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Averaged null energy condition violation in a conformally flat spacetime

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We show that the averaged null energy condition can be violated by a conformally coupled scalar field in a conformally flat spacetime in 3 + 1 dimensions. The violation is dependent on the quantum state and can be made as large as desired. It does not arise from the presence of anomalies, although anomalous violations are also possible. Since all geodesics in conformally flat spacetimes are achronal, the achronal averaged null energy condition is likewise violated.

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I. INTRODUCTION

Without any restriction on the states of matter that can act as sources, general relativity allows arbitrary spacetimes, which may contain closed timelike curves, wormholes, and other such exotic phenomena. To prevent their occurrence requires restrictions on the stress-energy tensor of matter, which are called *energy conditions*. For example, the usual classical fields obey the weak energy condition: the energy density seen by any (timelike) observer can never be negative.[1] From this condition, wormholes, superluminal travel, and construction of time machines can be ruled out [4–7].

Unfortunately, quantum fields can violate any restriction on the value of the stress-energy tensor T_{ab} at a point, so the above argument does not hold in semiclassical gravity. For example, a superposition of the vacuum and a two-photon state gives negative energy density at certain locations. To make progress in this case, one can go to averaged energy conditions which restrict only certain averages of T_{ab}. In particular, the exotic situations mentioned above could be ruled out by the averaged null energy condition (ANEC), which states that the projection of T_{ab} onto the tangent vector of a null geodesic cannot give a negative integral,

$$\int_\gamma T_{ab} l^a l^b \geq 0, \qquad (1)$$

where l^a is the tangent vector to the geodesic γ.

In Minkowski space, ANEC always holds [8,9]. It cannot be violated even if one allows arbitrary boundaries (generalizing the parallel plates of the Casimir effect), as long as these do not approach arbitrarily close to the geodesic [10]. The result of [10] applies also to spacetimes that are flat near the geodesic but have curvature in distant places, as long as that curvature does not change the causal structure near the geodesic. However, this result does not apply for null geodesics that are chronal, that is to say some of whose points are in the chronological future of others.

A simple example of ANEC violation for chronal geodesics is given by the Casimir-like system produced by compactifying one spatial dimension in Minkowski space. In this case both the energy density and the pressure in the compactified direction are negative everywhere, and ANEC is violated by geodesics going in the compact direction. Because of the compactification, all geodesics are chronal.

ANEC can also be violated in 3 + 1 dimensional curved space. An example is given by the Schwarzschild spacetime in the Boulware vacuum state [11]. All complete geodesics (i.e., those that avoid the singularity) violate ANEC, but all those geodesics are chronal.

In 1 + 1 dimensions, on the other hand, either all geodesics are chronal (if the spatial dimension is compactified) or all geodesics are achronal. In the latter case ANEC always holds [9].

The above considerations might lead one to guess that quantum fields always obey "achronal ANEC," i.e., they obey ANEC on any achronal geodesic. This condition is sufficient to rule out many exotic situations [12], but even it is violated. Visser [13] showed that in general spacetimes, one can always violate ANEC by rescaling (which does not change the chronality of a geodesic) However, his violation results from the anomalous transformation of the stress tensor. It involves the logarithm of the rescaling factor multiplied by a tiny number such as $1/(2880\pi^2)$. Thus, any realistic rescaling will have negligible effect. Reference [12] conjectured that a principle of self-consistency could rule out this violation. If one requires the spacetime to be generated self-consistently by the state of the quantum fields (perhaps with the addition of some classical matter), Visser's anomalous violation would be too small to lead to the curvature necessary to produce the violation.

In the next section, we exhibit a new curved-space ANEC violation. We study the conformally coupled scalar

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[1]Non-minimally coupled scalar fields are an exception [1–3]. In this case the classical field can easily violate all pointwise energy conditions. However, classical violations of ANEC [1,2] are possible only if the field takes on Planck-scale values, which lead the effective Newton's constant to first diverge and then assume negative values. This may mean that such states are not physically realizable.

field in conformally flat spacetimes and find a state-dependent violation which can be made arbitrarily large. The basic idea is to start with a Minkowski-space quantum state which obeys ANEC (as it must), but which violates the null energy condition (NEC), which requires that $T_{ab}l^a l^b \geq 0$ for every point and every null vector l^a. We then choose a conformal transformation which enhances the NEC-violating regions in the ANEC integral, so that in the conformally related spacetime ANEC is violated. By choosing appropriate quantum states, ANEC can be violated to any desired degree.

The violation we discuss here differs from that of Visser [13] in that it depends on the state, rather than arising from anomalous terms, which depend only on the spacetime curvature. Additionally, the conformally flat systems discussed here are a special case to which Visser's argument does not apply, and where the scaling anomaly he discusses does not occur.

In Sec. III, we construct another kind of ANEC violation arising only from the anomalous terms in the transformation of the stress-energy tensor, starting from the Minkowski-space vacuum. The violation depends on having an inhomogeneous conformal transformation, rather than simply a rescaling, and does not depend the choice of renormalization scale

In Sec. IV, we review explicitly why these approaches cannot be used to violate ANEC in $1 + 1$ dimensions. In this case the anomalous contribution is always positive and cancels the largest effect that can be generated by reweighting an NEC-violating Minkowski-space state.

Finally, we conclude in Sec. V with some possibilities for how one might rule out exotic phenomena even though ANEC does not always hold.

We work in units where $c = 1$ and $\hbar = 1$. Our sign conventions are $(+++)$ in the categorization of Misner, Thorne, and Wheeler [14].

II. NONANOMALOUS VIOLATION

We will construct our violation of ANEC as a conformal transformation of a spacetime that obeys ANEC but violates NEC, i.e., there is a geodesic γ with tangent vector l^a, such that $T_{ab}l^a l^b < 0$ in certain places but $\int_\gamma T_{ab}l^a l^b \geq 0$. For simplicity, we will take the untransformed spacetime to be Minkowski space. We will show that a conformal transformation can enhance the contribution to the integral in those places where NEC is violated, so that the overall integral is negative in the transformed spacetime.

We let our transformed metric be $\bar{g}_{ab} = \Omega^2(x)\eta_{ab}$. The stress-energy tensor then transforms as [15]

$$\bar{T}_{ab} = \Omega^{-2}T_{ab} + \text{anomaly}. \tag{2}$$

The anomalous contribution depends only on local curvature terms and is finite. A null geodesic remains a null geodesic under a conformal transformation, but the parameterization is no longer affine. A new affine parameterization is given by $d\bar{\lambda} = \Omega^2 d\lambda$, and so $\bar{l}^a = (dx^a/d\bar{\lambda}) = \Omega^{-2}l^a$. The ANEC integral then becomes

$$\int \bar{T}_{ab}\bar{l}^a\bar{l}^b d\bar{\lambda} = \int \Omega^{-4}T_{ab}l^a l^b d\lambda + \text{anomaly}. \tag{3}$$

For a given conformal transformation, we will exhibit a sequence of states in which the nonanomalous term becomes arbitrarily negative. Thus, even if the anomalous term is positive, there are states which overcome it and make the ANEC integral negative. In fact, it is possible to arrange the transformation so that the anomalous term also gives a negative contribution.

Our argument follows closely the work of Fewster and Roman [16] on null quantum inequalities. A *quantum inequality* is a restriction on the amount by which a weighted average of the stress-energy tensor can be negative. For example, for a minimally coupled massless scalar in Minkowski space, we have [17]

$$\int \frac{\tau_0}{\pi(\tau^2 + \tau_0^2)} T_{ab}V^a V^b d\tau \geq -\frac{1}{32\pi^2\tau_0^4}, \tag{4}$$

where the integral is taken over a timelike geodesic with tangent vector V^a, parameterized by proper time τ, and τ_0 is a arbitrary constant.

Fewster and Roman [16] showed that no inequality such as Eq. (4) can hold for null geodesics. Specifically, for any affinely parameterized null geodesic $\gamma(\lambda)$ with tangent vector l^a and any smooth, bounded, compactly supported function $f(\lambda)$, Fewster and Roman construct a sequence of states which make $\int f(\lambda)T_{ab}l^a l^b d\lambda$ unboundedly negative. We will use their construction to produce a Minkowski-space state which will violate ANEC when conformally transformed.

Consider a geodesic γ as above and a smooth conformal transformation $\Omega(x)$, with the properties that

$$\Omega(\gamma(\lambda)) \leq 1 \text{ everywhere on } \gamma \tag{5}$$

$$\Omega(\gamma(\lambda)) \text{ is bounded from below by some } \epsilon > 0 \text{and} \tag{6}$$

$$\Omega(\gamma(\lambda)) \text{ differs from 1 on a non-empty compact set of } \lambda. \tag{7}$$

The conformal transformation shrinks the spacetime by some bounded amount over some limited range of the geodesic. We can then define $g(\lambda) = \Omega(\gamma(\lambda))^{-4}$ and $f(\lambda) = g(\lambda) - 1$, and f will then be smooth, bounded, and of compact support.

The ANEC integral in the conformally flat spacetime is

$$\int \bar{T}_{ab}\bar{l}^a\bar{l}^b d\bar{\lambda} = \mathcal{E}[g] + \text{anomaly}, \tag{8}$$

where $\mathcal{E}[g]$ is defined as the flat-spacetime integral with sampling function g,

$$\mathcal{E}[g] = \int_\gamma g(\lambda) T_{ab} l^a l^b d\lambda. \tag{9}$$

Following [16], we will now exhibit a sequence of states ψ_α that will make the ANEC integral arbitrarily negative. Since we are concerned only with a counterexample to ANEC, we will not attempt to be general but opt instead for simplicity. Our procedure differs from that of [16] in that our field is conformally rather than minimally coupled, and our sampling function g is not compactly supported but rather goes to 1 at large distances.

A massless field ϕ is defined by

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3(2\omega)^{1/2}}(a(k)e^{-ik_a x^a} + a^\dagger(k)e^{ik_a x^a}). \tag{10}$$

We define a class of vacuum plus two particle state vectors, which depend on a parameter $\alpha \in (0, 1)$. First, given the function f, we will define a momentum parameter Λ_0 by a procedure to be described later. Then we define our states

$$\psi_\alpha = N_\alpha\left[|0\rangle + \frac{\alpha^{-1/4}}{\Lambda^4}\int_\Sigma \frac{d^3k d^3k'}{(2\pi)^3(2\pi)^3}\sqrt{kk'}|k, k'\rangle\right], \tag{11}$$

where $\Lambda = \Lambda_0/\alpha$ is a momentum cutoff, N_α is a normalization constant,

$$N_\alpha = \left(1 + \frac{\alpha^{3/2}}{128\pi^4}\right)^{-1/2}, \tag{12}$$

and

$$\int_\Sigma d^3k \text{ denotes } \int_0^\Lambda k^2 dk \int_{1-\alpha}^1 d\cos\theta \int_0^{2\pi} d\phi \tag{13}$$

where k is the magnitude of the vector $\mathbf{k}$, θ is the angle between $\mathbf{k}$ and the tangent vector $\boldsymbol{l}$, and ϕ is the azimuthal angle. These states excite only particles with momentum less than Λ, and directed inside an angle $\cos^{-1}(1-\alpha)$ from the null ray, which puts the four-momentum inside a tightening and lengthening cone as $\alpha \to 0$. Note that as α falls to zero, $N_\alpha \to 1$, and the excitation term in Eq. (11) goes to zero. Thus, the state approaches the vacuum, but we shall see that its stress-energy tensor does not.

In order to find the stress tensor, we need the normal ordered two point function [16]

$$\begin{aligned}&\langle\psi_\alpha|{:}\phi(x)\phi(x'){:}|\psi_\alpha\rangle\\ &= \frac{2N_\alpha^2}{\Lambda^4}\int_\Sigma \frac{d^3k d^3k'}{(2\pi)^6}\\ &\quad\times\left[\alpha^{-1/4}e^{-i(k\cdot x + k'\cdot x')} + \frac{\alpha^{1/2}}{8\pi^2}e^{i(-k\cdot x + k'\cdot x')}\right].\end{aligned} \tag{14}$$

The first term arises from the coupling of the two-particle states to the vacuum. The second arises from the coupling between the two-particle states. In the limit $\alpha \to 0$, the first term is dominant because the admixture of two-particle states becomes very small.

The stress tensor for a conformally coupled scalar field is

$$\begin{aligned}T_{ab} = {}&\tfrac{2}{3}\phi_{;a}\phi_{;b} - \tfrac{1}{3}\phi_{;ab}\phi - \tfrac{1}{6}g_{ab}g^{\rho\sigma}\phi_{;\rho}\phi_{;\sigma} + \tfrac{1}{12}g_{ab}\phi\Box\phi\\ &- \tfrac{1}{6}[R_{ab} - \tfrac{1}{4}Rg_{ab}]\phi^2.\end{aligned} \tag{15}$$

In flat space the curvature terms vanish, and terms involving g_{ab} vanish in the null projection, so

$$l^a l^b T_{ab} = \tfrac{2}{3}l^a l^b \phi_{,a}\phi_{,b} - \tfrac{1}{3}l^a l^b \phi_{,ab}\phi. \tag{16}$$

We take the expectation value in the state ψ_α and renormalize by subtracting the vacuum contribution (which is equivalent to normal ordering), then set $x' = x$. The first term becomes

$$\begin{aligned}\frac{2}{3}\langle{:}\phi_{,a}\phi_{,b}l^a l^b{:}\rangle_\alpha = {}&\frac{4N_\alpha^2}{3\Lambda^4}\int_\Sigma \frac{d^3k d^3k'}{(2\pi)^6} l^a k_a l^b k'_b\\ &\times\left[-\alpha^{-1/4}e^{-ix\cdot(k+k')} + \frac{\alpha^{1/2}}{8\pi^2}e^{ix\cdot(k-k')}\right].\end{aligned} \tag{17}$$

The other term is

$$\begin{aligned}-\frac{1}{3}\langle{:}\phi_{,ab}\phi l^a l^b{:}\rangle_\alpha = {}&\frac{2N_\alpha^2}{3\Lambda^4}\int_\Sigma \frac{d^3k d^3k'}{(2\pi)^6}(l^a k_a)^2\\ &\times\left[\alpha^{-1/4}e^{-ix\cdot(k+k')} + \frac{\alpha^{1/2}}{8\pi^2}e^{ix\cdot(k-k')}\right].\end{aligned} \tag{18}$$

We now specify a Fourier transform by

$$\hat{f}(u) = \int dt e^{-iut} f(t). \tag{19}$$

Since $g(t) = f(t) + 1$, $\hat{g}(u) = \hat{f}(u) + 2\pi\delta(u)$. From the properties of Ω, we see that f is bounded and has a well-defined, positive integral. Thus, $\hat{f}$ is continuous and $\hat{f}(0) > 0$.

For any fixed 4-vector K,

$$\int d\lambda g(\lambda) e^{-i\gamma(\lambda)^a K_a} = \hat{g}(l\cdot K), \tag{20}$$

so we can write $\mathcal{E}[g] = \mathcal{E}_1[g] + \mathcal{E}_2[g]$, where

$$\begin{aligned}\mathcal{E}_1[g] = {}&\frac{N_\alpha^2\alpha^{1/2}}{12\pi^2\Lambda^4}\int_\Sigma \frac{d^3k d^3k'}{(2\pi)^6}[(l\cdot k)^2 + 2(l\cdot k)(l\cdot k')]\\ &\times \hat{g}(l\cdot(k-k')),\end{aligned} \tag{21}$$

$$\begin{aligned}\mathcal{E}_2[g] = {}&\frac{2N_\alpha^2\alpha^{-1/4}}{3\Lambda^4}\int_\Sigma \frac{d^3k d^3k'}{(2\pi)^6}[(l\cdot k)^2 - 2(l\cdot k)(l\cdot k')]\\ &\times \hat{g}(l\cdot(k+k')).\end{aligned} \tag{22}$$

We will first consider $\mathcal{E}_2[f]$, following [16]. Since we are in flat space, the tangent vector l is constant. We can take it to have unit time component, so that $k \cdot l = k(1 - \cos\theta)$. We do the azimuthal integrations and change variables to $v = k\alpha$, $u = k \cdot l$, and similarly for v' and u'. We find

$$\mathcal{E}_2[f] = \frac{2N_\alpha^2 \alpha^{-1/4}}{3(2\pi)^4 \Lambda_0^4} \int_0^{\Lambda_0} dv \int_0^{\Lambda_0} dv' v v' \int_0^{v} du \times \int_0^{v'} du' [u^2 - 2uu'] \hat{f}(u + u'). \quad (23)$$

Now $\hat{f} > 0$. Since $\hat{f}$ is continuous, we can choose $\Lambda_0 > 0$ such that $\hat{f}(u)$ is arbitrarily close to $\hat{f}(0)$. Thus, we can make the integrals in Eq. (23) arbitrarily close to

$$\hat{f}(0) \int_0^{\Lambda_0} dv \int_0^{\Lambda_0} dv' v v' \int_0^{v} du \int_0^{v'} du' [u^2 - 2uu'] = -\frac{13}{1440} \hat{f}(0) < 0. \quad (24)$$

As $\alpha \to 0$, the prefactor in Eq. (23) goes to positive infinity, so we conclude that $\mathcal{E}_2[f] \to -\infty$ in this limit.

The rest of the terms are all finite. Equation (21) gives

$$\mathcal{E}_1[f] = \frac{N_\alpha^2 \alpha^{1/2}}{12\pi^2 \Lambda_0^4} \int_0^{\Lambda_0} dv \int_0^{\Lambda_0} dv' v v' \int_0^{v} du \times \int_0^{v'} du' [u^2 + 2uu'] \hat{f}(u - u'). \quad (25)$$

Since f has compact support, $\hat{f}$ is bounded and the integrals give some finite number independent of α. Since the power of α is positive in this case, we find that $\mathcal{E}_1[f] \to 0$ as $\alpha \to 0$.

In addition, we have the delta function in Eqs. (21) and (22), which gives the flat-spacetime ANEC integral discussed in Sec. II D of Ref. [16]. Since k is restricted to a cone around the direction of l, $l \cdot k \geq 0$. There is no contribution to $\mathcal{E}_2[\delta]$ except from $k = k' = 0$, in which case the term in brackets vanishes. Thus, $\mathcal{E}_2[\delta] = 0$.

Finally, we have

$$\mathcal{E}_1[\delta] = \frac{N_\alpha^2 \alpha^{1/2}}{12\pi^2 \Lambda_0^4} \int_0^{\Lambda_0} dv \int_0^{\Lambda_0} dv' v v' \int_0^{v} du \times \int_0^{v'} du' [u^2 + 2uu'] \delta(u - u'). \quad (26)$$

Again the integrals give a finite number, and the prefactor goes to zero, so $\mathcal{E}_1[f] \to 0$ as $\alpha \to 0$, and finally

$$\lim_{\alpha \to 0} \mathcal{E}[g] \to -\infty. \quad (27)$$

Thus, for a spacetime given by fixed conformal transformation Ω, we can find a quantum state such that $\mathcal{E}[g]$ is arbitrarily negative. In particular, any positive anomalous term can be overcome by large negative $\mathcal{E}[g]$, so that Eq. (8) is negative and ANEC is violated.

III. ANOMALOUS VIOLATION

The prior example constructs a violation of ANEC over a class of excited states. The contribution from the transformed $T_{\mu\nu}$ dominates the anomalous terms. It is also possible to construct a spacetime where the anomalous term is negative, and thus even for the vacuum state, with $T_{\mu\nu} = 0$, a violation can occur. In addition to the example found by Visser, we find cases which are conformally flat. In these, there is no dependence at all on the renormalization scale μ, only the geometry of the new space.

Conformal transformation properties are taken from [15,18]. The transformation is $\bar{g}_{ab} = \Omega^2 g_{ab}$. In the following, derivatives of barred quantities are always meant to be taken in the new metric, and unbarred quantities in the original metric. Derivatives of the transformation function Ω are also taken in the old coordinates.

The stress tensor is given by Eq. (15). For a conformally coupled scalar field the transformation properties are known [19]. We specialize to the case where the initial spacetime is Minkowski. Thus, curvature quantities in the untransformed spacetime all vanish, and the Weyl tensor vanishes even in the transformed spacetime. We have the particular transformation

$$\bar{T}^{\ a}_{\ b} = \Omega^{-4} T^a_b - 2\beta \bar{H}^a_b - \frac{\gamma}{6} \bar{I}^a_b. \quad (28)$$

The constants are dependent on the spin of the field; for a real scalar field,

$$\beta = -\frac{1}{5760\pi^2}, \quad (29)$$

$$\gamma = -2\beta. \quad (30)$$

The tensors $\bar{H}$ and $\bar{I}$ are given by

$$\bar{H}_{ab} = -\bar{R}^c_a \bar{R}_{cb} + \tfrac{2}{3}\bar{R}\bar{R}_{ab} + (\tfrac{1}{2}\bar{R}^c_d \bar{R}^d_c - \tfrac{1}{4}\bar{R}^2)\bar{g}_{ab}. \quad (31)$$

$$\bar{I}_{ab} = 2\bar{R}_{;ab} - 2\bar{R}\bar{R}_{ab} + (\tfrac{1}{2}\bar{R}^2 - 2\Box\bar{R})\bar{g}_{ab}. \quad (32)$$

We eliminate those terms which will not contribute due to the null projection as well as the $R_{;ab}$ in (32) which appears in ANEC as $l^b(R_{,a} l^a)_{,b}$ and thus vanishes upon integration. After this, and expressing γ in terms of β, we find

$$\bar{T}_{ab} = \Omega^{-2} T_{ab} + 2\beta[\bar{R}^c_a \bar{R}_{cb} - \bar{R}\bar{R}_{ab}]. \quad (33)$$

The curvatures in the new spacetime are given by

$$\bar{R}_{cb} = -2\omega_{,cb} - g_{cb}\Box\omega + 2\omega_{,b}\omega_{,c} - 2g_{cb}\omega_{,\rho}\omega^{,\rho}, \quad (34)$$

$$\bar{R} = \Omega^{-2}[-6\Box\omega - 6\omega_{,\rho}\omega^{,\rho}], \quad (35)$$

with $\omega = \ln\Omega$. Again dropping terms with g_{ab}, the stress tensor is given by

$$\bar{T}_{ab} = \Omega^{-2}T_{ab} + 8\beta\Omega^{-2}[\omega_a^{;c}\omega_{,cb} - 2(\Box\omega + \omega^{;c}\omega_{,c}) \times (\omega_{,ab} - \omega_{,a}\omega_{,b}) - \omega^{;c}\omega_{,a}\omega_{,cb} - \omega^{;c}\omega_{,b}\omega_{,ca}]. \tag{36}$$

Now we give a specific example of a transformation which violates ANEC. We take an initial state with $T_{\mu\nu} = 0$, so the state does not contribute to $\bar{T}_{\mu\nu}$. We will work in Minkowski space in null coordinates, with $u = (z - t)/\sqrt{2}$ and $v = (z + t)/\sqrt{2}$. We take our geodesic going in the v direction along the line $u = x = y = 0$. We choose the particular transformation

$$\omega = (a + bx^2r^{-2})e^{-(u^2+v^2+x^2+y^2)/r^2}. \tag{37}$$

This gives a localized transformation, so our spacetime is both conformally and asymptotically flat. We take a and b both much less than one, so we may ignore terms of order ω^3. That leaves us with only

$$\bar{T}_{vv} = 8\beta\Omega^{-2}[g^{cd}\omega_{,cv}\omega_{,dv} - 2\Box\omega\omega_{,vv}]. \tag{38}$$

The first term vanishes because $\omega_{,cv} = 0$ unless the index c is v, but $g^{vv} = 0$. The remaining term is the product of

$$\Box\omega = 2r^{-2}(b - 2a)e^{-v^2/r^2} \tag{39}$$

and

$$\omega_{,vv} = 2ar^{-2}(2v^2r^{-2} - 1)e^{-v^2/r^2}. \tag{40}$$

Now together we have

$$\bar{T}_{vv} = -64\beta ar^{-4}(2v^2r^{-2} - 1)(b - 2a)e^{-2v^2/r^2} \tag{41}$$

Integrating over the full geodesic gives

$$\int_{-\infty}^{+\infty} \bar{T}_{vv}dv = (ab - 2a^2)\frac{16\beta\sqrt{2\pi}}{r^3}. \tag{42}$$

The constant $\beta < 0$. We can choose $1 \gg b > 2a$ so that the ANEC integral is negative.

1 + 1 DIMENSIONS

It is interesting to compare the results of previous sections with the situation in 1 + 1 dimensions. In that case we know [9] that ANEC cannot be violated, even in curved space. What happens when we attempt to violate it using the techniques of previous sections?

First, our construction of a Minkowski-space state that violates a weighted average of NEC depended on a cone of momenta surrounding the tangent vector to our null geodesic. In 1 + 1 dimensions, there are no transverse directions, so that technique cannot work. In fact, unlike in 3 + 1 dimensions, there is a 1 + 1-dimensional quantum inequality derived by Flanagan [20],

$$\mathcal{E}[g] = \int_\gamma T_{ab}l^al^bd\lambda \geq -\frac{1}{48\pi}\int_\gamma \frac{g'(\lambda)^2}{g(\lambda)}d\lambda \tag{43}$$

for any smooth, non-negative function g.

Nevertheless, Eq. (43) still permits NEC violation, and we can still enhance that violation. In 1 + 1 dimensions, Eq. (2) becomes

$$\bar{T}_{ab} = T_{ab} + \text{anomaly}, \tag{44}$$

and the ANEC integral, Eq. (3), becomes

$$\int \bar{T}_{ab}\bar{l}^a\bar{l}^bd\bar{\lambda} = \int (T_{ab} + \text{anomaly})\Omega^{-2}l^al^bd\lambda. \tag{45}$$

Thus, if NEC is violated in certain locations in Minkowski space, we can choose $\Omega \ll 1$ there to enhance their contribution to Eq. (45). However, we cannot make this contribution arbitrarily large by the choice of states, because the NEC violation is restricted by Eq. (43).

The anomalous term in Eq. (44) (see Eq. (6.134) of [15]) is

$$\frac{1}{12\pi}[\Omega^{-1}\Omega_{,ab} - 2\Omega^{-2}\Omega_{,a}\Omega_{,b} + g_{ab}g^{cd}((3/2)\Omega^{-2}\Omega_{,b}\Omega_{,c}\Omega^{-1} - \Omega_{,bc})]. \tag{46}$$

The term proportional to g_{ab} does not contribute to NEC so

$$\int \bar{T}_{ab}\bar{l}^a\bar{l}^bd\bar{\lambda} = \int\bigg[\Omega^{-2}T_{ab} + \frac{1}{12\pi}(\Omega^{-3}\Omega_{,ab} - 2\Omega^{-4}\Omega_{,a}\Omega_{,b})\bigg]l^al^bd\lambda. \tag{47}$$

We can integrate the anomalous terms by parts. We write

$$(\Omega^{-3}\Omega_{,a})_{,b} = \Omega^{-3}\Omega_{,ab} - 3\Omega^{-4}\Omega_{,a}\Omega_{,b}. \tag{48}$$

In our situation, $\Omega \to 1$ as $\lambda \to \pm\infty$, and thus $\Omega_{,a} \to 0$ in that limit. So the total derivative does not contribute, and

$$\int \bar{T}_{ab}\bar{l}^a\bar{l}^bd\bar{\lambda} = \int\bigg(\Omega^{-2}T_{ab} + \frac{1}{12\pi}\Omega^{-4}\Omega_{,a}\Omega_{,b}\bigg)l^al^bd\lambda. \tag{49}$$

The anomalous term in Eq. (49) is manifestly positive, so in 1 + 1 dimensions there is no anomalous violation as in Sec. III. In fact, when we define $g(\lambda) = \Omega(\gamma(\lambda))^{-2}$, we find the anomalous contribution is just

$$\frac{1}{48\pi}\int \frac{g'(\lambda)^2}{g(\lambda)}d\lambda, \tag{50}$$

and so by Eq. (43),

$$\int \bar{T}_{ab}\bar{l}^a\bar{l}^bd\bar{\lambda} \geq 0; \tag{51}$$

ANEC is always obeyed.

This derivation is essentially the same used by Flanagan [21] to generalize his quantum inequality, Eq. (43), to curved spacetimes. We find it remarkable that Eq. (43), which is a statement entirely about quantum field theory in flat spacetime somehow "knows" about the anomalous transformation of T_{ab} in such a way that they together preserve ANEC in curved spacetime.

V. CONCLUSION

We have given two explicit violations of the achronal averaged null energy condition, both in spaces which are conformally and asymptotically flat. First we used a transformation which amplifies the NEC-violating portions of a sequence of excited states. As the momentum grows in magnitude and is constrained within a cone which increasingly narrows around the direction of the null geodesic, the ANEC integral becomes increasingly negative. This effect can be seen in a broad class of states and transformations; we gave a specific example for concreteness.

The second violation was constructed purely from the geometric anomalous terms in the stress tensor. In this way we find negative average energy in some conformally flat spaces even in the vacuum state. As the deviation from flat space becomes more sharply localized, the violation grows. Both of these violations can become arbitrarily negative.

We now wonder if there is any possibility to exclude exotic phenomena from general relativity with some weaker condition that would not be violated by quantum fields. One option is requiring an additional transverse average over a congruence of geodesics. Physically this is a natural requirement, as any nonzero-sized exotic feature, such as a wormhole or time machine, would require some certain level of ANEC violation over some nonzero range of geodesics.

It appears that both of the above violations could be softened by some transverse averaging. In the example of Sec. II, the stress-energy tensor oscillates rapidly in the transverse direction [16]. Likewise, the violation in Sec. III grows as $r \to 0$, where r parameterizes the width of the deviation from flatness. Averaging over a distance greater than r could cancel this effect. Timelike averages of null quantities have been considered in [16]; one could also consider spacetime averages.

Another possibility is the additional requirement of self-consistency, that is, that the field and geometry be a solution to the semiclassical Einstein equation $G_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle$, where $T_{\mu\nu}$ is the stress tensor of some state of a set of fields in the background whose Einstein tensor is $G_{\mu\nu}$. In both the above examples we have computed the stress tensor in a given background without attempting to impose self-consistency. Progress along this line has been made in [22] for perturbations of flat space. Ref. [23] finds state-dependent bounds on averaged energies, which may also be useful in this context. It is possible that self-consistency may be enough to enforce the energy conditions in the general case [12].

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