 28x^2}} \times \frac{\phi_\pi(0)}{(\phi(0))^2} e^{-b_N^2((\mathbf{q}-\mathbf{p}_N)^2/6)((4x^2)/(3+28x^2))} \tag{79}$$

or

$$J_{\mathrm{orb}} = g_r \frac{648\sqrt[4]{2}\sqrt{3}(2x^2 + 3)^{3/2}}{5\pi^{3/2}(28x^2 + 3)^{3/2}\phi^2(0) m_p \sqrt{m_\pi}} \times e^{-b_N^2((\mathbf{q}-\mathbf{p}_N)^2/6)((4x^2)/(3+28x^2))}. \tag{80}$$

In this instance the obtained results depend on the pion w.f. at the origin (via x).

V. RESULTS

Our main results are the coefficients $\alpha_{2\pi}$ and $\alpha_{1\pi}$, which multiply the standard nuclear matrix elements. We will not elaborate further on the new nonlocal terms (at the nucleon level).

A. The coupling coefficients $\alpha_{2\pi}$

Before presenting our results we should mention that in the elementary particle treatment [11] one can write

$$\alpha_{2\pi} = \frac{1}{6f_A^2} g_r^2 h_\pi^2 \left(\frac{m_\pi}{m_p}\right)^4, \tag{81}$$

obtained under the factorization approximation,

$$\langle \pi^+ | J_P j_P | \pi^- \rangle = \tfrac{5}{3} \langle \pi^+ | J_P | 0 \rangle \langle 0 | j_P | \pi^- \rangle, \quad \langle 0 | J_P | \pi^- \rangle = m_\pi^2 h_\pi. \tag{82}$$

The parameter h_π is given by

$$h_\pi = i\sqrt{2} \times 0.668 \frac{m_\pi}{m_d + m_u}. \tag{83}$$

Returning back to our approach we note that the nonrelativistic reduction is applicable in the constituent quark mass framework, $m_u = m_d \approx m_N/3$. In this case the pseudoscalar term contribution becomes

$$\alpha_{2\pi} = -0.0005 \quad (\text{for } x = 1.0),$$
$$\alpha_{2\pi} = -0.05 \quad (\text{for } x = 0.4).$$

We should compare this with the value obtained in V-A theory, see Eq. (24), using $f_{\pi NN}^2 = 0.08$ and $b_N = 1.0$ fm:

$$\alpha_{2\pi} = 0.013 \quad (x = 1) \quad \text{and} \quad \alpha_{2\pi} = 0.11 \quad (x = 1.0),$$

i.e. it is quite a bit smaller. It is also much smaller than the value 0.20 obtained in the elementary particle treatment [11] using current quark masses. This disagreement cannot be healed by the fact that in the present case we encounter a very strong dependence of the results on the pion size parameter, unless we use very unrealistic values of the pion size parameter. One expects, of course, an enhancement of the pseudoscalar contribution, if one uses the current quark masses, since they are assumed to be very small. Indeed in this way for typical values $x = 1$, $b_N = 1$ fm, $m_d = 5$ MeV, and $m_u = 10$ MeV we obtain $\alpha_{2\pi} = -1.3$ and $\alpha_{2\pi}(\Omega_{1\pi}) = 0.08$, which are very large. We should mention, however, that the validity of the nonrelativistic reduction at the quark level may be questionable in this case.

B. The coupling coefficients $\alpha_{1\pi}$

Before proceeding further we will briefly present how the coefficient $\alpha_{1\pi}$ was obtained in the context of the elementary particle treatment [11]:

$$\alpha_{1\pi} = -F_P \frac{1}{36 f_A^2} g_r h_\pi \left(\frac{m_\pi}{m_p}\right)^4. \tag{84}$$

The needed parameters were obtained using the factorization approximation one writes in the case of the $1-\pi$ mode,

$$\langle p|j_P J_P|n\pi\rangle = \tfrac{5}{3}\langle p|J_P|n\rangle\langle 0|J_P|\pi^-\rangle, \quad \langle p|J_P|n\rangle = F_P \approx 4.41. \tag{85}$$

The matrix element $\langle 0|J_P|\pi^-\rangle$ was given above [see Eq. (82)]. Thus these authors [11] find

$$\alpha_{1\pi} = -4.4 \times 10^{-2}. \tag{86}$$

Returning to our approach these coefficients are obtained in the following procedure: First we write

$$\frac{4\pi}{m_p^2}\mathcal{M} = c_{1\pi} g_r \frac{\sigma_N \cdot \mathbf{q}}{2m_p}. \tag{87}$$

Then, ignoring the momentum dependence in the exponential, we get

(1) Double beta decay only.
From Eqs. (69)–(71) we see that the only local contribution comes from the axial current:

$$c_{1\pi} = \frac{10\sqrt{2}\sqrt[4]{\pi}\sqrt{m_\pi}(m_d + m_u)}{9 g_r \sqrt{b_N^3} m_d m_N m_u} f_{1\pi}^A(x) \tag{88}$$

(current masses)

$$c_{1\pi} = \frac{20\sqrt{2}\sqrt[4]{\pi}\sqrt{m_\pi}}{3 g_r \sqrt{b_N^3} m_N^2} f_{1\pi}^A(x) \quad \text{(constituent masses)} \tag{89}$$

with

$$f_{1\pi}^A(x) = x^{3/2}. \tag{90}$$

Using $m_d = 5$ MeV, $m_u = 10$ MeV, and $g_r = 13.4$, we get

$$c_{1\pi} = 1.6 f_{1\pi}^A(x), \qquad \text{(current masses)}. \tag{91}$$

On the other hand, for the constituent masses we find

$$c_{1\pi} = 3.4 \times 10^{-2} f_{1\pi}^A(x), \quad \text{(constituent masses)}. \tag{92}$$

The corresponding coefficient that must multiply the nuclear matrix element is $\alpha_{1\pi}$:

$$\alpha_{1\pi} = c_{1\pi}\frac{f_{\pi NN}^2}{f_A^2} \tag{93}$$

$$\alpha_{1\pi} = 0.085 f_{1\pi}^A(x) \quad \text{(current masses)}, \qquad \alpha_{1\pi} = 1.8 \times 10^{-3} f_{1\pi}^A(x) \quad \text{(constituent masses)}. \tag{94}$$

(2) The direct term.
There is no contribution of the direct diagram if the nonlocal terms are ignored.

(3) The exchange term.
(i) In the case of the current quark masses we get the standard term:

$$c_{1\pi} = 1.0 \times 10^3 f_{1\pi}^{\text{cur}}(x). \tag{95}$$

In addition we have an operator which results from the term in the amplitude, which was cubic in q. Thus, we factor out the $q^2 m_\pi^2$ and absorb it in the effective transition operator. In the remaining coefficient we merely replace q^2 by m_π^2. Thus,

$$c_{1\pi} = 91.5 g_{1\pi}^{\text{cur}}(x). \tag{96}$$

Proceeding as above we get, respectively,

$$\alpha_{1\pi} = 51 f_{1\pi}^{\text{cur}}(x) \quad \text{or } \alpha_{1\pi} = 51 g_{1\pi}^{\text{cur}}(x). \tag{97}$$

The coefficient $f_{1\pi}^{\text{cur}}(x)$ is associated with the standard operator $\Omega_{1\pi}(x_\pi)$, while $g_{1\pi}^{\text{cur}}(x)$ must be linked with a new type of operator $\tilde{\Omega}_{1\pi}(x_\pi)$ with modified radial dependence, i.e. $F_i^{(k)}(x) \Rightarrow -\nabla^2 F_i^{(k)}(x)$ [see Eq. (10)]. Both coefficients are so normalized that $f_{1\pi}^{\text{cur}}(1) = g_{1\pi}^{\text{cur}}(1) = 1$. In any

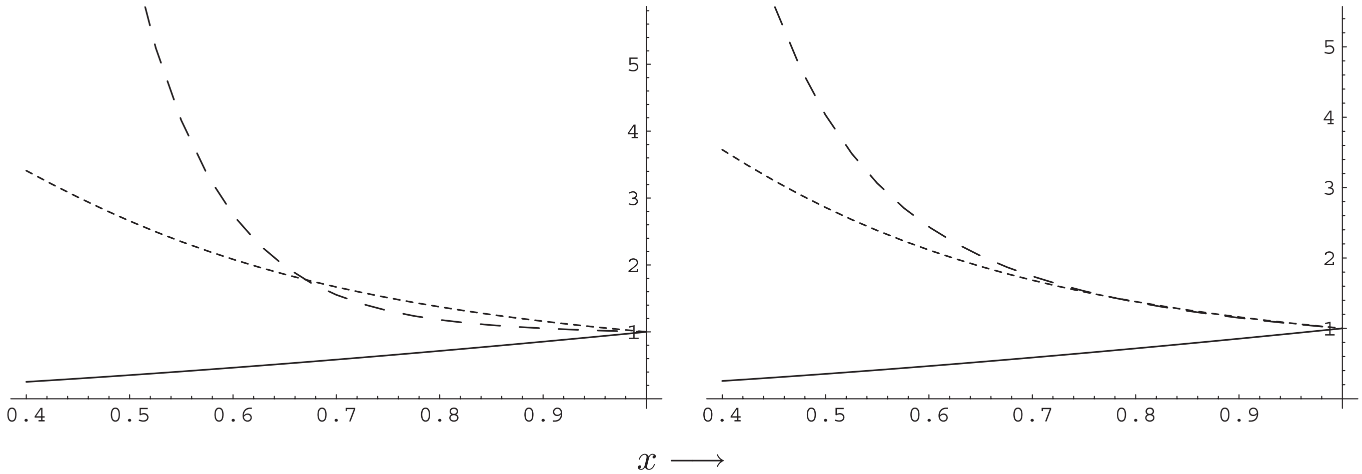

FIG. 7. The functions which provide the dependence of 1π amplitude on the pion size parameter through the variable $x = b_\pi/b_N$ are exhibited. On the left we show the relevant coefficients using the current quark masses. The continuous curve is associated with the coefficient $f^A_{1\pi}$ [see Eq. (94)], the long dash is associated with the exchange q-independent coefficient ($f^{\rm cur}_{1\pi}$) and the short dash with that of $g^{\rm cur}_{1\pi}$ [see Eq. (97)]. On the right we show the same quantities obtained with constituent quark masses.

case the use of current quark masses leads to very large values.

(ii) The constituent quark masses.
In this case we get:

$$c_{1\pi} = 1.37 f^{\rm con}_{1\pi}(x) \quad \text{or } c_{1\pi} = 1.72 g^{\rm con}_{1\pi}(x) \tag{98}$$

$$\alpha_{1\pi} = 0.071 f^{\rm con}_{1\pi}(x) \quad \text{or } \alpha_{1\pi} = 0.090 g^{\rm con}_{1\pi}(x). \tag{99}$$

Again the coefficient $f^{\rm con}_{1\pi}(x)$ is associated with the standard operator, while $g^{\rm con}_{1\pi}(x)$ must be linked with the operator $\tilde{\Omega}_{1\pi}(x_\pi)$, with $f^{\rm con}_{1\pi}(1) = g^{\rm con}_{1\pi}(1) = 1$.

The functions $f^A_{1\pi}$, $f^{\rm cur}1\pi(x)$, $g^{\rm cur}_{1\pi}(x)$ $f^A_{1\pi}$, $f^{\rm con}_{1\pi}(x)$, and $g^{\rm con}_{1\pi}(x)$ are shown in Fig. 7. For $x = 1$ for the standard local 1π operator considering all contributions mentioned above with constituent quark masses, we find $\alpha_{1\pi} = 7.3 \times 10^{-2}$, which is in size almost a factor of 2 larger than that obtained in the elementary particle treatment [11] [see Eq. (86)]. Note, however, that our results depend on the pion size parameter.

VI. DISCUSSION

In the present paper we have considered the effective $0\nu\beta\beta$ decay operator associated with the exchange of heavy particles mediated by pions in flight between nucleons. A harmonic oscillator nonrelativistic quark model in momentum space was employed for the pion and the nucleon. This allowed one to separate out the relative from the center of mass motion. The ratio of the pion to the nucleon harmonic oscillator parameter, $x = b_\pi/b_N$, was treated as a parameter. When needed, the constituent quark mass equal to $1/3$ of the nucleon mass was employed. The obtained results were compared to the elementary particle treatment, with current quark masses, previously employed. In the case of the two-pion mode we find a new term with different momentum dependence, which is not present in the elementary particle treatment. This gives rise to a new operator, which has the same structure as the one previously associated with the one-pion mechanism.

In connection with the one-pion mechanism, we found that there exist three diagrams, which cannot be distinguished in the elementary particle treatment, namely,

(1) Diagrams in which the $q\bar{q}$ is crated out of the vacuum via the strong interaction. In this case we employed the $^3P_0$ model. The strength of this interaction was fitted to the pion nucleon coupling g_r.
We distinguished two possibilities:
(i) The two interacting quarks participate only in the structure of the nucleon.
(ii) One of the interacting quarks participates in the structure of the pion.

(2) The $q\bar{q}$ is crated by the weak interaction itself.

Depending on the mechanism we encountered new nonlocal terms, i.e. terms which depend on the nucleon momentum. These will lead to new types of effective nuclear operators, which have not been examined up to now.

The results obtained in the present calculation depend among other things on the ratio of the pion to nucleon size parameters. Using reasonable values for this ratio, we obtain values of $\alpha_{1\pi}$, which are in good agreement with those obtained in the elementary particle treatment. Regarding the couplings $\alpha_{2\pi}$, however, we find that they are slightly smaller than those obtained in the elementary particle treatment in the case of the *V-A* theory. They are, however, quite a bit smaller than those obtained in the case of the pseudoscalar term, when the constituent quark masses are used. We can, of course, obtain much larger values for the pseudoscalar term, if the current quark masses are used. Admittedly, however, it may not be very consistent to do so in our approach, since it is essentially a nonrelativistic treatment. We thus suspect that the small current quark masses are behind the large values found in the elementary particle treatment.

In summary, taking into account the fact that a number of approximations are behind both approaches, we may say that there exists a reasonable agreement between them, which gives a degree of confidence in both. A more complete comparison can, of course, be made only after the inclusion in the calculation of the nuclear matrix elements of the new operators found in the present approach, namely: (i) the local operator $\tilde{\Omega}_{1\pi}(x_\pi)$ resulting from terms cubic in q and (ii) the nonlocal operators, which depend on the nucleon momentum.

ACKNOWLEDGMENTS

The work of one of us (J. D. V.) started while he was visiting RCNP, it was continued while he was in Tuebingen under a Alexander von Humboldt Research Award, and it was finished during a visit at CERN. He is indebted to these institutions as well as to the grant MRTN-CT-2006-035863 (UniverseNet) for their support and to Professor H. Toki, Professor A. Faessler, and Professor I. Antoniadis for their hospitality. He is also happy to acknowledge useful discussions with H. Ejiri.

APPENDIX

In this Appendix we will present the relevant formulas in the case the $q\bar{q}$ pair is produced in via the strong interaction. In the case of the one-pion contribution we get

(1) The direct term.

In this case we can write the amplitude as

$$\mathcal{M} = \frac{1}{(2\pi)^{3/2}} \frac{1}{3\sqrt{3}} \frac{\sqrt{2m_\pi}}{\sqrt{2\sqrt{2}}} \mathbf{A}_1 \cdot \mathbf{A}_2 J_{\rm orb}. \tag{A1}$$

In the case of the scalar contribution we find from Table I that

$$\mathbf{A}_1 \cdot \mathbf{A}_2 = -\frac{\sqrt{2}}{9} \mathbf{q} \cdot \sigma_N. \tag{A2}$$

In the case of the pseudoscalar contribution we find

$$\begin{aligned}\mathbf{A}_1 \cdot \mathbf{A}_2 = {}& \frac{(x^2+2)(m_d-m_u)p_N \cdot \sigma_1}{(2x^2+3)m_d m_u} \frac{(x^2+2)(m_d-m_u)p_N \cdot \sigma_2}{(2x^2+3)m_d m_u}\left(q + \frac{2p_N}{2x^2+3}\right) \cdot \sigma_4 \frac{1}{3} \frac{(x^2+2)(m_d-m_u)p_N\sigma_1}{(2x^2+3)m_d m_u} \\ & \times \frac{(m_d-m_u)}{\sqrt{2}\sqrt{2x^2+3}b_N m_d m_u} \frac{2\sqrt{2}}{\sqrt{2x^2+3}b_N} \sigma_2 \cdot \sigma_4 \frac{1}{3} \frac{(x^2+2)(m_d-m_u)p_N\sigma_2}{(2x^2+3)m_d m_u} \frac{(m_d-m_u)}{\sqrt{2}\sqrt{2x^2+3}b_N m_d m_u} \\ & \times \frac{2\sqrt{2}}{\sqrt{2x^2+3}b_N} \sigma_1 \cdot \sigma_4 + \frac{1}{3}\left(q + \frac{2p_N}{2x^2+3}\right) \cdot \sigma_4 \left(\frac{(m_d-m_u)}{\sqrt{2}\sqrt{2x^2+3}b_N m_d m_u}\right)^2 \sigma_1 \cdot \sigma_2. \end{aligned} \tag{A3}$$

In the limit $m_d = m_u$ the above expression vanishes. In the local approximation $\mathbf{p}_N = 0$, we find

$$\mathbf{A}_1 \cdot \mathbf{A}_2 = \frac{1}{3}\left(\frac{(m_d-m_u)}{\sqrt{4x^2+6}b_N m_d m_u}\right)^3 (\sigma_1 \cdot \sigma_2) q \cdot \sigma_4. \tag{A4}$$

We expect this to be a good approximation. In any event it makes the operator tractable. The corresponding orbital integral is given by Eq. (75).

(2) The exchange term.

The amplitude takes the form

$$\mathcal{M} = \frac{1}{(2\pi)^{3/2}} \frac{1}{3\sqrt{3}} \frac{\sqrt{2m_\pi}}{\sqrt{2\sqrt{2}}} \mathbf{A}_2 \cdot \mathbf{A}_3 J_{\rm orb}. \tag{A5}$$

Again there is no contribution in *V-A* theories, since the vector and the axial vector contributions are identical. In the case of the scalar contribution we find from I that

$$\mathbf{A}_2 \cdot \mathbf{A}_3 = \frac{\sqrt{2}}{9} \mathbf{q} \cdot \sigma_N. \tag{A6}$$

In the case of the pseudoscalar contribution we find

$$\mathbf{A}_2\cdot\mathbf{A}_3 = -\frac{(7x^2+1)(21(14x^2+1)m_d-(70x^2+9)m_u)((224x^2-9)m_d+3(224x^2+19)m_u)}{441(28x^2+3)^3m_d^2m_u^2}q\cdot\sigma_2 q\cdot\sigma_3 q\cdot\sigma_4$$
$$+\frac{((1400x^2+117)m_d-3(168x^2+23)m_u)((448x^2+45)m_u-21m_d)}{3528(28x^2+3)^3m_d^2m_u^2}p_N\cdot\sigma_2 p_N\cdot\sigma_3 p_N\cdot\sigma_4$$
$$+\frac{4(11m_d+5m_u)(21(14x^2+1)m_d-(70x^2+9)m_u)}{63(28x^2+3)^2b_N^2m_d^2m_u^2}\sigma_2\cdot\sigma_4 q\cdot\sigma_3$$
$$+\frac{2(11m_d+5m_u)(21m_d-(448x^2+45)m_u)}{63(28x^2+3)^2b_N^2m_d^2m_u^2}\sigma_2\cdot\sigma_4 p_N\cdot\sigma_3$$
$$+\frac{2q\cdot\sigma_2(7m_d+m_u)((224x^2-9)m_d+3(224x^2+19)m_u)}{63(28x^2+3)^2b_N^2m_d^2m_u^2}\sigma_3\cdot\sigma_4 q\cdot\sigma_2$$
$$-\frac{2(7m_d+m_u)((1400x^2+117)m_d-3(168x^2+23)m_u)}{63(28x^2+3)^2b_N^2m_d^2m_u^2}\sigma_3\cdot\sigma_4 p_N\cdot\sigma_2$$
$$-\frac{8(7x^2+1)(11m_d^2+(4x^2+7)m_u m_d+2(6x^2+1)m_u^2)\sigma_2\sigma_3}{3(28x^2+3)^2b_N^2m_d^2m_u^2}\sigma_2\cdot\sigma_3 q\cdot\sigma_4$$
$$+\frac{2p_N\cdot\sigma_4(11m_d^2+(4x^2+7)m_u m_d+2(6x^2+1)m_u^2)}{3(28x^2+3)^2b_N^2m_d^2m_u^2}\sigma_2\cdot\sigma_3 p_N\cdot\sigma_4. \tag{A7}$$

In the limit of ignoring the nonlocal terms we get

$$\mathbf{A}_2\cdot\mathbf{A}_3 = -\frac{(7x^2+1)(21(14x^2+1)m_d-(70x^2+9)m_u)((224x^2-9)m_d+3(224x^2+19)m_u)}{441(28x^2+3)^3m_d^2m_u^2}q\cdot\sigma_2 q\cdot\sigma_3 q\cdot\sigma_4$$
$$+\frac{4(11m_d+5m_u)(21(14x^2+1)m_d-(70x^2+9)m_u)}{63(28x^2+3)^2b_N^2m_d^2m_u^2}\sigma_2\cdot\sigma_4 q\cdot\sigma_3$$
$$+\frac{2q\cdot\sigma_2(7m_d+m_u)((224x^2-9)m_d+3(224x^2+19)m_u)}{63(28x^2+3)^2b_N^2m_d^2m_u^2}\sigma_3\cdot\sigma_4 q\cdot\sigma_2$$
$$-\frac{8(7x^2+1)(11m_d^2+(4x^2+7)m_u m_d+2(6x^2+1)m_u^2)\sigma_2\sigma_3}{3(28x^2+3)^2b_N^2m_d^2m_u^2}\sigma_2\cdot\sigma_3 q\cdot\sigma_4, \tag{A8}$$

where $x = b_\pi/b_N$. In the special case $m_u = m_d = \frac{m_N}{3}$ we get for the local terms,

$$\mathbf{A}_2\cdot\mathbf{A}_3 = -\frac{64(7x^2+1)(56x^2+3)^2}{441(28x^2+3)^3m_N^2}q\cdot\sigma_2 q\cdot\sigma_3 q\cdot\sigma_4 + \frac{256(56x^2+3)}{63(28x^2+3)^2b_N^2m_N^2}\sigma_2\cdot\sigma_4 q\cdot\sigma_3$$
$$+\frac{256(56x^2+3)}{63(28x^2+3)^2b_N^2m_N^2}\sigma_3\cdot\sigma_4 q\cdot\sigma_2 - \frac{32(4x^2+5)(7x^2+1)}{3(28x^2+3)^2b_N^2m_N^2}\sigma_2\cdot\sigma_3 q\cdot\sigma_4 \tag{A9}$$

while the nonlocal terms become

$$\mathbf{A}_2\cdot\mathbf{A}_3 = \frac{16(56x^2+3)^2}{441(28x^2+3)^3m_N^2}p_N\cdot\sigma_2 p_N\cdot\sigma_3 p_N\cdot\sigma_4 - \frac{256(56x^2+3)}{63(28x^2+3)^2b_N^2m_N^2}\sigma_2\cdot\sigma_4 p_N\cdot\sigma_3$$
$$-\frac{256(56x^2+3)}{63(28x^2+3)^2b_N^2m_N^2}\sigma_3\cdot\sigma_4 p_N\cdot\sigma_2 + \frac{8(4x^2+5)}{3(28x^2+3)^2b_N^2m_N^2}\sigma_2\cdot\sigma_3 p_N\cdot\sigma_4. \tag{A10}$$

The first term of the local equation can be cast in the more suitable form by noting that

$$\sigma_2\cdot q\sigma_3\cdot q\sigma_4\cdot q = \frac{q^2}{3}\left(\sigma_2\cdot\sigma_3\sigma_4\cdot q - \frac{2\sqrt{3}}{\sqrt{5}}[(\sigma_2\times\sigma_3)k_{12} = 2\times\sigma_4]^{k=1}\cdot q\right). \tag{A11}$$

Using the spin matrix elements of Table I, we finally get, using the current quark masses

$$\mathbf{A}_2 \cdot \mathbf{A}_3 = 5\sqrt{2}q^2\Bigg[\frac{(7x^2+1)(21(14x^2+1)m_d-(70x^2+9)m_u)((224x^2-9)m_d+3(224x^2+19)m_u)}{1323(28x^2+3)^3 m_d^2 m_u^2} + 26\sqrt{2}\frac{(-7(224x^2-75)m_d^2+2(1176x^4-938x^2+93)m_u m_d+(7056x^4+2212x^2+201)m_u^2)}{567(28x^2+3)^2 b_N^2 m_d^2 m_u^2}\Bigg] \times \sigma_N \cdot q, \tag{A12}$$

where σ_N is the nucleon spin, while for the constituent quark masses we get

$$\mathbf{A}_2 \cdot \mathbf{A}_3 = \Bigg[q^2\frac{320\sqrt{2}(7x^2+1)(56x^2+3)^2}{147(28x^2+3)^3 m_N^2} + \frac{416\sqrt{2}(588x^4-77x^2+57)}{63(28x^2+3)^2 b_N^2 m_N^2}\Bigg]\sigma_N \cdot q. \tag{A13}$$

Note the presence of the q^2 in the first term. This will lead to an operator with a different radial dependence, i.e. $F_i^{(k)}(x) \Rightarrow -\nabla^2 F_i^{(k)}(x)$ [see Eq. (10)].

---

[1] Y. Fukuda *et al.* (The Super-Kamiokande Collaboration), Phys. Rev. Lett. **86**, 5651 (2001); **81**, 1562 (1998); **81**, 1158 (1998); **82**, 1810 (1999); **85**, 3999 (2000).

[2] Q. R. Ahmad *et al.* (The SNO Collaboration), Phys. Rev. Lett. **89**, 011302 (2002); **89**, 011301 (2002); **87**, 071301 (2001); K. Lande *et al.* (Homestake Collaboration), Astrophys. J. **496**, 505 (1998); W. Hampel *et al.* (The Gallex Collaboration), Phys. Lett. B **447**, 127 (1999); J. N. Abdurashitov *et al.* (Sage Collaboration), Phys. Rev. C **59**, 2246 (1999); G. L Fogli, E. Lisi, A. Marrone, D. Montanino, and A. Palazzo, Phys. Rev. D **66**, 053010 (2002).

[3] K. Eguchi *et al.* (The KamLAND Collaboration), Phys. Rev. Lett. **90**, 021802 (2003).

[4] J. N. Bahcall, M. Gonzalez-Garcia, and C. Peã-Garay, J. High Energy Phys. 02 (2003) 009.

[5] V. Barger and D. Marfatia, Phys. Lett. B **555**, 144 (2003).

[6] J. D. Vergados, Phys. Rep. **361**, 1 (2002).

[7] J. D. Vergados, Phys. Rev. C **24**, 640 (1981).

[8] J. D. Vergados, Nucl. Phys. **B250**, 618 (1985).

[9] J. D. Vergados, Phys. Lett. B **184**, 55 (1987).

[10] A. Faessler, S. Kovalenko, F. Simkovic, and J. Schwieger, Phys. Rev. Lett. **78**, 183 (1997).

[11] A. Faessler, S. Kovalenko, and F. Simkovic, Phys. Rev. D **58**, 055004 (1998).

[12] A. Wodecki W. A. Kaminski, and F. Simkovic, Phys. Rev. D **60**, 115007 (1999).

[13] A. Faessler, T. Gutsche, S. Kovalenko, and F. Simkovic, Phys. Rev. D **77**, 113012 (2008).

[14] G. Prezeau, M. Ramsey-Musolf, and P. Vogel, Phys. Rev. D **68**, 034016 (2003).

[15] G. Pantis, F. Simkovic, J. D. Vergados, and A. Faessler, Phys. Rev. C **53**, 695 (1996).

[16] E. M. Henley, T. Oka, and J. Verhados, Phys. Lett. **166B**, 274 (1986).

[17] L. Micu, Nucl. Phys. **B10**, 521 (1969).

[18] A. Leyaouanc, L. Oliver, O. P‘ene, and J. Raynal, Phys. Rev. D **8**, 2223 (1973).

[19] A. Leyaouanc, L. Oliver, O. P‘ene, and J. Raynal, Phys. Rev. D **9**, 1415 (1974).

[20] A. Leyaouanc, L. Oliver, O. P‘ene, and J. Raynal, Phys. Rev. D **18**, 1591 (1978).

[21] D. T. da Silva, M. L. L. da Silva, J. N. de Qua, and D. Hadjimichef, Phys. Rev. D **78**, 076004 (2008).