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Semiclassical rigid strings with two spins in AdS_5

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Semiclassical spinning string states in AdS_5 are, in general, characterized by the three $SO(2,4)$ conserved charges: the energy E and the two spins S_1 and S_2. We discuss several examples of explicit classical solutions for rigid closed strings of (bended) circular shape with two nonzero spins. In particular, we identify a solution that should represent a state that has minimal energy for large values of the two equal spins. Similarly to the spiky string in AdS_3, in the large-spin limit this string develops long "arcs" that stretch towards the boundary of AdS_5. This allows the string to increase the spin while having the energy growing only logarithmically with $S = S_1 + S_2$. The large-spin asymptotics of such solutions is effectively controlled by their near-boundary parts which, as in the spiky string case, happen to be $SO(2,4)$ equivalent to segments of the straight folded spinning string. As a result, the coefficient of the $\log S$ term in the string energy should be given, up to an overall $3/2$ coefficient, by the same universal scaling function (cusp anomaly) as in the folded string case, to all orders in the inverse string tension or strong-coupling expansion.

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I. INTRODUCTION

Trying to achieve a better understanding of the spectrum of strings in $AdS_5 \times S^5$ and thus of strong-coupling expansion of the $\mathcal{N} = 4$ super Yang-Mills (SYM) anomalous dimensions it is of interest to study generalizations of folded [1,2] and spiky [3] strings with single spin in the AdS_3 part of AdS_5 to the case of strings moving in full AdS_5 and carrying two spins. The dimension of AdS_5 space implies that generic states may be labeled by the values of the three $SO(2,4)$ Cartan generators (E, S_1, S_2). Such semiclassical states should describe strong-coupling behavior of dimensions of gauge-theory operators outside the $SL(2)$ sector represented, e.g., by operators like[1] $\mathrm{Tr}[(D_0 + D_3)^{S_1}(D_1 + iD_2)^{S_2}\Phi^k]$.

One may expect that for large spins $S_1, S_2 \gg 1$ the string should stretch towards the boundary and the semiclassical states with minimal energy for given values of the spins should then again have the energy scaling logarithmically with the spins, $E - S \sim \ln S + \ldots$, $S = S_1 + S_2$. Indeed, as we shall discuss below, for the particular circular strings with $S_1 = S_2$ case one finds $E - S = \frac{3}{2} f(\lambda) \times \ln S + \ldots$, where $f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + \ldots$ is the same scaling function as in the 1-spin folded string case. The coefficient of the leading $\ln S$ term is controlled by the asymptotic large-spin limit of the solution which happens to be universal. The extra factor of $3/2$ is due to 3 "arcs" that the large-spin circular string has, compared to 2 arcs of the folded string spinning around zero. Remarkably, the same behavior of the energy was found very recently from the asymptotic Bethe ansatz approach at weak coupling in [4].

The large-spin limit corresponds to the case when some parts of the string approach the boundary of AdS_5; it may thus be of interest also for constructing new Wilson loop surfaces for open strings ending at the boundary. As was shown in [5], the large-spin limit of the folded string in AdS_3 is related via an analytic continuation and an $SO(2,4)$ transformation to the open-string solution ending on a null cusp at the boundary [6–8]. This suggests that asymptotic limits of more general solutions in AdS_5 may also be used for constructing interesting open-string solutions lying outside AdS_3 (cf. [9–11]).

Starting with the bosonic string in conformal gauge in AdS_5 space

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho(d\theta^2 + \cos^2\theta d\phi_1^2 + \sin^2\theta d\phi_2^2), \quad (1.1)$$

a rigid rotating 2-spin string ansatz is $(0 < \sigma \leq 2\pi)$ [12][2]

$$t = \kappa\tau, \qquad \rho = \rho(\sigma), \qquad \theta = \theta(\sigma), \qquad \phi_1 = \omega_1\tau, \qquad \phi_2 = \omega_2\tau. \quad (1.2)$$

The simplest circular solution of that type is a round string with $\rho = \rho_0 = \mathrm{const}$, $\theta = \frac{\pi}{4}$, $\omega_1 = \omega_2$ and thus with

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[1]Here Φ^k stands for some combination of fields of SYM theory (which may enter at different places under the trace) with k being small and fixed. In the semiclassical string limit in AdS_5 that we will consider below only the values of the spins S_1 and S_2 will matter, i.e., will be "visible" on the string theory side.

[2]An equivalent form of this ansatz is $t = \kappa\tau$, $\rho = \rho(\sigma)$, $\theta' = \frac{\pi}{4}$, $\phi_1' = \omega_1\tau + \theta(\sigma)$, $\phi_2' = \omega_2\tau - \theta(\sigma)$. This follows from writing this solution in embedding coordinates and applying a global $SO(4)$ rotation.

$S_1 = S_2$ found in [12]. It does not, however, represent a state with a minimal energy for given values of the spins (and is, indeed, unstable under small fluctuations for large enough value of the spin parameter [12]).[3] To get a stable lower-energy solution with $S_1 = S_2$ one is to relax the $\rho =$ const condition, allowing the string to develop, in the large-spin limit, long arcs stretching to infinity (i.e., the boundary of AdS_5) and carrying most of the energy.

A general approach to finding such rigid-string solutions in S^5 or AdS_5 was developed in [17] using the reduction of the conformal-gauge string sigma model to the 1D Neumann integrable model.[4] The solutions in S^5 and AdS_5 are closely related via an analytic continuation. Starting with the $R^{2,4}$ embedding coordinates satisfying $|Y_{05}|^2 - |Y_{12}|^2 - |Y_{34}|^2 = 1$ ($Y_{nm} \equiv Y_n + iY_m$)

$$Y_{05} = y_0 e^{it}, \qquad Y_{12} = y_1 e^{i\phi_1}, \qquad Y_{34} = y_2 e^{i\phi_2}, \qquad y_0^2 - y_1^2 - y_2^2 = 1, \tag{1.3}$$

where the choice of coordinates in (1.1) corresponds to

$$y_0 = \cosh\rho, \qquad y_1 = \sinh\rho\cos\theta, \qquad y_2 = \sinh\rho\sin\theta, \tag{1.4}$$

and assuming that $y_a = y_a(\sigma) = y_a(\sigma + 2\pi)$ and $t = \kappa\tau$, $\phi_i = \omega_i\tau$ one finds that the equations for y_a are those of a harmonic oscillator constrained to move on a 2D hyperboloid—an integrable system with 2 integrals of motion b_1, b_2 with $b_1 + b_2 = \kappa^2 + \omega_1^2 + \omega_2^2$.

Following the discussion in [17], the closed-string solutions will be parametrized by the three "frequencies" $\omega_a = (\omega_0, \omega_1, \omega_2)$, $\omega_0 \equiv \kappa$, as well as by the two integrals of motion b_i. Four of these parameters, say (ω_i, b_i), may be viewed as independent coordinates on the moduli space of such solitons. The closed-string periodicity condition in σ implies that solutions will be classified by two integer "winding numbers" n_i related to ω_a and b_i. In general, the energy E will be a function not only of S_1, S_2 but also of the values of n_i. Depending on the values of these parameters the string's shape may be of two distinct types: (i) *"folded"*, i.e., having topology of an interval, or (ii) *"circular"*, i.e., having topology of a circle. A folded string may be straight as in the 1-spin case [1] or bent. A circular string may be a round circle as in [12] or may have a more general "bent circle" shape. To have a folded string we need all derivatives y_a' vanishing at the two points of the σ interval. To have a bend we need only one out of the two independent coordinates having their derivative vanishing in a middle point of the σ interval.[5]

It is instructive to recall [17] how classical solutions with such shapes appear in the flat $R^{1,4}$ Minkowski space which corresponds to the $\rho \to 0$ limit of (1.1) or the limit of (1.3) when $y_0 \to 1$ and y_1, y_2 are small. The 5 independent string coordinates can thus be parametrized by $t = \kappa\tau$ and Y_{12} and Y_{34} in (1.3) and a solution of interest is given by

$$\omega_1 = n_1, \qquad \omega_2 = n_2, \qquad y_1 = a_1\sin(n_1\sigma), \qquad y_2 = a_2\sin[n_2(\sigma + \sigma_0)], \tag{1.5}$$

where n_i are integers, $\sigma_0 =$ const and $\kappa^2 = n_1^2 a_1^2 + n_2^2 a_2^2$. Then the energy and the two spins are $E = \frac{\kappa}{\alpha'}$, $S_i = \frac{n_i a_i^2}{2\alpha'}$, i.e., $E = \sqrt{\frac{2}{\alpha'}(n_1 S_1 + n_2 S_2)}$. To get the states on the leading Regge trajectory (having minimal energy for given values of the two spins) one is to choose $n_1 = n_2 = 1$.[6] The shape of the string depends on the values of σ_0 and n_1, n_2: it can be either circular or folded.[7]

The structure of the soliton strings in the curved $R_t \times S^5$ or AdS_5 case is analogous [17]. Indeed, the equations of motion of the Neumann system are linearized on the Jacobian of the hyperelliptic curve. The general solution for $y_a(\sigma)$ in (1.4) is then expressed in terms of hyperelliptic functions (theta functions defined on the Jacobian of the hyperelliptic genus 2 Riemann surface).[8] The shape of the physical string at fixed moment of time lying on the 2D hyperboloid described by y_a in (1.3) will depend on the values of n_1, n_2 and other moduli parameters and may be of the bent folded type or of the circular type.[9]

[3] As was suggested in [13], possible gauge-theory duals of such circular strings are operators built out of self-dual part of gauge field strength [14]. Like spiky strings corresponding to "excited" states in the band of states in the $sl(2)$ sector [15,16], such circular strings should correspond to higher-level or excited 2-spin states.

[4] A more general rigid-string ansatz where in addition to $\rho = \rho(\sigma)$, $\theta = \theta(\sigma)$ one has $\phi_1 = \omega_1\tau + \alpha_1(\sigma)$, $\phi_2 = \omega_2\tau + \alpha_2(\sigma)$ and where the corresponding 1D system is the Neumann-Rosochatius one was considered in [18]. Since in this case the string is stretched not only in θ but also in the other two angles, one expects that such solutions should have more energy for given values of the spins; we will not consider this more general case in what follows.

[5] If that vanishing happens in k points we will have a string with k bends.

[6] One is also to set $\sigma_0 = \frac{\pi}{2}$ since otherwise the $O(4)$ angular momentum has other nonzero components apart from $S_{12} = S_1$, $S_{34} = S_2$ and thus the solution can be rotated to a single-spin one. For example, for $\sigma_0 = 0$ the solution is equivalent to the straight folded string.

[7] If $\frac{\sigma_0}{\pi}$ is irrational then the string always has a circular shape and, in general, will not be lying in one plane, i.e., will have one or several bends [17]. For rational values of σ_0 the string can be either circular or folded, depending on the values of n_1, n_2. For $\sigma_0 = 0$ if both n_1 and n_2 are either even or odd and different then the string is folded and has several bends (in the 13 and 24 planes). If $\frac{\sigma_0}{\pi} = \frac{1}{2n_2}$ and $n_1 = n_2$ the string is an ellipsoid, becoming a round circle in the special case of $a_1 = a_2$. The string is also circular if n_1 is even and n_2 is odd. If, however, n_1 is odd and n_2 is even the string is folded and bent.

[8] The image of the string in the Jacobian (Liouville torus) winds around two nontrivial cycles with the winding numbers n_1 and n_2. The size and the shape of the Liouville torus are governed by the moduli (ω_i, b_i). For given n_1, n_2, two of the 4 independent parameters (ω_i, b_i) are then uniquely determined by the periodicity conditions.

[9] The simplest round-circle string solution [12] mentioned above corresponds to the case $\omega_1 = \omega_2$ and $b_1 = b_2$.

There are a few special cases when solutions simplify, i.e., when the hyperelliptic surface degenerates into an elliptic one so that $y_a(\sigma)$ can be expressed in terms of the standard elliptic functions (as in the 1-spin case [1,2,19]). Such special solutions are much easier to analyze and potentially compare to the corresponding states on the gauge-theory side. These are the cases when 2 of the 3 frequencies ($\kappa = \omega_0,\ \omega_1,\ \omega_2$) or 2 of the integrals of motion b_1, b_2 are equal. In Secs. II and IV below we shall consider two of such special cases:

$$\text{(i) } \omega_1 = \omega_2, \qquad \text{(ii) } \kappa = \omega_2. \tag{1.6}$$

As we shall see, in these cases the string is of circular type. In the first case $S_1 = S_2$ while in the second one $S_1 \neq S_2$.[10] The case of $b_1 = b_2$ will be discussed in Appendix B.

The solution corresponding to the first case in (1.6) was found in the S^5 setting in Sec. 4.2 of [17]; its direct AdS_5 counterpart which has $S_1 = S_2$ was implicit in Sec. 6 of [17] and was described explicitly in [20] (this solution was also generalized to a nonzero value of one angular momentum in S^5 in [21]).

In the first case with $\omega_1 = \omega_2 \equiv \omega$ the string sigma model equations corresponding to (1.1) can be readily solved by integrating the equation for θ. The integral of the equation for ρ is given by the conformal-gauge condition, so that we end up with

$$\theta' = \frac{c}{\sinh^2\rho}, \tag{1.7}$$

$$\rho'^2 = \kappa^2\cosh^2\rho - \frac{c^2}{\sinh^2\rho} - \omega^2\sinh^2\rho, \tag{1.8}$$

where c is an integration constant. When $c = 0$ the solution reduces to the single-spin folded string one. Since θ' does not vanish for $c \neq 0$ (unless at the points where $\rho \to \infty$ which correspond to the large-spin asymptotics) this solution has a circular shape. The simplest among such 2-spin solutions [12] has ($\sinh^2\rho_0 = \frac{c}{m}$)

$$\rho(\sigma) = \rho_0 = \text{const}, \qquad \theta(\sigma) = m\sigma, \qquad m = 1, 2, 3, \ldots \tag{1.9}$$

and describes a rigid circular string wrapped m times in θ and rotating in two planes with equal spins $S_1 = S_2 = S/2$. The energy of this solution for large $\mathcal{S} = \frac{S}{\sqrt{\lambda}} \gg 1$ scales is

$$E - S = \sqrt{\lambda}[\tfrac{3}{4}(2m^2\mathcal{S})^{1/3} + \ldots], \tag{1.10}$$

i.e., it grows faster than $\ln S$. This solution is unstable for large enough spin [12,13], suggesting that there should be a similar $S_1 = S_2$ solution having lower energy for given spins.

To find such a lower-energy 2-spin state one is to consider solutions of (1.8) with nonconstant ρ: that will allow one to increase the spin by stretching parts of the circular-shaped string towards the boundary. This is energetically more favorable than putting the whole round string at large value of ρ as in the case of the "round-circle" solution (1.9). In contrast to spikes [3], these stretched arcs will still have regular shape: the induced metric here $ds^2 = (\rho'^2 + \frac{c^2}{\sinh^2\rho})(-d\tau^2 + d\sigma^2)$ is everywhere smooth as long as $c \neq 0$.

We shall review and clarify the corresponding solution [17,20] of Eqs. (1.7) and (1.8) in Sec. II. As we shall discuss in Sec. III, its large-spin limit when its $E - S$ scales as $\ln S$ is effectively controlled by the asymptotic "single-arc" open-string solution corresponding to the case when $\kappa = \omega_1 = \omega_2$. This solution is found to be equivalent, by an $SO(2,4)$ transformation, to the asymptotic limit of the folded or spiky string. This implies that the coefficient of the leading $\ln S$ term should be proportional to the universal scaling (cusp anomaly) function; that should be true to all orders in the string $\alpha' = \frac{1}{\sqrt{\lambda}}$ expansion.

In Sec. II we shall also compute the first subleading coefficient in the large-spin expansion in the classical string energy and compare it to the one in the spiky string case [3,22]. Our result for the leading terms in large-spin expansion of the classical energy of a circle-shaped string with two equal spins ($S_1 = S_2 = \frac{1}{2}S$), winding number m and $n > \frac{m}{2}$ arcs is

$$E - S = \frac{n\sqrt{\lambda}}{2\pi}\left(\ln\frac{16\pi S}{n\sqrt{\lambda}} - 1 + 2\ln\sin\frac{\pi m}{n}\right) + \mathcal{O}\left(\frac{1}{S}\right). \tag{1.11}$$

The $\ln S$ large-spin asymptotics of the energy of this $S_1 = S_2$ solution was first observed in [20]. The minimal energy for given spins is found for $m = 1$, $n = 3$ when

$$E_{\min} - 2S_1 = \frac{3}{2} \times \frac{\sqrt{\lambda}}{\pi}\left(\ln\frac{8\pi S_1}{\sqrt{\lambda}} - 1\right) + \mathcal{O}\left(\frac{1}{S}\right). \tag{1.12}$$

This is $\frac{3}{2}$ times the expression for the folded string with a single spin $S_1 = \frac{1}{2}S$ which represents the minimal energy state for given spin in AdS_3 (or the ground state in the $sl(2)$ sector).

Remarkably, this matches the strong-coupling prediction following from the very recent analysis [4] of the full asymptotic Bethe ansatz (ABA) equations [23].[11] This agreement is, of course, not unexpected as the scaling ("thermodynamic") limit of the full version of the

[10] While the solution corresponding to (i) appears to have minimal energy for given $S_1 = S_2$ this is likely not to be the case for the solution in (ii): there should be a folded bended string solution that carries less energy for given $S_1 \neq S_2$. When discussing minimal energy for given spins we assume we also choose minimal possible values for the "winding numbers" n_1, n_2.

[11] We thank A. Rej for informing us about the results of [4].

strong-coupling limit of the Bethe ansatz equations [24] should reproduce finite-gap solutions of the classical string sigma model [15,25]. Still, the precise identification of a particular string solution that has a clear space-time interpretation with a particular Bethe root distribution is, in general, nontrivial, especially for states outside simplest rank-one sectors.

In [4] a 1-cut Bethe root distribution was found, which allowed the authors to compute the leading and subleading[12] terms in the large S expansion of the energy. To describe the strong-coupling solution for a finite value of semiclassical spin one should go beyond the 1-cut solution of the ABA equations, i.e., one should identify the 2-cut distributions with elliptic (genus 1) string solutions and 3-cut distributions with solutions associated to the hyperelliptic (genus 2) Riemann surface. The large S asymptotics correspond to the case when cuts collide. The rigid 2-spin solutions that we discuss here are generically hyperelliptic; it should be possible to determine which root density that solves integral equations following from the ABA should correspond to the generic hyperelliptic rigid-string solution with a finite (semiclassical) value of S.

In Sec. IV we shall consider the second special case—that of $\kappa = \omega_2$ in (1.6). This case is closely related to the previous one via an analytic continuation in which the roles of κ and ω_1 and y_0 and y_1 are interchanged (as implied by the general discussion in [17]). Here in general $S_1 \neq S_2$, but to allow for the existence of a large-spin limit one is to go back to the case of $S_1 = S_2$. The corresponding asymptotic solution is again the one of Sec. III.

In Sec. V we shall comment on large-spin behavior of more general solutions described by the ansatz (1.2) and make some concluding remarks.

In Appendix A we shall discuss a special case of the solution of Sec. IV. In Appendix B we shall review the approach of [17] to solution of equations corresponding to the rigid-string ansatz (1.2) and consider in detail the special case of $b_1 = b_2$ when the circular string solution is again expressed in terms of elliptic functions. In this case the energy is found to scale with the large total spin $S = S_1 + S_2$ as in the round-circle $S_1 = S_2$ case, i.e., $E - S \sim S^{1/3}$.

II. RIGID CIRCULAR $S_1 = S_2$ SOLUTION: $\omega_1 = \omega_2$

Setting $x \equiv y_0 = \cosh\rho$ in (1.8) we obtain the equation for $x(\sigma)$

$$x'^2 = \kappa^2 x^2(x^2 - 1) - c^2 - \omega^2(x^2 - 1)^2, \qquad x \equiv \cosh\rho, \tag{2.1}$$

or, equivalently [17,20]

$$x'^2 = (\omega^2 - \kappa^2)(x^2 - a_-)(a_+ - x^2), \tag{2.2}$$

where

$$a_\pm = \frac{2\omega^2 - \kappa^2 \pm \sqrt{\kappa^4 - 4c^2(\omega^2 - \kappa^2)}}{2(\omega^2 - \kappa^2)}. \tag{2.3}$$

Thus

$$c^2 = (a_+ - 1)(a_- - 1)(\omega^2 - \kappa^2),$$
$$\kappa^2 = \omega^2 \frac{a_+ + a_- - 2}{a_+ + a_- - 1} = \omega^2 \frac{\mu(2 - \nu)}{\nu + \mu - \nu\mu} \tag{2.4}$$

where we introduced the parameters μ and ν related to $a_\pm$ by $(0 < \mu < \nu \leq 1)$[13]

$$\mu = \frac{a_+ - a_-}{a_+}, \qquad \nu = \frac{a_+ - a_-}{a_+ - 1},$$
$$a_+ = \frac{\nu}{\nu - \mu}, \qquad a_- = \frac{\nu(1 - \mu)}{\nu - \mu} \tag{2.5}$$

$x(\sigma)$ takes values in $\sqrt{a_-} \leq x \leq \sqrt{a_+}$, i.e., the radial string coordinate ρ changes in the interval

$$\rho_- \leq \rho \leq \rho_+, \qquad a_\pm = \cosh^2\rho_\pm. \tag{2.6}$$

Since $x = \cosh\rho \geq 1$ we have $\omega \geq \kappa$ and $1 \leq a_- < a_+$. The solution of equation for ρ (2.2) is

$$x = \cosh\rho = \frac{\sqrt{a_-}}{\mathrm{dn}[c\sqrt{\frac{a_+}{(a_+ - 1)(a_- - 1)}}\sigma, \mu]}, \tag{2.7}$$

where dn is the Jacobi elliptic function.

We will assume that ρ starts at its minimum ρ_- at $\sigma = 0$ and goes to its maximum ρ_+ at $\sigma = \frac{\pi}{n}$ where n is an integer number. To get a closed string defined on $0 \leq \sigma \leq 2\pi$ we need to glue together $2n$ such segments (or n string arcs where ρ first grows to maximum and then comes back) imposing the periodicity condition $\rho(\sigma + 2\pi) = \rho(\sigma)$. Since the period of the $\mathrm{dn}[z, \mu]$ function is $2K[\mu]$[14] we have

$$c\sqrt{\frac{a_+}{(a_+ - 1)(a_- - 1)}} \frac{2\pi}{n} = 2K[\mu], \tag{2.8}$$

i.e., the periodic solution can be written as

$$x = \cosh\rho = \frac{\sqrt{a_-}}{\mathrm{dn}[\frac{K[\mu]n}{\pi}\sigma, \mu]}. \tag{2.9}$$

[12]To get the subleading correction in ABA one splits the root distribution $\rho(u) = \rho_0(u) + r(u)$, where ρ_0 is the root distribution with a 1-cut support. To get the correction $r(u)$, one solves the resulting integral equation on the whole real axis. Thus, the subleading large S result contains information from outside the actual 1-cut region. On the string side, this corresponds to the fact that to get the subleading in large S correction, one needs to use extra information about the exact solution, not only its leading asymptotic form.

[13]Our notation are related to those of [20,21] by $\mu \to m$, $\nu \to n$, $m \to M$, $n \to N$.

[14]Here $K[\mu] = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-\mu\sin^2\alpha}}$. Below we shall also use $E[\mu] = \int_0^{\pi/2} d\alpha\sqrt{1 - \mu\sin^2\alpha}$.

At $\sigma = \frac{\pi}{n}$ we indeed have $x = \sqrt{a_+}$ since $\mathrm{dn}[K[\mu], \mu] = \sqrt{1-\mu}$. Note that (2.8) and (2.4) imply

$$\omega = \sqrt{\frac{a_+ + a_- - 1}{a_+}}\frac{K[\mu]n}{\pi} = \sqrt{\frac{\nu + \mu - \nu\mu}{\nu}}\frac{K[\mu]n}{\pi}. \tag{2.10}$$

To find θ it is more convenient to use the equation for $x(\theta) = x(\theta(\sigma))$ which follows from (1.7) and (1.8)

$$\frac{dx}{d\theta} = \pm\frac{(x^2-1)\sqrt{(a_+ - x^2)(x^2 - a_-)}}{\sqrt{(a_+ - 1)(a_- - 1)}}. \tag{2.11}$$

Solving this equation for $\theta(x)$ with the initial condition $\theta(x_{\min} = \sqrt{a_-}) = 0$ [i.e., $\theta(\sigma = 0) = 0$] we obtain[15]

$$\theta(x) = \sqrt{\frac{a_- - 1}{a_+(a_+ - 1)}}\Big(\Pi[\nu, \mu] - \Pi\Big[\nu, \arcsin\sqrt{\frac{a_+ - x^2}{a_+ - a_-}}, \mu\Big]\Big). \tag{2.12}$$

Then

$$\theta(\sqrt{a_+}) = \sqrt{\frac{a_- - 1}{a_+(a_+ - 1)}}\Pi[\nu, \mu]. \tag{2.13}$$

The expression (2.12) is valid for one half-arc of the string with $\rho_- < \rho < \rho_+$, i.e., it gives $\theta(\sigma) = \theta(\cosh\rho(\sigma))$ for $0 \le \sigma \le \frac{\pi}{n}$. Full solution for $\theta(\sigma)$ can be easily obtained using (2.7). To cover the $(0, 2\pi)$ σ interval we should glue together $2n$ segments given by (2.12). The condition for having a closed string gives

$$\theta(2\pi) = 2\pi m = 2n\theta\Big(\frac{\pi}{n}\Big), \qquad m = 1, 2, 3, \ldots, \tag{2.14}$$

where we introduced an arbitrary winding number m. Plugging this into (2.12) gives

$$\pi\frac{m}{n} = \sqrt{\frac{a_- - 1}{a_+(a_+ - 1)}}\Pi[\nu, \mu] = \sqrt{\frac{(1-\nu)(\nu-\mu)}{\nu}}\Pi[\nu, \mu]. \tag{2.15}$$

We thus need $m \neq 0$ in order to satisfy this condition. One can show that the right-hand side in (2.15) is always smaller than $\frac{\pi}{2}$. This implies the condition on the parameters [20]

$$2m < n, \quad \text{i.e.,} \quad n \ge 3. \tag{2.16}$$

The minimal choice is thus $m = 1$, $n = 3$.

A. Energy and spins

The energy and the two spins are defined by ($E = \sqrt{\lambda}\mathcal{E}$, $S_i = \sqrt{\lambda}\mathcal{S}_i$)

$$\mathcal{E} = \kappa\int_0^{2\pi}\frac{d\sigma}{2\pi}\cosh^2\rho, \quad \mathcal{S}_1 = \omega_1\int_0^{2\pi}\frac{d\sigma}{2\pi}\sinh^2\rho\cos^2\theta, \quad \mathcal{S}_1 = \omega_2\int_0^{2\pi}\frac{d\sigma}{2\pi}\sinh^2\rho\sin^2\theta. \tag{2.17}$$

For $\omega_1 = \omega_2$, periodic $\rho(\sigma)$ and $\theta(\sigma + 2\pi) = \theta(\sigma) + 2\pi m$ one can argue that $\mathcal{S}_1 = \mathcal{S}_2$. This follows, e.g., from the vanishing of the integral $\int d\theta\sinh^4\rho\cos 2\theta$ obtained by converting the integral over σ into the integral over θ using (1.7). One can similarly show that other "non-Cartan" components of the $SO(2,4)$ angular momentum tensor vanish (cf. [17]), i.e., the corresponding semiclassical state has only E, S_1, S_2 as its global "quantum numbers."

The energy and the total spin $S = S_1 + S_2 = 2S_1 = \sqrt{\lambda}\mathcal{S}$ can be written as

$$\mathcal{E} = \frac{n}{\pi}\frac{E[\mu]}{\nu - \mu}\sqrt{\nu\mu(2-\nu)}, \quad \mathcal{S} = \frac{n}{\pi}\Big(\frac{\nu E[\mu]}{\nu - \mu} - K[\mu]\Big)\sqrt{\frac{\nu + \mu - \nu\mu}{\nu}}, \tag{2.18}$$

where E is the standard elliptic function. Then for given n, m, and $\mathcal{S}$ the energy takes the form

$$\mathcal{E} = n\bar{\mathcal{E}}\Big(\frac{\mathcal{S}}{n}, \frac{m}{n}\Big). \tag{2.19}$$

We have checked numerically that for the minimal choice $m = 1$, $n = 3$ Eq. (2.15), and the second equation in (2.18) admit solutions for ν, μ for various values of the spin $\mathcal{S}$.

In Fig. 1 we present the profile of the string in (ρ, θ) polar coordinates. Let us stress that the shape of the string is smooth (no spikes at maxima or minima of ρ) since $\frac{d\rho}{d\theta}$ is continuous. This case is thus "intermediate" between the one of the 1-spin spiky string[16] [3] and the one of the 2-spin circular string in [12].

Let us now look at some special cases.

B. "Round-circle" limit: $\rho_+ \approx \rho_-$

The limit when the variation of the radial distance ρ with σ (or, equivalently, with θ) is small, i.e., the shape of the string is close to a round circle, may be obtained by taking $a_+ \to a_-$, i.e., $\mu \ll 1$ [cf. (2.5)]. Using the expansion of the elliptic integral $\Pi[\nu, \mu]$ for small μ

[15] Here $\Pi[\nu, z, \mu] = \int_0^z \frac{d\alpha}{(1-n\sin^2\alpha)\sqrt{1-\mu\sin^2\alpha}}$, $\Pi[\nu, \mu] \equiv \Pi[\nu, \frac{\pi}{2}, \mu]$.

[16] A similar effect appears also in the spiky string when a S^5 spin J is included [26].

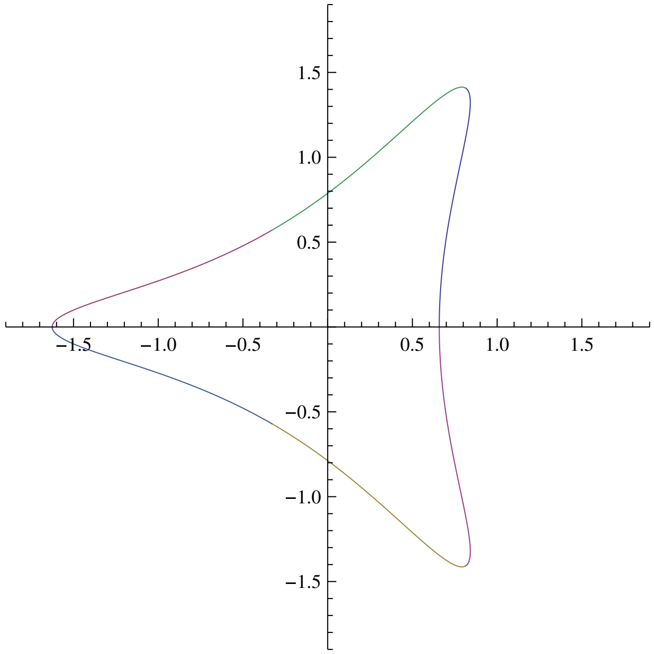

FIG. 1 (color online). Shape of $S_1 = S_2$ string with $m = 1$, $n = 3$, and $a_- = 1.5$, $a_+ = 6.97$.

$$\Pi[\nu, \mu] = \frac{\pi}{2\sqrt{1-\nu}} + \mathcal{O}(\mu) \tag{2.20}$$

the condition (2.15) becomes

$$\frac{2m}{n} \approx \sqrt{1 - \frac{\mu}{\nu}}. \tag{2.21}$$

Since $\nu, \mu > 0$ we again conclude that $2m < n$. The condition (2.21) can be satisfied for generic integer n, m if ν is also small, i.e., $\nu \approx \mu \ll 1$. In this limit we obtain

$$\begin{aligned} a_+ &= \cosh^2\rho_+ \approx \left(\frac{n}{2m}\right)^2, \\ a_- &= \cosh^2\rho_- \approx \left(\frac{n}{2m}\right)^2 (1-\mu). \end{aligned} \tag{2.22}$$

The shape of the string is close to a round circle located at a distance $\rho_+ \approx \rho_-$ from the center of AdS_5 (its length may be approximated by $2\pi m \sinh^2\rho_+$). The energy and total spin are found to be

$$\mathcal{E} = \frac{n^2}{4\sqrt{2}m^2}\sqrt{n^2 - 4m^2}, \qquad \mathcal{S} = \frac{n^2 - 4m^2}{4\sqrt{2}m^2}\sqrt{n^2 - 2m^2}. \tag{2.23}$$

The case of the circular string (where $\rho_+ = \rho_-$) corresponds to fixing, e.g., $m = 1$ and taking n large to recover the round shape of the string. Then solving for n in terms of $\mathcal{S}$ we find the same large-spin relation as for the circular string of [12]

$$\mathcal{E} = \mathcal{S} + \frac{3}{4}(2\mathcal{S})^{1/3} + \ldots \tag{2.24}$$

C. Large-spin limit

Let us now consider the limit when the variation of ρ with θ is large and the total spin is large. This corresponds to $\nu \approx \mu \approx 1$ with $\mu < \nu$. In this limit

$$K[\mu] \approx \frac{1}{2}\ln\frac{16}{1-\mu} \gg 1. \tag{2.25}$$

Then we can approximate the elliptic integral on the right-hand side of (2.15) as

$$\Pi[\nu, \mu] \approx \sqrt{\frac{\nu}{(1-\nu)(\nu-\mu)}}\left(\frac{\pi}{2} - \arcsin\sqrt{\frac{1-\nu}{1-\mu}}\right) \tag{2.26}$$

and Eq. (2.15) becomes

$$\cos\frac{\pi m}{n} \approx \sqrt{\frac{1-\nu}{1-\mu}} \tag{2.27}$$

which implies $2m < n$. This limit becomes

$$\kappa^2 = \omega^2\frac{\mu(2-\nu)}{\nu + \mu - \nu\mu} \approx \omega^2, \qquad \omega \approx K[\mu]\frac{n}{\pi} \gg 1 \tag{2.28}$$

where we used (2.10). The parameters κ and ω are thus approximately equal and large in this limit. At leading order in large a_+ Eq. (2.27) implies

$$a_- \approx \frac{1}{\sin^2\frac{\pi m}{n}}. \tag{2.29}$$

Then for $\mathcal{E} - \mathcal{S}$ we obtain at the leading order

$$\mathcal{E} - \mathcal{S} = \frac{n}{2\pi}\ln\frac{16}{1-\mu} + \ldots \tag{2.30}$$

The total spin $\mathcal{S} = 2\mathcal{S}_1 = 2\mathcal{S}_2$ can be written as

$$\mathcal{S} = \frac{n}{\pi}\frac{1}{(1-\mu)\sin^2\frac{\pi m}{n}} - \frac{n}{4\pi\sin^2\frac{\pi m}{n}}\left[2 + \cos\frac{2\pi m}{n} \times \left(1 + \ln\frac{1-\mu}{16}\right)\right] + \mathcal{O}(1-\mu) \tag{2.31}$$

and is large since $\mu \to 1$. Solving (2.31) for $1 - \mu$ in terms of $\mathcal{S} \gg 1$ and plugging it into the energy in (2.18) we find

$$\mathcal{E} - \mathcal{S} = \frac{n}{2\pi}\left[\ln\frac{16\pi\mathcal{S}}{n} - 1 + 2\ln\sin\frac{\pi m}{n}\right] + \mathcal{O}\left(\frac{1}{\mathcal{S}}\right). \tag{2.32}$$

The minimal energy solution is found for $m = 1$, $n = 3$ when we get[17] ($\mathcal{S} = 2\mathcal{S}_1$)

[17] To get the $\mathcal{O}(\frac{1}{\mathcal{S}_1})$ term we used the next order expansion $\Pi[\nu, \mu] \approx \sqrt{\frac{\nu}{(1-\nu)(\nu-\mu)}}(\frac{\pi}{2} - \arcsin\sqrt{\frac{1-\nu}{1-\mu}}) + [\frac{1}{2} - \frac{\sqrt{\nu}}{4}]\ln\frac{16}{1-\mu} + \frac{\sqrt{\nu}}{4} + \mathcal{O}(\mu - 1)$.

$$\mathcal{E} - 2\mathcal{S}_1 = \frac{3}{2\pi}[\ln(8\pi\mathcal{S}_1) - 1] + \frac{9}{8\pi^2\mathcal{S}_1}\left[\ln(8\pi\mathcal{S}_1) + \frac{1}{2}\right] + \mathcal{O}\left(\frac{1}{\mathcal{S}_1^2}\right). \quad (2.33)$$

Remarkably, the factor multiplying $\frac{3}{2}$ in the first term of the right-hand side in (2.33) is exactly the same as for the folded string with a single spin $\mathcal{S}_1 = \frac{1}{2}\mathcal{S}$. This conclusion is precisely the one following (in the strong-coupling limit) from the analysis of the asymptotic Bethe ansatz equations [4]. Let us also remark that the coefficient of the $\frac{\ln\mathcal{S}_1}{\mathcal{S}_1}$ term is again as in the folded case; more precisely, it is $\frac{1}{2}$ times the coefficient of the $\ln\mathcal{S}_1$ squared.[18]

For general n (and $m = 1$) the expression (2.32) may be compared to the one found for the single-spin string with n spikes [3,16,22][19]:

$$\mathcal{E}_{\text{spiky}} - \mathcal{S}_1 = \frac{n}{2\pi}\left(\ln\frac{16\pi\mathcal{S}_1}{n} - 1 + \ln\sin\frac{\pi}{n}\right) + \mathcal{O}\left(\frac{1}{\mathcal{S}}\right). \quad (2.34)$$

We observe that the subleading $\ln\sin\frac{\pi}{n}$ term in (2.32) differs from the one in the spiky string case one by an extra factor of 2. This difference might be attributed to the fact that the subleading term is sensitive to the near-boundary (turning point of ρ) region where the spiky string has a cusp while the present $S_1 = S_2$ solution has a regular "round arc" shape.

The asymptotic solution of (2.2) in this large-spin limit is given by

$$x = \cosh\rho \approx \sqrt{a_-}\cosh\left(\frac{K[\mu]n}{\pi}\sigma\right) \approx \cosh\rho_-\cosh(\kappa\sigma). \quad (2.35)$$

At $\sigma = 0$ the radial coordinate ρ reaches its minimum while at $\sigma = \frac{\pi}{n}$ it grows to infinity as κ is large, i.e., the end points of n arcs of the string approach the boundary of AdS_5. Assuming the initial condition $\theta(0) = 0$ we obtain the approximate solution for θ on the interval $0 < \sigma < \frac{\pi}{n}$:

$$\tan\theta = \frac{\tanh\kappa\sigma}{\sinh\rho_-}. \quad (2.36)$$

Since κ is large, the value of θ at the end of the arc is $\tan\theta = \frac{1}{\sinh\rho_-} = \frac{1}{\sqrt{a_- - 1}} = \tan\frac{\pi m}{n}$, which matches the value in (2.13) in the limit we are considering in this subsection [see (2.14)].

Combining (2.35) and (2.36) we can determine the shape of the string $\rho(\theta)$

$$\cosh\rho = \frac{\cosh\rho_-}{\sqrt{1 - \sinh^2\rho_-\tan^2\theta}}. \quad (2.37)$$

We conclude that like in the folded [12] and the spiky [28] string cases the solution in the large-spin limit is represented by a combination of n arcs stretching to the boundary with each arc described by a simple analytic expression.

In Sec. III we shall rederive this asymptotic solution by starting with the assumption (valid in the large-spin limit) that $\omega = \kappa$.

III. ASYMPTOTIC SOLUTION: RELATION TO SINGLE-SPIN STRING

Let us go back to the system (1.7) and (1.8) and try to solve it by assuming that $\omega_1 = \omega_2 = \kappa$. Then

$$\theta' = \frac{c}{\sinh^2\rho}, \qquad \rho'^2 = \kappa^2 - \frac{c^2}{\sinh^2\rho}. \quad (3.1)$$

This system will not have a finite-length closed-string solution. We may, however, relax the closed-string condition, find its "open-string" solution and then use it to "glue" an asymptotic closed-string solution that in the $\kappa = \omega \gg 1$ limit will coincide with the large-spin limit of the solution from the previous section.

A. Solution for $\omega_1 = \omega_2 = \kappa$

The equation for $\rho(\sigma)$ in (3.1)

$$\rho' = \frac{\kappa}{\sinh\rho}\sqrt{\sinh^2\rho - \sinh^2\rho_-} \quad (3.2)$$

has the following solution with the initial condition $\rho(0) = \rho_-$

$$\cosh\rho(\sigma) = \cosh\rho_-\cosh(\kappa\sigma). \quad (3.3)$$

Then

$$\cot\theta(\sigma) = \sinh\rho_-\coth(\kappa\sigma). \quad (3.4)$$

If σ changes in the interval $0 < \sigma < \sigma_0$ then the maximal value of ρ is given by $\cosh\rho_+ = \cosh\rho_-\cosh(\kappa\sigma_0)$.[20] For example, in the previous section we had $\sigma_0 = \frac{\pi}{n}$ for a half-arc of the string and $\kappa \gg 1$ in the large-spin limit. In this case $\rho_+ \to \infty$, i.e., the string stretches all the way to the boundary while θ changes from 0 to $\text{arccot}(\sinh\rho_-)$.

From (3.1) we get the following equation for $\rho = \rho(\theta)$

$$\frac{d\rho}{d\theta} = \pm\frac{\sinh\rho}{\sinh\rho_-}\sqrt{\sinh^2\rho - \sinh^2\rho_-}, \qquad \sinh\rho_- \equiv \frac{c}{\kappa}. \quad (3.5)$$

Here ρ will change from the minimal value ρ_- to some maximal value. The solution for $\theta(\rho)$ on the interval $\rho_- < \rho < \infty$ with the initial condition $\theta(\rho_-) = 0$ is [for the plus

[18]As in the folded string case this is the pattern implied by the functional relation [22], i.e., $E - S = f\ln(S + \frac{f}{2}\ln S + \ldots) + \ldots = f\ln S + \frac{f^2}{2}\frac{\ln S}{S} + \ldots$, where now $f = \frac{3}{2\pi}$.

[19]See also [27] for a derivation using the spectral curve approach.

[20]Note that here θ' never vanishes so this solution cannot be interpreted as a usual free open string.

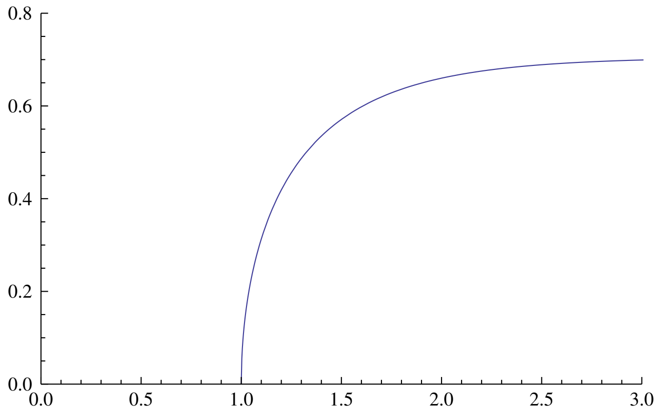

FIG. 2 (color online). Plot of $\theta(\rho)$ for $\rho_- = 1$: from 0 at $\rho = 1$ and to $\theta_0 \approx 0.705$ at $\rho = \infty$.

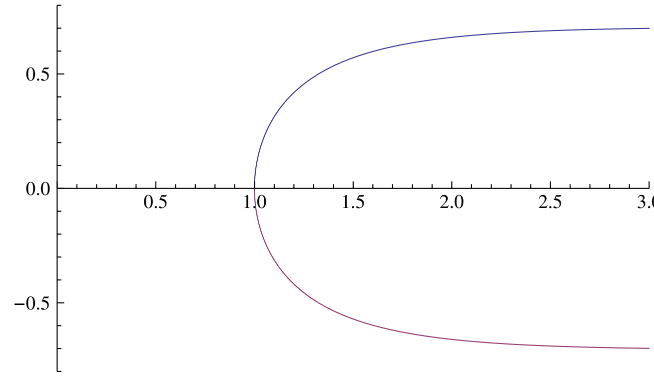

FIG. 3 (color online). String arc stretching to infinity: $\theta(\rho)$ changes from $\theta(\infty) = -\theta_0$ to $\theta(\infty) = \theta_0$ through $\theta(\rho_-) = 0$.

sign choice in (3.5)]

$$\cot\theta = \frac{\sinh\rho_- \cosh\rho}{\sqrt{\cosh^2\rho - \cosh^2\rho_-}}. \tag{3.6}$$

Thus $0 < \theta < \theta_0$, $\theta_0 = \mathrm{arccot}(\sinh\rho_-) < \frac{\pi}{2}$. In Fig. 2 we plotted $\theta(\rho)$ for $\rho_- = 1$.

Joining together two of such stretches of string (with ρ changing from infinity to ρ_- and then back to infinity while θ going from $-\theta_0$ to 0 and to θ_0)[21] we get a single-arc open-string solution (see Fig. 3). A doubled version of this arc might be interpreted[22] as a bended folded string anticipated in [17] to be the large-spin limit of a 2-spin configuration in AdS_5.

B. Relation to asymptotic limits of the folded, spiky, and 2-spin closed-string solutions

The "open-string" or "half-arc" solution constructed above is, in fact, equivalent to a similar single-spin solution in AdS_3. This can be seen by writing it in terms of the AdS_5 embedding coordinates (1.3) and (1.4). The solution (3.3) and (3.4) takes the form

$$Y_{05} = \cosh\rho_- \cosh\bar{\sigma} e^{i\bar{\tau}}, \qquad Y_{12} = \sinh\rho_- \cosh\bar{\sigma} e^{i\bar{\tau}},$$
$$Y_{34} = \sinh\bar{\sigma} e^{i\bar{\tau}}, \tag{3.7}$$

$$\bar{\tau} \equiv \kappa\tau, \qquad \bar{\sigma} \equiv \kappa\sigma. \tag{3.8}$$

Applying a global $SO(2,4)$ transformation with boost parameter ρ_- we can transform (3.7) into an AdS_3 solution:

$$Y'_{05} = \cosh\bar{\sigma} e^{i\bar{\tau}}, \qquad Y_{34} = \sinh\bar{\sigma} e^{i\bar{\tau}}, \qquad Y'_{12} = 0. \tag{3.9}$$

For $\bar{\sigma} = \kappa\sigma$ changing from 0 to ∞ (for $\kappa \to \infty$) this is the same solution as found (for a single stretch of the string or a half-spike) in the large-spin limit of the folded string [12] or the spiky string [28].

One can give a general proof that an open-string solution in AdS_5 described by the ansatz (1.2) with $\kappa = \omega_1 = \omega_2$ can be always $SO(2,4)$ transformed into an AdS_3 solution. Starting with the following ansatz for the embedding coordinates[23]

$$Y_{01} = y_0(\sigma)e^{i\kappa\tau}, \qquad Y_{12} = y_1(\sigma)e^{i\kappa\tau}, \qquad Y_{34} = y_2(\sigma)e^{i\kappa\tau} \tag{3.10}$$

we find (using the conformal-gauge constraint) that the equations of motion are $y''_a - \kappa^2 y_a = 0$ ($a = 0, 1, 2$), where the solutions should satisfy $y_0^2 - y_1^2 - y_2^2 = 1$. The most general solution is $y_a = A_a e^{\kappa\sigma} + B_a e^{-\kappa\sigma}$ and the constraints imply $A_0^2 = A_1^2 + A_2^2$, $B_0^2 = B_1^2 + B_2^2$, $A_0B_0 - A_1B_1 - A_2B_2 = \frac{1}{2}$. These relations determine three parameters, while one additional parameter can be fixed by a shift in σ, i.e., we are left with two free parameters. Since there are only two functionally-independent terms $e^{\kappa\sigma}$ and $e^{-\kappa\sigma}$ this implies that we can always set to zero, say, A_2 and B_2 by an $SO(1,2)$ transformation acting on y_a; this then puts the solution into AdS_3.[24]

[21]Here $\bar{\sigma} = \kappa\sigma$ may be assumed to be changing from $-\infty$ to $+\infty$ (corresponding to σ going from $-\pi$ to π for a single fold case and $\kappa \to \infty$).

[22]This interpretation makes sense if a 2-spin bended folded string solution can indeed reach the boundary, which is not *a priori* clear.

[23]One may consider also a more general solution with $Y_a = (Y_{05}, Y_{12}, Y_{34}) = z_a(\sigma + b\tau)e^{i\kappa\tau}$ where z_a are complex and σ is decompactified. One may redefine σ and τ preserving the conformal gauge to transform this into $Y_a = z_a(\sigma')e^{i\kappa'(\tau' + c\sigma')}$. Such a solution was found in an unpublished work of A. Irrgang and M. Kruczenski. It appears to be related to the one we discuss here by an $SO(2,4)$ transformation.

[24]For example, we can first do rotation in the (y_1, y_2) plane to make $A_2 = 0$. Then we get $y'_0 = A_0 e^{\kappa\sigma} + B_0 e^{-\kappa\sigma}$, $y'_1 = A_0 e^{\kappa\sigma} + B'_1 e^{-\kappa\sigma}$, $y'_2 = B'_2 e^{-\kappa\sigma}$. Then we can boost in the (0, 1) plane to make $B'_1 = 0$, getting $y''_0 = A_0 e^{\kappa\sigma} + B_0 e^{-\kappa\sigma}$, $y''_1 = A_0 e^{\kappa\sigma}$, $y''_2 = B_0 e^{-\kappa\sigma}$, with $A_0 B_0 = \frac{1}{2}$. Shifting σ by a constant we can set $A_0 = B_0 = \frac{1}{\sqrt{2}}$, i.e., end up with $y_0 = y_1 + y_2$, $y_1 = \frac{1}{\sqrt{2}} e^{\kappa\sigma}$, $y_2 = \frac{1}{\sqrt{2}} e^{-\kappa\sigma}$. Finally, we can make an $SO(1,2)$ transformation that sets $\tilde{y}_2 = 0$: $\tilde{y}_0 = \cosh\kappa\sigma = \sqrt{2}[y_0 - \frac{1}{2} \times (y_1 + y_2)]$, $\tilde{y}_1 = \sinh\kappa\sigma = \frac{1}{\sqrt{2}}(y_1 - y_2)$, $\tilde{y}_2 = 0 = -y_0 + y_1 + y_2$.

Since Y_{05} and Y_{12} in (3.7) are even under $\sigma \rightarrow -\sigma$ the global boost leading to (3.9) applies also to the full arc solution or its bended folded string generalization mentioned above. This transformation "straightens up" the folded string, i.e., removes the bend making string pass through $\rho = 0$ and thus making it equivalent to the large-spin limit of the single-spin folded string.

Before the $SO(2,4)$ rotation leading to (3.9) some of the "non-Cartan" components of the $SO(2,4)$ generators are nonzero so the right semiclassical state interpretation of this half-arc solution is that of a single-spin solution with spin equal to S. However, different ways of gluing single-arc solutions into a closed-string solution which do not, in general, commute with $SO(2,4)$ rotations may lead to inequivalent solutions representing inequivalent semiclassical string states once the strict large-spin limit is relaxed.

Indeed, the "straight" string in (3.9) [for which $\theta = 0$, $\phi_1 = 0$ in (1.3)] can be glued with itself into (an infinite spin limit of) the folded string. Applying a different global $SO(2,4)$ boost to (3.9) one finds a different "bended" form of the single-spin arc string; gluing together n such arcs gives [28] the asymptotic form of the n-spike solution of [3].

Let us now discuss more explicitly how to reproduce the asymptotic form of the 2-spin solution (2.35) and (2.36) by starting with (3.3) and (3.4), or (3.7). We need to glue together $2n$ such segments (with $0 \leq \sigma \leq \sigma_0 = \frac{\pi}{n}$) to form an n-arc closed string with end points of the arcs approaching the boundary. Allowing for the winding number m in θ, the closed-string condition is then

$$2n\,\mathrm{arccot}(\sinh\rho_-) = 2\pi m. \tag{3.11}$$

This implies a relation between c and κ, or, equivalently, fixes the minimal value of ρ

$$\cot\frac{\pi m}{n} = \sinh\rho_- = \frac{c}{\kappa}. \tag{3.12}$$

This is the same value (2.29) found by taking the large-spin limit of the 2-spin solution in the previous section. Unless $c = 0$ (when $\rho_- = 0$ we are back to the straight folded string case) we again get the condition $2m < n$, implying that the minimal values are $m = 1$, $n = 3$. The plot of such a solution obtained by gluing 3 arcs of the type in Fig. 3 is presented in Fig. 4. Viewed as a limit of the solution plotted in Fig. 1, the end points of the 3 arcs should be rounded. This is in contrast with the single-spin spiky string in AdS_3 [3] where the spikes were present for any value of the spin.

Assuming that $\kappa \rightarrow \infty$, we find for a half-arc solution (3.7) ($0 < \kappa\sigma < \frac{\pi}{n}\kappa \rightarrow \infty$)

$$\mathcal{E} - \mathcal{S} = \frac{1}{4\pi}\ln\mathcal{S} + \ldots, \tag{3.13}$$

where $\mathcal{S}$ is the total spin. This is to be multiplied by (i) 4 in the case of the folded single-spin string, (ii) $2n$ in the case of the single-spin n-spike solution, and also (iii) $2n$ in the

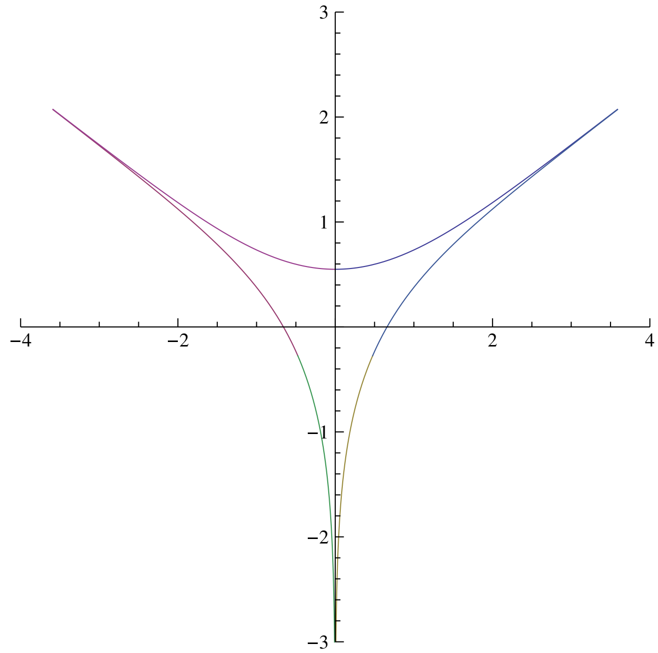

FIG. 4 (color online). Shape of $S_1 = S_2$ string with $m = 1$, $n = 3$ in the (ρ, θ) plane. The end points are, in fact, not cusps but rounded up as in Fig. 1.

case of the two equal-spin n-arc solutions of the previous section [cf. (2.32)].

The folded string is a special case of the spiky string and represents the lowest-energy state for given spin in the sector of 1-spin solutions (or, in particular, in the $sl(2)$ sector of the dual gauge theory). The "lightest" nontrivial spiky string with $n = 3$ has higher energy than the ground state (folded string) in the 1-spin set of states. The lightest nontrivial 2-spin solution with $m = 1$, $n = 3$ having

$$\mathcal{E}_{\min} - \mathcal{S} = \frac{3}{2\pi}\ln\mathcal{S} + \ldots \tag{3.14}$$

should be representing the "ground state" in the class of 2-spin solutions with equal spins; the same should apply also to the corresponding gauge-theory states.[25]

It was suggested in [28] that for the single-spin spiky string the coefficient of the $n\ln S$ term in the energy should be the same for all n to all orders in the $\frac{1}{\sqrt{\lambda}}$ expansion as it is determined by the same asymptotic solution as in the folded string case. The above conclusion about universality of the half-arc solution implies that the coefficient of the leading $n\ln S$ term in the $S_1 = S_2$ string energy should also be the same for any n and any value of spin. Thus the coefficient of the $n\ln\mathcal{S}$ term in the large-spin expansion of the string energy of the 2-spin solution of the previous section [cf. (2.32)] should be given by the same universal function of the string tension, i.e., the cusp anomaly function

[25] We thank A. Rej for a discussion of this point.

$$E - S = \frac{n}{2} f(\lambda) \ln S + \ldots, \qquad f(\lambda)_{\lambda \gg 1} = \frac{\sqrt{\lambda}}{\pi} + O(1), \qquad n = 3, 4, \ldots \tag{3.15}$$

C. Asymmetric gluing

Let us consider the possibility of gluing arcs asymmetrically to construct circular solutions with unequal spins. We concentrate on the minimal energy solution with $m = 1$ and number of arcs $n = 3$.[26] The asymptotic solution discussed below should correspond to the asymmetric solution of Subsec. II A in the large a_+ limit.

We would like to glue together three different arcs described by the asymptotic solutions (2.35) and (2.36)

$$\cosh\rho_i = \cosh\rho_{-i} \cosh\kappa\sigma, \qquad \cot\theta_i = \sinh\rho_{-i} \coth\kappa\sigma, \qquad i = 1, 2, 3 \tag{3.16}$$

with constant parameters ρ_{-i} (minimal values of ρ_i) being, in general, different. To explicitly construct the solution we need to split the $0 < \sigma \leq 2\pi$ interval into six $\frac{\pi}{3}$ intervals. The gluing condition [cf. (3.11)] is $\Delta\theta = 2\sum_i \text{arccot}(\sinh\rho_{-i}) = 2\pi$, which gives

$$\arctan(\sinh\rho_{-1}) + \arctan(\sinh\rho_{-2}) + \arctan(\sinh\rho_{-3}) = \frac{\pi}{2}. \tag{3.17}$$

Thus only two out of three parameters ρ_{-i} are independent, e.g.,

$$\sinh\rho_{-3} = \frac{1 - \sinh\rho_{-1} \sinh\rho_{-2}}{\sinh\rho_{-1} + \sinh\rho_{-2}}. \tag{3.18}$$

An example of the resulting solution is shown in Fig. 5.

Computing the spins at leading order in large κ expansion we get

$$\mathcal{S}_1 \approx \frac{e^{(2\pi/3)\kappa}}{16\pi} h_1(\rho_{-1}, \rho_{-2}), \qquad \mathcal{S}_2 \approx \frac{e^{(2\pi/3)\kappa}}{16\pi} h_2(\rho_{-1}, \rho_{-2}) \tag{3.19}$$

where we used (3.18) and the functions h_1, h_2 are, in general, different[27] (their explicit form is somewhat complicated). Hence in the large κ case $\mathcal{S}_1, \mathcal{S}_2 \gg 1$ while $\frac{\mathcal{S}_1}{\mathcal{S}_2}$ is fixed. The energy is then

$$\mathcal{E} - \mathcal{S}_1 - \mathcal{S}_2 = \kappa \approx \frac{3}{2\pi} \ln\mathcal{S}_1 + O(1) = \frac{3}{2\pi} \ln(\mathcal{S}_1 + \mathcal{S}_2) + O(1). \tag{3.20}$$

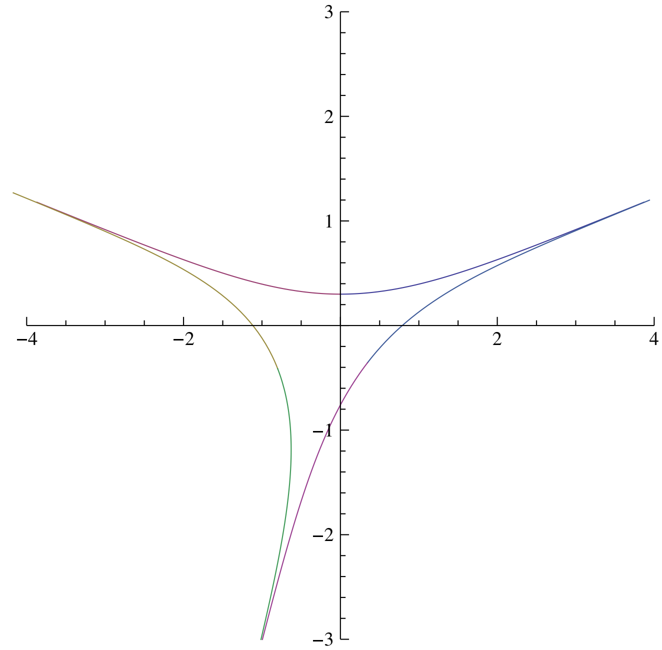

FIG. 5 (color online). Asymmetric circular string with $m = 1$, $n = 3$, $\rho_{1-} = 0.3$, $\rho_{2-} = 0.9$, $\rho_{3-} = 0.5$, $\frac{\mathcal{S}_1}{\mathcal{S}_2} = 0.64$ in the (ρ, θ) plane.

The coefficient of the leading $\ln\mathcal{S}$ term is thus independent of $\frac{\mathcal{S}_1}{\mathcal{S}_2}$, i.e., the same as in the symmetric gluing case.

A puzzling feature of the resulting solution is that some of the "off diagonal" components of the $SO(2, 4)$ angular momentum are apparently nonzero, suggesting that the gluing procedure should be more subtle (since otherwise this solution does not represent a highest-weight state and thus one should be able to rotate it into a simpler solution). One option is that this glued solution is not actually a limit of any regular closed-string solution. Indeed, as follows from (3.5), the derivative $\frac{d\rho}{d\theta}$ for large ρ goes as $\frac{\sinh^2\rho}{\sinh\rho_-}$; since it depends on ρ_- the derivatives from the "left" and from the "right" at the end of the string will not match and thus there will be a cusp.

IV. CIRCULAR SOLUTION WITH (S_1, S_2): $\kappa = \omega_2 \neq \omega_1$

Let us now present an explicit 2-spin solution with unequal spins by considering the second special case in (1.6), i.e., $\kappa = \omega_2 \neq \omega_1$. In this case the string will again have a topology of a circle. The large-spin limit in which the string can reach the boundary will be possible only for $S_1 = S_2$. In this case the asymptotic solution can be again glued out of parts equivalent to the one discussed in Sec. III.

A. Constructing solution

To integrate the system of equations for $\kappa = \omega_2$ it is useful to change the coordinates in (1.1) and (1.4) from

---

[26] Since $\text{arccot}(\sinh\rho_-) \leq \frac{\pi}{2}$ to satisfy the gluing condition for $m = 1$ we need at least $n = 3$.

[27] They are equal only in the symmetric case, which corresponds to $\sinh\rho_{-1} = \sinh\rho_{-2} = \sinh\rho_{-3} = \frac{1}{\sqrt{3}}$.

(ρ, θ) to (χ, ψ) defined according to

$$\cosh\rho = \cosh\chi\cosh\psi, \quad \sin\theta = \frac{\cosh\chi\sinh\psi}{\sqrt{\cosh^2\chi\sinh^2\psi + \sinh^2\chi}}. \tag{4.1}$$

Then the embedding coordinates in (1.3) and (1.4) take the following form:

$$y_0 = \cosh\chi\cosh\psi, \qquad y_1 = \sinh\chi, \qquad y_2 = \cosh\chi\sinh\psi \tag{4.2}$$

and the AdS_5 metric (1.1) becomes

$$ds^2 = -\cosh^2\chi\cosh^2\psi dt^2 + d\chi^2 + \cosh^2\chi d\psi^2 + \sinh^2\chi d\phi_1^2 + \cosh^2\chi\sinh^2\psi d\phi_2^2. \tag{4.3}$$

Assuming the ansatz $(\omega_1 \equiv \omega)$

$$t = \phi_2 = \kappa\tau, \qquad \phi_1 = \omega\tau, \qquad \chi = \chi(\sigma), \qquad \psi = \psi(\sigma), \tag{4.4}$$

the equation for ψ can be easily integrated [cf. (1.7)]

$$\psi' = \frac{c}{\cosh^2\chi}. \tag{4.5}$$

When $c = 0$ we have $\psi = \psi_0 = \text{const}$ and the resulting solution is related by a global $SO(2, 4)$ boost to the single-spin folded string solution [cf. (1.3) and (4.2)]. For $c \neq 0$ we have $\psi' \neq 0$ and so the string's shape cannot be of bended folded type, i.e., it should be of a circular type (see [17] and Sec. I).

The conformal constraint gives the following equation for χ:

$$x'^2 = \kappa^2(1 + x^2)^2 - c^2 - \omega^2 x^2(1 + x^2), \qquad x \equiv y_1 = \sinh\chi, \tag{4.6}$$

which is very similar to the one we had in (2.1). Equations (2.1) and (4.6) are related by the analytic continuation $x \to ix$, $\omega \to \kappa$. Indeed, the equation for $x \equiv y_0 = \cosh\rho$ in (2.1) we had in the $\omega_1 = \omega_2$ case in the present $\kappa = \omega_2$ case is replaced by the equation for $x \equiv y_1 = \sinh\chi$.[28] In this sense χ is now playing the role of ρ and the subsequent analysis of the solution of (4.6) will be similar to the one in Sec. II. Equation (4.6) can be written in the same way as (2.2)[29]:

$$x'^2 = (\omega^2 - \kappa^2)(x^2 - a_-)(a_+ - x^2), \tag{4.7}$$

where

$$a_\pm = \frac{2\kappa^2 - \omega^2 \pm \sqrt{\omega^4 - 4c^2(\omega^2 - \kappa^2)}}{2(\omega^2 - \kappa^2)}. \tag{4.8}$$

If $a_\pm > 0$ (as we shall assume in this section) then $\sqrt{2}\kappa > \omega > \kappa$ and $c^2 > \kappa^2$.[30] Then χ changes from a minimal to a maximal value. The case when $a_+ > 0$ and $a_- < 0$ will be discussed in Appendix A. Some useful relations are

$$\frac{\omega^2}{\kappa^2} = \frac{a_+ + a_- + 2}{a_+ + a_- + 1}, \qquad \omega^2 - \kappa^2 = \frac{\kappa^2}{a_+ + a_- + 1}, \qquad \frac{c^2}{\kappa^2} = 1 + \frac{a_+ a_-}{a_+ + a_- + 1}. \tag{4.9}$$

It is convenient to introduce the following parametrization of $a_\pm$ $(\mu > \nu)$:

$$\mu = \frac{a_+ - a_-}{a_+}, \qquad \nu = \frac{a_+ - a_-}{a_+ + 1}, \qquad a_+ = \frac{\nu}{\mu - \nu}, \qquad a_- = \frac{\nu(1 - \mu)}{\mu - \nu}. \tag{4.10}$$

Solving (4.7) with the initial condition $x(0) = \sqrt{a_-}$ on an interval $0 < \sigma \leq \sigma_0 = \frac{\pi}{n}$ we find

$$x = \sinh\chi = \pm \frac{\sqrt{a_-}}{\mathrm{dn}[\frac{\omega\sqrt{a_+}}{\sqrt{a_+ + a_- + 2}}\sigma, \mu]} \tag{4.11}$$

where $x(\sigma = \frac{\pi}{n}) = \sqrt{a_+}$. We will then need to glue together $2n$ of such segments to form a closed string. Because the period of the $\mathrm{dn}(z, \mu)$ Jacobi function is $2K[\mu]$ we need to satisfy

$$\omega\sqrt{\frac{a_+}{a_+ + a_- + 2}} = \frac{K[\mu]n}{\pi}. \tag{4.12}$$

This equation along with (4.9) can be used to eliminate one constant out of κ, ω, c. The corresponding equation for $x = x(\psi)$ that follows from (4.5) and (4.7) is

$$\left(\frac{dx}{d\psi}\right)^2 = \frac{1}{(a_+ + 1)(a_- + 1)}(1 + x^2)^2(x^2 - a_-)(a_+ - x^2). \tag{4.13}$$

Taking the square root there are two branches. Using the negative branch and the initial condition that at $\sigma = 0$ where $x = \sqrt{a_-}$ we should have $\psi = 0$ [i.e., $\psi(x = \sqrt{a_-}) = 0$]. We obtain the following solution:

[28] This effective replacement is implied by the general analysis of equations for $y_a(\sigma)$ in [17].

[29] In this section we shall use similar notation for the parameters as in Sec. II but their values will be of course different as the two solutions are different.

[30] For $\omega < \kappa$ there are no closed-string solutions. If $c = \kappa$ we get $a_- = 0$ and $a_+ = \frac{2\kappa^2 - \omega^2}{\omega^2 - \kappa^2}$. In this case the string can reach the center of AdS_5 and one may expect to get typical near flat space expressions. However, for $c = \kappa$ the motion is not periodic in σ: we get $x = \sinh\chi \sim \tanh\kappa\sigma$, i.e., there is only the "open-string" solution resembling the giant magnon [29] one.

$$\psi(x) = \frac{\sqrt{a_- + 1}}{\sqrt{a_+(a_+ + 1)}}\left(\Pi[\nu, \mu] - \Pi\left[\nu, \arcsin\sqrt{\frac{a_+ - x^2}{a_+ - a_-}}, \mu\right]\right). \quad (4.14)$$

At $\sigma = \frac{\pi}{n}$ where $x = \sqrt{a_+}$ we have

$$\psi(\sqrt{a_+}) \equiv \psi_0 = \frac{\sqrt{a_- + 1}}{\sqrt{a_+(a_+ + 1)}}\Pi[\nu, \mu]. \quad (4.15)$$

Thus ψ increases from 0 to ψ_0. The solution for the positive branch has the opposite sign, decreasing from 0 to $-\psi_0$. To get $\psi = \psi(\sigma)$ one may plug the solution for $x(\sigma)$ (4.11) into (4.14).

B. Closed-string condition, energy, and spins

To construct a closed-string solution that will have a circle shape we need to glue together $2n$ of the above string segments. We should first express the string segments obtained above in (χ, ψ) coordinates in terms of (ρ, θ) coordinates (with $0 < \theta \leq 2\pi$), and then glue them together in a similar way as was done in Sec. II.

When σ changes from 0 to $\sigma_0 = \frac{\pi}{n}$, i.e., when $x = \sinh\chi$ changes from $\sqrt{a_-}$ to $\sqrt{a_+}$ and ψ changes from 0 to ψ_0 in (4.15) we get, according to (4.1), that θ changes from 0 to θ_0 with

$$\sin\theta_0 = \frac{\sinh\psi_0}{\sqrt{\sinh^2\psi_0 + \frac{a_+}{1+a_+}}}. \quad (4.16)$$

To get a closed string with a circular shape in θ we need to demand (m is a winding number)

$$\theta_0 = \frac{\pi m}{n}, \qquad 2m < n. \quad (4.17)$$

We found (using numerical methods) that gluing $2n$ segments into such a closed-string solution is indeed possible, and also checked that all non-Cartan $SO(2,4)$ charges then vanish, while in general $S_1 \neq S_2$. For $m = 1$, $n = 3$ the resulting string shape is shown in Fig. 6 in polar coordinates (ρ, θ). As a_+ gets larger, the spins and ρ grow. We found that to have a limit when the string touches the boundary ($\kappa \approx \omega$) is possible only when the two spins are actually equal $S_1 = S_2$ (see below).

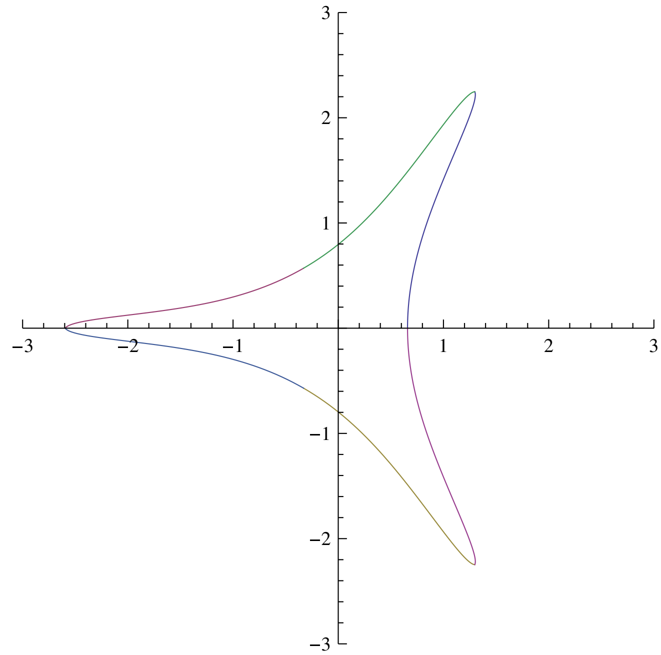

FIG. 6 (color online). Shape of (S_1, S_2) string for $m = 1$, $n = 3$ in polar coordinates (ρ, θ) for $a_- = 0.5$, $a_+ = 11.095$. In this case $\frac{S_1}{S_2} = 1.039$.

Let us describe the gluing procedure in more detail. Given $\chi(\sigma)$, $\psi(\sigma)$ in (4.11) and (4.14) we may use (4.1) to compute the corresponding $\bar{\rho}(\sigma)$, $\bar{\theta}(\sigma)$ for $0 < \sigma \leq \frac{\pi}{n}$. To find the full closed-string solution $\rho(\sigma)$, $\theta(\sigma)$ defined on $0 < \sigma \leq 2\pi$ we attach together $\rho(\sigma)$ intervals as in Sec. II while θ is obtained by combining its values on separate intervals as follows:

$$\theta = \begin{cases} (k-1)\theta_0 + \bar{\theta}(\sigma), & \frac{(k-1)\pi}{n} \leq \sigma \leq \frac{k\pi}{n}, \quad k = 1, 3, 5, \ldots, 2n-1 \\ k\theta_0 - \bar{\theta}(\sigma), & \frac{(k-1)\pi}{n} \leq \sigma \leq \frac{k\pi}{n}, \quad k = 2, 4, 6, \ldots, 2n \end{cases}. \quad (4.18)$$

$\theta(\sigma)$ defined in this way has period $\frac{2\pi}{n}$. The energy and the spins are found from (2.17) with $\omega_1 = \omega$, $\omega_2 = \kappa$. They thus satisfy $\frac{\mathcal{E}-\mathcal{S}_2}{\kappa} - \frac{\mathcal{S}_1}{\omega} = 1$. Since the energy is given by an integral of a function of ρ only, it may be computed directly using the expressions for $\chi(\sigma)$, $\psi(\sigma)$ in

$$\mathcal{E} = \frac{2n\kappa}{2\pi}\int_0^{\pi/n} d\sigma \cosh^2\chi \cosh^2\psi. \quad (4.19)$$

The two spins are given by

$$\mathcal{S}_1 = \frac{\omega}{2\pi}\sum_{k=2,4,\ldots}^{2n}\left(\int_0^{\pi/n} d\sigma \sinh^2\rho(\sigma)\cos^2[(k-2)\theta_0 + \bar{\theta}(\sigma)] + \int_{\pi/n}^{(2\pi)/n} d\sigma \sinh^2\rho(\sigma)\cos^2[k\theta_0 - \bar{\theta}(\sigma)]\right)\mathcal{S}_2$$
$$= \frac{\kappa}{2\pi}\sum_{k=2,4,\ldots}^{2n}\left(\int_0^{\pi/n} d\sigma \sinh^2\rho(\sigma)\sin^2[(k-2)\theta_0 + \bar{\theta}(\sigma)] + \int_{\pi/n}^{(2\pi)/n} d\sigma \sinh^2\rho(\sigma)\sin^2[k\theta_0 - \bar{\theta}(\sigma)]\right).$$

Like (4.19) these integrals are complicated (it does not seem possible to express them in terms of elliptic functions). The only integrals that simplify are those along the first segment since in that case we can map back to the χ, ψ functions

$$\int_0^{\pi/n} d\sigma \sinh^2\rho\cos^2\theta = \int_0^{\pi/n} d\sigma \sinh^2\chi = \frac{\pi a_+}{n}\frac{E[\mu]}{K[\mu]} \tag{4.20}$$

$$\int_0^{\pi/n} d\sigma(\cosh^2\rho - \sinh^2\rho\sin^2\theta) = \int_0^{\pi/n} d\sigma\cosh^2\chi = \frac{\pi}{n}\left(a_+\frac{E[\mu]}{K[\mu]} + 1\right). \tag{4.21}$$

The above expressions produce, in principle, the functions $\mathcal{E} = \mathcal{E}(a_+, a_-, n)$, $\mathcal{S}_1 = \mathcal{S}_1(a_+, a_-, n)$, $\mathcal{S}_2 = \mathcal{S}_2(a_+, a_-, n)$. The condition (4.17) with θ_0 given by (4.15) and (4.16) gives one relation between a_+, a_-, n, and integer m. This implies a relation between $\mathcal{S}_1$, $\mathcal{S}_2$, n, and m. As a result, one may determine (at least numerically) the energy as a function of the two spins: $\mathcal{E} = \mathcal{E}(\mathcal{S}_1, \mathcal{S}_2, n)$.

From a numerical analysis we concluded that for an arbitrary value of $\frac{\mathcal{S}_1}{\mathcal{S}_2}$, the solution for a_+ is bounded, and the spins cannot be large. It is only if $\mathcal{S}_1 = \mathcal{S}_2$ that we can have a solution with large a_+ (i.e., large χ and thus large ρ), so that the string may touch the boundary and the spins can take large values.

C. Large-spin limit

Let us consider the case when the limit of $a_+ \gg 1$ is possible. This limit corresponds to $\nu \approx \mu \approx 1$ when

$$\omega \approx \kappa \approx \frac{n}{2\pi}\ln\frac{16}{1-\mu} \gg 1. \tag{4.22}$$

The solution for $x = \sinh\chi$ in (4.11) becomes

$$\sinh\chi = \sqrt{a_-}\cosh\left(\frac{K[\mu]n}{\pi}\sigma\right) = \sqrt{a_-}\cosh(\kappa\sigma). \tag{4.23}$$

Here we again relax the periodicity condition in σ and consider only one interval $0 < \kappa\sigma < \kappa\frac{\pi}{n} \gg 1$. Then Eq. (4.13) for ψ leads to

$$\tanh\psi = \frac{1}{\sqrt{1+a_-}}\tanh(\kappa\sigma). \tag{4.24}$$

In this limit we can use the following approximation for $\Pi[\nu, \mu]$ for $\nu < \mu$[31]

$$\Pi[\nu, \mu] = \sqrt{\frac{\nu}{(1-\nu)(\mu-\nu)}}\,\mathrm{arcsinh}\sqrt{\frac{\mu-\nu}{1-\mu}}, \tag{4.25}$$

implying that

$$\sinh\psi_0 = \sqrt{\frac{\mu-\nu}{1-\mu}}. \tag{4.26}$$

The condition (4.16) becomes

$$\cos\frac{\pi m}{n} = \sqrt{\frac{1-\mu}{1-\nu}}. \tag{4.27}$$

Here $2m < n$, i.e., the minimal choice is again $m = 1$, $n = 3$. Equation (4.27) implies $a_- = \cot^2\frac{\pi m}{n}$. Using (4.1) we then conclude that the resulting asymptotic solution is the same as found in Sec. II C [cf. (2.35) and (2.36)]

$$\cosh\rho = \cosh\rho_-\cosh(\kappa\sigma), \qquad \cosh\rho_- = \frac{1}{\sin\frac{\pi m}{n}} \tag{4.28}$$

$$\tan\theta = \tan\theta_0\tanh(\kappa\sigma), \qquad \tan\theta_0 = \tan\frac{\pi m}{n} = \frac{1}{\sinh\rho_-}. \tag{4.29}$$

As we have seen in Sec. III, starting with such an asymptotic solution where string is stretching towards the boundary we may glue such arcs together to get a closed-string solution with two equal spins $\mathcal{S}_1 = \mathcal{S}_2 \gg 1$ and $\ln\mathcal{S}$ scaling of the energy. If we would try to continue this asymptotic solution to finite values of the spins we would end up with the solution of Sec. II A.

V. COMMENTS ON GENERAL SOLUTIONS WITH $\mathcal{S}_1 \neq \mathcal{S}_2 \gg 1$

Let us now relax the assumptions about the frequencies like in (1.6) and try to determine the general properties of a solution that may allow the large-spin limit, i.e., for which parts of the closed string may stretch towards the boundary ($\rho \to \infty$). This may be either a bended folded string [17] or a circular string of the type described in the previous sections.

A. Universal ln$S$ scaling

Let us start with the string equations of motion and the conformal constraint (their first integral) for the ansatz (1.2) with generic κ, ω_1, ω_2[32]:

$$(\theta'\sinh^2\rho)' = (\omega_1^2 - \omega_2^2)\sin\theta\cos\theta\sinh^2\rho \tag{5.1}$$

[31] This is different from (2.26) as the parameters here are different.

[32] More general ansatz for infinite (open) strings ending on the boundary was considered in [10].

$$\rho'' - \cosh\rho \sinh\rho(\kappa^2 + \theta'^2 - \omega_1^2\cos^2\theta - \omega_2^2\sin^2\theta) = 0 \quad (5.2)$$

$$\rho'^2 - \kappa^2\cosh^2\rho + \sinh^2\rho\theta'^2 + \omega_1^2\sinh^2\rho\cos^2\theta + \omega_2^2\sinh^2\rho\sin^2\theta = 0. \quad (5.3)$$

Being interested in solutions which have parts stretching towards large ρ, let us focus on such an asymptotic region where we may set $\sinh\rho \approx \frac{1}{2}e^\rho$. Then Eqs. (5.1) and (5.3) reduce to

$$(\theta' e^{2\rho})' = (\omega_1^2 - \omega_2^2)\sin\theta\cos\theta e^{2\rho} \quad (5.4)$$

$$\rho'^2 - \kappa^2\left(1 + \frac{e^{2\rho}}{4}\right) + \frac{e^{2\rho}}{4}\theta'^2 + \frac{e^{2\rho}}{4}(\omega_1^2\cos^2\theta + \omega_2^2\sin^2\theta) = 0. \quad (5.5)$$

Let us rescale σ by κ, i.e., introduce $\bar\sigma \equiv \kappa\sigma$ and also define $w_i \equiv \frac{\omega_i}{\kappa}$. Then (now prime will be a derivative over $\bar\sigma$)

$$(e^{2\rho}\theta')' = (w_1^2 - w_2^2)e^{2\rho}\sin\theta\cos\theta \quad (5.6)$$

$$4(\rho'^2 - 1) - e^{2\rho} + e^{2\rho}\theta'^2 + e^{2\rho}(w_1^2\cos^2\theta + w_2^2\sin^2\theta) = 0. \quad (5.7)$$

To get $\ln S$ behavior of the energy at large spin we shall assume that $\kappa \gg 1$ so that $\bar\sigma$ can take large values and will require e^ρ to increase exponentially with $\bar\sigma$. Setting $e^\rho = u(\bar\sigma)e^{\bar\sigma}$ we get

$$2\theta' + 2u\theta' + u^2\theta'' = (w_1^2 - w_2^2)u\sin\theta\cos\theta \quad (5.8)$$

$$4\frac{u'^2 + 2uu'}{u^2} - u^2e^{2\sigma} + u^2e^{2\sigma}\theta'^2 + u^2e^{2\sigma}(w_1^2\cos^2\theta + w_2^2\sin^2\theta) = 0. \quad (5.9)$$

These equations are not readily solved but we observe that $\mathcal{E} - \mathcal{S} \approx \frac{\kappa}{2\pi} \gg 1$, where $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ and κ plays the role of a cutoff on $\bar\sigma$. As in the previous sections, the asymptotic solution reaching the boundary may be built out of several segments. For such a segment for any $u(\bar\sigma)$ at the leading order we have $\kappa \approx \frac{1}{2}\ln\mathcal{S}$ and thus

$$\mathcal{E} - \mathcal{S} = \frac{1}{4\pi}\ln\mathcal{S} + \ldots \quad (5.10)$$

To get a closed-string solution we would need to combine together several such segments; we need at least four of them as in the case of the folded string.

The general conclusion appears to be that for spinning strings with two large spins and having minimal energy for given spins so that only parts of them are reaching the boundary the $\ln\mathcal{S}$ behavior of the energy in the large-spin limit appears to be *universal*. It remains an open problem to systematically classify such string solutions with unequal ($S_1 \neq S_2$) spins which may be taken to be large with their ratio fixed.

B. $S_1 \neq S_2$ deformation of the solution of Sec. II

Let us now discuss how one may construct an example of a solution with $E \sim \ln S$ and $S_1 \neq S_2$ by perturbing the equal-spin circular solution of Sec. II. Let us go back to Eqs. (5.1) and (5.3) writing them in terms of the rescaled parameters $w_i = \frac{\omega_i}{\kappa}$ and $\bar\sigma = \kappa\sigma$

$$(\theta'\sinh^2\rho)' = (w_1^2 - w_2^2)\sin\theta\cos\theta\sinh^2\rho \quad (5.11)$$

$$\rho'^2 - \cosh^2\rho + \sinh^2\rho\theta'^2 + \sinh^2\rho(w_1^2\cos^2\theta + w_2^2\sin^2\theta) = 0 \quad (5.12)$$

and now taking $w_1 = w$ and $w_2 = w - \epsilon$ with $\epsilon \ll 1$. Thus we will have $1 \leq w_2^2 \leq w_1^2$. At leading order in small ϵ expansion when $w_1 = w_2$ when this system of equations decouples it was already solved in Sec. II. Expanding near this solution ρ_0, θ_0 given in (2.7) and (2.12) with [see (2.4)]

$$w^2 = \frac{a_+ + a_- - 1}{a_+ + a_- - 2} \quad (5.13)$$

we have

$$\rho(\bar\sigma, a_\pm) = \rho_0(\bar\sigma, a_\pm) + \epsilon\rho_1(\bar\sigma, a_\pm), \quad \theta(\bar\sigma, a_\pm) = \theta_0(\bar\sigma, a_\pm) + \epsilon\theta_1(\bar\sigma, a_\pm) \quad (5.14)$$

where from (5.11) and (5.12) the linear perturbations should satisfy

$$\sinh\rho_0[2(\rho_1'\theta_0' + \rho_0'\theta_1')\cosh\rho_0 + \theta_1''\sinh\rho_0] - 2w\sin\theta_0\cos\theta_0\sinh^2\rho_0 = 0 \quad (5.15)$$

$$\rho_0'\rho_1' + \rho_1\sinh\rho_0\cosh\rho_0(w^2 - 1 + \theta_0'^2) + \sinh^2\rho_0(\theta_0'\theta_1' - w\sin^2\theta_0) = 0. \quad (5.16)$$

We can then solve the second equation for θ_1' in terms of ρ_1, ρ_1' and plug it into the first. This gives the following equation for ρ_1

$$\begin{aligned} &A\rho_1'' + B\rho_1' + C\rho_1 + D = 0\\ &A = -2\rho_0''\theta_0',\\ &B = \theta_0'[(1 - w^2 + \theta_0'^2)\sinh(2\rho_0) - 2\rho_0''] + 2\theta_0''\rho_0'\\ &C = \theta_0''(w^2 - 1 + \theta_0'^2)\sinh(2\rho_0),\\ &D = 2w\theta_0'\rho_0'\sin^2\theta_0\sinh(2\rho_0) - 2w\theta_0''\sin^2\theta_0\sinh^2\rho_0. \end{aligned} \quad (5.17)$$

Given the complicated form of the functions ρ_0, θ_0 we cannot solve for ρ_1 but we have checked numerically that

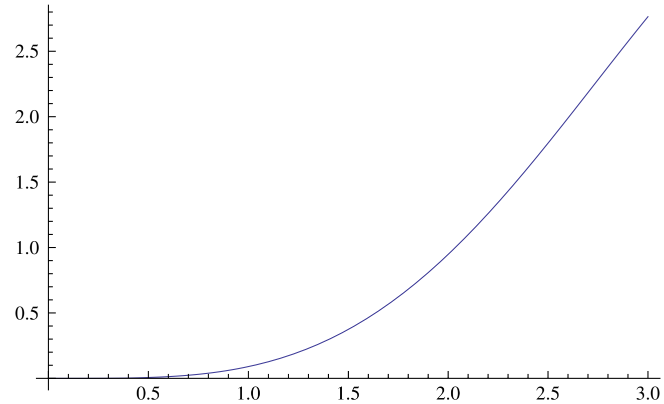

FIG. 7 (color online). Solution for $\rho_1(\bar{\sigma})$ for $a_- = 1.5$, $a_+ = 6.98$ with satisfying $\rho_1(0) = 0$, $\rho_1'(0) = 0$.

Eq. (5.17) does have a solution with trivial initial conditions.[33] It is plotted in Fig. 7. Using the solution for ρ_1 one can then find the solution for θ_1 from (5.15).

As in Sec. II, interpreting this solution as describing a segment of a string with $0 < \bar{\sigma} \le \frac{\pi\kappa}{n}$ we may then glue $2n$ such pieces together by imposing the condition $\Delta\theta_0 + \epsilon\Delta\theta_1 = \frac{\pi m}{n}$ on the change of the angle. This leads to a closed-string solution with unequal spins. One can see that the spins indeed differ by an $O(\epsilon)$ term by using the ζ coordinates (B1) and the expressions for the spins in (B16). Here the two spins remain different in the large-spin limit. Since the change in θ is small, the minimal energy choice for this perturbed solution will be again $m = 1$, $n = 3$.

As a result, we find a circular-shaped solution with different spins and $\ln S$ asymptotics of the energy. From the leading contribution to the energy from one string arc (5.10), which is controlled by the asymptotic (large ρ) region of the solution, we conclude that for the minimal solution with $m = 1$, $n = 3$ we get like in (2.33)

$$\mathcal{E} - \mathcal{S} = \frac{3}{2\pi}\ln\mathcal{S} + \ldots, \qquad \mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2, \qquad (5.18)$$

[33]This equation can be simplified by considering the large-spin limit when the string touches the boundary ($a_+ \to \infty$). Then $\kappa \to \infty$, i.e., the range of $\bar{\sigma}$ is $0 \le \bar{\sigma} < \infty$. We found in Sec. II that in this limit (we use 00 subscript to denote the asymptotic form of the leading-order $S_1 = S_2$ solution) $w = 1$, $\cosh\rho_{00} = \cosh\rho_- \cosh\bar{\sigma}$, $\cot\theta_{00} = \sinh\rho_- \coth\bar{\sigma}$. Then

$$\frac{1 - a_-\cosh^2\bar{\sigma}}{\sinh\bar{\sigma}}\rho_{10}'' - 2a_-\cosh\bar{\sigma}\rho_{10}' - \frac{2(a_- - 1)}{[a_-\cosh(2\bar{\sigma}) + a_- - 2]\sinh\bar{\sigma}}\rho_{10} + 2\sqrt{a_-}\sqrt{a_-\cosh^2\bar{\sigma} - 1}\sinh(2\bar{\sigma}) = 0.$$

This equation still seems hard to solve analytically. Numerically we again find a solution with the initial conditions $\rho_{10}(0) = 0$, $\rho_{10}'(0) = 0$.

but now the subleading terms are expected to depend on $\frac{S_1}{S_2}$. They may be, in principle, determined numerically.

The above perturbative solution thus provides evidence of existence of a circular-shaped solution with different spins and $\ln S$ asymptotics of the energy (5.18). To construct it explicitly it appears necessary to study the general form of the Neumann system solution in [17] expressed in terms of hyperelliptic integrals.

VI. CONCLUSIONS

In this paper we have studied rigid semiclassical string solutions with two spins in AdS_5. We explicitly wrote down solutions that can be expressed in terms of the elliptic integrals, i.e., the solutions that correspond to the cases when the general hyperelliptic surface associated to a Neumann system solution degenerates into an elliptic one.

We identified a particular solution that should represent a state that has minimal energy for large values of the two equal spins. We found that the energy of this solution grows logarithmically with the total spin S for large S. Since the asymptotics of this 3-arc solution is controlled by the near-boundary parts of the string, we concluded that the coefficient of the $\ln S$ term should be given, up to a $3/2$ coefficient, by the same universal scaling function as in the folded string case.

ACKNOWLEDGMENTS

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APPENDIX A: SPECIAL CASE OF THE $\kappa = \omega_2 \neq \omega_1$ SOLUTION

The possibility omitted in Sec. IVA is when $a_\pm$ in (4.8) do not have the same sign. At least one of them should be positive in order to have a periodic solution for x in (4.7). The resulting solution will not, however, admit a large-spin limit. If $\omega > \sqrt{2}\kappa$ and $\kappa^2 > c^2$ we get that $a_+ > 0$ and $-1 < a_- < 0$. The relations (4.9) are still valid. Note that from (4.8) we have

$$a_+ + a_- = \frac{2\kappa^2 - \omega^2}{\omega^2 - \kappa^2}. \qquad (A1)$$

To have a long string/large-spin limit we should allow a_+ to be large. There are two choices of parameters when this might be possible. The first one we considered already in Sec. IVA—to take $\omega \approx \kappa$ while keeping a_- fixed. Since here we assume $2\kappa^2 < \omega^2$, i.e., $a_+ + a_- < 0$, we may have $a_+ \gg 1$ only if a_- is allowed to take large negative values. However, from above we have $a_- > -1$, i.e., the maximal value of a_+ is one, which can be reached only if

$c = 0$. The solution discussed below thus has a finite "size," not reaching the boundary.

Here it is convenient to define the parameters μ, ν as follows [cf. (4.10)]:

$$\mu = \frac{a_+}{a_+ - a_-}, \qquad \nu = \frac{a_+(a_- + 1)}{a_+ - a_-}, \\ a_+ = \frac{\mu - \nu}{1 - \mu}, \qquad a_- = \frac{\nu - \mu}{\mu}. \tag{A2}$$

They satisfy $0 < \nu$, $\mu < 1$, and $\mu > \nu$. In this case x changes in the interval: $0 < x < \sqrt{a_+}$. Solving Eq. (4.7) with the initial condition $x(0) = 0$ we find

$$x = \sinh\chi = \pm \frac{\sqrt{-a_+ a_-}\,\mathrm{sn}[\frac{\omega\sqrt{a_+ - a_-}}{\sqrt{a_+ + a_- + 2}}\sigma, \mu]}{\sqrt{a_+ - a_-}\,\mathrm{dn}[\frac{\omega\sqrt{a_+ - a_-}}{\sqrt{a_+ + a_- + 2}}\sigma, \mu]}. \tag{A3}$$

Assuming again that here $0 < \sigma \leq \frac{\pi}{n}$ we get

$$\frac{\omega\sqrt{a_+ - a_-}}{\sqrt{a_+ + a_- + 2}} = \frac{K[\mu]n}{\pi}. \tag{A4}$$

Solving the equation for ψ (4.13) with the initial condition $\psi(x = \sqrt{a_+}) = 0$ at $\sigma = \frac{\pi}{n}$ we obtain

$$\psi(x) = \sqrt{\frac{a_+ + 1}{(a_- + 1)(a_+ - a_-)}}\Bigg[a_-\Bigg(\Pi[\nu, \mu] - \Pi\Bigg[\nu, \arcsin\Bigg(\frac{x}{\sqrt{a_+}}\sqrt{\frac{a_+ - a_-}{x^2 - a_-}}\Bigg), \mu\Bigg]\Bigg) + K[\mu] - F\Bigg[\arcsin\Bigg(\frac{x}{\sqrt{a_+}}\sqrt{\frac{a_+ - a_-}{x^2 - a_-}}\Bigg), \mu\Bigg]\Bigg], \tag{A5}$$

where $F[z, \mu] = \int_0^z \frac{d\alpha}{\sqrt{1 - \mu \sin^2\alpha}}$ is the elliptic F function. At $\sigma = 0$ we have $x = 0$ and

$$\psi(0) = \psi_0 = \sqrt{\frac{a_+ + 1}{(a_- + 1)(a_+ - a_-)}}(a_-\Pi[\nu, \mu] + K[\mu]). \tag{A6}$$

Using (4.1) we obtain $\theta(\sigma = 0) = \frac{\pi}{2}$ and $\theta(\sigma = \frac{\pi}{n}) = 0$.

Thus here we get $\triangle\theta = \frac{\pi}{2}$. This is different from the situation studied in Sec. IVA since now the motion takes place in the interval $0 < x < \sqrt{a_+}$. After mapping a segment in the interval $0 < \sigma < \frac{\pi}{n}$ from (χ, ψ) coordinates into (ρ, θ) coordinates we can glue several of them together to form again a circular closed string. Because for each arc $\triangle\theta = \frac{\pi}{2}$ we simply have the condition $2n = m$, where m is the winding number. One can again solve the equations $\mathcal{S}_1 = \mathcal{S}_1(a_+, a_-, n)$, $\mathcal{S}_2 = \mathcal{S}_2(a_+, a_-, n)$ for $a_\pm$ and obtain the energy $\mathcal{E} = \mathcal{E}(\mathcal{S}_1, \mathcal{S}_2, n)$. We checked numerically that such solutions indeed exist. The shape of the solution for $m = 1$ is illustrated in Fig. 8.

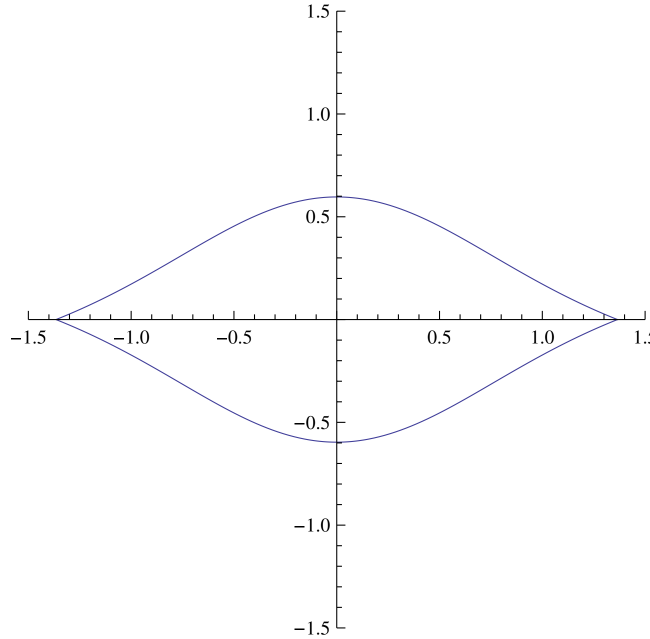

FIG. 8 (color online). (S_1, S_2) string for $m = 1$, $n = 2$ in polar coordinates (ρ, θ) for $a_- = -0.5$, $a_+ = 0.4$. In this case $\frac{S_1}{S_2} = 6.614$.

Since $a_\pm$ change within a finite range and a_- is never close to zero, the parameters ω, κ cannot be large, and thus the spins here cannot take large values.

APPENDIX B: CIRCULAR SOLUTION IN THE SPECIAL CASE $b_1 = b_2$

In this Appendix we shall consider another particular case in which the solution can be expressed in terms of the elliptic integrals. Let us briefly recall the approach of [17] to the solution of string equations for the rigid-string ansatz (1.2) based on the reduction to the 1D Neumann integrable model. Introducing the two independent "hyperbolic" coordinates ζ_i related to y_a in (1.3) by[34]

$$y_1^2 = \frac{(\omega_1^2 - \zeta_1)(\omega_1^2 - \zeta_2)}{\omega_{12}^2\omega_{01}^2}, \\ y_2^2 = \frac{(\omega_2^2 - \zeta_1)(\omega_2^2 - \zeta_2)}{\omega_{12}^2\omega_{20}^2}, \\ y_0^2 = \frac{(\omega_0^2 - \zeta_1)(\omega_0^2 - \zeta_2)}{\omega_{02}^2\omega_{01}^2} \tag{B1}$$

where $\omega_a = (\omega_0, \omega_1, \omega_2)$, $\omega_0 = \kappa$, and $\omega_{ab}^2 = \omega_a^2 - \omega_b^2$, it was shown in [17] that the string equations of motion expressed in terms of $\zeta_i(\sigma)$ take the form

[34]We follow the notation in [17] apart from interchanging the roles of y_1 and y_2 and using ω_0 instead of ω_3. Continuation from S^5 to AdS_5 replaces y_3 by y_0 and ω_3 by ω_0.

$$\zeta_1'^2 = -4\frac{P(\zeta_1)}{(\zeta_2-\zeta_1)^2}, \qquad \zeta_2'^2 = -4\frac{P(\zeta_2)}{(\zeta_2-\zeta_1)^2} \tag{B2}$$

where

$$P(\zeta) = (\zeta-\omega_1^2)(\zeta-\omega_2^2)(\zeta-\omega_0^2)(\zeta-b_1)(\zeta-b_2). \tag{B3}$$

Here b_1, b_2 are two constants of motion that satisfy $b_1 + b_2 = \omega_0^2 + \omega_1^2 + \omega_2^2$. They are related to the integrals of motion of the Neumann system F_a satisfying $F_1 + F_2 + F_0 = 1$ by

$$b_1 + b_2 = (\omega_0^2+\omega_1^2)F_1 + (\omega_0^2+\omega_2^2)F_2 + (\omega_1^2+\omega_2^2)F_0,$$
$$b_1 b_2 = \omega_0^2\omega_1^2 F_1 + \omega_0^2\omega_2^2 F_2 + \omega_1^2\omega_2^2 F_0. \tag{B4}$$

The integrals of motion satisfy $F_1 + F_2 + F_0 = 1$. The integrals of motion can be expressed as

$$F_1 = \frac{(b_1-\omega_2^2)(b_2-\omega_2^2)}{(\omega_0^2-\omega_2^2)(\omega_1^2-\omega_2^2)},$$
$$F_2 = \frac{(b_1-\omega_1^2)(\omega_1^2-b_2)}{(\omega_0^2-\omega_1^2)(\omega_1^2-\omega_2^2)}, \tag{B5}$$
$$F_0 = \frac{(b_1-\omega_0^2)(b_2-\omega_0^2)}{(\omega_0^2-\omega_1^2)(\omega_0^2-\omega_2^2)}.$$

It was argued in [17] that to get a 2-spin solution of a circular type one needs to assume

$$\omega_0^2 \le \omega_2^2 \le \zeta_1 \le \omega_1^2 \le b_1 \le \zeta_2 \le b_2 \tag{B6}$$

while to get a folded string solution one needs

$$\omega_0^2 \le \omega_2^2 \le \zeta_1 \le b_1 \le \omega_1^2 \le \zeta_2 \le b_2. \tag{B7}$$

Since $P(\zeta)$ is a degree 5 polynomial, the only way to obtain a solution expressed in terms of elliptic integrals is to assume that at least two of the five parameters in (B6) and (B7) are equal.

For the circular string choice (B6) the possible special cases are:

(i) $\omega_1^2 = \omega_2^2$ which we considered in Sec. II;
(i′) $\omega_1^2 = \omega_2^2 = \omega_0^2$ which we considered in Sec. III;
(ii) $\omega_0^2 = \omega_2^2$ which we considered in Sec. IV;
(iii) $b_1 = b_2$ which we shall analyze below;
(iv) $\omega_1^2 = b_1$, which reduces to the 1-spin case.

Note that the case of $\omega_1^2 = b_1 = b_2$ reduces to the 1-spin case as then $y_1 = 0$. In the folded string case (B7) potentially nontrivial special choices are $\omega_2^2 = b_1$ and $\omega_1^2 = b_2$; however, they reduce to the 1-spin solution. Note that in these cases and the case (iv) one of the F_a's in (B5) is equal to zero.

Let us now study in detail the case of $b_1 = b_2 \equiv b$. Then $2b = \omega_1^2 + \omega_2^2 + \omega_0^2$ and $\zeta_2 = b$. Equation (B2) becomes

$$z'^2 = 4(z-\omega_2^2)(z-\omega_0^2)(\omega_1^2-z), \qquad z \equiv \zeta_1. \tag{B8}$$

With the initial condition $z(0) = \omega_2^2$ its solution is

$$z(\sigma) = \frac{\omega_2^2 - p\omega_0^2 \mathrm{sn}^2[\omega_{10}\sigma, p]}{\mathrm{dn}^2[\omega_{10}\sigma, p]}, \qquad p \equiv \frac{\omega_{12}^2}{\omega_{10}^2}, \qquad 0 < p < 1. \tag{B9}$$

We want to have a periodic motion of $z(\sigma)$ between ω_2^2 and ω_1^2. As in the previous sections, to get a closed-string solution on the interval $0 < \sigma \le 2\pi$ we should start with (B9) on $0 < \sigma \le \frac{\pi}{n}$ with $z(0) = \omega_2^2$, $z(\sigma = \frac{\pi}{n}) = \omega_1^2$, and then glue together $2n$ such segments. The periodicity condition implies

$$\omega_{10} = \frac{nK[p]}{\pi} \tag{B10}$$

which gives a relation between ω_a's. The resulting expressions for the string coordinates y_a in (1.3) and (B1) are

$$y_1^2 = (\omega_1^2 - z)C_1, \qquad y_2^2 = (\omega_2^2 - z)C_2, \qquad y_0^2 = (\omega_0^2 - z)C_0 \tag{B11}$$

where

$$C_1 = \frac{\omega_1^2-\omega_2^2-\omega_0^2}{2\omega_{12}^2\omega_{01}^2}, \qquad C_2 = \frac{\omega_2^2-\omega_1^2-\omega_0^2}{2\omega_{12}^2\omega_{20}^2},$$
$$C_0 = \frac{\omega_0^2-\omega_1^2-\omega_2^2}{2\omega_{02}^2\omega_{01}^2}. \tag{B12}$$

We need $C_1 > 0$, $C_2 < 0$, and $C_0 < 0$. This requirement along with the condition $y_0^2 > 1$ [cf. (1.3)] gives the possible range of the frequencies

$$\omega_1^2 - \omega_2^2 < \omega_0^2 < \omega_1^2 + \omega_2^2. \tag{B13}$$

Using the relation between y_a and angular coordinates in (1.1) one obtains

$$\tan\theta = \sqrt{\frac{(\omega_2^2 - z)C_2}{(\omega_1^2 - z)C_1}} \tag{B14}$$

which implies $\theta(\sigma = 0) = \frac{\pi}{2}$ and $\theta(\sigma = \frac{\pi}{n}) = 0$. If $m = 1, 2, \ldots$ is the winding number in θ then

$$n = 2m = 2, 4, \ldots \tag{B15}$$

For the conserved charges [cf. (2.17)]

$$\mathcal{E} = \omega_0 \int_0^{2\pi} \frac{d\sigma}{2\pi} y_0^2, \qquad \mathcal{S}_1 = \omega_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} y_1^2,$$
$$\mathcal{S}_2 = \omega_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} y_2^2 \tag{B16}$$

we then find

$$\mathcal{E} = -\frac{n\omega_0 C_0 \omega_{10}}{\pi} E[p],$$
$$\mathcal{S}_1 = \frac{n\omega_1 C_1 \omega_{10}}{\pi}(K[p] - E[p]), \tag{B17}$$
$$\mathcal{S}_2 = \frac{n\omega_2 C_2 \omega_{10}}{\pi}((1-p)K[p] - E[p]).$$

Given $\mathcal{S}_1 = \mathcal{S}_1(\omega_1, \omega_2, \omega_0)$, $\mathcal{S}_2 = \mathcal{S}_2(\omega_1, \omega_2, \omega_0)$ and the periodicity condition (B10) we may solve for ω_a and find the energy $\mathcal{E} = \mathcal{E}(\mathcal{S}_1, \mathcal{S}_2, n)$.

We have checked numerically that such closed-string solutions indeed exist. We illustrate the shape of the string with $n = 2$ in Fig. 9. In what follows we shall look for solutions that admit the possibility of having both spins being large. For finite ω_a, the only limit that may give large spins is $\omega_2 \to \omega_0$. But then $p \to 1$ and the periodicity condition (B10) cannot be solved since $K[1]$ diverges.

The other possibility is to take ω_1 large; from (B13) this means that ω_2 is also large, so that p is small and (B10) does not have a solution for ω_0. However, we can take ω_0 also to be large; then p will be a finite number in the interval (0, 1). Assuming

$$\omega_1 = \omega, \qquad \omega_2 = a\omega, \qquad \omega_0 = b\omega, \qquad 0 < b \le a \le 1 \tag{B18}$$

and expanding (B10) in large ω we get

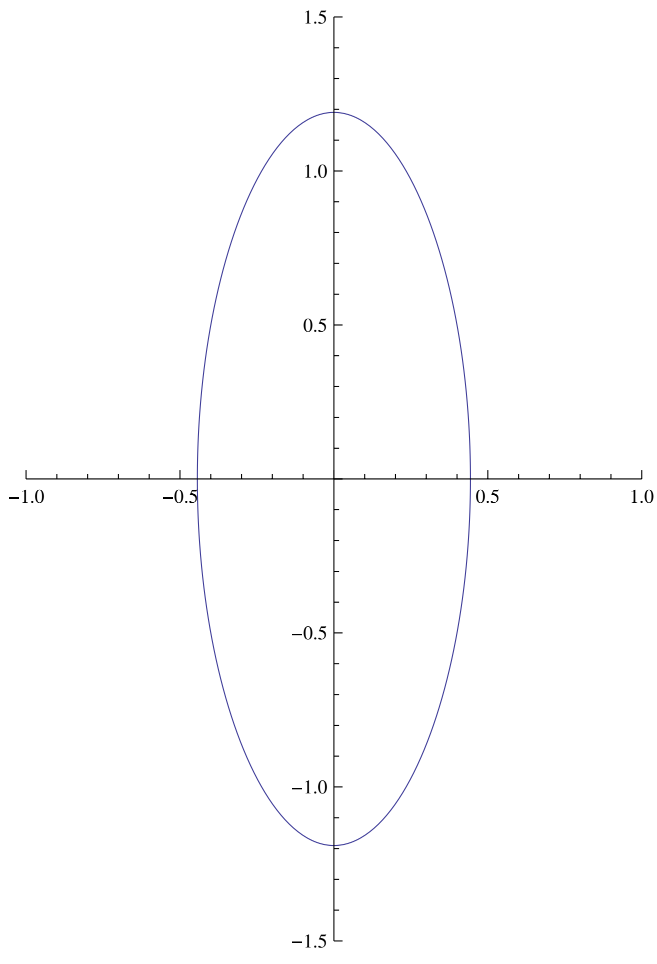

FIG. 9 (color online). (S_1, S_2) string for $m = 1$, $n = 2$ in polar coordinates (ρ, θ) for $\omega_1 = 1.8$, $\omega_2 = 1.5$, $\omega_0 = 1.2867$. In this case $\frac{S_1}{S_2} = 0.144$.

$$\omega_{10} - \frac{nK[p]}{\pi} = \sqrt{1 - b^2}\omega - \frac{n}{\pi}K\left[\frac{a^2 - 1}{b^2 - 1}\right] + \mathcal{O}\left(\frac{1}{\omega}\right) = 0 \tag{B19}$$

which leads, to leading order in $\omega \gg 1$, to $a = b = 1$. This suggests to set[35]

$$\omega_0 = \omega - \frac{c}{\omega} + \ldots, \qquad \omega_2 = \omega - \frac{d}{\omega} + \ldots, \qquad \omega_1 = \omega \gg 1. \tag{B20}$$

Then the periodicity condition (B10) gives

$$\sqrt{2c} - \frac{n}{\pi}K\left[\frac{d}{c}\right] + \mathcal{O}\left(\frac{1}{\omega}\right) = 0. \tag{B21}$$

Let us define $q = \frac{d}{c} < 1$, so that $p = q + \mathcal{O}(\frac{1}{\omega^2})$. Expanding the spins and the energy in large ω gives

$$\mathcal{S}_1 = \frac{n\omega^3}{4\pi\sqrt{2c}}\frac{E[q] - (1-q)K[q]}{d(1-q)} + \mathcal{O}(\omega), \qquad \mathcal{S}_2 = \frac{n\omega^3}{4\pi\sqrt{2c}}\frac{K[q] - E[q]}{d} + \mathcal{O}(\omega) \tag{B22}$$

$$\mathcal{E} = \frac{n\omega^3}{4\pi\sqrt{2c}}\frac{E[q]}{c(1-q)} + \mathcal{O}(\omega). \tag{B23}$$

For this solution both spins are thus large (scaling as $\omega^{1/3}$) and different with [cf. (B21)]

$$\mathcal{E} - \mathcal{S}_1 - \mathcal{S}_2 = \frac{3n\omega}{4\pi\sqrt{2c}}K[q] + \mathcal{O}\left(\frac{1}{\omega}\right) = \frac{3}{4}\omega + \mathcal{O}\left(\frac{1}{\omega}\right), \tag{B24}$$

$$\frac{\mathcal{S}_1}{\mathcal{S}_2} = \frac{E[q] - (1-q)K[q]}{(1-q)(K[q] - E[q])} + \mathcal{O}\left(\frac{1}{\omega^2}\right). \tag{B25}$$

We have checked that if $\frac{\mathcal{S}_1}{\mathcal{S}_2} \ge 1$ Eq. (B25) has a solution for q. Then we can use (B21) to find c. At large ω we have

$$\omega^3 = 4c(1-q)\frac{K[q]}{E[q]}\mathcal{S}, \qquad \mathcal{S} \equiv \mathcal{S}_1 + \mathcal{S}_2 \tag{B26}$$

which leads to the following dependence of the energy on the total spin:

$$\mathcal{E} - \mathcal{S} = \frac{3}{4}\left(4c(1-q)\frac{K[q]}{E[q]}\right)^{1/3}\mathcal{S}^{1/3} + \ldots, \qquad \mathcal{S} \gg 1, \tag{B27}$$

where c, q are functions of $\frac{\mathcal{S}_1}{\mathcal{S}_2}$ and n.

The particular case $\mathcal{S}_1 = \mathcal{S}_2$ corresponds to $q = 0$, i.e., $d = 0$ and $c = \frac{n^2}{8} = \frac{m^2}{2}$. Then

$$\mathcal{E} - \mathcal{S} = \tfrac{3}{4}(2m^2)^{1/3}\mathcal{S}^{1/3} + \ldots \tag{B28}$$

i.e., we recover the round circular string [12] expression (1.10) with the winding number $m = \frac{n}{2} = 1, 2, \ldots$

[35] Including $\mathcal{O}(\omega^0)$ terms in ω_i does not lead to a new solution.

[1] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Nucl. Phys. **B636**, 99 (2002).
[2] H. J. de Vega and I. L. Egusquiza, Phys. Rev. D **54**, 7513 (1996).
[3] M. Kruczenski, J. High Energy Phys. 08 (2005) 014.
[4] L. Freyhult, A. Rej, and S. Zieme, arXiv:0911.2458.
[5] M. Kruczenski, R. Roiban, A. Tirziu, and A. A. Tseytlin, Nucl. Phys. **B791**, 93 (2008).
[6] M. Kruczenski, J. High Energy Phys. 12 (2002) 024.
[7] L. F. Alday and J. M. Maldacena, J. High Energy Phys. 06 (2007) 064.
[8] L. F. Alday and J. M. Maldacena, J. High Energy Phys. 11 (2007) 019.
[9] L. F. Alday and J. Maldacena, J. High Energy Phys. 11 (2009) 082.
[10] A. Irrgang and M. Kruczenski, J. High Energy Phys. 12 (2009) 014.
[11] N. Drukker and B. Fiol, J. High Energy Phys. 01 (2006) 056.
[12] S. Frolov and A. A. Tseytlin, Nucl. Phys. **B668**, 77 (2003).
[13] I. Y. Park, A. Tirziu, and A. A. Tseytlin, Phys. Rev. D **71**, 126008 (2005).
[14] G. Ferretti, R. Heise, and K. Zarembo, Phys. Rev. D **70**, 074024 (2004); N. Beisert, G. Ferretti, R. Heise, and K. Zarembo, Nucl. Phys. **B717**, 137 (2005).
[15] A. V. Belitsky, A. S. Gorsky, and G. P. Korchemsky, Nucl. Phys. **B748**, 24 (2006).
[16] L. Freyhult, M. Kruczenski, and A. Tirziu, J. High Energy Phys. 07 (2009) 038.
[17] G. Arutyunov, S. Frolov, J. Russo, and A. A. Tseytlin, Nucl. Phys. **B671**, 3 (2003).
[18] G. Arutyunov, J. Russo, and A. A. Tseytlin, Phys. Rev. D **69**, 086009 (2004).
[19] S. Frolov and A. A. Tseytlin, J. High Energy Phys. 06 (2002) 007.
[20] A. L. Larsen and A. Khan, Nucl. Phys. **B686**, 75 (2004).
[21] S. Ryang, J. High Energy Phys. 04 (2004) 053.
[22] M. Beccaria, V. Forini, A. Tirziu, and A. A. Tseytlin, Nucl. Phys. **B812**, 144 (2009).
[23] N. Beisert and M. Staudacher, Nucl. Phys. **B727**, 1 (2005); N. Beisert, B. Eden, and M. Staudacher, J. Stat. Mech. 0701 (2007) P021.
[24] G. Arutyunov, S. Frolov, and M. Staudacher, J. High Energy Phys. 10 (2004) 016.
[25] N. Beisert, V. A. Kazakov, K. Sakai, and K. Zarembo, Commun. Math. Phys. **263**, 659 (2006); N. Beisert, V. A. Kazakov, and K. Sakai, Commun. Math. Phys. **263**, 611 (2006); V. A. Kazakov, A. Marshakov, J. A. Minahan, and K. Zarembo, J. High Energy Phys. 05 (2004) 024.
[26] R. Ishizeki, M. Kruczenski, A. Tirziu, and A. A. Tseytlin, Phys. Rev. D **79**, 026006 (2009).
[27] N. Dorey and M. Losi, arXiv:0812.1704.
[28] M. Kruczenski and A. A. Tseytlin, Phys. Rev. D **77**, 126005 (2008).
[29] D. M. Hofman and J. M. Maldacena, J. Phys. A **39**, 13095 (2006).