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Geometry and dynamics of a tidally deformed black hole

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The metric of a nonrotating black hole deformed by a tidal interaction is calculated and expressed as an expansion in the strength of the tidal coupling. The expansion parameter is the inverse length scale $\mathcal{R}^{-1}$, where $\mathcal{R}$ is the radius of curvature of the external spacetime in which the black hole moves. The expansion begins at order $\mathcal{R}^{-2}$, and it is carried out through order $\mathcal{R}^{-4}$. The metric is parametrized by a number of tidal multipole moments, which specify the black hole's tidal environment. The tidal moments are freely-specifiable functions of time that are related to the Weyl tensor of the external spacetime. At order $\mathcal{R}^{-2}$ the metric involves the tidal quadrupole moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$. At order $\mathcal{R}^{-3}$ it involves the time derivative of the quadrupole moments and the tidal octupole moments $\mathcal{E}_{abc}$ and $\mathcal{B}_{abc}$. At order $\mathcal{R}^{-4}$ the metric involves the second time derivative of the quadrupole moments, the first time derivative of the octupole moments, the tidal hexadecapole moments $\mathcal{E}_{abcd}$ and $\mathcal{B}_{abcd}$, and bilinear combinations of the quadrupole moments. The metric is presented in a light-cone coordinate system that possesses a clear geometrical meaning: The advanced-time coordinate v is constant on past light cones that converge toward the black hole; the angles θ and ϕ are constant on the null generators of each light cone; and the radial coordinate r is an affine parameter on each generator, which decreases as the light cones converge toward the black hole. The coordinates are well-behaved on the black-hole horizon, and they are adjusted so that the coordinate description of the horizon is the same as in the Schwarzschild geometry: $r = 2M + O(\mathcal{R}^{-5})$. At the order of accuracy maintained in this work, the horizon is a stationary null hypersurface foliated by apparent horizons; it is an isolated horizon in the sense of Ashtekar and Krishnan. As an application of our results we examine the induced geometry and dynamics of the horizon, and calculate the rate at which the black-hole surface area increases as a result of the tidal interaction.

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I. INTRODUCTION AND OVERVIEW

A. This work and its context

Our main goal in this work is to calculate the gravitational field of a nonrotating black hole that is deformed by a tidal interaction with external bodies. We assume that the tidal interaction is weak, and that it changes slowly; we make no other assumptions, and describe the tidal environment in the most general terms compatible with the main assumptions. This work is a continuation of a line of inquiry that was initiated by Manasse [1] in the early 1960s; our implementation is much more general, and much more accurate, than Manasse's original work.

We first introduce the two length scales that are relevant to this problem. (Throughout the paper we use relativistic units and set $G = c = 1$.) The first is M, the mass of the black hole. The second is $\mathcal{R}$, the radius of curvature of the external spacetime associated with the external bodies, evaluated at the black hole's position. We assume that the scales are widely separated, so that

$$M \ll \mathcal{R}. \tag{1.1}$$

This condition ensures that the tidal interaction is weak, and allows us to speak meaningfully of a black hole moving in an external spacetime; when M is comparable to $\mathcal{R}$, no clear distinction can be made between the "black hole" and the "external spacetime." We refer to the approximation scheme based on the condition $M/\mathcal{R} \ll 1$ as the *small-tide approximation*.

As a concrete example we may consider a situation in which the black hole is a member of a binary system. Let M' denote the mass of the external body, b the separation between the companions, and $V \sim \sqrt{(M + M')/b}$ the orbital velocity. The radius of curvature is then $\mathcal{R} \sim \sqrt{b^3/(M + M')}$, and

$$\frac{M}{\mathcal{R}} \sim \frac{M}{M + M'} V^3. \tag{1.2}$$

We demand that this be a small quantity. There are two particular ways to achieve this. In the *small-hole approximation* the black-hole mass is assumed to be much smaller than the external mass, so that $M/(M + M') \ll 1$; then $M/\mathcal{R}$ is small irrespective of the size of V, and the binary system can be strongly relativistic. In the *weak-field approximation* it is V that is assumed to be small, while the mass ratio is left unconstrained; here the two companions can have comparable masses, or the black hole can be much larger than its companion, but the mutual gravity between the bodies must be weak. The small-hole and weak-field approximations are particular instances of the more general requirement that $M/\mathcal{R} \ll 1$; they are both incorporated within our small-tide approximation.

The effects of a tidal field on the structure of spacetime around a black hole were first investigated by Manasse [1], in the specific context of the small-hole approximation. Using techniques similar to those exploited in this paper, Manasse calculated the metric of a small black hole that falls radially toward a much larger black hole. Each black hole was taken to be nonrotating, and the small hole was taken to move on a geodesic of the (unperturbed) Schwarzschild spacetime of the large hole. The case of circular motion around a large Schwarzschild black hole was treated much later by Poisson [2], and Comeau and Poisson [3] examined the case of circular motion around a Kerr black hole.

The methods employed by Manasse could be applied beyond the small-hole approximation. Alvi [4,5] realized that they could be seamlessly extended to the more general context of the small-tide approximation of Eq. (1.1). Alvi exploited this insight to calculate the tidal fields acting on a black hole in a post-Newtonian binary system. In Alvi's work, the two bodies have comparable masses and the black hole has a significant influence on the geometry of the external spacetime. Alvi calculated the tidal fields to leading (Newtonian) order in the post-Newtonian approximation to general relativity, and specialized the orbital motion to circular orbits. His work was later generalized to first post-Newtonian order, and to generic orbits, by Taylor and Poisson [6].

Alvi's work motivated an effort to improve our understanding of the tidal interaction of black holes by constructing the black-hole metric to high order in the coupling strength $M/\mathcal{R}$. More precisely, the metric is calculated in the black hole's local neighborhood, and expressed as an expansion in powers of $r/\mathcal{R} \ll 1$, where r is the distance to the black hole; the metric is valid to all orders in M/r. This program was initiated by Detweiler [7,8], who calculated the metric through order $(r/\mathcal{R})^2$, the lowest order at which tidal effects appear. It was pursued by Poisson [9], who calculated the metric through order $(r/\mathcal{R})^3$. In this paper we improve on the earlier work by calculating the metric through order $(r/\mathcal{R})^4$, and present the most accurate version ever produced of the metric of a tidally deformed black hole. The improvement of the metric, from order $(r/\mathcal{R})^3$ to order $(r/\mathcal{R})^4$, represents a significant challenge, and the work presented here goes much beyond the work contained in Ref. [9].

Other authors have also contributed to this effort. Frolov and his collaborators [10,11] examined the internal geometry of a tidally deformed black hole, and Damour and Lecian [12] characterized the tidal deformation in terms of a polarizability and "shape Love numbers"; these works were restricted to axisymmetric tidal fields, a restriction that is not made in this paper. In a genuine *tour de force*, Yunes and Gonzalez [13] calculated the tidal deformation of a rapidly rotating black hole to order $(r/\mathcal{R})^2$. The tidal deformation of neutron stars (and other types of compact bodies) has also been the subject of recent investigations [14–17].

B. Tidal quadrupole moments

In the work of Detweiler [7,8] and Poisson [9] reviewed previously, and in the work presented here, the tidal environment is described in the most general terms compatible with the Einstein field equations. The metric is parametrized by freely-specifiable functions of time that serve the specific purpose of specifying the tidal environment; these are packaged in symmetric tracefree (STF) tensors $\mathcal{E}_{a_1 a_2 \cdots a_l}$ and $\mathcal{B}_{a_1 a_2 \cdots a_l}$ [18], which we refer to as the *tidal multipole moments* of the black-hole spacetime. (The tidal moments can be thought of as Cartesian tensors; they are symmetric and tracefree in all pairs of indices.) At order $(r/\mathcal{R})^2$ the metric involves the quadrupole moments $\mathcal{E}_{ab}$ (5 functions) and $\mathcal{B}_{ab}$ (5 functions). At order $(r/\mathcal{R})^3$ the metric involves the time derivative of the quadrupole moments, as well as the octupole moments $\mathcal{E}_{abc}$ (7 functions) and $\mathcal{B}_{abc}$ (7 functions). And at order $(r/\mathcal{R})^4$ the metric involves the second time derivative of the quadrupole moments, the first time derivative of the octupole moments, the hexadecapole moments $\mathcal{E}_{abcd}$ (9 functions) and $\mathcal{B}_{abcd}$ (9 functions), and bilinear combinations of the quadrupole moments. (The calculational challenge of this work resides mostly with the inclusion of the bilinear terms, which are supplied by the nonlinearities of the Einstein field equations.) This generality is a substantial virtue of our work, which unifies and extends earlier works [4,5,19,20] that examined special cases.

The metric of a tidally deformed black hole is obtained by integrating the vacuum field equations in the local neighborhood of the black hole. The field equations leave the tidal moments undetermined, and the metric is presented as a functional of these arbitrary moments. In applications of this framework the tidal environment must be specified, and this is achieved by making appropriate choices for the tidal moments. In practice this typically requires matching the local metric to a global metric that includes the black hole and the external bodies that are responsible for the tidal interaction. For example, Taylor and Poisson [6] and Johnson-McDaniel *et al.* [21] carried out such a matching in the context of the slow-motion approximation, in which the mutual gravity between the black hole and the external bodies is weak. The global metric was expressed as a post-Newtonian expansion, within which the black hole can justifiably be represented as a point particle. These authors determined the quadrupole moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ to first post-Newtonian order, and their work could easily be extended to obtain the higher multipole moments that also appear in the black-hole metric. In general the nonlinearities of the field equations imply that tidal fields depend on the black-hole mass M. In the small-hole approximation, however, the black hole can be treated as a test body, and this dependence disappears; in this case the determination of the tidal moments is simplified [1–3].

C. Light-cone coordinates

The choice of coordinates is often critical in the construction of a metric and the exploration of its properties. This is all the more true in the case of black-hole spacetimes, which require coordinates that are well-behaved on the horizon. We have given a lot of attention to the selection of coordinates, and have chosen to work with a system (v, r, θ, ϕ) that is specifically tailored to describe the geometry of past light cones. We refer to these as *light-cone coordinates*. The coordinates have a clear geometrical meaning: The advanced-time coordinate v is constant on past light cones that converge toward the black hole; the angles θ and ϕ (which we collectively denote θ^A, with the upper-case Latin index A running from 2 to 3) are constant on the null generators of each light cone; and the radial coordinate r is an affine parameter on each generator, which decreases as the light cones converge toward the black hole. Through order $(r/\mathcal{R})^3$ the radial coordinate doubles as an areal radius, in the sense that the area of each two-surface $(v, r) = \text{constant}$ is equal to $4\pi r^2[1 + O(r^4/\mathcal{R}^4)]$; this property is lost at order $(r/\mathcal{R})^4$. In addition, the radial coordinate is tuned so that the coordinate description of the black-hole horizon is the same as in the Schwarzschild spacetime: $r = 2M[1 + O(M^5/\mathcal{R}^5)]$. The light-cone coordinates are well-behaved across the horizon.

The choice of coordinates is inspired from the work of Bondi *et al.* [22] and Sachs [23], in which light-cone coordinates were utilized to construct the metric of an asymptotically-flat spacetime. (Here the coordinates were based on future light cones that expand toward future null infinity.) It is inspired also by the work of Ellis and collaborators on observational cosmology [24–29], in which the metric of an expanding universe is constructed from observations made by a typical cosmological observer. Light-cone coordinates were also introduced by Synge in his classic textbook on general relativity [30]; we followed his methods closely in this work and its precursors [2,9,31,32].

In addition to the quasispherical system of light-cone coordinates (v, r, θ^A), we find it convenient to introduce also a quasi-Cartesian variant (v, x^a). The spatial coordinates x^a (with the lower-case Latin index a running from 1 to 3) are constructed in the usual way from the quasi-spherical coordinates (r, θ^A); we have the relations $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$, which we collectively denote $x^a = r\Omega^a(\theta^A)$.

D. Black-hole metric

The metric of a tidally deformed black hole is obtained by constructing a perturbation of the Schwarzschild solution, which describes a nonrotating black hole in complete isolation. The Schwarzschild metric is presented in the Eddington-Finkelstein coordinates (v, r, θ^A), and the perturbation is presented in a light-cone gauge that preserves the geometrical meaning of the background coordinates. To construct the perturbation we rely on the covariant and gauge-invariant formalism of Martel and Poisson [33], and on the formulation of the light-cone gauge by Preston and Poisson [34]. At orders $(r/\mathcal{R})^2$ and $(r/\mathcal{R})^3$ the perturbation satisfies the vacuum field equations linearized with respect to the Schwarzschild solution. At order $(r/\mathcal{R})^4$ the perturbation satisfies nonlinear field equations.

In the quasi-Cartesian coordinates (v, x^a) the building blocks of the metric are the tidal potentials introduced in Tables I, II, III, IV, V, and VI, of Sec. II; these are generated by the tidal multipole moments introduced previously. The black-hole metric appears in Eqs. (3.5) of Sec. III C, and it is expressed in terms of the radial functions listed in Tables XIV and XV. In the quasispherical coordinates (v, r, θ^A) the tidal potentials are listed in Tables VIII, IX, X, XI, XII, and XIII, of Sec. II, and expressed as expansions in spherical-harmonic functions (see Table VII). The black-hole metric appears in Eqs. (3.7) of Sec. III C, and it involves the same set of radial functions.

An important property of the black-hole metric is the fact, mentioned previously, that in our light-cone coordinates, the position of the horizon is given by

$$r_{\text{horizon}} = 2M[1 + O(M^5/\mathcal{R}^5)], \tag{1.3}$$

the same relation as in the unperturbed Schwarzschild spacetime. This information allows us to investigate the nature and dynamics of the horizon's geometry, which is determined by the horizon's induced metric as well as the expansion and shear of the horizon's generators. An important outcome of this investigation is the statement that

$$\Theta = O(M^5/\mathcal{R}^6). \tag{1.4}$$

This represents the rate at which the congruence of null generators expand, and we find that within the degree of accuracy maintained in this work, the generators are stationary. This implies that the horizon of a tidally deformed black hole is foliated by apparent horizons. The black-hole horizon is an *isolated horizon* in the sense of Ashtekar and Krishnan [35–37].

E. Tidal heating

A dynamical consequence of the tidal interaction is the fact that the black hole grows in size at a rate described by

$$\frac{\kappa_0}{8\pi}\dot{\mathcal{A}} = \frac{16}{45}M^6(\dot{\mathcal{E}}_{ab}\dot{\mathcal{E}}^{ab} + \dot{\mathcal{B}}_{ab}\dot{\mathcal{B}}^{ab}) + \frac{16}{4725}M^8\Big(\dot{\mathcal{E}}_{abc}\dot{\mathcal{E}}^{abc} + \frac{16}{9}\dot{\mathcal{B}}_{abc}\dot{\mathcal{B}}^{abc}\Big) + O(M^9/\mathcal{R}^9). \tag{1.5}$$

Here $\mathcal{A}$ is the surface area of a cross-section $v =$ constant of the black-hole horizon, and $\kappa_0 := (4M)^{-1}$ is the surface gravity of the unperturbed horizon. An overdot indicates differentiation with respect to v, and the right-hand side of the equation involves the tidal moments $\mathcal{E}_{ab}$, $\mathcal{E}_{abc}$, $\mathcal{B}_{ab}$, and $\mathcal{B}_{abc}$. We refer to the growth of area that results from the tidal interaction as the *tidal heating* of the black hole by the external bodies. This choice of terminology deserves an explanation.

The changes in black-hole parameters that result from time-dependent, external processes have been the subject of investigation by many researchers, starting from the pioneering work of Teukolsky and Press [38]. It is useful to classify the physical processes that alter the configuration of a black hole as *fast processes* on the one hand, and *slow processes* on the other. Fast processes occur on a time scale that is shorter than or comparable to the black-hole mass M, and these produce (electromagnetic and/or gravitational) radiation that is partially absorbed by the black hole; in this case the changes in black-hole parameters that result from the interaction are radiative changes. Slow processes, on the other hand, occur on a time scale that is long compared with the black-hole mass; while these continue to produce changes in the black-hole parameters, the phenomenon no longer possesses a radiative character, and the phrase ‘‘tidal heating’’ captures the physics better than the phrase ‘‘black-hole absorption.’’ It is good to point out that the mathematical formulation of the phenomenon by Teukolsky and Press [38] (see also Ref. [39]) is valid both for fast and slow processes, and is insensitive to matters of interpretation.

The tidal heating of black holes was recently investigated by Alvi [40], Poisson [9,39], Taylor and Poisson [6], and Comeau and Poisson [3], building on earlier work by Poisson and Sasaki [41] and Tagoshi, Mano, and Takasugi [42]. The notion of tidal work, tidal torque, and tidal heating was put on a firm relativistic footing by Purdue [43], Favata [44], and Booth and Creighton [45]. In a recent work [46], Poisson compared the equations that describe the rate of change of the black-hole mass, angular momentum, and surface area that result from a tidal interaction with external bodies, with the equations that describe how tidal forces do work, torque, and produce heat in a Newtonian, viscous body; the equations are strikingly similar, and the correspondence between the Newtonian-body and black-hole results is revealed to hold in near-quantitative detail. The tidal heating and torquing of black holes was incorporated in an effective theory of point particles by Goldberger and Rothstein [47] and Porto [48].

In favorable circumstances the tidal heating and torquing of a black hole can be relevant to astrophysical sources of gravitational waves [49]. In particular, it is likely to be significant in the generation of low-frequency waves that would be measured by a space-based detector such as LISA [50]. For example, Martel [51] showed that during a close encounter between a massive black hole and a compact body of a much smaller mass, up to approximately 5% of the lost orbital energy goes toward the tidal heating of the black hole; the rest is carried off to infinity by the gravitational waves. Hughes [52] calculated that when the massive black hole is rapidly rotating, the tidal heating slows down the inspiral of the orbiting body, and therefore increases the duration of the gravitational-wave signal. These conclusions are supported by Hughes *et al.* [53] and Drasco and Hughes [54].

There are indications that the tidal heating and torquing of black holes may have been seen in accurate numerical simulations of the inspiral and merger of binary black holes [55,56]. It is conceivable that as the precision of these simulations continues to improve in the future, the tidal heating will be exploited as a diagnostic of numerical accuracy; the surface area of each simulated black hole should be seen to grow in accordance to Eq. (1.5) instead of staying constant in time.

F. Other applications

The work presented here can serve as a foundation for the construction of initial data sets for the numerical simulation of black-hole inspirals (see Ref. [57] for a review of the current state of the art). Simulations carried out thus far have largely relied on initial data [58,59] that were adopted more for their flexibility and convenience than their astrophysical realism; these initial data tend to contain a large amount of spurious radiation and produce unwanted eccentric orbital motion. A promising alternative strategy is to rely on a post-Newtonian metric to construct initial data sets that give a faithful (though approximate) description of two widely separated black holes. Early implementations [60–62] of this idea treated the black holes as post-Newtonian point masses, and produced initial data that were unreliable close to each body. An improvement was proposed by Alvi [4,5], who recognized that the post-Newtonian metric should be replaced by a different representation of the gravitational field in the local neighborhood of each black hole; he therefore patched the post-Newtonian metric to the metric of a tidally deformed black hole in a buffer region $M \ll r \ll \mathcal{R}$, in which each metric gives an acceptable representation of the true gravitational field. Alvi’s construction was perfected by Yunes and his collaborators [63–65], and the most mature implementation of this idea is contained in the recent work of Johnson-McDaniel *et al.* [21]. Because the black-hole metric presented in this paper is more accurate than the Detweiler metric [7,8] used by Johnson-McDaniel *et al.*, it could be involved in an improved version of their construction.

Another interesting avenue of application for our metric would be to involve it in the dynamical-horizon formalism of Ashtekar and Krishnan [35–37], which provides a purely local characterization of the structure and dynamics of black-hole horizons. Our work could contribute new insights to this effort by allowing the horizon quantities to be expressed in terms of the tidal moments, which encode information about the external universe. In this way, our metric would play the role of messenger between the outside world and the horizon. Initial steps along those lines were taken by Kavanagh and Booth [66]. A particularly interesting question to investigate is whether the dynamical-horizon notions of mass and current multipole moments [67] are compatible with the recent observation that the tidal Love numbers of a nonrotating black hole must all be zero [17].

G. Organization of the paper

The remainder of the paper is divided into five sections and four appendixes. We describe our results in Secs. II, III, and IV, and provide derivations of these results in Secs. V and VI, and the appendixes.

We begin in Sec. II by providing definitions for the tidal multipole moments $\mathcal{E}_{ab}$, $\mathcal{E}_{abc}$, $\mathcal{E}_{abcd}$, $\mathcal{B}_{ab}$, $\mathcal{B}_{abc}$, and $\mathcal{B}_{abcd}$. The tidal moments allow us to introduce the length scale $\mathcal{R}$, and they give rise to tidal potentials that form the building blocks for the construction of the black-hole metric.

In Sec. III we display our expressions for the metric of a tidally deformed black hole. We proceed in two steps. We first consider a smooth timelike geodesic γ in a vacuum region of an arbitrary spacetime, and we construct the metric of this spacetime in a neighborhood of the world line. We refer to this spacetime as the *background spacetime*, and its metric is displayed in Eqs. (3.3) and (3.4); the metric is presented in the light-cone coordinates (v, r, θ^A) and (v, x^a) introduced previously. We next insert a black hole of mass M into the background spacetime, place it on the world line γ, and recalculate the metric. The metric of the black-hole spacetime is displayed in Eqs. (3.7) and (3.5).

In Sec. IV we describe the consequences of the tidal deformation on the structure and dynamics of the black-hole horizon. We establish the statements of Eqs. (1.3), (1.4), and (1.5).

In Sec. V we provide a derivation of the background metric of Eqs. (3.3) and (3.4). In Sec. VI we present a derivation of the black-hole metric of Eqs. (3.5) and (3.7).

In Appendix A we describe how the tidal potentials can be decomposed in (scalar, vector, and tensor) spherical harmonics. In Appendix B we calculate the determinant of the induced metric on the black-hole horizon; the result is displayed (but not derived) in Sec. IV. In Appendix C we provide calculational details relevant to the computation of the tidal heating in Sec. IV. And in Appendix D we sketch a second, alternative derivation of the background metric of Eqs. (3.3) and (3.4).

II. TIDAL MOMENTS AND POTENTIALS

The work of Zhang [18] reveals that the metric of any vacuum spacetime can be constructed in the neighborhood of any geodesic world line and expressed in terms of two sets of tidal multipole moments. We begin our exploration of the geometry of a tidally deformed black hole in Sec. II A with a formal definition of these moments. We use them in Sec. II B to specify the length and time scales associated with the tidal environment, and in Sec. II C we describe their properties under parity transformations. We conclude in Secs. II D and II E with the introduction of potentials that can be constructed from the tidal moments.

A. Definition of tidal moments

We consider a vacuum region of spacetime in a neighborhood of a smooth timelike geodesic γ. The world line is described by the parametric relations $z^\alpha(\tau)$ in an arbitrary coordinate system x^α, and it is parametrized by proper time τ. The velocity vector $u^\alpha := dz^\alpha/d\tau$ is tangent to the world line, and we construct a vectorial basis by adding to u^α an orthonormal triad of vectors $e^\alpha_a(\tau)$, which we assume to be orthogonal to u^α and parallel-transported on the world line; the Latin index a labels the three members of the triad.

We use the basis (u^α, e^α_a) to decompose tensors that are evaluated on the world line. For example,

$$C_{a0b0} := C_{\alpha\mu\beta\nu} e^\alpha_a u^\mu e^\beta_b u^\nu, \tag{2.1a}$$

$$C_{abc0} := C_{\alpha\beta\gamma\mu} e^\alpha_a e^\beta_b e^\gamma_c u^\mu, \tag{2.1b}$$

$$C_{abcd} := C_{\alpha\beta\gamma\delta} e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d \tag{2.1c}$$

are the frame components of the Weyl tensor evaluated on γ; these are functions of proper time τ. We shall also need the frame components of its first covariant derivative,

$$C_{a0b0|c} := C_{\alpha\mu\beta\nu;\gamma} e^\alpha_a u^\mu e^\beta_b u^\nu e^\gamma_c, \tag{2.2a}$$

$$C_{abc0|d} := C_{\alpha\beta\gamma\mu;\delta} e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d u^\mu, \tag{2.2b}$$

$$C_{abcd|e} := C_{\alpha\beta\gamma\delta;\epsilon} e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d e^\epsilon_e, \tag{2.2c}$$

as well as

$$C_{a0b0|cd} := C_{\alpha\mu\beta\nu;\gamma\delta} e^\alpha_a u^\mu e^\beta_b u^\nu e^\gamma_c e^\delta_d, \tag{2.3a}$$

$$C_{abc0|de} := C_{\alpha\beta\gamma\mu;\delta\epsilon} e^\alpha_a e^\beta_b e^\gamma_c u^\mu e^\delta_d e^\epsilon_e, \tag{2.3b}$$

$$C_{abcd|ef} := C_{\alpha\beta\gamma\delta;\epsilon\eta} e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d e^\epsilon_e e^\eta_f, \tag{2.3c}$$

the frame components of its second covariant derivatives. We manipulate frame indices as if they were associated with Cartesian tensors; we lower them with the Kronecker delta δ_{ab}, and we raise them with δ^{ab}.

The symmetries of the Weyl tensor imply that it possesses 10 algebraically-independent components, and these can be encoded in the two symmetric-tracefree (STF) tensors

$$\mathcal{E}_{ab} := (C_{a0b0})^{\rm STF}, \tag{2.4a}$$

$$\mathcal{B}_{ab} := \frac{1}{2}(\epsilon_{apq}C^{pq}{}_{b0})^{\rm STF}. \tag{2.4b}$$

Here ϵ_{abc} is the permutation symbol, and the STF sign instructs us to symmetrize all free indices and remove all traces. We also use an angular-bracket notation to indicate the same operation: If A_{abcd} is an arbitrary tensor, then

$$A_{\langle abcd\rangle} := (A_{abcd})^{\rm STF}. \tag{2.5}$$

In the case of Eq. (2.4) the STF operation is superfluous, because C_{a0b0} and $\epsilon_{apq}C^{pq}{}_{b0}$ are already symmetric and tracefree. We refer to $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ as the *tidal quadrupole moments* associated with the world line γ. Each STF tensor contains 5 independent components, and these are functions of proper time τ. We use $\dot{\mathcal{E}}_{ab}$ and $\dot{\mathcal{B}}_{ab}$ to denote the derivative of each moment with respect to τ, and $\ddot{\mathcal{E}}_{ab}$ and $\ddot{\mathcal{B}}_{ab}$ to denote the second derivatives. The derivatives of the Weyl tensor in the direction of u^α can be expressed directly in terms of these quantities.

The symmetries of the Weyl tensor and the Bianchi identities imply that the spatial derivatives of the Weyl tensor—those listed in Eq. (2.2)—possess 24 algebraically-independent components. These are encoded in the STF tensors

$$\mathcal{E}_{abc} := (C_{a0b0|c})^{\rm STF}, \tag{2.6a}$$

$$\mathcal{B}_{abc} := \frac{3}{8}(\epsilon_{apq}C^{pq}{}_{b0|c})^{\rm STF}, \tag{2.6b}$$

and in $\dot{\mathcal{E}}_{ab}$ and $\dot{\mathcal{B}}_{ab}$. We refer to $\mathcal{E}_{abc}$ and $\mathcal{B}_{abc}$ as the *tidal octupole moments* associated with the world line γ. Each tensor contains 7 independent components, and these are functions of proper time τ. We use $\dot{\mathcal{E}}_{abc}$ and $\dot{\mathcal{B}}_{abc}$ to denote the derivative of each moment with respect to τ. The derivatives of the spatially-differentiated Weyl tensor in the direction of u^α can be expressed directly in terms of these quantities.

We shall also require the second spatial derivatives of the Weyl tensor, as listed in Eqs. (2.3). In this case there are 62 algebraically-independent components, and these are encoded in $\ddot{\mathcal{E}}_{ab}$, $\ddot{\mathcal{B}}_{ab}$, $\dot{\mathcal{E}}_{abc}$, $\dot{\mathcal{B}}_{abc}$, and in the new STF tensors

$$\mathcal{E}_{abcd} := \frac{1}{2}(C_{a0b0|cd})^{\rm STF}, \tag{2.7a}$$

$$\mathcal{B}_{abcd} := \frac{3}{20}(\epsilon_{apq}C^{pq}{}_{b0|cd})^{\rm STF}. \tag{2.7b}$$

We refer to $\mathcal{E}_{abcd}$ and $\mathcal{B}_{abcd}$ as the *tidal hexadecapole moments* associated with the world line γ. Each tensor contains 9 independent components, and these are functions of proper time τ.

The decomposition of the Weyl tensor and its derivatives in terms of tidal moments is displayed in Eqs. (D15), (D17), and (D19) of Appendix D. The numerical factors inserted in Eqs. (2.4), (2.6), and (2.7), are inherited from Zhang's choice of normalization [18].

The black-hole metric of Sec. III will be expressed in terms of the tidal moments $\mathcal{E}_{ab}$, $\mathcal{E}_{abc}$, $\mathcal{E}_{abcd}$, $\mathcal{B}_{ab}$, $\mathcal{B}_{abc}$, and $\mathcal{B}_{abcd}$, which are treated as freely-specifiable functions of time. As we shall explain in Sec. III, the relation between the tidal moments and the Weyl tensor will be more subtle than what was described here.

B. Tidal scales

The tidal moments allow us to specify length and time scales that characterize the tidal environment around the world line γ. We thus introduce the length scales $\mathcal{R}$ and $\mathcal{L}$, the time scale $\mathcal{T}$, and the velocity scale $\mathcal{V}$. Our discussion here follows Thorne and Hartle [68] and Zhang [18].

The length $\mathcal{R}$ is the local radius of curvature, and represents the strength of the Weyl tensor evaluated on the world line. We express this as

$$\mathcal{E}_{ab} \sim \frac{1}{\mathcal{R}^2}, \qquad \mathcal{B}_{ab} \sim \frac{\mathcal{V}}{\mathcal{R}^2}. \tag{2.8}$$

The first equation indicates that a typical component of $\mathcal{E}_{ab}$ will have a magnitude comparable to $\mathcal{R}^{-2}$. The second equation indicates that a typical component of $\mathcal{B}_{ab}$ will differ from this by a factor of $\mathcal{V}$, and this defines the velocity scale.

The length $\mathcal{L}$ is the inhomogeneity scale, and it measures the degree of spatial variation of the Weyl tensor. It is defined through the relations

$$\mathcal{E}_{abc} \sim \frac{1}{\mathcal{R}^2\mathcal{L}}, \qquad \mathcal{B}_{abc} \sim \frac{\mathcal{V}}{\mathcal{R}^2\mathcal{L}} \tag{2.9}$$

involving the tidal octupole moments. We expect that the hexadecapole moments will be suppressed by an additional factor of $\mathcal{L}$:

$$\mathcal{E}_{abcd} \sim \frac{1}{\mathcal{R}^2\mathcal{L}^2}, \qquad \mathcal{B}_{abcd} \sim \frac{\mathcal{V}}{\mathcal{R}^2\mathcal{L}^2}. \tag{2.10}$$

The time $\mathcal{T}$ is the scale associated with changes in the behavior of the Weyl tensor. This is defined by

$$\dot{\mathcal{E}}_{ab} \sim \frac{1}{\mathcal{R}^2\mathcal{T}}, \qquad \dot{\mathcal{B}}_{ab} \sim \frac{\mathcal{V}}{\mathcal{R}^2\mathcal{T}}. \tag{2.11}$$

We also have $\ddot{\mathcal{E}}_{ab} = \mathcal{R}^{-2}\mathcal{T}^{-2}$ and $\dot{\mathcal{E}}_{abc} = \mathcal{R}^{-2}\mathcal{L}^{-1}\mathcal{T}^{-1}$, as well as $\ddot{\mathcal{B}}_{ab} = \mathcal{V}\mathcal{R}^{-2}\mathcal{T}^{-2}$ and $\dot{\mathcal{B}}_{abc} = \mathcal{V}\mathcal{R}^{-2}\mathcal{L}^{-1}\mathcal{T}^{-1}$.

To illustrate the meaning of these tidal scales, let us consider as an example the world line of an observer moving on a circular orbit of radius b around a body of mass M'. In this case the velocity scale is $\mathcal{V} \sim \sqrt{M'/b}$, the radius of curvature is $\mathcal{R} \sim \sqrt{b^3/M'}$, the inhomogeneity scale is $\mathcal{L} \sim b$, and the time scale is $\mathcal{T} \sim \sqrt{b^3/M'}$. In this example we have that $\mathcal{T} \sim \mathcal{R}$, $\mathcal{V} \sim \mathcal{L}/\mathcal{T}$, and $\mathcal{L} \sim \mathcal{V}\mathcal{R}$. When the motion is slow we have that $\mathcal{L} \ll \mathcal{R}$, and there is a wide separation between the two length scales. When the motion is relativistic, however, all scales are comparable to each other.

To simplify our notation in later portions of the paper, we choose to eliminate the distinction between the different tidal scales. We therefore set $\mathcal{V} \sim 1$, $\mathcal{L} \sim \mathcal{T} \sim \mathcal{R}$, and adopt $\mathcal{R}$ as the single scale associated with the tidal environment. In this sloppy notation we write

$$\mathcal{E}_{ab} \sim \mathcal{B}_{ab} \sim \frac{1}{\mathcal{R}^2}, \tag{2.12a}$$

$$\mathcal{E}_{abc} \sim \mathcal{B}_{abc} \sim \dot{\mathcal{E}}_{ab} \sim \dot{\mathcal{B}}_{ab} \sim \frac{1}{\mathcal{R}^3}, \tag{2.12b}$$

$$\mathcal{E}_{abcd} \sim \mathcal{B}_{abcd} \sim \dot{\mathcal{E}}_{abc} \sim \dot{\mathcal{B}}_{abc} \sim \ddot{\mathcal{E}}_{ab} \sim \ddot{\mathcal{B}}_{ab} \sim \frac{1}{\mathcal{R}^4}. \tag{2.12c}$$

We emphasize that this relaxation of our notation is simply to save writing. For example, an error term that should be written $O(r^5\mathcal{R}^{-2}\mathcal{L}^{-2}\mathcal{T}^{-1})$ will be condensed to the simpler expression $O(r^5/\mathcal{R}^5)$. The form of the equations will always allow us to determine the order of magnitude of each term in relation to the complete set of scaling quantities.

C. Parity rules

In the context of this work, a parity transformation is a change of tetrad vectors described by $u^\alpha \to u^\alpha$ and $e^\alpha_a \to -e^\alpha_a$; the transformation keeps the permutation symbol unchanged: $\epsilon_{abc} \to \epsilon_{abc}$. Under the transformation the frame components of the Weyl tensor change according to $C_{a0b0} \to C_{a0b0}$ and $C_{abc0} \to -C_{abc0}$. We also have $C_{a0b0|c} \to -C_{a0b0|c}$, $C_{abc0|d} \to C_{abc0|d}$, and $C_{a0b0|cd} \to C_{a0b0|cd}$, $C_{abc0|de} \to -C_{abc0|de}$. From Eqs. (2.4), (2.6), and (2.7) we deduce the transformation rules

$$\mathcal{E}_{ab} \to \mathcal{E}_{ab}, \qquad \mathcal{E}_{abc} \to -\mathcal{E}_{abc}, \qquad \mathcal{E}_{abcd} \to \mathcal{E}_{abcd}, \tag{2.13a}$$

$$\mathcal{B}_{ab} \to -\mathcal{B}_{ab}, \qquad \mathcal{B}_{abc} \to \mathcal{B}_{abc}, \qquad \mathcal{B}_{abcd} \to -\mathcal{B}_{abcd} \tag{2.13b}$$

for the tidal moments. We say that $\mathcal{E}_{ab}$, $\mathcal{E}_{abc}$, and $\mathcal{E}_{abcd}$ have even parity, because they transform as ordinary Cartesian tensors under a parity transformation. And we say that $\mathcal{B}_{ab}$, $\mathcal{B}_{abc}$, and $\mathcal{B}_{abcd}$ have odd parity, because they transform as pseudotensors.

D. Tidal potentials: Cartesian coordinates

For the purposes of writing down the black-hole metric in Sec. III, we involve the tidal moments in the construction of *tidal potentials* that will form the main building blocks for the metric. We first achieve this with the help of a system of Cartesian coordinates x^a that we assume is at our disposal. In the following subsection we convert the potentials to spherical coordinates (r, θ, ϕ).

We introduce

$$\Omega^a := x^a/r \tag{2.14}$$

as the radial unit vector, with $r := \sqrt{\delta_{ab}x^a x^b}$ denoting the usual Euclidean distance. We refer to the radial direction as the *longitudinal* direction, and to the orthogonal space as the *transverse* directions.

We wish to combine the tidal moments with Ω^a so as to form scalar, vector, and (rank-two, symmetric) tensor potentials that satisfy the following properties:

(1) Each potential is an element of an irreducible representation of the rotation group labeled by multipole order l.
(2) Each scalar potential transforms as such under a parity transformation.
(3) Each vector potential transforms as such under a parity transformation, and is purely transverse, in the sense of being orthogonal to Ω^a.
(4) Each tensor potential transforms as such under a parity transformation, and is transverse-tracefree, in the sense of being orthogonal to Ω^a and having a vanishing trace.

The first property implies that each potential will satisfy an appropriate eigenvalue equation that depends on the multipole order and the tensorial rank of the potential; these are displayed in Eqs. (A3) and (A10) of Appendix A. Under a parity transformation the longitudinal vector transforms as $\Omega^a \to -\Omega^a$, a scalar potential remains invariant, a vector potential changes sign, and a tensor potential remains invariant. To aid the construction of the potentials we introduce

$$\gamma^a{}_b := \delta^a{}_b - \Omega^a\Omega_b \tag{2.15}$$

as a projector to the transverse space orthogonal to Ω^a; this transforms as $\gamma_{ab} \to \gamma_{ab}$ under a parity transformation.

The required potentials are displayed in a number of tables. In Table I we list the tidal potentials that are constructed from the even-parity tidal moments $\mathcal{E}_{ab}$, $\mathcal{E}_{abc}$, and $\mathcal{E}_{abcd}$. In Table II we have the tidal potentials that are constructed from the odd-parity tidal moments $\mathcal{B}_{ab}$, $\mathcal{B}_{abc}$, and $\mathcal{B}_{abcd}$. In Table III we list the potentials that arise from the bilinear combination $\mathcal{E}_{ab}\mathcal{E}_{cd}$ of even-parity moments. In Table IV we have the potentials that arise from the bilinear combination $\mathcal{B}_{ab}\mathcal{B}_{cd}$ of odd-parity moments. In Table V we list potentials that arise when we combine $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ into an even-parity structure. And finally, in Table VI we combine them into an odd-parity structure.

We illustrate the construction of the potentials by examining a few examples. The simplest cases involve the tidal quadrupole moment $\mathcal{E}_{ab}$, which transforms as a tensor

TABLE I. Irreducible tidal potentials: type-$\mathcal{E}$, even-parity. The superscripts q, o, and h stand for "quadrupole," "octupole," and "hexadecapole," respectively. The tidal moments $\mathcal{E}_{ab}$, $\mathcal{E}_{abc}$, and $\mathcal{E}_{abcd}$ are the STF tensors defined in the text. The extra factors of 2 in the hexadecapole potentials are inserted to respect Zhang's normalization convention for $\mathcal{E}_{abcd}$; see Eqs. (2.7).

$$\mathcal{E}^{\mathsf{q}} = \mathcal{E}_{cd}\Omega^c\Omega^d$$
$$\mathcal{E}^{\mathsf{q}}_a = \gamma_a{}^c\mathcal{E}_{cd}\Omega^d$$
$$\mathcal{E}^{\mathsf{q}}_{ab} = 2\gamma_a{}^c\gamma_b{}^d\mathcal{E}_{cd} + \gamma_{ab}\mathcal{E}^{\mathsf{q}}$$
$$\mathcal{E}^{\mathsf{o}} = \mathcal{E}_{cde}\Omega^c\Omega^d\Omega^e$$
$$\mathcal{E}^{\mathsf{o}}_a = \gamma_a{}^c\mathcal{E}_{cde}\Omega^d\Omega^e$$
$$\mathcal{E}^{\mathsf{o}}_{ab} = 2\gamma_a{}^c\gamma_b{}^d\mathcal{E}_{cde}\Omega^e + \gamma_{ab}\mathcal{E}^{\mathsf{o}}$$
$$\mathcal{E}^{\mathsf{h}} = 2\mathcal{E}_{cdef}\Omega^c\Omega^d\Omega^e\Omega^f$$
$$\mathcal{E}^{\mathsf{h}}_a = 2\gamma_a{}^c\mathcal{E}_{cdef}\Omega^d\Omega^e\Omega^f$$
$$\mathcal{E}^{\mathsf{h}}_{ab} = 4\gamma_a{}^c\gamma_b^d\mathcal{E}_{cdef}\Omega^e\Omega^f + \gamma_{ab}\mathcal{E}^{\mathsf{h}}$$

TABLE II. Irreducible tidal potentials: type-$\mathcal{B}$, odd-parity. The superscripts q, o, and h stand for "quadrupole," "octupole," and "hexadecapole," respectively. The tidal moments $\mathcal{B}_{ab}$, $\mathcal{B}_{abc}$, and $\mathcal{B}_{abcd}$ are the STF tensors defined in the text. The factors of $\frac{4}{3}$ and $\frac{10}{3}$ are inserted to respect Zhang's normalization convention for $\mathcal{B}_{abc}$ and $\mathcal{B}_{abcd}$; see Eqs. (2.6) and (2.7).

$$\mathcal{B}^{\mathsf{q}}_a = \epsilon_{apq}\Omega^p\mathcal{B}^q{}_c\Omega^c$$
$$\mathcal{B}^{\mathsf{q}}_{ab} = \epsilon_{apq}\Omega^p\mathcal{B}^q{}_d\gamma^d{}_b + \epsilon_{bpq}\Omega^p\mathcal{B}^q{}_c\gamma^c{}_a$$
$$\mathcal{B}^{\mathsf{o}}_a = \tfrac{4}{3}\epsilon_{apq}\Omega^p\mathcal{B}^q{}_{cd}\Omega^c\Omega^d$$
$$\mathcal{B}^{\mathsf{o}}_{ab} = \tfrac{4}{3}(\epsilon_{apq}\Omega^p\mathcal{B}^q{}_{de}\gamma^d{}_b + \epsilon_{bpq}\Omega^p\mathcal{B}^q{}_{ce}\gamma^c{}_a)\Omega^e$$
$$\mathcal{B}^{\mathsf{h}}_a = \tfrac{10}{3}\epsilon_{apq}\Omega^p\mathcal{B}^q{}_{cde}\Omega^c\Omega^d\Omega^e$$
$$\mathcal{B}^{\mathsf{h}}_{ab} = \tfrac{10}{3}(\epsilon_{apq}\Omega^p\mathcal{B}^q{}_{def}\gamma^d_b + \epsilon_{bpq}\Omega^p\mathcal{B}^q{}_{cef}\gamma^c{}_a)\Omega^e\Omega^f$$

TABLE III. Irreducible tidal potentials: type-$\mathcal{E}\mathcal{E}$, even-parity. The superscripts m, q, and h stand for "monopole," "quadrupole," and "hexadecapole," respectively.

$$\mathcal{P}^{\mathsf{m}} = \mathcal{E}_{pq}\mathcal{E}^{pq}$$
$$\mathcal{P}^{\mathsf{q}} = \mathcal{E}_{p\langle c}\mathcal{E}^p{}_{d\rangle}\Omega^c\Omega^d$$
$$\mathcal{P}^{\mathsf{q}}_a = \gamma_a{}^c\mathcal{E}_{p\langle c}\mathcal{E}^p{}_{d\rangle}\Omega^d$$
$$\mathcal{P}^{\mathsf{q}}_{ab} = 2\gamma_a{}^c\gamma_b{}^d\mathcal{E}_{p\langle c}\mathcal{E}^p{}_{d\rangle} + \gamma_{ab}\mathcal{P}^{\mathsf{q}}$$
$$\mathcal{P}^{\mathsf{h}} = \mathcal{E}_{\langle cd}\mathcal{E}_{ef\rangle}\Omega^c\Omega^d\Omega^e\Omega^f$$
$$\mathcal{P}^{\mathsf{h}}_a = \gamma_a{}^c\mathcal{E}_{\langle cd}\mathcal{E}_{ef\rangle}\Omega^d\Omega^e\Omega^f$$
$$\mathcal{P}^{\mathsf{h}}_{ab} = 2\gamma_a{}^c\gamma_b{}^d\mathcal{E}_{\langle cd}\mathcal{E}_{ef\rangle}\Omega^e\Omega^f + \gamma_{ab}\mathcal{P}^{\mathsf{h}}$$

TABLE IV. Irreducible tidal potentials: type-$\mathcal{B}\mathcal{B}$, even-parity. The superscripts m, q, and h stand for "monopole," "quadrupole," and "hexadecapole," respectively.

$$\mathcal{Q}^{\mathsf{m}} = \mathcal{B}_{pq}\mathcal{B}^{pq}$$
$$\mathcal{Q}^{\mathsf{q}} = \mathcal{B}_{p\langle c}\mathcal{B}^p{}_{d\rangle}\Omega^c\Omega^d$$
$$\mathcal{Q}^{\mathsf{q}}_a = \gamma_a{}^c\mathcal{B}_{p\langle c}\mathcal{B}^p{}_{d\rangle}\Omega^d$$
$$\mathcal{Q}^{\mathsf{q}}_{ab} = 2\gamma_a{}^c\gamma_b{}^d\mathcal{B}_{p\langle c}\mathcal{B}^p{}_{d\rangle} + \gamma_{ab}\mathcal{Q}^{\mathsf{q}}$$
$$\mathcal{Q}^{\mathsf{h}} = \mathcal{B}_{\langle cd}\mathcal{B}_{ef\rangle}\Omega^c\Omega^d\Omega^e\Omega^f$$
$$\mathcal{Q}^{\mathsf{h}}_a = \gamma_a{}^c\mathcal{B}_{\langle cd}\mathcal{B}_{ef\rangle}\Omega^d\Omega^e\Omega^f$$
$$\mathcal{Q}^{\mathsf{h}}_{ab} = 2\gamma_a{}^c\gamma_b{}^d\mathcal{B}_{\langle cd}\mathcal{B}_{ef\rangle}\Omega^e\Omega^f + \gamma_{ab}\mathcal{Q}^{\mathsf{h}}$$

TABLE V. Irreducible tidal potentials: type-$\mathcal{E}\mathcal{B}$, even-parity. The superscripts d and o stand for "dipole" and "octupole," respectively.

$$\mathcal{G}^{\mathsf{d}} = \epsilon_{cpq}\mathcal{E}^p{}_r\mathcal{B}^{rq}\Omega^c$$
$$\mathcal{G}^{\mathsf{d}}_a = \gamma_a{}^c\epsilon_{cpq}\mathcal{E}^p{}_r\mathcal{B}^{rq}$$
$$\mathcal{G}^{\mathsf{o}} = \epsilon_{pq\langle c}\mathcal{E}^p{}_d\mathcal{B}^q{}_{e\rangle}\Omega^c\Omega^d\Omega^e$$
$$\mathcal{G}^{\mathsf{o}}_a = \gamma_a{}^c\epsilon_{pq\langle c}\mathcal{E}^p{}_d\mathcal{B}^q{}_{e\rangle}\Omega^d\Omega^e$$
$$\mathcal{G}^{\mathsf{o}}_{ab} = 2\gamma_a{}^c\gamma_b{}^d\epsilon_{pq\langle c}\mathcal{E}^p{}_d\mathcal{B}^q{}_{e\rangle}\Omega^e + \gamma_{ab}\mathcal{G}^{\mathsf{o}}$$

TABLE VI. Irreducible tidal potentials: type-$\mathcal{E}\mathcal{B}$, odd-parity. The superscripts q and h stand for "quadrupole" and "hexadecapole," respectively.

$$\mathcal{H}^{\mathsf{q}}_a = \epsilon_a{}^{pq}\Omega_p\mathcal{E}_{r\langle q}\mathcal{B}^r{}_{c\rangle}\Omega^c$$
$$\mathcal{H}^{\mathsf{q}}_{ab} = \epsilon_a{}^{pq}\Omega_p\mathcal{E}_{r\langle q}\mathcal{B}^r{}_{d\rangle}\gamma^d{}_b + \epsilon_b{}^{pq}\Omega_p\mathcal{E}_{r\langle q}\mathcal{B}^r{}_{c\rangle}\gamma^c{}_a$$
$$\mathcal{H}^{\mathsf{h}}_a = \epsilon_a{}^{pq}\Omega_p\mathcal{E}_{\langle qc}\mathcal{B}_{de\rangle}\Omega^c\Omega^d\Omega^e$$
$$\mathcal{H}^{\mathsf{h}}_{ab} = (\epsilon_a{}^{pq}\Omega_p\mathcal{E}_{\langle qd}\mathcal{B}_{ef\rangle}\gamma^d{}_b + \epsilon_b{}^{pq}\Omega_p\mathcal{E}_{\langle qc}\mathcal{B}_{ef\rangle}\gamma^c{}_a)\Omega^e\Omega^f$$

under a parity transformation. The associated scalar potential is $\mathcal{E}_{ab}\Omega^a\Omega^b$; this transforms appropriately under a parity transformation, and because $\mathcal{E}_{ab}$ is tracefree this satisfies the scalar eigenvalue equation with $l = 2$. To get a vector potential we first form $\mathcal{E}_{cb}\Omega^b$, which has the appropriate number of free indices. We next multiply this by $\gamma^c{}_a$ to make the vector transverse, and verify that the final object satisfies the vectorial eigenvalue equation with $l = 2$. To get a transverse tensor potential we first form $\gamma^c{}_a\gamma^d{}_b\mathcal{E}_{cd}$, and we remove its trace by adding $\frac{1}{2}\gamma_{ab}(\mathcal{E}_{cd}\Omega^c\Omega^d)$; the result satisfies the appropriate eigenvalue equation with $l = 2$. The method generalizes easily to octupole and hexadecapole potentials, and the final results are displayed in Table I.

We next examine the potentials associated with the odd-parity quadrupole moment $\mathcal{B}_{ab}$. The combination $\mathcal{B}_{ab}\Omega^a\Omega^b$ transforms as a pseudoscalar under a parity transformation and does not, therefore, satisfy the criteria that were formulated previously. To obtain a suitable potential we must involve the permutation symbol, and the simplest allowed combination is $\epsilon_{apq}\Omega^p\mathcal{B}^q{}_c\Omega^c$. This transforms as a vector under a parity transformation and satisfies the appropriate eigenvalue equation with $l = 2$; we therefore include it as one of our building blocks. To form a tensor potential we remove Ω^c, replace it with the projector $\gamma^c{}_b$, and symmetrize the indices; the end result satisfies all the properties required of a tensor potential. The method generalizes easily to octupole and hexadecapole potentials, and these are listed in Table II.

The bilinear potentials are constructed by forming STF products of the quadrupole moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ and combining these with appropriate factors of Ω^a, $\gamma^a{}_b$, and ϵ_{abc}. For example, the STF products that can be formed from two factors of $\mathcal{E}_{ab}$ are the scalar $\mathcal{E}_{pq}\mathcal{E}^{pq}$, the rank-two tensor $\mathcal{E}_{p\langle a}\mathcal{E}^p{}_{b\rangle}$, and the rank-four tensor $\mathcal{E}_{\langle ab}\mathcal{E}_{cd\rangle}$; the associated potentials are listed in Table III, and those of Table IV follow from similar manipulations. When $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ are both involved we must be mindful of the parity rules; for example, the hexadecapole potential $\mathcal{E}_{\langle ab}\mathcal{B}_{cd\rangle}\Omega^a\Omega^b\Omega^c\Omega^d$ is ruled out because it transforms as a pseudoscalar under a parity transformation. The appropriate combinations must involve the permutation symbol, and these are displayed in Tables V and VI.

E. Tidal potentials: Angular coordinates

A transformation from Cartesian coordinates x^a to spherical coordinates (r, θ, ϕ) is effected by

$$x^a = r\Omega^a(\theta^A), \tag{2.16}$$

in which the longitudinal vector Ω^a is now parametrized by two polar angles $\theta^A = (\theta, \phi)$. Explicitly, we have that $\Omega^a = [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]$. The transformation implies that $\partial x^a/\partial r = \Omega^a$ and $\partial x^a/\partial\theta^A = r\Omega^a_A$, with

$$\Omega^a_A := \frac{\partial\Omega^a}{\partial\theta^A}. \tag{2.17}$$

We note the useful identities

$$\Omega_a\Omega^a_A = 0, \tag{2.18a}$$

$$\Omega_{AB} = \gamma_{ab}\Omega^a_A\Omega^b_B, \tag{2.18b}$$

$$\Omega^{AB}\Omega^a_A\Omega^b_B = \gamma^{ab}. \tag{2.18c}$$

Here $\Omega_{AB} = \mathrm{diag}[1, \sin^2\theta]$ is the metric on the unit two-sphere, and Ω^{AB} is its inverse. We introduce D_A as the covariant-derivative operator compatible with Ω_{AB}, and ϵ_{AB} as the Levi-Civita tensor on the unit two-sphere (with nonvanishing components $\epsilon_{\theta\phi} = -\epsilon_{\phi\theta} = \sin\theta$). We adopt the convention that upper-case Latin indices are raised and lowered with Ω^{AB} and Ω_{AB}, respectively. Finally, we note that $D_C\Omega_{AB} = D_C\epsilon_{AB} = 0$.

We convert the vector and tensor potentials from their initial Cartesian incarnations to angular-coordinate versions by making use of the transformation matrix Ω^a_A.

TABLE VII. Spherical-harmonic functions Y^{lm} and harmonic components $\mathcal{A}^{(l)}_m$ involved in the decomposition of $\mathcal{A}^{(l)} := \mathcal{A}_{k_1\cdots k_l}\Omega^{k_1}\cdots\Omega^{k_l} = \sum_m \mathcal{A}^{(l)}_m Y^{lm}$. The functions are real, and they are listed for the relevant modes $l = 1$ (dipole), $l = 2$ (quadrupole), $l = 3$ (octupole), and $l = 4$ (hexadecapole). The abstract index m describes the dependence of these functions on the angle ϕ; for example $Y^{l,2s}$ is proportional to $\sin2\phi$. To simplify the expressions we write $C := \cos\theta$ and $S := \sin\theta$. The harmonic components are expressed in terms of the independent components of the STF tensor $\mathcal{A}_{k_1\cdots k_l}$.

$Y^{1,0} = C$	$\mathcal{A}^{\mathsf{d}}_0 = \mathcal{A}_3$
$Y^{1,1c} = S\cos\phi$	$\mathcal{A}^{\mathsf{d}}_{1c} = \mathcal{A}_1$
$Y^{1,1s} = S\sin\phi$	$\mathcal{A}^{\mathsf{d}}_{1s} = \mathcal{A}_2$
$Y^{2,0} = 1 - 3C^2$	$\mathcal{A}^{\mathsf{q}}_0 = \frac{1}{2}(\mathcal{A}_{11} + \mathcal{A}_{22})$
$Y^{2,1c} = 2SC\cos\phi$	$\mathcal{A}^{\mathsf{q}}_{1c} = \mathcal{A}_{13}$
$Y^{2,1s} = 2SC\sin\phi$	$\mathcal{A}^{\mathsf{q}}_{1s} = \mathcal{A}_{23}$
$Y^{2,2c} = S^2\cos2\phi$	$\mathcal{A}^{\mathsf{q}}_{2c} = \frac{1}{2}(\mathcal{A}_{11} - \mathcal{A}_{22})$
$Y^{2,2s} = S^2\sin2\phi$	$\mathcal{A}^{\mathsf{q}}_{2s} = \mathcal{A}_{12}$
$Y^{3,0} = C(3 - 5C^2)$	$\mathcal{A}^{\mathsf{o}}_0 = \frac{1}{2}(\mathcal{A}_{113} + \mathcal{A}_{223})$
$Y^{3,1c} = \frac{3}{2}S(1 - 5C^2)\cos\phi$	$\mathcal{A}^{\mathsf{o}}_{1c} = \frac{1}{2}(\mathcal{A}_{111} + \mathcal{A}_{122})$
$Y^{3,1s} = \frac{3}{2}S(1 - 5C^2)\sin\phi$	$\mathcal{A}^{\mathsf{o}}_{1s} = \frac{1}{2}(\mathcal{A}_{112} + \mathcal{A}_{222})$
$Y^{3,2c} = 3S^2C\cos2\phi$	$\mathcal{A}^{\mathsf{o}}_{2c} = \frac{1}{2}(\mathcal{A}_{113} - \mathcal{A}_{223})$
$Y^{3,2s} = 3S^2C\sin2\phi$	$\mathcal{A}^{\mathsf{o}}_{2s} = \mathcal{A}_{123}$
$Y^{3,3c} = S^3\cos3\phi$	$\mathcal{A}^{\mathsf{o}}_{3c} = \frac{1}{4}(\mathcal{A}_{111} - 3\mathcal{A}_{122})$
$Y^{3,3s} = S^3\sin3\phi$	$\mathcal{A}^{\mathsf{o}}_{3s} = \frac{1}{4}(3\mathcal{A}_{112} - \mathcal{A}_{222})$
$Y^{4,0} = \frac{1}{2}(3 - 30C^2 + 35C^4)$	$\mathcal{A}^{\mathsf{h}}_0 = \frac{1}{4}(\mathcal{A}_{1111} + 2\mathcal{A}_{1122} + \mathcal{A}_{2222})$
$Y^{4,1c} = 2SC(3 - 7C^2)\cos\phi$	$\mathcal{A}^{\mathsf{h}}_{1c} = \frac{1}{2}(\mathcal{A}_{1113} + \mathcal{A}_{1223})$
$Y^{4,1s} = 2SC(3 - 7C^2)\sin\phi$	$\mathcal{A}^{\mathsf{h}}_{1s} = \frac{1}{2}(\mathcal{A}_{1123} + \mathcal{A}_{2223})$
$Y^{4,2c} = S^2(1 - 7C^2)\cos2\phi$	$\mathcal{A}^{\mathsf{h}}_{2c} = \frac{1}{2}(\mathcal{A}_{1111} - \mathcal{A}_{2222})$
$Y^{4,2s} = S^2(1 - 7C^2)\sin2\phi$	$\mathcal{A}^{\mathsf{h}}_{2s} = \mathcal{A}_{1112} + \mathcal{A}_{1222}$
$Y^{4,3c} = 4S^3C\cos3\phi$	$\mathcal{A}^{\mathsf{h}}_{3c} = \frac{1}{4}(\mathcal{A}_{1113} - 3\mathcal{A}_{1223})$
$Y^{4,3s} = 4S^3C\sin3\phi$	$\mathcal{A}^{\mathsf{h}}_{3s} = \frac{1}{4}(3\mathcal{A}_{1123} - \mathcal{A}_{2223})$
$Y^{4,4c} = S^4\cos4\phi$	$\mathcal{A}^{\mathsf{h}}_{4c} = \frac{1}{8}(\mathcal{A}_{1111} - 6\mathcal{A}_{1122} + \mathcal{A}_{2222})$
$Y^{4,4s} = S^4\sin4\phi$	$\mathcal{A}^{\mathsf{h}}_{4s} = \frac{1}{2}(\mathcal{A}_{1112} - \mathcal{A}_{1222})$

TABLE VIII. The first column lists the harmonic components of type-$\mathcal{E}$, even-parity tidal potentials, as defined in Table I. The second column lists their expansions in scalar, vector, and tensor harmonics. The components of $\mathcal{E}^{\mathsf{h}}$ come with an additional factor of 2 to accommodate Zhang's choice of normalization; see Table I.

$\mathcal{E}^{\mathsf{q}}_0 = \frac{1}{2}(\mathcal{E}_{11} + \mathcal{E}_{22})$	
$\mathcal{E}^{\mathsf{q}}_{1c} = \mathcal{E}_{13}$	$\mathcal{E}^{\mathsf{q}} = \sum_m \mathcal{E}^{\mathsf{q}}_m Y^{2,m}$
$\mathcal{E}^{\mathsf{q}}_{1s} = \mathcal{E}_{23}$	$\mathcal{E}^{\mathsf{q}}_A = \frac{1}{2}\sum_m \mathcal{E}^{\mathsf{q}}_m Y^{2,m}_A$
$\mathcal{E}^{\mathsf{q}}_{2c} = \frac{1}{2}(\mathcal{E}_{11} - \mathcal{E}_{22})$	$\mathcal{E}^{\mathsf{q}}_{AB} = \sum_m \mathcal{E}^{\mathsf{q}}_m Y^{2,m}_{AB}$
$\mathcal{E}^{\mathsf{q}}_{2s} = \mathcal{E}_{12}$	
$\mathcal{E}^{\mathsf{o}}_0 = \frac{1}{2}(\mathcal{E}_{113} + \mathcal{E}_{223})$	
$\mathcal{E}^{\mathsf{o}}_{1c} = \frac{1}{2}(\mathcal{E}_{111} + \mathcal{E}_{122})$	
$\mathcal{E}^{\mathsf{o}}_{1s} = \frac{1}{2}(\mathcal{E}_{112} + \mathcal{E}_{222})$	$\mathcal{E}^{\mathsf{o}} = \sum_m \mathcal{E}^{\mathsf{o}}_m Y^{3,m}$
$\mathcal{E}^{\mathsf{o}}_{2c} = \frac{1}{2}(\mathcal{E}_{113} - \mathcal{E}_{223})$	$\mathcal{E}^{\mathsf{o}}_A = \frac{1}{3}\sum_m \mathcal{E}^{\mathsf{o}}_m Y^{3,m}_A$
$\mathcal{E}^{\mathsf{o}}_{2s} = \mathcal{E}_{123}$	$\mathcal{E}^{\mathsf{o}}_{AB} = \frac{1}{3}\sum_m \mathcal{E}^{\mathsf{o}}_m Y^{3,m}_{AB}$
$\mathcal{E}^{\mathsf{o}}_{3c} = \frac{1}{4}(\mathcal{E}_{111} - 3\mathcal{E}_{122})$	
$\mathcal{E}^{\mathsf{o}}_{3s} = \frac{1}{4}(3\mathcal{E}_{112} - \mathcal{E}_{222})$	
$\mathcal{E}^{\mathsf{h}}_0 = \frac{1}{2}(\mathcal{E}_{1111} + 2\mathcal{E}_{1122} + \mathcal{E}_{2222})$	
$\mathcal{E}^{\mathsf{h}}_{1c} = \mathcal{E}_{1113} + \mathcal{E}_{1223}$	
$\mathcal{E}^{\mathsf{h}}_{1s} = \mathcal{E}_{1123} + \mathcal{E}_{2223}$	
$\mathcal{E}^{\mathsf{h}}_{2c} = \mathcal{E}_{1111} - \mathcal{E}_{2222}$	$\mathcal{E}^{\mathsf{h}} = \sum_m \mathcal{E}^{\mathsf{h}}_m Y^{4,m}$
$\mathcal{E}^{\mathsf{h}}_{2s} = 2(\mathcal{E}_{1112} + \mathcal{E}_{1222})$	$\mathcal{E}^{\mathsf{h}}_A = \frac{1}{4}\sum_m \mathcal{E}^{\mathsf{h}}_m Y^{4,m}_A$
$\mathcal{E}^{\mathsf{h}}_{3c} = \frac{1}{2}(\mathcal{E}_{1113} - 3\mathcal{E}_{1223})$	$\mathcal{E}^{\mathsf{h}}_{AB} = \frac{1}{6}\sum_m \mathcal{E}^{\mathsf{h}}_m Y^{4,m}_{AB}$
$\mathcal{E}^{\mathsf{h}}_{3s} = \frac{1}{2}(3\mathcal{E}_{1123} - \mathcal{E}_{2223})$	
$\mathcal{E}^{\mathsf{h}}_{4c} = \frac{1}{4}(\mathcal{E}_{1111} - 6\mathcal{E}_{1122} + \mathcal{E}_{2222})$	
$\mathcal{E}^{\mathsf{h}}_{4s} = \mathcal{E}_{1112} - \mathcal{E}_{1222}$	

We thus define

$$\mathcal{E}^{\mathsf{q}}_A := \mathcal{E}^{\mathsf{q}}_a \Omega^a_A, \qquad \mathcal{E}^{\mathsf{q}}_{AB} := \mathcal{E}^{\mathsf{q}}_{ab}\Omega^a_A\Omega^b_A, \tag{2.19}$$

and apply the same rule to all other potentials; for example, $\mathcal{G}^{\mathsf{o}}_{AB} := \mathcal{G}^{\mathsf{o}}_{ab}\Omega^a_A\Omega^b_B$. After this conversion the tidal potentials become scalar, vector, and tensor fields on the unit two-sphere, and they depend on the angular coordinates θ^A only. It is easy to show that the conversion can be undone; for example $\mathcal{E}^{\mathsf{q}}_a = \mathcal{E}^{\mathsf{q}}_A \Omega^A_a$, with $\Omega^A_a := \delta_{ab}\Omega^{AB}\Omega^b_B$.

The tidal potentials can all be expressed in terms of (scalar, vector, and tensor) spherical harmonics. Let Y^{lm} be real-valued spherical-harmonic functions (as defined in Table VII). The relevant vectorial and tensorial harmonics of even parity are

$$Y^{lm}_A := D_A Y^{lm}, \tag{2.20a}$$

$$Y^{lm}_{AB} := \left[D_A D_B + \frac{1}{2}l(l+1)\Omega_{AB}\right]Y^{lm}; \tag{2.20b}$$

notice that $\Omega^{AB}Y^{lm}_{AB} = 0$ by virtue of the eigenvalue equation satisfied by the spherical harmonics. The vectorial and tensorial harmonics of odd parity are

TABLE IX. The first column lists the harmonic components of type-$\mathcal{B}$, odd-parity tidal potentials, as defined in Table II. The second column lists their expansions in scalar, vector, and tensor harmonics. The components of $\mathcal{B}^{\mathsf{o}}$ come with an additional factor of $\frac{4}{3}$ to accommodate Zhang's choice of normalization, and those of $\mathcal{B}^{\mathsf{h}}$ come with an additional factor $\frac{10}{3}$; see Table II.

$\mathcal{B}^{\mathsf{q}}_0 = \frac{1}{2}(\mathcal{B}_{11} + \mathcal{B}_{22})$	
$\mathcal{B}^{\mathsf{q}}_{1c} = \mathcal{B}_{13}$	$\mathcal{B}^{\mathsf{q}} = \sum_m \mathcal{B}^{\mathsf{q}}_m Y^{2,m}$
$\mathcal{B}^{\mathsf{q}}_{1s} = \mathcal{B}_{23}$	$\mathcal{B}^{\mathsf{q}}_A = \frac{1}{2}\sum_m \mathcal{B}^{\mathsf{q}}_m X^{2,m}_A$
$\mathcal{B}^{\mathsf{q}}_{2c} = \frac{1}{2}(\mathcal{B}_{11} - \mathcal{B}_{22})$	$\mathcal{B}^{\mathsf{q}}_{AB} = \sum_m \mathcal{B}^{\mathsf{q}}_m X^{2,m}_{AB}$
$\mathcal{B}^{\mathsf{q}}_{2s} = \mathcal{B}_{12}$	
$\mathcal{B}^{\mathsf{o}}_0 = \frac{2}{3}(\mathcal{B}_{113} + \mathcal{B}_{223})$	
$\mathcal{B}^{\mathsf{o}}_{1c} = \frac{2}{3}(\mathcal{B}_{111} + \mathcal{B}_{122})$	
$\mathcal{B}^{\mathsf{o}}_{1s} = \frac{2}{3}(\mathcal{B}_{112} + \mathcal{B}_{222})$	$\mathcal{B}^{\mathsf{o}} = \sum_m \mathcal{B}^{\mathsf{o}}_m Y^{3,m}$
$\mathcal{B}^{\mathsf{o}}_{2c} = \frac{2}{3}(\mathcal{B}_{113} - \mathcal{B}_{223})$	$\mathcal{B}^{\mathsf{o}}_A = \frac{1}{3}\sum_m \mathcal{B}^{\mathsf{o}}_m X^{3,m}_A$
$\mathcal{B}^{\mathsf{o}}_{2s} = \frac{4}{3}\mathcal{B}_{123}$	$\mathcal{B}^{\mathsf{o}}_{AB} = \frac{1}{3}\sum_m \mathcal{B}^{\mathsf{o}}_m X^{3,m}_{AB}$
$\mathcal{B}^{\mathsf{o}}_{3c} = \frac{1}{3}(\mathcal{B}_{111} - 3\mathcal{B}_{122})$	
$\mathcal{B}^{\mathsf{o}}_{3s} = \frac{1}{3}(3\mathcal{B}_{112} - \mathcal{B}_{222})$	
$\mathcal{B}^{\mathsf{h}}_0 = \frac{5}{6}(\mathcal{B}_{1111} + 2\mathcal{B}_{1122} + \mathcal{B}_{2222})$	
$\mathcal{B}^{\mathsf{h}}_{1c} = \frac{5}{3}(\mathcal{B}_{1113} + \mathcal{B}_{1223})$	
$\mathcal{B}^{\mathsf{h}}_{1s} = \frac{5}{3}(\mathcal{B}_{1123} + \mathcal{B}_{2223})$	
$\mathcal{B}^{\mathsf{h}}_{2c} = \frac{5}{3}(\mathcal{B}_{1111} - \mathcal{B}_{2222})$	$\mathcal{B}^{\mathsf{h}} = \sum_m \mathcal{B}^{\mathsf{h}}_m Y^{4,m}$
$\mathcal{B}^{\mathsf{h}}_{2s} = \frac{10}{3}(\mathcal{B}_{1112} + \mathcal{B}_{1222})$	$\mathcal{B}^{\mathsf{h}}_A = \frac{1}{4}\sum_m \mathcal{B}^{\mathsf{h}}_m X^{4,m}_A$
$\mathcal{B}^{\mathsf{h}}_{3c} = \frac{5}{6}(\mathcal{B}_{1113} - 3\mathcal{B}_{1223})$	$\mathcal{B}^{\mathsf{h}}_{AB} = \frac{1}{6}\sum_m \mathcal{B}^{\mathsf{h}}_m X^{4,m}_{AB}$
$\mathcal{B}^{\mathsf{h}}_{3s} = \frac{5}{6}(3\mathcal{B}_{1123} - \mathcal{B}_{2223})$	
$\mathcal{B}^{\mathsf{h}}_{4c} = \frac{5}{12}(\mathcal{B}_{1111} - 6\mathcal{B}_{1122} + \mathcal{B}_{2222})$	
$\mathcal{B}^{\mathsf{h}}_{4s} = \frac{5}{3}(\mathcal{B}_{1112} - \mathcal{B}_{1222})$	

TABLE X. Harmonic components of type-$\mathcal{E}\mathcal{E}$, even-parity tidal potentials, as defined in Table III. The spherical-harmonic decompositions are $\mathcal{P}^{\mathsf{q}} = \sum_m \mathcal{P}^{\mathsf{q}}_m Y^{2,m}$, $\mathcal{P}^{\mathsf{q}}_A = \frac{1}{2}\sum_m \mathcal{P}^{\mathsf{q}}_m Y^{2,m}_A$, $\mathcal{P}^{\mathsf{q}}_{AB} = \sum_m \mathcal{P}^{\mathsf{q}}_m Y^{2,m}_{AB}$, $\mathcal{P}^{\mathsf{h}} = \sum_m \mathcal{P}^{\mathsf{h}}_m Y^{4,m}$, $\mathcal{P}^{\mathsf{h}}_A = \frac{1}{4}\sum_m \mathcal{P}^{\mathsf{h}}_m Y^{4,m}_A$, and $\mathcal{P}^{\mathsf{h}}_{AB} = \frac{1}{6}\sum_m \mathcal{P}^{\mathsf{h}}_m Y^{4,m}_{AB}$.

$\mathcal{P}^{\mathsf{m}} = 6(\mathcal{E}^{\mathsf{q}}_0)^2 + 2(\mathcal{E}^{\mathsf{q}}_{1c})^2 + 2(\mathcal{E}^{\mathsf{q}}_{1s})^2 + 2(\mathcal{E}^{\mathsf{q}}_{2c})^2 + 2(\mathcal{E}^{\mathsf{q}}_{2s})^2$
$\mathcal{P}^{\mathsf{q}}_0 = -(\mathcal{E}^{\mathsf{q}}_0)^2 - \frac{1}{6}(\mathcal{E}^{\mathsf{q}}_{1c})^2 - \frac{1}{6}(\mathcal{E}^{\mathsf{q}}_{1s})^2 + \frac{1}{3}(\mathcal{E}^{\mathsf{q}}_{2c})^2 + \frac{1}{3}(\mathcal{E}^{\mathsf{q}}_{2s})^2$
$\mathcal{P}^{\mathsf{q}}_{1c} = -\mathcal{E}^{\mathsf{q}}_0\mathcal{E}^{\mathsf{q}}_{1c} + \mathcal{E}^{\mathsf{q}}_{1c}\mathcal{E}^{\mathsf{q}}_{2c} + \mathcal{E}^{\mathsf{q}}_{1s}\mathcal{E}^{\mathsf{q}}_{2s}$
$\mathcal{P}^{\mathsf{q}}_{1s} = -\mathcal{E}^{\mathsf{q}}_0\mathcal{E}^{\mathsf{q}}_{1s} + \mathcal{E}^{\mathsf{q}}_{1c}\mathcal{E}^{\mathsf{q}}_{2s} - \mathcal{E}^{\mathsf{q}}_{1s}\mathcal{E}^{\mathsf{q}}_{2c}$
$\mathcal{P}^{\mathsf{q}}_{2c} = 2\mathcal{E}^{\mathsf{q}}_0\mathcal{E}^{\mathsf{q}}_{2c} + \frac{1}{2}(\mathcal{E}^{\mathsf{q}}_{1c})^2 - \frac{1}{2}(\mathcal{E}^{\mathsf{q}}_{1s})^2$
$\mathcal{P}^{\mathsf{q}}_{2s} = 2\mathcal{E}^{\mathsf{q}}_0\mathcal{E}^{\mathsf{q}}_{2s} + \mathcal{E}^{\mathsf{q}}_{1c}\mathcal{E}^{\mathsf{q}}_{1s}$
$\mathcal{P}^{\mathsf{h}}_0 = \frac{18}{35}(\mathcal{E}^{\mathsf{q}}_0)^2 - \frac{4}{35}(\mathcal{E}^{\mathsf{q}}_{1c})^2 - \frac{4}{35}(\mathcal{E}^{\mathsf{q}}_{1s})^2 + \frac{1}{35}(\mathcal{E}^{\mathsf{q}}_{2c})^2 + \frac{1}{35}(\mathcal{E}^{\mathsf{q}}_{2s})^2$
$\mathcal{P}^{\mathsf{h}}_{1c} = \frac{6}{7}\mathcal{E}^{\mathsf{q}}_0\mathcal{E}^{\mathsf{q}}_{1c} + \frac{1}{7}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{E}^{\mathsf{q}}_{2c} + \frac{1}{7}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{E}^{\mathsf{q}}_{2s}$
$\mathcal{P}^{\mathsf{h}}_{1s} = \frac{6}{7}\mathcal{E}^{\mathsf{q}}_0\mathcal{E}^{\mathsf{q}}_{1s} + \frac{1}{7}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{E}^{\mathsf{q}}_{2s} - \frac{1}{7}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{E}^{\mathsf{q}}_{2c}$
$\mathcal{P}^{\mathsf{h}}_{2c} = \frac{6}{7}\mathcal{E}^{\mathsf{q}}_0\mathcal{E}^{\mathsf{q}}_{2c} - \frac{2}{7}(\mathcal{E}^{\mathsf{q}}_{1c})^2 + \frac{2}{7}(\mathcal{E}^{\mathsf{q}}_{1s})^2$
$\mathcal{P}^{\mathsf{h}}_{2s} = \frac{6}{7}\mathcal{E}^{\mathsf{q}}_0\mathcal{E}^{\mathsf{q}}_{2s} - \frac{4}{7}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{E}^{\mathsf{q}}_{1s}$
$\mathcal{P}^{\mathsf{h}}_{3c} = \frac{1}{2}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{E}^{\mathsf{q}}_{2c} - \frac{1}{2}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{E}^{\mathsf{q}}_{2s}$
$\mathcal{P}^{\mathsf{h}}_{3s} = \frac{1}{2}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{E}^{\mathsf{q}}_{2s} + \frac{1}{2}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{E}^{\mathsf{q}}_{2c}$
$\mathcal{P}^{\mathsf{h}}_{4c} = \frac{1}{2}(\mathcal{E}^{\mathsf{q}}_{2c})^2 - \frac{1}{2}(\mathcal{E}^{\mathsf{q}}_{2s})^2$
$\mathcal{P}^{\mathsf{h}}_{4s} = \mathcal{E}^{\mathsf{q}}_{2c}\mathcal{E}^{\mathsf{q}}_{2s}$

TABLE XI. Harmonic components of type-$\mathcal{B}\mathcal{B}$, even-parity tidal potentials, as defined in Table IV. The spherical-harmonic decompositions are $\mathcal{Q}^{\mathsf{q}} = \sum_m \mathcal{Q}^{\mathsf{q}}_m Y^{2,m}$, $\mathcal{Q}^{\mathsf{q}}_A = \frac{1}{2}\sum_m \mathcal{Q}^{\mathsf{q}}_m Y^{2,m}_A$, $\mathcal{Q}^{\mathsf{q}}_{AB} = \sum_m \mathcal{Q}^{\mathsf{q}}_m Y^{2,m}_{AB}$, $\mathcal{Q}^{\mathsf{h}} = \sum_m \mathcal{Q}^{\mathsf{h}}_m Y^{4,m}$, $\mathcal{Q}^{\mathsf{h}}_A = \frac{1}{4}\sum_m \mathcal{Q}^{\mathsf{h}}_m Y^{4,m}_A$, and $\mathcal{Q}^{\mathsf{h}}_{AB} = \frac{1}{6}\sum_m \mathcal{Q}^{\mathsf{h}}_m Y^{4,m}_{AB}$.

$$\mathcal{Q}^{\mathsf{m}} = 6(\mathcal{B}^{\mathsf{q}}_0)^2 + 2(\mathcal{B}^{\mathsf{q}}_{1c})^2 + 2(\mathcal{B}^{\mathsf{q}}_{1s})^2 + 2(\mathcal{B}^{\mathsf{q}}_{2c})^2 + 2(\mathcal{B}^{\mathsf{q}}_{2s})^2$$

$$\mathcal{Q}^{\mathsf{q}}_0 = -(\mathcal{B}^{\mathsf{q}}_0)^2 - \tfrac{1}{6}(\mathcal{B}^{\mathsf{q}}_{1c})^2 - \tfrac{1}{6}(\mathcal{B}^{\mathsf{q}}_{1s})^2 + \tfrac{1}{3}(\mathcal{B}^{\mathsf{q}}_{2c})^2 + \tfrac{1}{3}(\mathcal{B}^{\mathsf{q}}_{2s})^2$$
$$\mathcal{Q}^{\mathsf{q}}_{1c} = -\mathcal{B}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{1c} + \mathcal{B}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{2c} + \mathcal{B}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{2s}$$
$$\mathcal{Q}^{\mathsf{q}}_{1s} = -\mathcal{B}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{1s} + \mathcal{B}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{2s} - \mathcal{B}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{2c}$$
$$\mathcal{Q}^{\mathsf{q}}_{2c} = 2\mathcal{B}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{2c} + \tfrac{1}{2}(\mathcal{B}^{\mathsf{q}}_{1c})^2 - \tfrac{1}{2}(\mathcal{B}^{\mathsf{q}}_{1s})^2$$
$$\mathcal{Q}^{\mathsf{q}}_{2s} = 2\mathcal{B}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{2s} + \mathcal{B}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{1s}$$

$$\mathcal{Q}^{\mathsf{h}}_0 = \tfrac{18}{35}(\mathcal{B}^{\mathsf{q}}_0)^2 - \tfrac{4}{35}(\mathcal{B}^{\mathsf{q}}_{1c})^2 - \tfrac{4}{35}(\mathcal{B}^{\mathsf{q}}_{1s})^2 + \tfrac{1}{35}(\mathcal{B}^{\mathsf{q}}_{2c})^2 + \tfrac{1}{35}(\mathcal{B}^{\mathsf{q}}_{2s})^2$$
$$\mathcal{Q}^{\mathsf{h}}_{1c} = \tfrac{6}{7}\mathcal{B}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{1c} + \tfrac{1}{7}\mathcal{B}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{2c} + \tfrac{1}{7}\mathcal{B}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{2s}$$
$$\mathcal{Q}^{\mathsf{h}}_{1s} = \tfrac{6}{7}\mathcal{B}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{1s} + \tfrac{1}{7}\mathcal{B}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{2s} - \tfrac{1}{7}\mathcal{B}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{2c}$$
$$\mathcal{Q}^{\mathsf{h}}_{2c} = \tfrac{6}{7}\mathcal{B}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{2c} - \tfrac{2}{7}(\mathcal{B}^{\mathsf{q}}_{1c})^2 + \tfrac{2}{7}(\mathcal{B}^{\mathsf{q}}_{1s})^2$$
$$\mathcal{Q}^{\mathsf{h}}_{2s} = \tfrac{6}{7}\mathcal{B}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{2s} - \tfrac{4}{7}\mathcal{B}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{1s}$$
$$\mathcal{Q}^{\mathsf{h}}_{3c} = \tfrac{1}{2}\mathcal{B}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{2c} - \tfrac{1}{2}\mathcal{B}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{2s}$$
$$\mathcal{Q}^{\mathsf{h}}_{3s} = \tfrac{1}{2}\mathcal{B}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{2s} + \tfrac{1}{2}\mathcal{B}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{2c}$$
$$\mathcal{Q}^{\mathsf{h}}_{4c} = \tfrac{1}{2}(\mathcal{B}^{\mathsf{q}}_{2c})^2 - \tfrac{1}{2}(\mathcal{B}^{\mathsf{q}}_{2s})^2$$
$$\mathcal{Q}^{\mathsf{h}}_{4s} = \mathcal{B}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{2s}$$

$$X^{lm}_A := -\epsilon_A{}^B D_B Y^{lm}, \tag{2.21a}$$

$$X^{lm}_{AB} := -\frac{1}{2}(\epsilon_A{}^C D_B + \epsilon_B{}^C D_A) D_C Y^{lm} = 0; \tag{2.21b}$$

the tensorial harmonics X^{lm}_{AB} also are tracefree: $\Omega^{AB} X^{lm}_{AB} = 0$. The decomposition of the tidal potentials in spherical harmonics is presented in Tables VIII, IX, X, XI, XII, and XIII. A derivation of these results is presented in Appendix A.

III. GEOMETRY OF A DEFORMED BLACK HOLE

We construct the metric of a tidally deformed black hole in two steps. In the first step we continue to think of a smooth timelike geodesic γ in a vacuum region of an arbitrary spacetime, and we construct the metric of this (background) spacetime in a neighborhood of the world line. We denote this neighborhood by $\mathcal{N}$, and we require that it be small in comparison with the length scale $\mathcal{R}$ that characterizes the tidal environment; we demand that

$$r \ll \mathcal{R}, \tag{3.1}$$

where r is a radial coordinate (to be introduced below) that measures distance to the world line. In the second step we insert a black hole of mass M into the background spacetime, and place it on the world line γ. For the construction to be successful it is necessary that the world tube traced by the black hole fit comfortably within $\mathcal{N}$, and this can be achieved when

$$M \ll \mathcal{R}. \tag{3.2}$$

This condition implies both that the black hole is weakly perturbed by the tidal environment, and that the world tube is small when viewed on the scale $\mathcal{R}$ of the external spacetime. In this situation it makes (approximate) sense to say that the black hole moves on a world line γ.

The metric of the background spacetime is presented in Sec. III B, and the metric of the black-hole spacetime is presented in Sec. III C. These subsections contain what is truly a presentation of the metrics, and there the reader will find no trace of a derivation of our results. The calculations that lead to the metrics are quite lengthy, and derivations are relegated to Secs. V and VI, and Appendix D. We begin in Sec. III A with a description of our coordinate systems.

A. Light-cone coordinates

We work with a system (v, r, θ, ϕ) that is specifically tailored to describe the geometry of light-cone surfaces. We refer to these as *light-cone coordinates*. In the case of the background spacetime (Fig. 1), we consider past light cones that converge toward the world line γ, so that the apex of each cone coincides with a point on the world line. In $\mathcal{N}$ the light cones provide spacetime with a foliation by null hypersurfaces, and each light cone is generated by a

TABLE XII. Harmonic components of type-$\mathcal{E}\mathcal{B}$, even-parity tidal potentials, as defined in Table V. The spherical-harmonic decompositions are $\mathcal{G}^{\mathsf{d}} = \sum_m \mathcal{G}^{\mathsf{d}}_m Y^{1,m}$, $\mathcal{G}^{\mathsf{d}}_A = \sum_m \mathcal{G}^{\mathsf{d}}_m Y^{1,m}_A$, $\mathcal{G}^{\mathsf{o}} = \sum_m \mathcal{G}^{\mathsf{o}}_m Y^{3,m}$, $\mathcal{G}^{\mathsf{o}}_A = \frac{1}{3}\sum_m \mathcal{G}^{\mathsf{o}}_m Y^{3,m}_A$, and $\mathcal{G}^{\mathsf{o}}_{AB} = \frac{1}{3}\sum_m \mathcal{G}^{\mathsf{o}}_m Y^{3,m}_{AB}$.

$$\mathcal{G}^{\mathsf{d}}_0 = \mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{1s} - \mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{1c} + 2\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{2s} - 2\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{2c}$$
$$\mathcal{G}^{\mathsf{d}}_{1c} = (3\mathcal{E}^{\mathsf{q}}_0 - \mathcal{E}^{\mathsf{q}}_{2c})\mathcal{B}^{\mathsf{q}}_{1s} - \mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{2s} - \mathcal{E}^{\mathsf{q}}_{1s}(3\mathcal{B}^{\mathsf{q}}_0 - \mathcal{B}^{\mathsf{q}}_{2c}) + \mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{1c}$$
$$\mathcal{G}^{\mathsf{d}}_{1s} = -(3\mathcal{E}^{\mathsf{q}}_0 + \mathcal{E}^{\mathsf{q}}_{2c})\mathcal{B}^{\mathsf{q}}_{1c} + \mathcal{E}^{\mathsf{q}}_{1c}(3\mathcal{B}^{\mathsf{q}}_0 + \mathcal{B}^{\mathsf{q}}_{2c}) + \mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{2s} - \mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{1s}$$

$$\mathcal{G}^{\mathsf{o}}_0 = -\tfrac{2}{5}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{1s} + \tfrac{2}{5}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{1c} + \tfrac{1}{5}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{2s} - \tfrac{1}{5}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{2c}$$
$$\mathcal{G}^{\mathsf{o}}_{1c} = -\tfrac{2}{5}\mathcal{E}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{1s} - \tfrac{1}{5}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{2s} + \tfrac{1}{5}\mathcal{E}^{\mathsf{q}}_{1s}(2\mathcal{B}^{\mathsf{q}}_0 + \mathcal{B}^{\mathsf{q}}_{2c}) - \tfrac{1}{5}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{1s} + \tfrac{1}{5}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{1c}$$
$$\mathcal{G}^{\mathsf{o}}_{1s} = \tfrac{2}{5}\mathcal{E}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{1c} - \tfrac{1}{5}\mathcal{E}^{\mathsf{q}}_{1c}(2\mathcal{B}^{\mathsf{q}}_0 - \mathcal{B}^{\mathsf{q}}_{2c}) + \tfrac{1}{5}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{2s} - \tfrac{1}{5}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{1c} - \tfrac{1}{5}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{1s}$$
$$\mathcal{G}^{\mathsf{o}}_{2c} = \mathcal{E}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{2s} - \mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_0$$
$$\mathcal{G}^{\mathsf{o}}_{2s} = -\mathcal{E}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{2c} + \mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_0$$
$$\mathcal{G}^{\mathsf{o}}_{3c} = -\tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{2s} - \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{2c} + \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{1s} + \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{1c}$$
$$\mathcal{G}^{\mathsf{o}}_{3s} = \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{2c} - \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{2s} - \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{1c} + \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{1s}$$

TABLE XIII. Harmonic components of type-$\mathcal{E}\mathcal{B}$, odd-parity tidal potentials, as defined in Table VI. The spherical-harmonic decompositions are $\mathcal{H}^{\mathsf{q}}_A = \frac{1}{2}\sum_m \mathcal{H}^{\mathsf{q}}_m X^{2,m}_A$, $\mathcal{H}^{\mathsf{q}}_{AB} = \sum_m \mathcal{H}^{\mathsf{q}}_m X^{2,m}_{AB}$, $\mathcal{H}^{\mathsf{h}}_A = \frac{1}{4}\sum_m \mathcal{H}^{\mathsf{h}}_m X^{4,m}_A$, and $\mathcal{H}^{\mathsf{h}}_{AB} = \frac{1}{6}\sum_m \mathcal{H}^{\mathsf{h}}_m X^{4,m}_{AB}$.

$$\begin{aligned}
\mathcal{H}^{\mathsf{q}}_0 &= -\mathcal{E}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_0 - \tfrac{1}{6}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{1c} - \tfrac{1}{6}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{1s} + \tfrac{1}{3}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{2c} + \tfrac{1}{3}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{2s} \\
\mathcal{H}^{\mathsf{q}}_{1c} &= -\tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{1c} - \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{1c}(\mathcal{B}^{\mathsf{q}}_0 - \mathcal{B}^{\mathsf{q}}_{2c}) + \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{2s} + \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{1c} + \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{1s} \\
\mathcal{H}^{\mathsf{q}}_{1s} &= -\tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{1s} + \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{2s} - \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{1s}(\mathcal{B}^{\mathsf{q}}_0 + \mathcal{B}^{\mathsf{q}}_{2c}) - \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{1s} + \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{1c} \\
\mathcal{H}^{\mathsf{q}}_{2c} &= \mathcal{E}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{2c} + \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{1c} - \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{1s} + \mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_0 \\
\mathcal{H}^{\mathsf{q}}_{2s} &= \mathcal{E}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{2s} + \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{1s} + \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{1c} + \mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_0
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}^{\mathsf{h}}_0 &= \tfrac{18}{35}\mathcal{E}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_0 - \tfrac{4}{35}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{1c} - \tfrac{4}{35}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{1s} + \tfrac{1}{35}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{2c} + \tfrac{1}{35}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{2s} \\
\mathcal{H}^{\mathsf{h}}_{1c} &= \tfrac{3}{7}\mathcal{E}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{1c} + \tfrac{1}{14}\mathcal{E}^{\mathsf{q}}_{1c}(6\mathcal{B}^{\mathsf{q}}_0 + \mathcal{B}^{\mathsf{q}}_{2c}) + \tfrac{1}{14}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{2s} + \tfrac{1}{14}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{1c} + \tfrac{1}{14}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{1s} \\
\mathcal{H}^{\mathsf{h}}_{1s} &= \tfrac{3}{7}\mathcal{E}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{1s} + \tfrac{1}{14}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{2s} + \tfrac{1}{14}\mathcal{E}^{\mathsf{q}}_{1s}(6\mathcal{B}^{\mathsf{q}}_0 - \mathcal{B}^{\mathsf{q}}_{2c}) - \tfrac{1}{14}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{1s} + \tfrac{1}{14}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{1c} \\
\mathcal{H}^{\mathsf{h}}_{2c} &= \tfrac{3}{7}\mathcal{E}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{2c} - \tfrac{2}{7}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{1c} + \tfrac{2}{7}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{1s} + \tfrac{3}{7}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_0 \\
\mathcal{H}^{\mathsf{h}}_{2s} &= \tfrac{3}{7}\mathcal{E}^{\mathsf{q}}_0\mathcal{B}^{\mathsf{q}}_{2s} - \tfrac{2}{7}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{1s} - \tfrac{2}{7}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{1c} + \tfrac{3}{7}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_0 \\
\mathcal{H}^{\mathsf{h}}_{3c} &= \tfrac{1}{4}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{2c} - \tfrac{1}{4}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{2s} + \tfrac{1}{4}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{1c} - \tfrac{1}{4}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{1s} \\
\mathcal{H}^{\mathsf{h}}_{3s} &= \tfrac{1}{4}\mathcal{E}^{\mathsf{q}}_{1c}\mathcal{B}^{\mathsf{q}}_{2s} + \tfrac{1}{4}\mathcal{E}^{\mathsf{q}}_{1s}\mathcal{B}^{\mathsf{q}}_{2c} + \tfrac{1}{4}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{1s} + \tfrac{1}{4}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{1c} \\
\mathcal{H}^{\mathsf{h}}_{4c} &= \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{2c} - \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{2s} \\
\mathcal{H}^{\mathsf{h}}_{4s} &= \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{2c}\mathcal{B}^{\mathsf{q}}_{2s} + \tfrac{1}{2}\mathcal{E}^{\mathsf{q}}_{2s}\mathcal{B}^{\mathsf{q}}_{2c}
\end{aligned}$$

congruence of null geodesics. (We assume that caustics do not develop within $\mathcal{N}$, except at the apex of each light cone.) The coordinates are intimately tied to the light cones and their generators:

(1) The advanced-time coordinate v is constant on each light cone, and its value on a given light cone is equal to proper time τ on the corresponding point of the world line.

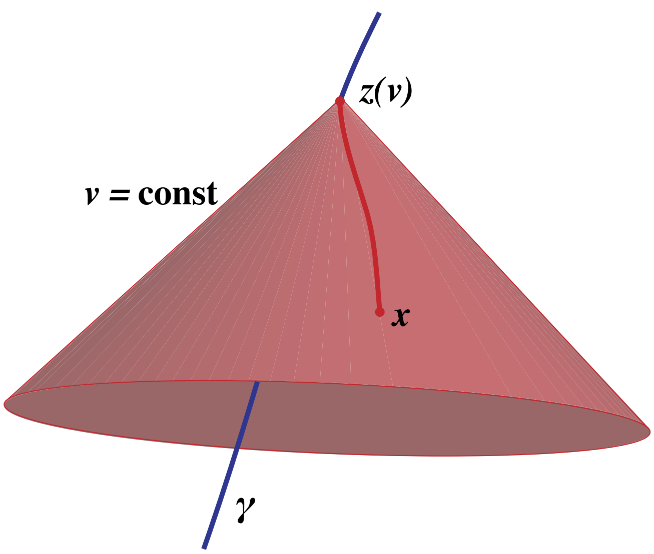

FIG. 1 (color online). Light-cone coordinates centered on a world line γ. The figure shows a light cone $v =$ constant that intersects the world line at the point $z(v)$. It shows also one of the light cone's generators, along which θ^A is constant; the affine parameter r decreases to zero as the generator approaches the world line.

(2) The angular coordinates $\theta^A = (\theta, \phi)$ are constant on the null generators of each light cone; the angles refer to a set of axes that are aligned with the basis vectors e^α_a introduced in Sec. II A.

(3) The radial coordinate r is an affine parameter on the null generators of each light cone, normalized in such a way that the metric takes a Minkowski form in the immediate vicinity of the world line; note that r decreases to zero as the generators converge toward the world line.

To illustrate the meaning of the coordinates we examine the simple case of an observer at rest in flat spacetime. The observer's proper time τ is equal to coordinate time t, the advanced-time coordinate is obtained by the transformation $v = t + r$, and (r, θ, ϕ) are obtained in the usual way from the Lorentzian coordinates x^a. The metric is $ds^2 = -dv^2 + 2dvdr + r^2 d\Omega^2$, with $d\Omega^2 := d\theta^2 + \sin^2\theta d\phi^2$ denoting the metric on the unit two-sphere. It is easy to verify that the light-cone coordinates satisfy all the properties listed previously.

In the case of the black hole spacetime (Fig. 2), the light cones no longer converge toward a world line. Instead they converge toward the horizon, which traces a world tube in spacetime. They still, however, provide $\mathcal{N}$ with a foliation by null hypersurfaces, and each light cone is still generated by a congruence of null geodesics. The light-cone coordinates keep most of the properties listed previously:

(1) The advanced-time coordinate v is constant on each light cone.

(2) The angular coordinates $\theta^A = (\theta, \phi)$ are constant on the null generators of each light cone.

(3) The radial coordinate r is an affine parameter on the null generators of each light cone.

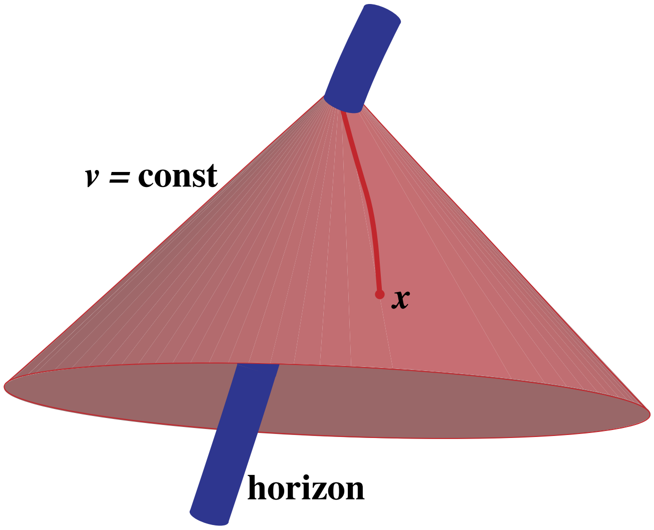

FIG. 2 (color online). Light-cone coordinates centered on a black hole. The figure shows the world tube traced by the black-hole horizon. It shows also the light cone $v =$ constant and one of its generators. As in Fig. 1, the angles θ^A are constant and the affine parameter r decreases on the generator.

Since the calibration by the world line is no longer available, the coordinates have lost some of their rigidity. We can, however, restore most of this rigidity by imagining two spacetimes foliated by past light cones. The first is the background spacetime, with its fully specified set of light-cone coordinates. The second is the black-hole spacetime, with its own set of light-cone coordinates. Far from the black hole, where r is much larger than M but still much smaller than $\mathcal{R}$, the gravitational influence of the black hole is small, and light rays behave there just as they do in the background spacetime. We can therefore tune the black-hole coordinates so that the asymptotic description of the null generators agrees with the background description. This correspondence does not permit the complete specification of the coordinates; there remains a limited freedom to redefine the coordinates without changing their geometrical meaning. We have thoroughly exploited this freedom to simplify the form of the black-hole metric.

The light-cone coordinates are best described in terms of the radial distance r and the angles θ^A. It is useful, however, to introduce also a variant of the light-cone coordinates that involves the quasi-Cartesian system x^a instead of the quasispherical system (r, θ^A). The coordinates x^a are defined in the usual way by Eq. (2.16). In the flat-spacetime example examined previously, we find that the components of the Minkowski metric are given by $\eta_{vv} = -1$, $\eta_{va} = \Omega_a$, and $\eta_{ab} = \gamma_{ab}$ when presented in the light-cone coordinates (v, x^a). It should be noted that the metric is mildly singular at $x^a = 0$, because its value depends on the ambiguous direction of the longitudinal vector Ω^a. This singularity persists in the background spacetime, and dealing with it requires some care; but it is not a serious obstacle to any computation.

B. Background spacetime

The metric of a vacuum region of spacetime that surrounds a timelike geodesic γ can be presented in the light-cone coordinates (v, x^a) introduced in the preceding subsection and expressed as an expansion in powers of $r/\mathcal{R} \ll 1$. It is given by

$$\begin{aligned} g_{vv} = {}& -1 - r^2\mathcal{E}^{\mathsf{q}} + \frac{1}{3}r^3\dot{\mathcal{E}}^{\mathsf{q}} - \frac{1}{3}r^3\mathcal{E}^{\mathsf{o}} - \frac{2}{21}r^4\ddot{\mathcal{E}}^{\mathsf{q}} + \frac{1}{6}r^4\dot{\mathcal{E}}^{\mathsf{o}} \\ & - \frac{1}{12}r^4\mathcal{E}^{\mathsf{h}} + \frac{1}{15}r^4(\mathcal{P}^{\mathsf{m}} + \mathcal{Q}^{\mathsf{m}}) + \frac{2}{15}r^4\mathcal{G}^{\mathsf{d}} + \frac{2}{7}r^4\mathcal{Q}^{\mathsf{q}} \\ & + \frac{2}{3}r^4\mathcal{G}^{\mathsf{o}} - \frac{1}{3}r^4(\mathcal{P}^{\mathsf{h}} + \mathcal{Q}^{\mathsf{h}}) + O(5), \end{aligned} \tag{3.3a}$$

$$\begin{aligned} g_{va} = {}& \Omega_a - \frac{2}{3}r^2(\mathcal{E}^{\mathsf{q}}_a - \mathcal{B}^{\mathsf{q}}_a) + \frac{1}{3}r^3(\dot{\mathcal{E}}^{\mathsf{q}}_a - \dot{\mathcal{B}}^{\mathsf{q}}_a) \\ & - \frac{1}{4}r^3(\mathcal{E}^{\mathsf{o}}_a - \mathcal{B}^{\mathsf{o}}_a) - \frac{8}{63}r^4(\ddot{\mathcal{E}}^{\mathsf{q}}_a - \ddot{\mathcal{B}}^{\mathsf{q}}_a) \\ & + \frac{1}{6}r^4(\dot{\mathcal{E}}^{\mathsf{o}}_a - \dot{\mathcal{B}}^{\mathsf{o}}_a) - \frac{1}{15}r^4(\mathcal{E}^{\mathsf{h}}_a - \mathcal{B}^{\mathsf{h}}_a) - \frac{8}{75}r^4\mathcal{G}^{\mathsf{d}}_a \\ & + \frac{8}{21}r^4\mathcal{H}^{\mathsf{q}}_a + \frac{4}{105}r^4(\mathcal{P}^{\mathsf{q}}_a + 11\mathcal{Q}^{\mathsf{q}}_a) + \frac{2}{5}r^4\mathcal{G}^{\mathsf{o}}_a \\ & - \frac{2}{15}r^4(\mathcal{P}^{\mathsf{h}}_a + \mathcal{Q}^{\mathsf{h}}_a) + O(5), \end{aligned} \tag{3.3b}$$

$$\begin{aligned} g_{ab} = {}& \gamma_{ab} - \frac{1}{3}r^2(\mathcal{E}^{\mathsf{q}}_{ab} - \mathcal{B}^{\mathsf{q}}_{ab}) + \frac{5}{18}r^3(\dot{\mathcal{E}}^{\mathsf{q}}_{ab} - \dot{\mathcal{B}}^{\mathsf{q}}_{ab}) \\ & - \frac{1}{6}r^3(\mathcal{E}^{\mathsf{o}}_{ab} - \mathcal{B}^{\mathsf{o}}_{ab}) - \frac{1}{7}r^4(\ddot{\mathcal{E}}^{\mathsf{q}}_{ab} - \ddot{\mathcal{B}}^{\mathsf{q}}_{ab}) \\ & + \frac{3}{20}r^4(\dot{\mathcal{E}}^{\mathsf{o}}_{ab} - \dot{\mathcal{B}}^{\mathsf{o}}_{ab}) - \frac{1}{20}r^4(\mathcal{E}^{\mathsf{h}}_{ab} - \mathcal{B}^{\mathsf{h}}_{ab}) \\ & + \frac{8}{225}r^4\gamma_{ab}(\mathcal{P}^{\mathsf{m}} + \mathcal{Q}^{\mathsf{m}}) + \frac{32}{225}r^4\gamma_{ab}\mathcal{G}^{\mathsf{d}} \\ & - \frac{16}{105}r^4\gamma_{ab}(\mathcal{P}^{\mathsf{q}} + \mathcal{Q}^{\mathsf{q}}) - \frac{3}{14}r^4(\mathcal{P}^{\mathsf{q}}_{ab} - \mathcal{Q}^{\mathsf{q}}_{ab}) \\ & + \frac{3}{7}r^4\mathcal{H}^{\mathsf{q}}_{ab} - \frac{8}{45}r^4\gamma_{ab}\mathcal{G}^{\mathsf{o}} + \frac{2}{45}r^4\gamma_{ab}(\mathcal{P}^{\mathsf{h}} + \mathcal{Q}^{\mathsf{h}}) \\ & + O(5). \end{aligned} \tag{3.3c}$$

The metric features the tidal potentials encountered in Sec. II D, and the notation $O(5)$ indicates that the error terms are of order $(r/\mathcal{R})^5$. The world line is situated at $x^a = 0$, and v is proper time on γ.

The terms of order $(r/\mathcal{R})^2$ involve the quadrupole tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$, which are now expressed as functions of advanced time v instead of proper time τ. At order $(r/\mathcal{R})^3$ we find terms involving $\dot{\mathcal{E}}_{ab}$ and $\dot{\mathcal{B}}_{ab}$, the time derivatives of the quadrupole moments, as well as the octupole tidal moments $\mathcal{E}_{abc}$ and $\mathcal{B}_{abc}$. And at order $(r/\mathcal{R})^4$ we see occurrences of $\ddot{\mathcal{E}}_{ab}$, $\ddot{\mathcal{B}}_{ab}$, $\dot{\mathcal{E}}_{abc}$, $\dot{\mathcal{B}}_{abc}$, the hexadecapole tidal moments $\mathcal{E}_{abcd}$, $\mathcal{B}_{abcd}$, and bilinear combinations of the quadrupole moments. We see that the metric involves the majority of the tidal potentials listed in Tables I, II, III, IV, V, and VI. Exceptions are $\mathcal{P}^{\mathsf{q}}$, $\mathcal{H}^{\mathsf{h}}_a$, $\mathcal{G}^{\mathsf{o}}_{ab}$, $\mathcal{P}^{\mathsf{h}}_{ab}$, $\mathcal{Q}^{\mathsf{h}}_{ab}$, and $\mathcal{H}^{\mathsf{h}}_{ab}$; these could have occurred in the metric, but they happen to be ruled out by the Einstein field equations.

A useful way to view the metric of Eqs. (3.3) is to recognize that it possesses the correct number of freely-specifiable functions of time to describe correctly, and in sufficient generality, the geometry of a vacuum region of spacetime in a neighborhood $\mathcal{N}$ of a timelike geodesic. The functions of time are contained in the tidal moments; we have 10 of them in $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$, 14 in $\mathcal{E}_{abc}$ and $\mathcal{B}_{abc}$, and 18 in $\mathcal{E}_{abcd}$ and $\mathcal{B}_{abcd}$. The total is 42, the correct number for a metric constructed through order $(r/\mathcal{R})^4$. These functions encode meaningful information about the spacetime geometry; as we saw in Sec. II A they represent components of the Weyl tensor and its derivatives evaluated on the world line γ.

In quasispherical coordinates (v, r, θ^A) the nonvanishing components of the metric are

$$g_{vv} = -1 - r^2\mathcal{E}^{\mathsf{q}} + \frac{1}{3}r^3\dot{\mathcal{E}}^{\mathsf{q}} - \frac{1}{3}r^3\mathcal{E}^{\mathsf{o}} - \frac{2}{21}r^4\ddot{\mathcal{E}}^{\mathsf{q}} + \frac{1}{6}r^4\dot{\mathcal{E}}^{\mathsf{o}} - \frac{1}{12}r^4\mathcal{E}^{\mathsf{h}} + \frac{1}{15}r^4(\mathcal{P}^{\mathsf{m}} + \mathcal{Q}^{\mathsf{m}}) + \frac{2}{15}r^4\mathcal{G}^{\mathsf{d}} + \frac{2}{7}r^4\mathcal{Q}^{\mathsf{q}} + \frac{2}{3}r^4\mathcal{G}^{\mathsf{o}} - \frac{1}{3}r^4(\mathcal{P}^{\mathsf{h}} + \mathcal{Q}^{\mathsf{h}}) + O(5), \tag{3.4a}$$

$$g_{vr} = 1, \tag{3.4b}$$

$$g_{vA} = -\frac{2}{3}r^3(\mathcal{E}^{\mathsf{q}}_A - \mathcal{B}^{\mathsf{q}}_A) + \frac{1}{3}r^4(\dot{\mathcal{E}}^{\mathsf{q}}_A - \dot{\mathcal{B}}^{\mathsf{q}}_A) - \frac{1}{4}r^4(\mathcal{E}^{\mathsf{o}}_A - \mathcal{B}^{\mathsf{o}}_A) - \frac{8}{63}r^5(\ddot{\mathcal{E}}^{\mathsf{q}}_A - \ddot{\mathcal{B}}^{\mathsf{q}}_A) + \frac{1}{6}r^5(\dot{\mathcal{E}}^{\mathsf{o}}_A - \dot{\mathcal{B}}^{\mathsf{o}}_A) - \frac{1}{15}r^5(\mathcal{E}^{\mathsf{h}}_A - \mathcal{B}^{\mathsf{h}}_A) - \frac{8}{75}r^5\mathcal{G}^{\mathsf{d}}_A + \frac{8}{21}r^5\mathcal{H}^{\mathsf{q}}_A + \frac{4}{105}r^5(\mathcal{P}^{\mathsf{q}}_A + 11\mathcal{Q}^{\mathsf{q}}_A) + \frac{2}{5}r^5\mathcal{G}^{\mathsf{o}}_A - \frac{2}{15}r^5(\mathcal{P}^{\mathsf{h}}_A + \mathcal{Q}^{\mathsf{h}}_A) + rO(5), \tag{3.4c}$$

$$g_{AB} = r^2\Omega_{AB} - \frac{1}{3}r^4(\mathcal{E}^{\mathsf{q}}_{AB} - \mathcal{B}^{\mathsf{q}}_{AB}) + \frac{5}{18}r^5(\dot{\mathcal{E}}^{\mathsf{q}}_{AB} - \dot{\mathcal{B}}^{\mathsf{q}}_{AB}) - \frac{1}{6}r^5(\mathcal{E}^{\mathsf{o}}_{AB} - \mathcal{B}^{\mathsf{o}}_{AB}) - \frac{1}{7}r^6(\ddot{\mathcal{E}}^{\mathsf{q}}_{AB} - \ddot{\mathcal{B}}^{\mathsf{q}}_{AB}) + \frac{3}{20}r^6(\dot{\mathcal{E}}^{\mathsf{o}}_{AB} - \dot{\mathcal{B}}^{\mathsf{o}}_{AB}) - \frac{1}{20}r^6(\mathcal{E}^{\mathsf{h}}_{AB} - \mathcal{B}^{\mathsf{h}}_{AB}) + \frac{8}{225}r^6\Omega_{AB}(\mathcal{P}^{\mathsf{m}} + \mathcal{Q}^{\mathsf{m}}) + \frac{32}{225}r^6\Omega_{AB}\mathcal{G}^{\mathsf{d}} - \frac{16}{105}r^6\Omega_{AB}(\mathcal{P}^{\mathsf{q}} + \mathcal{Q}^{\mathsf{q}}) - \frac{3}{14}r^6(\mathcal{P}^{\mathsf{q}}_{AB} - \mathcal{Q}^{\mathsf{q}}_{AB}) + \frac{3}{7}r^6\mathcal{H}^{\mathsf{q}}_{AB} - \frac{8}{45}r^6\Omega_{AB}\mathcal{G}^{\mathsf{o}} + \frac{2}{45}r^6\Omega_{AB}(\mathcal{P}^{\mathsf{h}} + \mathcal{Q}^{\mathsf{h}}) + r^2O(5). \tag{3.4d}$$

This can be obtained from Eq. (3.3) by applying the transformation rules described in Sec. II E. The statements that $g_{vr} = 1$ and $g_{rA} = 0$ are exact, and they follow from the light-cone nature of the coordinate system. Here the metric features the tidal potentials listed in Tables VIII, IX, X, XI, XII, and XIII. When dealing with the angular coordinates it is convenient to rely on the decomposition of the potentials in scalar, vector, and tensor harmonics.

C. Black-hole spacetime

The metric of a tidally deformed black hole resembles closely the background metric presented in the preceding subsection. In the quasi-Cartesian coordinates (v, x^a) it is given by $(f := 1 - 2M/r)$

$$g_{vv} = -f - r^2e^{\mathsf{q}}_1\mathcal{E}^{\mathsf{q}} + \frac{1}{3}r^3e^{\mathsf{q}}_2\dot{\mathcal{E}}^{\mathsf{q}} - \frac{1}{3}r^3e^{\mathsf{o}}_1\mathcal{E}^{\mathsf{o}} - \frac{2}{21}r^4e^{\mathsf{q}}_3\ddot{\mathcal{E}}^{\mathsf{q}} + \frac{1}{6}r^4e^{\mathsf{o}}_2\dot{\mathcal{E}}^{\mathsf{o}} - \frac{1}{12}r^4e^{\mathsf{h}}_1\mathcal{E}^{\mathsf{h}} + \frac{1}{15}r^4(p^{\mathsf{m}}_1\mathcal{P}^{\mathsf{m}} + q^{\mathsf{m}}_1\mathcal{Q}^{\mathsf{m}}) + \frac{2}{15}r^4g^{\mathsf{d}}_1\mathcal{G}^{\mathsf{d}} + \frac{2}{7}r^4(p^{\mathsf{q}}_1\mathcal{P}^{\mathsf{q}} + q^{\mathsf{q}}_1\mathcal{Q}^{\mathsf{q}}) + \frac{2}{3}r^4g^{\mathsf{o}}_1\mathcal{G}^{\mathsf{o}} - \frac{1}{3}r^4(p^{\mathsf{h}}_1\mathcal{P}^{\mathsf{h}} + q^{\mathsf{h}}_1\mathcal{Q}^{\mathsf{h}}) + O(5), \tag{3.5a}$$

$$g_{va} = \Omega_a - \frac{2}{3}r^2(e^{\mathsf{q}}_4\mathcal{E}^{\mathsf{q}}_a - b^{\mathsf{q}}_4\mathcal{B}^{\mathsf{q}}_a) + \frac{1}{3}r^3(e^{\mathsf{q}}_5\dot{\mathcal{E}}^{\mathsf{q}}_a - b^{\mathsf{q}}_5\dot{\mathcal{B}}^{\mathsf{q}}_a) - \frac{1}{4}r^3(e^{\mathsf{o}}_4\mathcal{E}^{\mathsf{o}}_a - b^{\mathsf{o}}_4\mathcal{B}^{\mathsf{o}}_a) - \frac{8}{63}r^4(e^{\mathsf{q}}_6\ddot{\mathcal{E}}^{\mathsf{q}}_a - b^{\mathsf{q}}_6\ddot{\mathcal{B}}^{\mathsf{q}}_a) + \frac{1}{6}r^4(e^{\mathsf{o}}_5\dot{\mathcal{E}}^{\mathsf{o}}_a - b^{\mathsf{o}}_5\dot{\mathcal{B}}^{\mathsf{o}}_a) - \frac{1}{15}r^4(e^{\mathsf{h}}_4\mathcal{E}^{\mathsf{h}}_a - b^{\mathsf{h}}_4\mathcal{B}^{\mathsf{h}}_a) - \frac{8}{75}r^4g^{\mathsf{d}}_2\mathcal{G}^{\mathsf{d}}_a + \frac{8}{21}r^4h^{\mathsf{q}}_2\mathcal{H}^{\mathsf{q}}_a + \frac{4}{105}r^4(p^{\mathsf{q}}_2\mathcal{P}^{\mathsf{q}}_a + 11q^{\mathsf{q}}_2\mathcal{Q}^{\mathsf{q}}_a) + \frac{2}{5}r^4g^{\mathsf{o}}_2\mathcal{G}^{\mathsf{o}}_a + r^4h^{\mathsf{h}}_2\mathcal{H}^{\mathsf{h}}_a - \frac{2}{15}r^4(p^{\mathsf{h}}_2\mathcal{P}^{\mathsf{h}}_a + q^{\mathsf{h}}_2\mathcal{Q}^{\mathsf{h}}_a) + O(5), \tag{3.5b}$$

$$g_{ab} = \gamma_{ab} - \frac{1}{3}r^2(e^{\mathsf{q}}_7\mathcal{E}^{\mathsf{q}}_{ab} - b^{\mathsf{q}}_7\mathcal{B}^{\mathsf{q}}_{ab}) + \frac{5}{18}r^3(e^{\mathsf{q}}_8\dot{\mathcal{E}}^{\mathsf{q}}_{ab} - b^{\mathsf{q}}_8\dot{\mathcal{B}}^{\mathsf{q}}_{ab}) - \frac{1}{6}r^3(e^{\mathsf{o}}_7\mathcal{E}^{\mathsf{o}}_{ab} - b^{\mathsf{o}}_7\mathcal{B}^{\mathsf{o}}_{ab}) - \frac{1}{7}r^4(e^{\mathsf{q}}_9\ddot{\mathcal{E}}^{\mathsf{q}}_{ab} - b^{\mathsf{q}}_9\ddot{\mathcal{B}}^{\mathsf{q}}_{ab}) + \frac{3}{20}r^4(e^{\mathsf{o}}_8\dot{\mathcal{E}}^{\mathsf{o}}_{ab} - b^{\mathsf{o}}_8\dot{\mathcal{B}}^{\mathsf{o}}_{ab}) - \frac{1}{20}r^4(e^{\mathsf{h}}_7\mathcal{E}^{\mathsf{h}}_{ab} - b^{\mathsf{h}}_7\mathcal{B}^{\mathsf{h}}_{ab}) + \frac{8}{225}r^4\gamma_{ab}(p^{\mathsf{m}}_3\mathcal{P}^{\mathsf{m}} + q^{\mathsf{m}}_3\mathcal{Q}^{\mathsf{m}}) + \frac{32}{225}r^4\gamma_{ab}g^{\mathsf{d}}_3\mathcal{G}^{\mathsf{d}} - \frac{16}{105}r^4\gamma_{ab}(p^{\mathsf{q}}_3\mathcal{P}^{\mathsf{q}} + q^{\mathsf{q}}_3\mathcal{Q}^{\mathsf{q}}) - \frac{3}{14}r^4(p^{\mathsf{q}}_4\mathcal{P}^{\mathsf{q}}_{ab} - q^{\mathsf{q}}_4\mathcal{Q}^{\mathsf{q}}_{ab}) + \frac{3}{7}r^4h^{\mathsf{q}}_3\mathcal{H}^{\mathsf{q}}_{ab} - \frac{8}{45}r^4\gamma_{ab}g^{\mathsf{o}}_3\mathcal{G}^{\mathsf{o}} + r^4g^{\mathsf{o}}_4\mathcal{G}^{\mathsf{o}}_{ab} + \frac{2}{45}r^4\gamma_{ab}(p^{\mathsf{h}}_3\mathcal{P}^{\mathsf{h}} + q^{\mathsf{h}}_3\mathcal{Q}^{\mathsf{h}}) + r^4(p^{\mathsf{h}}_4\mathcal{P}^{\mathsf{h}}_{ab} + q^{\mathsf{h}}_4\mathcal{Q}^{\mathsf{h}}_{ab}) + r^4h^{\mathsf{h}}_3\mathcal{H}^{\mathsf{h}}_{ab} + O(5). \tag{3.5c}$$

TABLE XIV. Radial functions: linear potentials. The radial functions are expressed in terms of $x := r/(2M)$ and $f := 1 - 1/x$. The dilogarithm function is defined as $\mathrm{dilog}x = \int_1^x dt \log(t)/(1-t)$, with $\log x$ denoting the natural logarithm. All radial functions vanish at $r = 2M$, except for $e_7^{\mathsf{q}} = \frac{1}{2}$, $e_7^{\mathsf{o}} = \frac{1}{10}$, $e_7^{\mathsf{h}} = \frac{1}{42}$, $b_7^{\mathsf{q}} = -\frac{1}{2}$, $b_7^{\mathsf{o}} = -\frac{1}{10}$, $b_7^{\mathsf{h}} = -\frac{1}{42}$.

$$e_1^{\mathsf{q}} = f^2$$
$$e_2^{\mathsf{q}} = f[1 + \tfrac{1}{4x}(5 + 12\log x) - \tfrac{1}{4x^2}(27 + 12\log x) + \tfrac{7}{4x^3} + \tfrac{3}{4x^4}]$$
$$e_3^{\mathsf{q}} = 1 + \tfrac{1}{24x}(89 + 84\log x) + \tfrac{1}{160x^2}(431 + 996\log x - 1680\mathrm{dilog}x) - \tfrac{1}{10x^3}(315 + 282\log x - 210\mathrm{dilog}x) + \tfrac{1}{120x^4}(4183 + 2322\log x - 1260\mathrm{dilog}x) - \tfrac{363}{40x^5} - \tfrac{809}{480x^6}$$
$$e_4^{\mathsf{q}} = f$$
$$e_5^{\mathsf{q}} = f[1 + \tfrac{1}{6x}(13 + 12\log x) - \tfrac{5}{2x^2} - \tfrac{3}{2x^3} - \tfrac{1}{2x^4}]$$
$$e_6^{\mathsf{q}} = 1 + \tfrac{1}{32x}(117 + 84\log x) + \tfrac{1}{320x^2}(929 + 1836\log x - 1680\mathrm{dilog}x) - \tfrac{1}{40x^3}(597 + 387\log x - 210\mathrm{dilog}x) + \tfrac{387}{80x^4} + \tfrac{809}{480x^5} + \tfrac{809}{960x^6}$$
$$e_7^{\mathsf{q}} = 1 - \tfrac{1}{2x^2}$$
$$e_8^{\mathsf{q}} = 1 + \tfrac{2}{5x}(4 + 3\log x) - \tfrac{9}{5x^2} - \tfrac{1}{5x^3}(7 + 3\log x) + \tfrac{3}{5x^4}$$
$$e_9^{\mathsf{q}} = 1 + \tfrac{1}{216x}(739 + 420\log x) + \tfrac{1}{90x^2}(262 + 387\log x - 210\mathrm{dilog}x) - \tfrac{1}{60x^3}(317 + 70\log x) - \tfrac{1}{540x^4}(1511 + 1161\log x - 630\mathrm{dilog}x) + \tfrac{809}{1080x^5}$$

$$e_1^{\mathsf{o}} = f^2(1 - \tfrac{1}{2x})$$
$$e_2^{\mathsf{o}} = f[1 + \tfrac{1}{30x}(73 + 60\log x) - \tfrac{1}{60x^2}(479 + 180\log x) + \tfrac{1}{20x^3}(87 + 20\log x) - \tfrac{3}{20x^4} + \tfrac{1}{60x^5}]$$
$$e_4^{\mathsf{o}} = f(1 - \tfrac{2}{3x})$$
$$e_5^{\mathsf{o}} = f[1 + \tfrac{1}{40x}(103 + 60\log x) - \tfrac{1}{60x^2}(233 + 60\log x) + \tfrac{1}{5x^3} - \tfrac{1}{20x^4} - \tfrac{1}{120x^5}]$$
$$e_7^{\mathsf{o}} = f + \tfrac{1}{10x^3}$$
$$e_8^{\mathsf{o}} = 1 + \tfrac{1}{54x}(85 + 60\log x) - \tfrac{10}{27x^2}(10 + 3\log x) + \tfrac{2}{3x^3} + \tfrac{1}{9x^4}(4 + \log x) + \tfrac{1}{54x^5}$$

$$e_1^{\mathsf{h}} = f^2(f + \tfrac{3}{14x^2})$$
$$e_4^{\mathsf{h}} = f(1 - \tfrac{5}{4x} + \tfrac{5}{14x^2})$$
$$e_7^{\mathsf{h}} = 1 - \tfrac{5}{3x} + \tfrac{5}{7x^2} - \tfrac{1}{42x^4}$$

$$b_4^{\mathsf{q}} = f$$
$$b_5^{\mathsf{q}} = f[1 + \tfrac{1}{6x}(7 + 12\log x) - \tfrac{3}{2x^2} - \tfrac{1}{2x^3} - \tfrac{1}{6x^4}]$$
$$b_6^{\mathsf{q}} = 1 + \tfrac{1}{32x}(75 + 84\log x) + \tfrac{1}{320x^2}(649 + 996\log x - 1680\mathrm{dilog}x) - \tfrac{1}{80x^3}(879 + 564\log x - 420\mathrm{dilog}x) + \tfrac{141}{40x^4} + \tfrac{223}{160x^5} + \tfrac{223}{320x^6}$$
$$b_7^{\mathsf{q}} = 1 - \tfrac{3}{2x^2}$$
$$b_8^{\mathsf{q}} = 1 + \tfrac{1}{5x}(5 + 6\log x) - \tfrac{9}{5x^2} - \tfrac{1}{5x^3}(2 + 3\log x) + \tfrac{1}{5x^4}$$
$$b_9^{\mathsf{q}} = 1 + \tfrac{1}{216x}(529 + 420\log x) + \tfrac{1}{360x^2}(593 + 1128\log x - 840\mathrm{dilog}x) - \tfrac{1}{30x^3}(106 + 35\log x) - \tfrac{1}{1080x^4}(2357 + 1692\log x - 1260\mathrm{dilog}x) + \tfrac{223}{360x^5}$$

$$b_4^{\mathsf{o}} = f(1 - \tfrac{2}{3x})$$
$$b_5^{\mathsf{o}} = f[1 + \tfrac{1}{40x}(97 + 60\log x) - \tfrac{1}{60x^2}(227 + 60\log x) + \tfrac{3}{10x^3} + \tfrac{1}{20x^4} + \tfrac{1}{120x^5}]$$
$$b_7^{\mathsf{o}} = f - \tfrac{1}{10x^3}$$
$$b_8^{\mathsf{o}} = 1 + \tfrac{1}{54x}(79 + 60\log x) - \tfrac{1}{27x^2}(97 + 30\log x) + \tfrac{2}{3x^3} + \tfrac{1}{27x^4}(13 + 3\log x) - \tfrac{1}{54x^5}$$

$$b_4^{\mathsf{h}} = f(1 - \tfrac{5}{4x} + \tfrac{5}{14x^2})$$
$$b_7^{\mathsf{h}} = 1 - \tfrac{5}{3x} + \tfrac{5}{7x^2} - \tfrac{1}{14x^4}$$

A comparison with Eqs. (3.3) reveals that there are two main differences between the metrics. The first is that the black-hole metric involves the radial functions e_n^{q}, e_n^{o}, e_n^{h}, b_n^{q}, b_n^{o}, and b_n^{h} that are listed in Table XIV, as well as the radial functions p_n^{m}, p_n^{q}, p_n^{h}, q_n^{m}, q_n^{q}, q_n^{h}, g_n^{d}, g_n^{o}, h_n^{q}, and h_n^{o} that are listed in Table XV. The second is that the tidal potentials $\mathcal{P}^{\mathsf{q}}$, $\mathcal{H}_a^{\mathsf{h}}$, $\mathcal{G}_{ab}^{\mathsf{o}}$, $\mathcal{P}_{ab}^{\mathsf{h}}$, $\mathcal{Q}_{ab}^{\mathsf{h}}$, and $\mathcal{H}_{ab}^{\mathsf{h}}$ now make an appearance in the black-hole metric. (Recall that they were absent in the background metric.) The black-hole metric contains the same number of freely-specifiable functions of time as the background metric; these are contained in the tidal moments $\mathcal{E}_{ab}$, $\mathcal{B}_{ab}$, $\mathcal{E}_{abc}$, $\mathcal{B}_{abc}$, $\mathcal{E}_{abcd}$, and $\mathcal{B}_{abcd}$, which all depend on advanced time v.

The metric of Eqs. (3.5) contains two fundamental length scales, the black-hole mass M and the tidal radius $\mathcal{R}$. It is assumed that $M \ll \mathcal{R}$, and the metric is valid when $r \ll \mathcal{R}$. When $M \to 0$ the metric becomes the background metric of Eqs. (3.3). This can be seen from the fact that most of the radial functions approach unity when $2M/r \to 0$. Exceptions are p_1^{q}, g_4^{o}, p_4^{h}, q_4^{h}, h_2^{h}, and h_3^{h}, which all approach zero; these functions come with the tidal potentials that were absent in the background metric.

TABLE XV. Radial functions: bilinear potentials. The radial functions are expressed in terms of $x := r/(2M)$ and $f := 1 - 1/x$. All radial functions vanish at $r = 2M$, except for $p_3^{\mathsf{m}} = q_3^{\mathsf{m}} = p_3^{\mathsf{q}} = q_3^{\mathsf{q}} = p_3^{\mathsf{h}} = q_3^{\mathsf{h}} = \frac{5}{16}$, $g_3^{\mathsf{d}} = g_3^{\mathsf{o}} = -\frac{5}{16}$, $p_4^{\mathsf{h}} = \frac{1}{84}$, $q_4^{\mathsf{h}} = -\frac{5}{126}$, and $h_3^{\mathsf{h}} = \frac{13}{252}$. All functions approach unity as $x \to \infty$, except for p_1^{q}, p_4^{h}, q_4^{h}, g_4^{o}, h_2^{h}, and h_3^{h}, which all approach zero.

$p_1^{\mathsf{m}} = f(1 - \frac{19}{15x} + \frac{1}{15x^2} + \frac{1}{15x^3} + \frac{1}{5x^4})$	$q_1^{\mathsf{m}} = f(1 - \frac{19}{15x} + \frac{1}{15x^2} + \frac{1}{15x^3} + \frac{1}{5x^4})$
$p_3^{\mathsf{m}} = 1 - \frac{5}{4x^2} + \frac{9}{16x^4}$	$q_3^{\mathsf{m}} = 1 - \frac{15}{4x^2} + \frac{49}{16x^4}$
$p_1^{\mathsf{q}} = f^2(-\frac{26}{15x} + \frac{19}{20x^2} + \frac{2}{15x^3} + \frac{1}{15x^4})$	$q_1^{\mathsf{q}} = f^2(1 + \frac{4}{15x} - \frac{41}{20x^2} + \frac{2}{15x^3} + \frac{1}{15x^4})$
$p_2^{\mathsf{q}} = f(1 - \frac{9}{x} + \frac{19}{4x^2} + \frac{1}{x^3} + \frac{1}{x^4})$	$q_2^{\mathsf{q}} = f(1 + \frac{1}{11x} - \frac{101}{44x^2} + \frac{1}{11x^3} + \frac{1}{11x^4})$
$p_3^{\mathsf{q}} = 1 - \frac{5}{4x^2} + \frac{5}{16x^4} + \frac{1}{4x^5}$	$q_3^{\mathsf{q}} = 1 - \frac{15}{4x^2} + \frac{45}{16x^4} + \frac{1}{4x^5}$
$p_4^{\mathsf{q}} = f(1 + \frac{1}{x})$	$q_4^{\mathsf{q}} = f(1 + \frac{1}{x} - \frac{2}{x^2} - \frac{2}{x^3})$
$p_1^{\mathsf{h}} = f^2(1 - \frac{121}{60x} + \frac{39}{280x^2} + \frac{1}{30x^3} + \frac{1}{60x^4})$	$q_1^{\mathsf{h}} = f^2(1 + \frac{29}{60x} - \frac{11}{280x^2} + \frac{1}{30x^3} + \frac{1}{60x^4})$
$p_2^{\mathsf{h}} = f(1 - \frac{61}{24x} + \frac{13}{28x^2} + \frac{1}{6x^3} + \frac{1}{6x^4})$	$q_2^{\mathsf{h}} = f(1 + \frac{29}{24x} - \frac{221}{84x^2} + \frac{1}{6x^3} + \frac{1}{6x^4})$
$p_3^{\mathsf{h}} = 1 - \frac{5}{4x^2} + \frac{5}{16x^4} + \frac{1}{4x^5}$	$q_3^{\mathsf{h}} = 1 - \frac{15}{4x^2} + \frac{45}{16x^4} + \frac{1}{4x^5}$
$p_4^{\mathsf{h}} = \frac{5}{36x} - \frac{29}{252x^2} - \frac{1}{84x^4}$	$q_4^{\mathsf{h}} = -\frac{5}{36x} + \frac{19}{84x^2} - \frac{8}{63x^4}$
$g_1^{\mathsf{d}} = f^2(1 + \frac{2}{15x} - \frac{7}{5x^2} + \frac{1}{15x^3} + \frac{1}{30x^4})$	
$g_2^{\mathsf{d}} = f(1 - \frac{2}{3x} + \frac{1}{6x^2} - \frac{1}{24x^3} - \frac{1}{24x^4})$	$h_2^{\mathsf{q}} = f^2(1 + \frac{2}{x})$
$g_3^{\mathsf{d}} = 1 - \frac{5}{2x^2} + \frac{5}{4x^4} - \frac{1}{16x^5}$	$h_3^{\mathsf{q}} = f^2(1 + \frac{1}{x})^2$
$g_1^{\mathsf{o}} = f^2(1 + \frac{79}{30x} - \frac{53}{20x^2} + \frac{1}{15x^3} + \frac{1}{30x^4})$	
$g_2^{\mathsf{o}} = f(1 + \frac{89}{24x} - \frac{29}{6x^2} + \frac{1}{6x^3} + \frac{1}{6x^4})$	$h_2^{\mathsf{h}} = -\frac{1}{2x} + \frac{29}{42x^2} - \frac{4}{21x^3}$
$g_3^{\mathsf{o}} = 1 - \frac{5}{2x^2} + \frac{15}{16x^4} + \frac{1}{4x^5}$	$h_3^{\mathsf{h}} = -\frac{5}{18x} + \frac{43}{126x^2} - \frac{1}{84x^4}$
$g_4^{\mathsf{o}} = f(\frac{7}{6x} - \frac{1}{3x^2} - \frac{1}{3x^3})$	

In this limit the tidal moments acquire their precise relation with the Weyl tensor (and its derivatives) evaluated on the world line $x^a = 0$.

When $M \neq 0$ the world line disappears and is replaced by the world tube traced by the black-hole horizon. But the black-hole metric continues to approach the background metric when r is very large compared with M (but still much smaller than $\mathcal{R}$). In this case the interpretation of the tidal moments is more subtle. They still provide a complete characterization of the tidal environment, but they are no longer associated with the Weyl tensor evaluated on a world line.

To elucidate the meaning of the tidal moments it is helpful to return to the two spacetimes of Sec. III A. The first is the background spacetime with its central world line, and the second is the black-hole spacetime in which no such world line exists. We assume that the spacetimes share the same set of tidal moments: The functions of time contained in $\mathcal{E}_{ab}$, $\mathcal{B}_{ab}$, $\mathcal{E}_{abc}$, $\mathcal{B}_{abc}$, $\mathcal{E}_{abcd}$, and $\mathcal{B}_{abcd}$ are the same in each spacetime. An observer in the first spacetime can measure the tidal moments anywhere and relate them to the Weyl tensor (and its derivatives) evaluated on the central world line. An observer in the second spacetime can still measure the tidal moments, but is prevented by the black hole from relating the results of his measurements to a world-line Weyl tensor. But the second observer is free to imagine that the black hole could be removed from the spacetime without altering the conditions in the asymptotic region $r \gg M$, so that his measurements could, after all, be related to a world-line Weyl tensor.[1] This thought experiment provides him with a loose interpretation of the tidal moments: They can still be related to a world-line Weyl tensor, but the Weyl tensor and the world line are fictitious extrapolations to $r = 0$ obtained from information available in the $r \gg M$ portion of the black-hole spacetime.

When the tidal perturbation is turned off (by putting all the tidal moments to zero, thereby sending $\mathcal{R}$ off to infinity), the black-hole metric becomes

$$g_{vv} = -f, \qquad g_{va} = \Omega_a, \qquad g_{ab} = \gamma_{ab}, \tag{3.6}$$

where, we recall, $f = 1 - 2M/r$. This is the well-known Schwarzschild solution expressed in the light-cone coordinates (v, x^a).

In quasispherical coordinates (v, r, θ^A) the nonvanishing components of the black-hole metric are

[1]This is a thought experiment that could not be realized physically. It is meant to reflect a mathematical procedure in which M is taken to zero while keeping the tidal moments fixed. The procedure disregards the fact that in most physical applications, the tidal moments carry a dependence on M that is revealed by matching the local black-hole metric to a global metric that contains the black hole and the external bodies.

$$g_{vv} = -f - r^2 e_1^{\mathsf{q}}\mathcal{E}^{\mathsf{q}} + \frac{1}{3}r^3 e_2^{\mathsf{q}}\dot{\mathcal{E}}^{\mathsf{q}} - \frac{1}{3}r^3 e_1^{\mathsf{o}}\mathcal{E}^{\mathsf{o}} - \frac{2}{21}r^4 e_3^{\mathsf{q}}\ddot{\mathcal{E}}^{\mathsf{q}} + \frac{1}{6}r^4 e_2^{\mathsf{o}}\dot{\mathcal{E}}^{\mathsf{o}} - \frac{1}{12}r^4 e_1^{\mathsf{h}}\mathcal{E}^{\mathsf{h}} + \frac{1}{15}r^4(p_1^{\mathsf{m}}\mathcal{P}^{\mathsf{m}} + q_1^{\mathsf{m}}\mathcal{Q}^{\mathsf{m}}) + \frac{2}{15}r^4 g_1^{\mathsf{d}}\mathcal{G}^{\mathsf{d}} + \frac{2}{7}r^4(p_1^{\mathsf{q}}\mathcal{P}^{\mathsf{q}} + q_1^{\mathsf{q}}\mathcal{Q}^{\mathsf{q}}) + \frac{2}{3}r^4 g_1^{\mathsf{o}}\mathcal{G}^{\mathsf{o}} - \frac{1}{3}r^4(p_1^{\mathsf{h}}\mathcal{P}^{\mathsf{h}} + q_1^{\mathsf{h}}\mathcal{Q}^{\mathsf{h}}) + O(5), \tag{3.7a}$$

$$g_{vr} = 1, \tag{3.7b}$$

$$g_{vA} = -\frac{2}{3}r^3(e_4^{\mathsf{q}}\mathcal{E}_A^{\mathsf{q}} - b_4^{\mathsf{q}}\mathcal{B}_A^{\mathsf{q}}) + \frac{1}{3}r^4(e_5^{\mathsf{q}}\dot{\mathcal{E}}_A^{\mathsf{q}} - b_5^{\mathsf{q}}\dot{\mathcal{B}}_A^{\mathsf{q}}) - \frac{1}{4}r^4(e_4^{\mathsf{o}}\mathcal{E}_A^{\mathsf{o}} - b_4^{\mathsf{o}}\mathcal{B}_A^{\mathsf{o}}) - \frac{8}{63}r^5(e_6^{\mathsf{q}}\ddot{\mathcal{E}}_A^{\mathsf{q}} - b_6^{\mathsf{q}}\ddot{\mathcal{B}}_A^{\mathsf{q}}) + \frac{1}{6}r^5(e_5^{\mathsf{o}}\dot{\mathcal{E}}_A^{\mathsf{o}} - b_5^{\mathsf{o}}\dot{\mathcal{B}}_A^{\mathsf{o}}) - \frac{1}{15}r^5(e_4^{\mathsf{h}}\mathcal{E}_A^{\mathsf{h}} - b_4^{\mathsf{h}}\mathcal{B}_A^{\mathsf{h}}) - \frac{8}{75}r^5 g_2^{\mathsf{d}}\mathcal{G}_A^{\mathsf{d}} + \frac{8}{21}r^5 h_2^{\mathsf{q}}\mathcal{H}_A^{\mathsf{q}} + \frac{4}{105}r^5(p_2^{\mathsf{q}}\mathcal{P}_A^{\mathsf{q}} + 11 q_2^{\mathsf{q}}\mathcal{Q}_A^{\mathsf{q}}) + \frac{2}{5}r^5 g_2^{\mathsf{o}}\mathcal{G}_A^{\mathsf{o}} + r^5 h_2^{\mathsf{h}}\mathcal{H}_A^{\mathsf{h}} - \frac{2}{15}r^5(p_2^{\mathsf{h}}\mathcal{P}_A^{\mathsf{h}} + q_2^{\mathsf{h}}\mathcal{Q}_A^{\mathsf{h}}) + rO(5), \tag{3.7c}$$

$$g_{AB} = r^2\Omega_{AB} - \frac{1}{3}r^4(e_7^{\mathsf{q}}\mathcal{E}_{AB}^{\mathsf{q}} - b_7^{\mathsf{q}}\mathcal{B}_{AB}^{\mathsf{q}}) + \frac{5}{18}r^5(e_8^{\mathsf{q}}\dot{\mathcal{E}}_{AB}^{\mathsf{q}} - b_8^{\mathsf{q}}\dot{\mathcal{B}}_{AB}^{\mathsf{q}}) - \frac{1}{6}r^5(e_7^{\mathsf{o}}\mathcal{E}_{AB}^{\mathsf{o}} - b_7^{\mathsf{o}}\mathcal{B}_{AB}^{\mathsf{o}}) - \frac{1}{7}r^6(e_9^{\mathsf{q}}\ddot{\mathcal{E}}_{AB}^{\mathsf{q}} - b_9^{\mathsf{q}}\ddot{\mathcal{B}}_{AB}^{\mathsf{q}}) + \frac{3}{20}r^6(e_8^{\mathsf{o}}\dot{\mathcal{E}}_{AB}^{\mathsf{o}} - b_8^{\mathsf{o}}\dot{\mathcal{B}}_{AB}^{\mathsf{o}}) - \frac{1}{20}r^6(e_7^{\mathsf{h}}\mathcal{E}_{AB}^{\mathsf{h}} - b_7^{\mathsf{h}}\mathcal{B}_{AB}^{\mathsf{h}}) + \frac{8}{225}r^6\Omega_{AB}(p_3^{\mathsf{m}}\mathcal{P}^{\mathsf{m}} + q_3^{\mathsf{m}}\mathcal{Q}^{\mathsf{m}}) + \frac{32}{225}r^6\Omega_{AB}g_3^{\mathsf{d}}\mathcal{G}^{\mathsf{d}} - \frac{16}{105}r^6\Omega_{AB}(p_3^{\mathsf{q}}\mathcal{P}^{\mathsf{q}} + q_3^{\mathsf{q}}\mathcal{Q}^{\mathsf{q}}) - \frac{3}{14}r^6(p_4^{\mathsf{q}}\mathcal{P}_{AB}^{\mathsf{q}} - q_4^{\mathsf{q}}\mathcal{Q}_{AB}^{\mathsf{q}}) + \frac{3}{7}r^6 h_3^{\mathsf{q}}\mathcal{H}_{AB}^{\mathsf{q}} - \frac{8}{45}r^6\Omega_{AB}g_3^{\mathsf{o}}\mathcal{G}^{\mathsf{o}} + r^6 g_4^{\mathsf{o}}\mathcal{G}_{AB}^{\mathsf{o}} + \frac{2}{45}r^6\Omega_{AB}(p_3^{\mathsf{h}}\mathcal{P}^{\mathsf{h}} + q_3^{\mathsf{h}}\mathcal{Q}^{\mathsf{h}}) + r^6(p_4^{\mathsf{h}}\mathcal{P}_{AB}^{\mathsf{h}} + q_4^{\mathsf{h}}\mathcal{Q}_{AB}^{\mathsf{h}}) + r^6 h_3^{\mathsf{h}}\mathcal{H}_{AB}^{\mathsf{h}} + r^2 O(5). \tag{3.7d}$$

In these coordinates the $\mathcal{R} \to \infty$ limit of the metric is $ds^2 = f dv^2 + 2dvdr + r^2 d\Omega^2$, the Eddington-Finkelstein form of the Schwarzschild solution.

IV. GEOMETRY AND DYNAMICS OF THE DEFORMED HORIZON

In this section we extract the consequences of the black-hole metric of Eqs. (3.7) on the structure and dynamics of the tidally deformed horizon. We begin in Sec. IV A with a proof that, in the coordinate system adopted here, the deformed horizon continues to be described by $r = 2M$. In Sec. IV B we display the components of the induced metric on the horizon. In Sec. IV C we examine the congruence of null geodesics that generates the horizon, and derive expressions for its expansion scalar and shear tensor. And finally, in Sec. IV D we integrate Raychaudhuri's equation and calculate the rate at which the horizon grows as a result of the tidal interaction.

A. Position of the horizon

The black-hole metric of Eqs. (3.7) is presented in light-cone coordinates (v, r, θ^A) whose geometrical meaning was described in Sec. III A. As we saw, the coordinates are tied to the behavior of incoming light rays that mesh together to form light cones that converge toward the black hole. And as we saw, the light-cone coordinates are not fully specified; the limited coordinate freedom that remains was exploited to simplify the form of the metric.

Concretely, the coordinate freedom was utilized to impose a set of *horizon-locking conditions* that are designed to keep the black-hole horizon in its usual place. As a result, *the horizon of a tidally deformed black hole is situated at*

$$r = 2M[1 + O(5)] \tag{4.1}$$

in the light-cone coordinates that give rise to the metric of Eqs. (3.7). In Eq. (4.1), and in all equations below, the symbol $O(5)$ means that the correction terms are of order $(M/\mathcal{R})^5$. The horizon-locking conditions are

$$g_{vv} = 0 = g_{vA} \quad \text{at } r = 2M, \tag{4.2}$$

and it is easy to verify that the metric of Eqs. (3.7) satisfies this property.

The horizon-locking conditions imply that the surface $r = 2M$ is a null hypersurface in the black-hole spacetime. This is seen from the fact that $g^{rr} = 0$ at $r = 2M$, which follows from the statement that thanks to Eq. (4.2), $g_{vr} = 1$ and g_{AB} are the only nonvanishing components of the metric at $r = 2M$. The null hypersurface is generated by a congruence of null geodesics, and in Sec. IV C we show that the expansion of this congruence vanishes. [The expansion scalar is denoted Θ, and we show more precisely that $\Theta = O(M^5/\mathcal{R}^6)$. This implies that the expansion vanishes through order $\mathcal{R}^{-4}$, which is the order of accuracy maintained in the metric and the description of the surface.] The surface $r = 2M$ is therefore a *stationary null surface*, and it is in this sense that it describes a black-hole horizon. It is an *isolated horizon* in the sense of Ashtekar and Krishnan [35–37].

In general we cannot state that $r = 2M$ is an event horizon, because the exact position of the event horizon depends on the entire future history of the spacetime, and is located by tracing light rays backward in time from future null infinity. Because our metric may not be accurate for all times, and because it applies only to a neighborhood of the black hole, it does not supply us with the tools to locate the event horizon. Under restrictive assumptions, however, we can go beyond these limitations and prove that $r = 2M$ describes the position of the event horizon.

We consider a situation in which the tidal perturbation is switched off after a time $v = v_1$, so that the metric returns to its usual Schwarzschild form when $v > v_1$. This final Schwarzschild metric possesses the same mass parameter M as the original black-hole metric, because (as we shall see in Sec. IV D) the total change in mass that can result from tidal heating over a time $v_1 \sim \mathcal{R}$ is of order $M^6/\mathcal{R}^5$; this effect is too small to be described by our metric. It follows that the event horizon is described by $r = 2M$ when $v > v_1$. At earlier times the horizon is described by a null hypersurface that joins smoothly with $r = 2M$ at $v = v_1$. Since $r = 2M$ is null at all times, we conclude that the event horizon is always situated at $r = 2M$.

B. Horizon metric and surface gravity

The horizon metric is the specialization of the black-hole metric to the null hypersurface $r = 2M[1 + O(5)]$. Because its component along the v-direction vanishes, the horizon metric is degenerate and explicitly two-dimensional. We denote its nonvanishing components by γ_{AB}, and these are obtained by inserting $r = 2M$ within Eqs. (3.7). We get

$$\begin{aligned}\gamma_{AB} = {} & 4M^2\Omega_{AB} - \frac{8}{3}M^4(\mathcal{E}^{\mathsf{q}}_{AB} + \mathcal{B}^{\mathsf{q}}_{AB}) \\ & - \frac{8}{15}M^5(\mathcal{E}^{\mathsf{o}}_{AB} + \mathcal{B}^{\mathsf{o}}_{AB}) - \frac{8}{105}M^6(\mathcal{E}^{\mathsf{h}}_{AB} + \mathcal{B}^{\mathsf{h}}_{AB}) \\ & + \frac{32}{45}M^6\Omega_{AB}(\mathcal{P}^{\mathsf{m}} + \mathcal{Q}^{\mathsf{m}}) - \frac{128}{45}M^6\Omega_{AB}\mathcal{G}^{\mathsf{d}} \\ & - \frac{64}{21}M^6\Omega_{AB}(\mathcal{P}^{\mathsf{q}} + \mathcal{Q}^{\mathsf{q}}) + \frac{32}{9}M^6\Omega_{AB}\mathcal{G}^{\mathsf{o}} \\ & + \frac{8}{9}M^6\Omega_{AB}(\mathcal{P}^{\mathsf{h}} + \mathcal{Q}^{\mathsf{h}}) + \frac{16}{21}M^6\mathcal{P}^{\mathsf{h}}_{AB} \\ & - \frac{160}{63}M^6\mathcal{Q}^{\mathsf{h}}_{AB} + \frac{208}{63}M^6\mathcal{H}^{\mathsf{h}}_{AB} + M^2O(5). \qquad (4.3)\end{aligned}$$

In Appendix B we calculate the determinant γ of the horizon metric. We find the simple result

$$\sqrt{\gamma} = 4M^2\sin\theta[1 + O(5)]; \qquad (4.4)$$

the tidal potentials do not give rise to corrections to the horizon's surface element, $\sqrt{\gamma}d\theta d\phi$.

The surface gravity κ of the perturbed horizon is defined by the relation $k^\beta\nabla_\beta k^\alpha = \kappa k^\alpha$, where the vector $k^\alpha := [1, 0, 0, 0]$ is tangent to the horizon's null generators. A calculation based on the metric of Eqs. (3.7) reveals that

$$\begin{aligned}\kappa = \frac{1}{4M}\bigg[& 1 + \frac{16}{3}M^3\dot{\mathcal{E}}_{ab}\Omega^a\Omega^b + \frac{32}{9}M^4\ddot{\mathcal{E}}_{ab}\Omega^a\Omega^b \\ & + \frac{8}{9}M^4\dot{\mathcal{E}}_{abc}\Omega^a\Omega^b\Omega^c - \frac{16}{225}M^4(\mathcal{E}_{ab}\mathcal{E}^{ab} + \mathcal{B}_{ab}\mathcal{B}^{ab}) \\ & + O(5)\bigg]. \qquad (4.5)\end{aligned}$$

We see that the surface gravity is no longer uniform on the horizon; the tidal perturbation introduces a variation of order $(M/\mathcal{R})^3$ over its surface. Notice that this variation is associated with *changes* in the tidal environment. When the perturbation is stationary the correction to the surface gravity comes from the last term, which is of order $(M/\mathcal{R})^4$ and uniform over the horizon. These observations are compatible with the zeroth law of black-hole mechanics.

C. Expansion scalar and shear tensor

The null generators of the horizon form a congruence whose behavior is described by an expansion scalar Θ and a shear tensor σ_{AB}; these are defined by (see, for example, Sec. III of Ref. [39])

$$\partial_v\gamma_{AB} = \Theta\gamma_{AB} + 2\sigma_{AB}, \qquad (4.6)$$

together with the requirement that the shear tensor be tracefree: $\gamma^{AB}\sigma_{AB} = 0$. The expansion scalar is then equal to the trace of $\partial_v\gamma_{AB}$, so that $\Theta = \frac{1}{2}\gamma^{-1}\partial_v\gamma$, where γ is the metric determinant. Equation (4.4) implies that

$$\Theta = O(M^5/\mathcal{R}^6). \qquad (4.7)$$

This means that up to this level of accuracy, the surface $r = 2M$ is foliated by apparent horizons.

With this we find that Eq. (4.6) reduces to $\sigma_{AB} = \frac{1}{2}\partial_v\gamma_{AB} + O(M^7/\mathcal{R}^6)$. This is

$$\begin{aligned}\sigma_{AB} = {} & -\frac{4}{3}M^4(\dot{\mathcal{E}}^{\mathsf{q}}_{AB} + \dot{\mathcal{B}}^{\mathsf{q}}_{AB}) - \frac{4}{15}M^5(\dot{\mathcal{E}}^{\mathsf{o}}_{AB} + \dot{\mathcal{B}}^{\mathsf{o}}_{AB}) - \frac{4}{105}M^6(\dot{\mathcal{E}}^{\mathsf{h}}_{AB} + \dot{\mathcal{B}}^{\mathsf{h}}_{AB}) + \frac{16}{45}M^6\Omega_{AB}(\dot{\mathcal{P}}^{\mathsf{m}} + \dot{\mathcal{Q}}^{\mathsf{m}}) - \frac{64}{45}M^6\Omega_{AB}\dot{\mathcal{G}}^{\mathsf{d}} \\ & - \frac{32}{21}M^6\Omega_{AB}(\dot{\mathcal{P}}^{\mathsf{q}} + \dot{\mathcal{Q}}^{\mathsf{q}}) + \frac{16}{9}M^6\Omega_{AB}\dot{\mathcal{G}}^{\mathsf{o}} + \frac{4}{9}M^6\Omega_{AB}(\dot{\mathcal{P}}^{\mathsf{h}} + \dot{\mathcal{Q}}^{\mathsf{h}}) + \frac{8}{21}M^6\dot{\mathcal{P}}^{\mathsf{h}}_{AB} - \frac{80}{63}M^6\dot{\mathcal{Q}}^{\mathsf{h}}_{AB} \\ & + \frac{104}{63}M^6\dot{\mathcal{H}}^{\mathsf{h}}_{AB} + MO(6). \qquad (4.8)\end{aligned}$$

D. Tidal heating

More information about the expansion scalar can be obtained by integrating Raychaudhuri's equation (see, for example, Sec. III of Ref. [39])

$$\partial_v \Theta = \kappa\Theta - \frac{1}{2}\Theta^2 - \sigma_{AB}\sigma^{AB}, \tag{4.9}$$

in which we can neglect $\Theta^2 = O(M^{10}/\mathcal{R}^{12})$. The squared shear contains terms that begin at order $M^4/\mathcal{R}^6$, and the neglected terms are of order $M^7/\mathcal{R}^9$ and higher. The solution to Eq. (4.9) will be an expression for Θ that begins at order $M^5/\mathcal{R}^6$, and neglects terms of order $M^8/\mathcal{R}^9$ and higher. Given this degree of available accuracy, it is appropriate to set

$$\kappa \simeq \kappa_0 := \frac{1}{4M} \tag{4.10}$$

in Eq. (4.9); the fractional corrections of order $(M/\mathcal{R})^3$ and $(M/\mathcal{R})^4$ displayed in Eq. (4.5) are not required in the determination of Θ.

Instead of proceeding directly with Eq. (4.9), we integrate it over a cross section of the horizon and work out a differential equation for the surface area

$$\mathcal{A}(v) := \int \sqrt{\gamma}\, d\theta d\phi. \tag{4.11}$$

Its rate of change is $\dot{\mathcal{A}} = \int \Theta\sqrt{\gamma}\, d\theta d\phi$, and the second rate of change is $\ddot{\mathcal{A}} = \int \partial_v\Theta\sqrt{\gamma}\, d\theta d\phi + \int \Theta^2\sqrt{\gamma}\, d\theta d\phi$; here we may once more ignore the second integral involving Θ^2.

The differential equation that governs the behavior of the area function is

$$\kappa_0\dot{\mathcal{A}} - \ddot{\mathcal{A}} = \int \sigma_{AB}\sigma^{AB}\sqrt{\gamma}\, d\theta d\phi + O(9). \tag{4.12}$$

The steps required in the evaluation of the right-hand side are described in Appendix C, and the end result is

$$\kappa_0\dot{\mathcal{A}} - \ddot{\mathcal{A}} = 8\pi\mathcal{F}, \tag{4.13}$$

where

$$\mathcal{F}(v) := \frac{16}{45}M^6(\dot{\mathcal{E}}_{ab}\dot{\mathcal{E}}^{ab} + \dot{\mathcal{B}}_{ab}\dot{\mathcal{B}}^{ab}) + \frac{16}{4725}M^8\left(\dot{\mathcal{E}}_{abc}\dot{\mathcal{E}}^{abc} + \frac{16}{9}\dot{\mathcal{B}}_{abc}\dot{\mathcal{B}}^{abc}\right) + O(9) \tag{4.14}$$

might be called the *flux function*. We notice that $\mathcal{F}$ is positive-definite, and that it cannot be expressed as a total derivative with respect to v.

As we show in Appendix C, the solution to Eq. (4.13) is

$$\frac{\kappa_0}{8\pi}\dot{\mathcal{A}}(v) = \mathcal{F}(v) + \frac{\dot{\mathcal{F}}(v)}{\kappa_0} + \frac{\ddot{\mathcal{F}}(v)}{\kappa_0^2} + O(9). \tag{4.15}$$

We observe that the second and third terms on the right-hand side are total time derivatives, and that it is natural to bring them over to the left-hand side of the equation. We write our final result as

$$\frac{\kappa_0}{8\pi}\dot{\mathcal{A}}^* = \mathcal{F} + O(9), \tag{4.16}$$

in terms of a "renormalized area function" defined by $\mathcal{A}^* = \mathcal{A} - 8\pi\mathcal{F}/\kappa_0^2 - 8\pi\dot{\mathcal{F}}/\kappa_0^3 + O(M^{10}/\mathcal{R}^8)$. The shift in area is of order $M^8/\mathcal{R}^6$ and corresponds to a correction to the relation $r = 2M$ of fractional order $(M/\mathcal{R})^6$; this is well beyond the level of accuracy of Eq. (4.1).

Equation (4.16) describes the rate at which the renormalized black-hole area $\mathcal{A}^*$ increases as a result of the tidal interaction. We refer to this phenomenon as *tidal heating*, recalling the deep analogy that exists between the relativistic tidal dynamics of black holes and the Newtonian tidal dynamics of viscous bodies [46,69,70]. The phrase also recalls the fact that in black-hole thermodynamics, the area plays the role of entropy S, while the surface gravity plays the role of (Hawking) temperature T; in this context we write $(\kappa_0/8\pi)\dot{\mathcal{A}}^* = T\dot{S}$ and interpret Eq. (4.16) as describing a flow of heat across the horizon. By the first law of black-hole mechanics, $dM = (\kappa/8\pi)dA$, the equation also describes the rate at which the black-hole mass increases during the tidal interaction.

V. BACKGROUND METRIC: DERIVATION

Our main results were presented in the preceding sections, and we now turn to a detailed derivation of these results. In this section we provide a derivation of the background metric of Eqs. (3.3) and (3.4); an alternative derivation is sketched in Appendix D. In Sec. VI we present a derivation of the black-hole metric of Eqs. (3.5) and (3.7).

A. Kinematical properties of the metric

We construct the metric of a vacuum region of spacetime that surrounds a timelike geodesic γ, and we adopt the light-cone coordinates (v, r, θ^A). These are centered on the world line, and the geometrical meaning of the coordinates was specified in Sec. III A. The three defining properties listed there give rise to four important conditions on the metric.

To spell them out we introduce the dual vector $\ell_\alpha := -\partial_\alpha v = [-1, 0, 0, 0]$, which is normal to hypersurfaces of constant advanced time v. According to the first property, this dual vector is null—$g^{\alpha\beta}\ell_\alpha\ell_\beta = 0$—and this condition immediately implies that $g^{vv} = 0$. The vector ℓ^α is tangent to the null generators of the light cones $v =$ constant, and the second and third properties imply that its components in the light-cone coordinates must be given by $\ell^\alpha = [0, -1, 0, 0]$; that $\ell^v = \ell^A = 0$ means that v and θ^A are constant on the generators, and $\ell^r = -1$ means that r is an affine parameter that decreases as the generators converge toward the world line. The relation $\ell^\alpha = g^{\alpha\beta}\ell_\beta$ then implies that the inverse metric must satisfy the conditions

$$g^{vv} = 0 = g^{vA}, \qquad g^{vr} = 1. \tag{5.1}$$

Calculating the inverse, we find that the metric must satisfy

$$g_{rr} = 0 = g_{rA}, \qquad g_{vr} = 1. \tag{5.2}$$

These statements are exact, and follow from the light-cone nature of the coordinates. The other nonvanishing components of the metric are g_{vv}, g_{vA}, and g_{AB}. We have not yet made a choice of normalization for the radial coordinate r, nor a choice of axes for the angular coordinates θ^A.

From the quasispherical coordinates (r, θ^A) we construct a system of quasi-Cartesian coordinates $x^a = r\Omega^a(\theta^A)$ and transform the spatial components of the inverse metric according to the rules described in Sec. II E. We find that $g^{va} = \Omega^a$, but the form of g^{ab} is not constrained by the conditions of Eq. (5.1).

At this stage we impose the additional condition that the inverse metric should be locally flat near the world line, so that $g^{ab} = \delta^{ab} + h^{ab}$, with h^{ab} going to zero (as r^2) when $r \to 0$. This condition implicitly specifies a normalization for r, which reduces to the usual Euclidean distance in the immediate vicinity of the world line. And it specifies a set of reference axes for the angular coordinates, which are aligned with the Cartesian directions associated with the coordinates x^a; the fact that h^{ab} scales as r^2 (instead of r) when $r \to 0$ implies that the world line is unaccelerated and that the axes are locally nonrotating.

Our expression for the inverse metric in quasi-Cartesian coordinates is therefore

$$g^{vv} = 0, \tag{5.3a}$$
$$g^{va} = \Omega^a, \tag{5.3b}$$
$$g^{ab} = \delta^{ab} + h^{ab}, \tag{5.3c}$$

with

$$h^{ab} = r^2 h_2^{ab} + r^3 h_3^{ab} + r^4 h_4^{ab} + O(r^5). \tag{5.4}$$

The last equation indicates that as $r \to 0$, $r^{-2}h^{ab}$ approaches the (direction-dependent) limit h_2^{ab}, $r^{-1}(r^{-2}h^{ab} - h_2^{ab})$ approaches h_3^{ab}, and so on.

To proceed it is useful to introduce a decomposition of h^{ab} into longitudinal and transverse pieces. We write

$$h^{ab} = \Omega^a\Omega^b A + \Omega^a A^b + A^a\Omega^b + A^{ab} \tag{5.5}$$

with

$$\Omega_a A^a = 0 = \Omega_a A^{ab}. \tag{5.6}$$

Here and below, indices on Ω^a are lowered with the Euclidean metric δ_{ab}. The first term in Eq. (5.5) is the longitudinal piece of h^{ab}, and $A := h^{ab}\Omega_a\Omega_b$ is its component in the direction of the unit radial vector Ω^a. The last term is the transverse piece of h^{ab}, and $A^{ab} := \gamma^a{}_c\gamma^b{}_d h^{cd}$ are the components in the directions orthogonal to Ω^a; $\gamma^a{}_c$ is the projector of Eq. (2.15). The second and third terms are the longitudinal-transverse piece of h^{ab}, and $A^a := \gamma^a{}_c h^{cb}\Omega_b$ contains the relevant components. The six independent components of h^{ab} are contained in A (one component), A^a (two independent components), and A^{ab} (three independent components). It is sometimes useful to further decompose A^{ab} into trace and tracefree pieces, but we choose not to do so at this stage; we will find in due course that the Einstein field equations automatically enforce $\delta_{ab}A^{ab} = 0$. The decomposition of Eq. (5.5) can be applied individually to each h_n^{ab} that appears in Eq. (5.4); this defines A_n, A_n^a, and A_n^{ab}.

We write the inverse metric of Eq. (5.3) as $g^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta}$, with $\eta^{\alpha\beta}$ denoting the inverse of the Minkowski metric in light-cone coordinates (with components $\eta^{vv} = 0$, $\eta^{va} = \Omega^a$, and $\eta^{ab} = \delta^{ab}$) and $h^{\alpha\beta} = O(r^2)$ denoting a perturbation (with components $h^{vv} = 0$, $h^{va} = 0$, and h^{ab}). The metric is then $g_{\alpha\beta} = \eta_{\alpha\beta} - h_{\alpha\beta} + h_{\alpha\mu}h^\mu{}_\beta + O(r^5)$, where all indices are lowered with the Minkowski metric $\eta_{\alpha\beta}$ (with components $\eta_{vv} = -1$, $\eta_{va} = \Omega_a$, and $\eta_{ab} = \gamma_{ab}$). A straightforward calculation using Eq. (5.5) reveals that

$$g_{vv} = -1 - A + A_a A^a + O(r^5), \tag{5.7a}$$
$$g_{va} = \Omega_a - A_a + A_{ab}A^b + O(r^5), \tag{5.7b}$$
$$g_{ab} = \gamma_{ab} - A_{ab} + A_{ac}A^c{}_b + O(r^5). \tag{5.7c}$$

In these expressions it is understood that indices on A^a and A^{ab} are lowered with δ_{ab}; by virtue of Eqs. (5.6) this operation is equivalent to lowering indices with $\eta_{ab} = \gamma_{ab}$. It is also understood that A, A^a, and A^{ab} can be expanded in powers of r as in Eq. (5.4); we have

$$A = r^2 A_2 + r^3 A_3 + r^4 A_4 + O(r^5), \tag{5.8a}$$
$$A_a = r^2 A_{2a} + r^3 A_{3a} + r^4 A_{4a} + O(r^5), \tag{5.8b}$$
$$A_{ab} = r^2 A_{2ab} + r^3 A_{3ab} + r^4 A_{4ab} + O(r^5), \tag{5.8c}$$

and a term like $A_a A^a$ reduces to $r^4 A_{2a} A_2^a + O(r^5)$.

We now return to the quasispherical coordinates (r, θ^A). We define the angular version of the vector potential A_a by

$$A_A := A_a \Omega_A^a, \tag{5.9}$$

where $\Omega_A^a := \partial_A \Omega^a$, and we define the angular version of the tensor potential A_{ab} by

$$A_{AB} := A_{ab}\Omega_A^a \Omega_B^b. \tag{5.10}$$

In spite of the suggestive notation, these are *not* the components of the Cartesian tensors A_a and A_{ab} in spherical coordinates; for these we have $A_a \partial x^a/\partial r = A_a\Omega^a = 0$ and $A_a \partial x^a/\partial\theta^A = rA_a\Omega_A^a = rA_A$, with similar equations holding for A_{ab}.

Using the identities of Eqs. (2.18) we find that the spherical-coordinate form of the metric is

$$g_{vv} = -1 - A + A_A A^A + O(r^5), \tag{5.11a}$$
$$g_{vr} = 1, \tag{5.11b}$$
$$g_{vA} = -rA_A + rA_{AB}A^B + O(r^6), \tag{5.11c}$$
$$g_{AB} = r^2\Omega_{AB} - r^2 A_{AB} + r^2 A_{AC}A^C{}_B + O(r^7). \tag{5.11d}$$

It is understood that, in these expressions, indices on A_A and A_{AB} are raised with Ω^{AB}. As we saw previously, each potential can be expanded in powers of r, so that

$$A = r^2 A_2 + r^3 A_3 + r^4 A_4 + O(r^5), \tag{5.12a}$$
$$A_A = r^2 A_{2A} + r^3 A_{3A} + r^4 A_{4A} + O(r^5), \tag{5.12b}$$
$$A_{AB} = r^2 A_{2AB} + r^3 A_{3AB} + r^4 A_{4AB} + O(r^5). \tag{5.12c}$$

The coefficients A_n, A_{nA}, and A_{nAB} are assumed to depend on θ^A only, so that the dependence of the metric on r is contained explicitly in the power expansion.

B. Field equations I

The metric forms of Eqs. (5.7) and (5.11) follow directly from the light-cone nature of the coordinate systems, and they embody the purely kinematical requirements imposed by the choice of coordinates. To obtain more information we must impose the Einstein field equations.

In this first stage we involve the metric of Eqs. (5.11) in a computation of R^{vv}, the time-time component of the Ricci tensor. Setting this to zero order-by-order in r reveals that

$$\Omega^{AB}A_{2AB} = 0, \tag{5.13a}$$
$$\Omega^{AB}A_{3AB} = 0, \tag{5.13b}$$
$$\Omega^{AB}A_{4AB} = \frac{3}{5}A_{2AB}A_2^{AB}. \tag{5.13c}$$

In quasi-Cartesian coordinates these equations read

$$\gamma^{ab}A_{2ab} = 0, \tag{5.14a}$$
$$\gamma^{ab}A_{3ab} = 0, \tag{5.14b}$$
$$\gamma^{ab}A_{4ab} = \frac{3}{5}A_{2ab}A_2^{ab}. \tag{5.14c}$$

These equations mean that A_{ab} is tracefree through order r^3, and that its trace is given by $\gamma^{ab}A_{ab} = \frac{3}{5}r^4 A_{2ab}A_2^{ab} + O(r^5)$.

We use this observation to simplify the form of the metric. Continuing to work in the quasi-Cartesian coordinates x^a, we reexpress A_{4ab} as a sum of trace and tracefree pieces:

$$A_{4ab} \to A_{4ab} + \frac{3}{10}\gamma_{ab}A_{2cd}A_2^{cd}, \tag{5.15}$$

where the new A_{4ab}, like A_{2ab} and A_{3ab}, is now known to be tracefree. We also exploit the identity

$$A_{2ac}A_{2b}^c = \frac{1}{2}\gamma_{ab}A_{2cd}A_2^{cd}, \tag{5.16}$$

which holds for any symmetric-tracefree tensor A_{2ab}.

Making these substitutions in Eq. (5.11), we arrive at

$$g_{vv} = -1 - r^2 A_2 - r^3 A_3 - r^4 A_4 + r^4 A_{2a}A_2^a + O(r^5), \tag{5.17a}$$
$$g_{va} = \Omega_a - r^2 A_{2a} - r^3 A_{3a} - r^4 A_{4a} + r^4 A_{2ab}A_2^b + O(r^5), \tag{5.17b}$$
$$g_{ab} = \gamma_{ab} - r^2 A_{2ab} - r^3 A_{3ab} - r^4 A_{4ab} + \frac{1}{5}r^4\gamma_{ab}A_{2cd}A_2^{cd} + O(r^5). \tag{5.17c}$$

It is understood that the vector potentials A_{2a}, A_{3a}, A_{4a}, and $A_{2ab}A_2^b$ are transverse, in the sense that they are all orthogonal to Ω^a. And it is now understood that the tensor potentials A_{2ab}, A_{3ab}, and A_{4ab} are *both* transverse *and* tracefree; the potential $\gamma_{ab}A_{2cd}A_2^{cd}$ is transverse and pure trace.

In quasispherical coordinates the metric is

$$g_{vv} = -1 - r^2 A_2 - r^3 A_3 - r^4 A_4 + r^4 A_{2A}A_2^A + O(r^5), \tag{5.18a}$$
$$g_{vr} = 1, \tag{5.18b}$$
$$g_{vA} = -r^3 A_{2A} - r^4 A_{3A} - r^5 A_{4A} + r^5 A_{2AB}A_2^B + O(r^6), \tag{5.18c}$$
$$g_{AB} = r^2\Omega_{AB} - r^4 A_{2AB} - r^5 A_{3AB} - r^6 A_{4AB} + \frac{1}{5}r^6\Omega_{AB}A_{2CD}A_2^{CD} + O(r^7). \tag{5.18d}$$

Here also the tensor potentials are tracefree, except for the term proportional to Ω_{AB}, which is pure trace.

C. Field equations II

The metric of Eqs. (5.17) and (5.18) is a partial solution to the Einstein field equations. In its quasi-Cartesian form the metric involves the transverse potentials A_{2a}, A_{3a}, A_{4a}, A_{2ab}, A_{3ab}, and A_{4ab}, and imposing the vacuum equation $R^{vv} = 0$ has revealed the important fact that the tensor potentials are all tracefree. In the quasispherical form of the metric, the potentials have components in the angular directions only, and these depend on θ^A only. To obtain a complete solution to the field equations we must now determine the potentials.

We rely on Zhang's work [18], which allows us to state that *the metric of a vacuum region of spacetime around a timelike geodesic is a functional of two (and only two) sets of tidal moments* $\mathcal{E}_{a_1a_2\cdots a_l}$ *and* $\mathcal{B}_{a_1a_2\cdots a_l}$; *the tidal moments are STF tensors that depend on proper time on the world line, and they are related to components of the Weyl tensor (and its derivatives) evaluated on the world line*. For a complete description of the metric one requires an infinite number of tidal moments; for an approximate description one requires a finite number. In our case the construction of the metric shall involve the quadrupole moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$, the octupole moments $\mathcal{E}_{abc}$ and $\mathcal{B}_{abc}$, and the hexadecapole moments $\mathcal{E}_{abcd}$ and $\mathcal{B}_{abcd}$. These were introduced in Sec. II A, and their scaling properties are described in Sec. II B.

Collecting the observations summarized in the preceding two paragraphs, we obtain three important guiding rules for the construction of the metric:

(1) The metric is constructed from scalar potentials, vector potentials that are transverse, and tensor potentials that are transverse and tracefree.
(2) The potentials depend on the angles θ^A only; they are independent of r, which appears as a multiplicative factor in front of the potentials.
(3) The potentials depend on two sets of tidal moments.

The rules imply that the metric must be constructed from the tidal potentials introduced in Secs. II D and II E. There are no other possible building blocks for the metric.

We begin the construction of the metric with the determination of A_2, A_{2A}, and A_{2AB}. (In practical matters it is convenient to deal with the quasispherical representation of the metric, because the r-dependence is then explicitly known.) These terms occur at order r^2 in g_{vv}, r^3 in g_{vA}, and r^4 in g_{AB}, and proper dimensionality requires that the potentials scale as $\mathcal{R}^{-2}$. Equation (2.12) then implies that the potentials must be constructed from $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$. The possible building blocks are listed in Tables VIII and IX, and we write $A_2 = a\mathcal{E}^{\mathsf{q}}$, $A_{2A} = b\mathcal{E}^{\mathsf{q}}_A + p\mathcal{B}^{\mathsf{q}}_A$, $A_{2AB} = c\mathcal{E}^{\mathsf{q}}_{AB} + q\mathcal{B}^{\mathsf{q}}_{AB}$, where a, b, c, p, and q are undetermined numerical coefficients. From this we form the metric $g_{vv} = -1 - r^2A_2 + O(r^3)$, $g_{vr} = 1$, $g_{vA} = -r^3A_{2A} + O(r^4)$, and $g_{AB} = r^2\Omega_{AB} - r^4A_{2AB} + O(r^5)$, which we substitute into the vacuum field equations. (At this stage of the computations the v-dependence of the tidal potentials can be ignored, because their v-derivatives are suppressed by a factor of order $\mathcal{R}^{-1}$ relative to the spatial derivatives. Another simplifying move is momentarily to switch off the ϕ-dependence of the metric by adopting $\mathcal{E}^{\mathsf{q}}_0$ and $\mathcal{B}^{\mathsf{q}}_0$ as the only nonvanishing components of the tidal moments.) The exercise returns the relations $b = \frac{2}{3}a$, $c = \frac{1}{3}a$, and $q = \frac{1}{2}p$, which leaves a and p as undetermined constants. To obtain a and p we compute the frame components C_{a0b0} and C_{abc0} of the Weyl tensor in the limit $r \to 0$, and demand that the results agree with Eqs. (2.4). From this we find that $a = 1$ and $p = -\frac{2}{3}$, and the metric is known through order $\mathcal{R}^{-2}$.

For the computation of the frame components of the Weyl tensor we need

$$u^v = 1, \qquad u^r = 0, \qquad u^A = 0, \tag{5.19a}$$

$$e^v_a = \Omega_a, \qquad e^r_a = \Omega_a, \qquad e^A_a = \frac{1}{r}\Omega^A_a, \tag{5.19b}$$

the components of the tetrad vectors in the light-cone coordinates. Here $\Omega^A_a := \delta_{ab}\Omega^{AB}\Omega^b_B$, and the factor of r^{-1} compensates for factors of r in the angular components of the metric. It is easy to show that the vectors are orthonormal and parallel-transported along u^α in the limit $r \to 0$. We observe that while some components of the Weyl tensor are ambiguous or go to zero in the limit $r \to 0$, the frame components are well-behaved and have a well-defined limit.

We continue with the determination of A_3, A_{3A}, and A_{3AB}. Here the potentials must scale as $\mathcal{R}^{-3}$, and Eq. (2.12) implies that they must be constructed from $\dot{\mathcal{E}}_{ab}$, $\dot{\mathcal{B}}_{ab}$, $\mathcal{E}_{abc}$, and $\mathcal{B}_{abc}$. The possible building blocks are listed in Tables VIII and IX, and we express A_3 as a linear combination of $\dot{\mathcal{E}}^{\mathsf{q}}$ and $\mathcal{E}^{\mathsf{o}}$, A_{3A} as a linear combination of $\dot{\mathcal{E}}^{\mathsf{q}}_A$, $\dot{\mathcal{B}}^{\mathsf{q}}_A$, $\mathcal{E}^{\mathsf{o}}_A$, and $\mathcal{B}^{\mathsf{o}}_A$, and A_{3AB} as a linear combination of $\dot{\mathcal{E}}^{\mathsf{q}}_{AB}$, $\dot{\mathcal{B}}^{\mathsf{q}}_{AB}$, $\mathcal{E}^{\mathsf{o}}_{AB}$, and $\mathcal{B}^{\mathsf{o}}_{AB}$. The numerical coefficients are determined in two steps. First, we construct an improved metric by appending the terms of order $\mathcal{R}^{-3}$, and we substitute it into the vacuum field equations. (In this step the v-dependence of the quadrupole tidal potentials must be taken into account, again keeping in mind that the v-derivatives are suppressed by a factor of order $\mathcal{R}^{-1}$ relative to the spatial derivatives. It is still possible momentarily to switch off the ϕ-dependence of the metric by adopting $\mathcal{E}^{\mathsf{q}}_0$, $\mathcal{B}^{\mathsf{q}}_0$, $\mathcal{E}^{\mathsf{o}}_0$, and $\mathcal{B}^{\mathsf{o}}_0$ as the only nonvanishing components of the tidal moments.) The first step leaves two coefficients undetermined, a common multiplicative factor in front of $\mathcal{E}_{abc}$, and another common factor in front of $\mathcal{B}_{abc}$. These are determined in the second step, in which we compute the frame components $C_{a0b0|c}$ and $C_{abc0|d}$ of the covariant derivatives of the Weyl tensor in the limit $r \to 0$, and demand that the results agree with Eqs. (2.6). The metric is now known through order $\mathcal{R}^{-3}$.

We complete the computation of the metric with the determination of A_4, A_{4A}, and A_{4AB}. Here the potentials must scale as $\mathcal{R}^{-4}$, and Eq. (2.12) implies that they must be constructed from $\ddot{\mathcal{E}}_{ab}$, $\ddot{\mathcal{B}}_{ab}$, $\dot{\mathcal{E}}_{abc}$, $\dot{\mathcal{B}}_{abc}$, $\mathcal{E}_{abcd}$, and $\mathcal{B}_{abcd}$. That is not all, however, because at order $\mathcal{R}^{-4}$ we must also include potentials that are generated by quadratic combinations of $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$; the list of potentials is long, and this makes the computations much more involved than in the previous cases. The possible building blocks are listed in Tables VIII, IX, X, XI, XII, and XIII. We express A_4 as a linear combination of $\ddot{\mathcal{E}}^{\mathsf{q}}$, $\dot{\mathcal{E}}^{\mathsf{o}}$, $\mathcal{E}^{\mathsf{h}}$, $\mathcal{P}^{\mathsf{m}}$, $\mathcal{P}^{\mathsf{q}}$, $\mathcal{P}^{\mathsf{h}}$, $\mathcal{Q}^{\mathsf{m}}$, $\mathcal{Q}^{\mathsf{q}}$, $\mathcal{Q}^{\mathsf{h}}$, $\mathcal{G}^{\mathsf{d}}$, and $\mathcal{G}^{\mathsf{o}}$. We express A_{4A} as a linear combination of $\ddot{\mathcal{E}}^{\mathsf{q}}_A$, $\dot{\mathcal{E}}^{\mathsf{o}}_A$, $\mathcal{E}^{\mathsf{h}}_A$, $\ddot{\mathcal{B}}^{\mathsf{q}}_A$, $\dot{\mathcal{B}}^{\mathsf{o}}_A$, $\mathcal{B}^{\mathsf{h}}_A$, $\mathcal{P}^{\mathsf{q}}_A$, $\mathcal{P}^{\mathsf{h}}_A$, $\mathcal{Q}^{\mathsf{q}}_A$, $\mathcal{Q}^{\mathsf{h}}_A$, $\mathcal{G}^{\mathsf{d}}_A$, $\mathcal{G}^{\mathsf{o}}_A$, $\mathcal{H}^{\mathsf{q}}_A$, and $\mathcal{H}^{\mathsf{h}}_A$. And we express A_{4AB} as a linear combination of $\ddot{\mathcal{E}}^{\mathsf{q}}_{AB}$, $\dot{\mathcal{E}}^{\mathsf{o}}_{AB}$, $\mathcal{E}^{\mathsf{h}}_{AB}$, $\ddot{\mathcal{B}}^{\mathsf{q}}_{AB}$, $\dot{\mathcal{B}}^{\mathsf{o}}_{AB}$, $\mathcal{B}^{\mathsf{h}}_{AB}$, $\mathcal{P}^{\mathsf{q}}_{AB}$, $\mathcal{P}^{\mathsf{h}}_{AB}$, $\mathcal{Q}^{\mathsf{q}}_{AB}$, $\mathcal{Q}^{\mathsf{h}}_{AB}$, $\mathcal{G}^{\mathsf{d}}_{AB}$, $\mathcal{G}^{\mathsf{o}}_{AB}$, $\mathcal{H}^{\mathsf{q}}_{AB}$, and $\mathcal{H}^{\mathsf{h}}_{AB}$. In addition to all this the angular components of the metric contain a bilinear term proportional to $A_{2AB}A_2^{AB} = \frac{1}{9}(\mathcal{E}^{\mathsf{q}}_{AB} - \mathcal{B}^{\mathsf{q}}_{AB}) \times (\mathcal{E}^{\mathsf{q}AB} - \mathcal{B}^{\mathsf{q}AB})$, which can be simplified with the help of Eqs. (B2). The many numerical coefficients that appear in the metric at order $\mathcal{R}^{-4}$ are determined as we did previously. Substitution of the metric in the vacuum field equations determines all but two coefficients, the common multiplicative factors in front of $\mathcal{E}_{abcd}$ and $\mathcal{B}_{abcd}$. These are determined by computing the frame components

$C_{a0b0|cd}$ and $C_{abc0|de}$ of the second covariant derivatives of the Weyl tensor in the limit $r \rightarrow 0$, and demanding that the results agree with Eqs. (2.7). The metric is now known through order $\mathcal{R}^{-4}$, and the end result is displayed in Eqs. (3.4). The quasi-Cartesian representation of Eqs. (3.3) can be obtained directly from this.

This completes the derivation of the background metric. An alternative derivation is presented in Appendix D.

VI. BLACK-HOLE METRIC: DERIVATION

We next turn to a derivation of the black-hole metric of Eqs. (3.7), a vacuum perturbation of the Schwarzschild solution

$$ds^2 = -f dv^2 + 2dvdr + r^2 d\Omega^2, \tag{6.1}$$

where $f = 1 - 2M/r$ and $d\Omega^2 = \Omega_{AB} d\theta^A d\theta^B = d\theta^2 + \sin^2\theta d\phi^2$. To construct the perturbation we rely heavily on the formalism of Martel and Poisson [33], and we implement the light-cone gauge of Preston and Poisson [34]. The computations that lead to the black-hole metric are extremely lengthy, and are presented in four highly technical subsections. In Secs. VI A and VI B we construct the linear piece of the metric perturbation, and in Secs. VI C and VI D we turn to the bilinear piece. The results are collected in Sec. VI E.

A. Linear perturbation: Preparation

1. Form of the metric perturbation and considerations of gauge

The metric of Eqs. (3.4) specifies the asymptotic conditions (when $r \gg 2M$) for the metric of a tidally deformed black hole. Focusing our attention on the linearly perturbed piece of the metric, we write it as

$$g_{vv} = -f - r^2 e_1^{\mathsf{q}} \mathcal{E}^{\mathsf{q}} + \frac{1}{3} r^3 e_2^{\mathsf{q}} \dot{\mathcal{E}}^{\mathsf{q}} - \frac{2}{21} r^4 e_3^{\mathsf{q}} \ddot{\mathcal{E}}^{\mathsf{q}} - \frac{1}{3} r^3 e_1^{\mathsf{o}} \mathcal{E}^{\mathsf{o}} + \frac{1}{6} r^4 e_2^{\mathsf{o}} \dot{\mathcal{E}}^{\mathsf{o}} - \frac{1}{12} r^4 e_1^{\mathsf{h}} \mathcal{E}^{\mathsf{h}} + \text{bilinear}, \tag{6.2a}$$

$$g_{vr} = 1, \tag{6.2b}$$

$$\begin{aligned} g_{vA} = {} & -\frac{2}{3} r^3 (e_4^{\mathsf{q}} \mathcal{E}_A^{\mathsf{q}} - b_4^{\mathsf{q}} \mathcal{B}_A^{\mathsf{q}}) + \frac{1}{3} r^4 (e_5^{\mathsf{q}} \dot{\mathcal{E}}_A^{\mathsf{q}} - b_5^{\mathsf{q}} \dot{\mathcal{B}}_A^{\mathsf{q}}) - \frac{8}{63} r^5 (e_6^{\mathsf{q}} \ddot{\mathcal{E}}_A^{\mathsf{q}} - b_6^{\mathsf{q}} \ddot{\mathcal{B}}_A^{\mathsf{q}}) - \frac{1}{4} r^4 (e_4^{\mathsf{o}} \mathcal{E}_A^{\mathsf{o}} - b_4^{\mathsf{o}} \mathcal{B}_A^{\mathsf{o}}) + \frac{1}{6} r^5 (e_5^{\mathsf{o}} \dot{\mathcal{E}}_A^{\mathsf{o}} - b_5^{\mathsf{o}} \dot{\mathcal{B}}_A^{\mathsf{o}}) \\ & - \frac{1}{15} r^5 (e_4^{\mathsf{h}} \mathcal{E}_A^{\mathsf{h}} - b_4^{\mathsf{h}} \mathcal{B}_A^{\mathsf{h}}) + \text{bilinear}, \end{aligned} \tag{6.2c}$$

$$\begin{aligned} g_{AB} = {} & r^2 \Omega_{AB} - \frac{1}{3} r^4 (e_7^{\mathsf{q}} \mathcal{E}_{AB}^{\mathsf{q}} - b_7^{\mathsf{q}} \mathcal{B}_{AB}^{\mathsf{q}}) + \frac{5}{18} r^5 (e_8^{\mathsf{q}} \dot{\mathcal{E}}_{AB}^{\mathsf{q}} - b_8^{\mathsf{q}} \dot{\mathcal{B}}_{AB}^{\mathsf{q}}) - \frac{1}{7} r^6 (e_9^{\mathsf{q}} \ddot{\mathcal{E}}_{AB}^{\mathsf{q}} - b_9^{\mathsf{q}} \ddot{\mathcal{B}}_{AB}^{\mathsf{q}}) - \frac{1}{6} r^5 (e_7^{\mathsf{o}} \mathcal{E}_{AB}^{\mathsf{o}} - b_7^{\mathsf{o}} \mathcal{B}_{AB}^{\mathsf{o}}) \\ & + \frac{3}{20} r^6 (e_8^{\mathsf{o}} \dot{\mathcal{E}}_{AB}^{\mathsf{o}} - b_8^{\mathsf{o}} \dot{\mathcal{B}}_{AB}^{\mathsf{o}}) - \frac{1}{20} r^6 (e_7^{\mathsf{h}} \mathcal{E}_{AB}^{\mathsf{h}} - b_7^{\mathsf{h}} \mathcal{B}_{AB}^{\mathsf{h}}) + \text{bilinear}, \end{aligned} \tag{6.2d}$$

where the undetermined radial functions $e_n^{\mathsf{q}}(r)$, $e_n^{\mathsf{o}}(r)$, $e_n^{\mathsf{h}}(r)$, $b_n^{\mathsf{q}}(r)$, $b_n^{\mathsf{o}}(r)$, and $b_n^{\mathsf{h}}(r)$ are all required to approach unity when $2M/r \rightarrow 0$. This ansatz for the metric is motivated by the facts that (i) it reduces to the Schwarzschild metric when the tidal fields are turned off; (ii) it reduces to the (linear piece of the) background metric when $M \rightarrow 0$ (so that the radial functions all become equal to unity); and (iii) its expansion in terms of tidal potentials constitutes a decomposition of the metric perturbation into a complete basis of spherical-harmonic modes. An important aspect of the metric is that its v-dependence is assumed to be slow (with a time scale of the order of $\mathcal{R}$) and contained in the tidal moments; we rule out time-dependent processes that take place over time scales comparable to $2M$.

We write the black-hole metric as $g_{\alpha\beta} = \hat{g}_{\alpha\beta} + p_{\alpha\beta}$, in which $\hat{g}_{\alpha\beta}$ is the Schwarzschild metric of Eq. (6.1) and $p_{\alpha\beta}$ is the tidal perturbation. In Eq. (6.2) the perturbation is presented in the Preston-Poisson light-cone gauge, which enforces the conditions $p_{vr} = p_{rr} = p_{rA} = 0$. The even-parity sector of the perturbation is

$$p_{vv} = \sum_{lm} h_{vv}^{lm} Y^{lm}, \tag{6.3a}$$

$$p_{vA} = \sum_{lm} j_v^{lm} Y_A^{lm}, \tag{6.3b}$$

$$p_{AB} = r^2 \sum_{lm} (K^{lm} \Omega_{AB} Y^{lm} + G^{lm} Y_{AB}^{lm}), \tag{6.3c}$$

where the reduced perturbations h_{vv}^{lm}, j_v^{lm}, K^{lm}, and G^{lm} depend on v and r only. Preston and Poisson show that for vacuum perturbations, the gauge can be refined to also enforce $K^{lm} = 0$. This leaves h_{vv}^{lm}, j_v^{lm}, and G^{lm} as non-vanishing perturbations, and these can be read off from Eqs. (6.2). The odd-parity sector of the perturbation is

$$p_{vA} = \sum_{lm} h_v^{lm} X_A^{lm}, \tag{6.4a}$$

$$p_{AB} = \sum_{lm} h_2^{lm} X_{AB}^{lm}, \tag{6.4b}$$

in which h_v^{lm} and h_2^{lm} depend on v and r only; these also can be read off from Eqs. (6.2).

Preston and Poisson show that the residual gauge freedom that remains within the even-parity sector of the perturbation is a family characterized by an arbitrary function $a^{lm}(v)$—one function for each l and m. Under such a gauge transformation the perturbations change according to

$$h_{vv} \to h_{vv} + l(l+1)\frac{M}{r^2}a - 2\left[\frac{1}{2}l(l+1) - 1 + \frac{3M}{r}\right]\dot{a} + 2r\ddot{a}, \tag{6.5a}$$

$$j_v \to j_v - \left[\frac{1}{2}l(l+1) - f\right]a + \frac{2r^2}{l(l+1)}\ddot{a}, \tag{6.5b}$$

$$G \to G - \frac{2}{r}a + \frac{4}{l(l+1)}\dot{a}, \tag{6.5c}$$

where the lm label was omitted for ease of notation. In addition, Preston and Poisson show that the residual gauge freedom that remains within the odd-parity sector of the perturbation is a family characterized by an arbitrary function $\alpha^{lm}(v)$—one function for each l and m. Under such a gauge transformation the perturbations change according to

$$h_v \to h_v - r^2\dot{\alpha}, \tag{6.6a}$$

$$h_2 \to h_2 - 2r^2\alpha. \tag{6.6b}$$

We shall use the residual gauge freedom to specialize the light-cone gauge to a *horizon-locking gauge* defined by the requirements $p_{vv} = p_{vr} = p_{vA} = 0$ at $r = 2M$. Since the light-cone gauge already enforces $p_{vr} = 0$ everywhere, the horizon-locking gauge requires

$$h^{lm}_{vv} = j^{lm}_v = 0 = h^{lm}_v \quad \text{at } r = 2M. \tag{6.7}$$

Because the residual gauge freedom is limited to two functions $a^{lm}(v)$ and $\alpha^{lm}(v)$, it may seem doubtful that the three conditions of Eq. (6.7) can be imposed. Nevertheless, we shall see that the Einstein field equations do allow the specialization of the light-cone gauge to the horizon-locking gauge. We saw in Sec. IV A that the conditions of Eq. (6.7) ensure that the horizon keeps its coordinate description $r = 2M$ in the perturbed spacetime.

The horizon-locking gauge, together with the vacuum field equations, imply the existence of a useful constraint on the values of h^{lm}_v and h^{lm}_2 at $r = 2M$. We examine the perturbation equation $P^r_{lm} = 0$ (in the notation of Martel and Poisson) and evaluate it at $r = 2M$. Making use of the statement $h^{lm}_v = 0$ returns

$$0 = -\frac{(l-1)(l+2)}{8M^2}\frac{\partial h_2}{\partial v} - \frac{\partial}{\partial v}\frac{\partial h_v}{\partial r}. \tag{6.8}$$

Integration with respect to v, keeping r anchored at $r = 2M$, yields

$$h^{lm}_2(v, r = 2M) = -\frac{8M^2}{(l-1)(l+2)}\frac{\partial h^{lm}_v}{\partial r}\bigg|_{r=2M}. \tag{6.9}$$

The constant of integration was set equal to zero to respect the ansatz of Eqs. (6.2), which implies that there is no v-independent term in the perturbation. We shall refer to Eq. (6.9) as the *horizon-locking constraint*; it refers to the odd-parity sector of the perturbation only.

2. Redefinition of tidal moments

In addition to making use of the residual gauge freedom, the form of the metric perturbation can be adjusted by redefining the tidal moments $\mathcal{E}_{ab}$, $\mathcal{E}_{abc}$, $\mathcal{B}_{ab}$, and $\mathcal{B}_{abc}$ according to $\mathcal{E} \to \mathcal{E} + p_1 M\dot{\mathcal{E}} + p_2 M^2\ddot{\mathcal{E}} + \cdots$ and $\mathcal{B} \to \mathcal{B} + q_1 M\dot{\mathcal{B}} + q_2 M^2\ddot{\mathcal{B}} + \cdots$. As we shall see below, these redefinitions have the effect of inducing changes in the radial functions e^{q}_n, e^{o}_n, b^{q}_n, and b^{o}_n.

3. Even-parity sector, $l = 2$

According to Eqs. (6.2) and Table VIII, the even-parity, $l = 2$ piece of the perturbation is described by

$$h_{vv} = -r^2 e_1\mathcal{E} + \frac{1}{3}r^3 e_2\dot{\mathcal{E}} - \frac{2}{21}r^4 e_3\ddot{\mathcal{E}} + \cdots, \tag{6.10a}$$

$$j_v = -\frac{1}{3}r^3 e_4\mathcal{E} + \frac{1}{6}r^4 e_5\dot{\mathcal{E}} - \frac{4}{63}r^5 e_6\ddot{\mathcal{E}} + \cdots, \tag{6.10b}$$

$$G = -\frac{1}{3}r^2 e_7\mathcal{E} + \frac{5}{18}r^3 e_8\dot{\mathcal{E}} - \frac{1}{7}r^4 e_9\ddot{\mathcal{E}} + \cdots, \tag{6.10c}$$

in which $e_n := e^{\mathsf{q}}_n(r)$ and $\mathcal{E} := \mathcal{E}^{\mathsf{q}}_m(v)$. For ease of notation we have omitted the label $lm = 2m$ on the perturbation functions; the radial functions do not depend on m. Under a residual gauge transformation generated by the function $a(v) = -\frac{1}{6}c_1 M^3\mathcal{E} - \frac{2}{9}c_2 M^4\dot{\mathcal{E}} - \frac{1}{63}c_3 M^5\ddot{\mathcal{E}} + \cdots$, the radial functions change according to

$$e_1 \to e_1 + c_1\frac{M^4}{r^4}, \tag{6.11a}$$

$$e_2 \to e_2 + 2c_1\frac{M^3}{r^3} + 3c_1\frac{M^4}{r^4} - 4c_2\frac{M^5}{r^5}, \tag{6.11b}$$

$$e_3 \to e_3 + \frac{7}{2}c_1\frac{M^3}{r^3} - \frac{28}{3}c_2\frac{M^4}{r^4} - 14c_2\frac{M^5}{r^5} + c_3\frac{M^6}{r^6}, \tag{6.11c}$$

$$e_4 \to e_4 - c_1\frac{M^3}{r^3} - c_1\frac{M^4}{r^4}, \tag{6.11d}$$

$$e_5 \to e_5 + \frac{8}{3}c_2\frac{M^4}{r^4} + \frac{8}{3}c_2\frac{M^5}{r^5}, \tag{6.11e}$$

$$e_6 \to e_6 + \frac{7}{8}c_1\frac{M^3}{r^3} - \frac{1}{2}c_3\frac{M^5}{r^5} - \frac{1}{2}c_3\frac{M^6}{r^6}, \tag{6.11f}$$

$$e_7 \to e_7 - c_1\frac{M^3}{r^3}, \tag{6.11g}$$

$$e_8 \to e_8 - \frac{2}{5}c_1\frac{M^3}{r^3} + \frac{8}{5}c_2\frac{M^4}{r^4}, \tag{6.11h}$$

$$e_9 \to e_9 + \frac{28}{27}c_2\frac{M^4}{r^4} - \frac{2}{9}c_3\frac{M^5}{r^5}. \tag{6.11i}$$

Here the residual gauge freedom was reduced from a functional family characterized by an arbitrary function $a(v)$ to a three-parameter family (with parameters c_1, c_2, and c_3). This loss of generality is a choice that is motivated by the observation that the even-parity, $l = 2$ piece of the perturbed metric should be driven by a single function $\mathcal{E}(v)$, so that $a(v)$ should involve only $\mathcal{E}(v)$ and its derivatives. Below we shall seek choices for c_1, c_2, and c_3 that enforce the horizon-locking conditions $e_1 = e_2 = e_3 = e_4 = e_5 = e_6 = 0$ at $r = 2M$.

Implementing the redefinition $\mathcal{E} \to \mathcal{E} - \frac{1}{3} p_1 M\dot{\mathcal{E}} + \frac{2}{21} p_2 M^2 \ddot{\mathcal{E}} + \cdots$ in Eqs. (6.10) has the effect of changing the identity of the radial functions $e_1 \cdots e_9$. They become

$$e_1 \to e_1, \tag{6.12a}$$

$$e_2 \to e_2 + p_1 \frac{M}{r} e_1, \tag{6.12b}$$

$$e_3 \to e_3 + \frac{7}{6} p_1 \frac{M}{r} e_2 + p_2 \frac{M^2}{r^2} e_1, \tag{6.12c}$$

$$e_4 \to e_4, \tag{6.12d}$$

$$e_5 \to e_5 + \frac{2}{3} p_1 \frac{M}{r} e_4, \tag{6.12e}$$

$$e_6 \to e_6 + \frac{7}{8} p_1 \frac{M}{r} e_5 + \frac{1}{2} p_2 \frac{M^2}{r^2} e_4, \tag{6.12f}$$

$$e_7 \to e_7, \tag{6.12g}$$

$$e_8 \to e_8 + \frac{2}{5} p_1 \frac{M}{r} e_7, \tag{6.12h}$$

$$e_9 \to e_9 + \frac{35}{54} p_1 \frac{M}{r} e_8 + \frac{2}{9} p_2 \frac{M^2}{r^2} e_7. \tag{6.12i}$$

Below we shall seek choices for p_1 and p_2 that enforce $e_8 = e_9 = 0$ at $r = 2M$; this does not alter the conditions already imposed by the horizon-locking gauge. With all these choices implemented, only e_7 will be nonvanishing at $r = 2M$.

4. Even-parity sector, $l = 3$

According to Eqs. (6.2) and Table VIII, the even-parity, $l = 3$ piece of the perturbation is described by

$$h_{vv} = -\frac{1}{3} r^3 e_1 \mathcal{E} + \frac{1}{6} r^4 e_2 \dot{\mathcal{E}} + \cdots, \tag{6.13a}$$

$$j_v = -\frac{1}{12} r^4 e_4 \mathcal{E} + \frac{1}{18} r^5 e_5 \dot{\mathcal{E}} + \cdots, \tag{6.13b}$$

$$G = -\frac{1}{18} r^3 e_7 \mathcal{E} + \frac{1}{20} r^4 e_8 \dot{\mathcal{E}} + \cdots, \tag{6.13c}$$

in which $e_n := e_n^{\mathsf{o}}(r)$ and $\mathcal{E} := \mathcal{E}_m^{\mathsf{o}}(v)$. For ease of notation we have omitted the label $lm = 3m$ on the perturbation functions; the radial functions do not depend on m. Under a residual gauge transformation generated by the function $a(v) = -\frac{1}{36} c_1 M^4 \mathcal{E} - \frac{1}{72} c_2 M^5 \dot{\mathcal{E}} + \cdots$, the radial functions change according to

$$e_1 \to e_1 + c_1 \frac{M^5}{r^5}, \tag{6.14a}$$

$$e_2 \to e_2 + \frac{5}{3} c_1 \frac{M^4}{r^4} + c_1 \frac{M^5}{r^5} - c_2 \frac{M^6}{r^6}, \tag{6.14b}$$

$$e_4 \to e_4 - \frac{5}{3} c_1 \frac{M^4}{r^4} - \frac{2}{3} c_1 \frac{M^5}{r^5}, \tag{6.14c}$$

$$e_5 \to e_5 + \frac{5}{4} c_2 \frac{M^5}{r^5} + \frac{1}{2} c_2 \frac{M^6}{r^6}, \tag{6.14d}$$

$$e_7 \to e_7 - c_1 \frac{M^4}{r^4}, \tag{6.14e}$$

$$e_8 \to e_8 - \frac{5}{27} c_1 \frac{M^4}{r^4} + \frac{5}{9} c_2 \frac{M^5}{r^5}. \tag{6.14f}$$

Here (as before) the residual gauge freedom was reduced from a functional family to a two-parameter family. We shall seek choices for c_1 and c_2 that enforce the horizon-locking conditions $e_1 = e_2 = e_4 = e_5 = 0$ at $r = 2M$.

Implementing the redefinition $\mathcal{E} \to \mathcal{E} - \frac{1}{2} p_1 M\dot{\mathcal{E}} + \cdots$ in Eqs. (6.13) has once more the effect of changing the identity of the radial functions. They become

$$e_1 \to e_1, \tag{6.15a}$$

$$e_2 \to e_2 + p_1 \frac{M}{r} e_1, \tag{6.15b}$$

$$e_4 \to e_4, \tag{6.15c}$$

$$e_5 \to e_5 + \frac{3}{4} p_1 \frac{M}{r} e_4, \tag{6.15d}$$

$$e_7 \to e_7, \tag{6.15e}$$

$$e_8 \to e_8 + \frac{5}{9} p_1 \frac{M}{r} e_7. \tag{6.15f}$$

Below we shall seek a choice for p_1 that enforces $e_8 = 0$ at $r = 2M$; this does not alter the conditions already imposed by the horizon-locking gauge. With all these choices implemented, only e_7 will be nonvanishing at $r = 2M$.

5. Even-parity sector, $l = 4$

According to Eqs. (6.2) and Table VIII, the even-parity, $l = 4$ piece of the perturbation is described by

$$h_{vv} = -\frac{1}{12} r^4 e_1 \mathcal{E} + \cdots, \tag{6.16a}$$

$$j_v = -\frac{1}{60} r^5 e_4 \mathcal{E} + \cdots, \tag{6.16b}$$

$$G = -\frac{1}{120} r^4 e_7 \mathcal{E} + \cdots, \tag{6.16c}$$

in which $e_n := e_n^{\mathsf{h}}(r)$ and $\mathcal{E} := \mathcal{E}_m^{\mathsf{h}}(v)$. For ease of notation we have omitted the label $lm = 4m$ on the perturbation functions; the radial functions do not depend on m. Under a residual gauge transformation generated by the function $a(v) = -\frac{1}{240} c_1 M^5 \mathcal{E} + \cdots$, the radial functions change according to

$$e_1 \rightarrow e_1 + c_1 \frac{M^6}{r^6}, \tag{6.17a}$$

$$e_4 \rightarrow e_4 - \frac{9}{4} c_1 \frac{M^5}{r^5} - \frac{1}{2} c_1 \frac{M^6}{r^6}, \tag{6.17b}$$

$$e_7 \rightarrow e_7 - c_1 \frac{M^5}{r^5}. \tag{6.17c}$$

We shall seek a choice for c_1 that enforces the horizon-locking conditions $e_1 = e_4 = 0$ at $r = 2M$. This leaves e_7 as the only nonvanishing function at $r = 2M$.

6. *Odd-parity sector,* $l = 2$

According to Eqs. (6.2) and Table IX, the odd-parity, $l = 2$ piece of the perturbation is described by

$$h_v = \frac{1}{3} r^3 b_4 \mathcal{B} - \frac{1}{6} r^4 b_5 \dot{\mathcal{B}} + \frac{4}{63} r^5 b_6 \ddot{\mathcal{B}} + \cdots, \tag{6.18a}$$

$$h_2 = \frac{1}{3} r^4 b_7 \mathcal{B} - \frac{5}{18} r^5 b_8 \dot{\mathcal{B}} + \frac{1}{7} r^6 b_9 \ddot{\mathcal{B}} + \cdots, \tag{6.18b}$$

in which $b_n := b_n^{\mathsf{q}}(r)$ and $\mathcal{B} := \mathcal{B}_m^{\mathsf{q}}(v)$. For ease of notation we have omitted the label $lm = 2m$ on the perturbation functions; the radial functions do not depend on m. Under a residual gauge transformation generated by the function $\alpha(v) = -\frac{1}{6} k_1 M^2 \mathcal{B} + \frac{5}{36} k_2 M^3 \dot{\mathcal{B}} - \frac{1}{14} k_3 M^4 \ddot{\mathcal{B}} + \cdots$, the radial functions change according to

$$b_4 \rightarrow b_4, \tag{6.19a}$$

$$b_5 \rightarrow b_5 - k_1 \frac{M^2}{r^2}, \tag{6.19b}$$

$$b_6 \rightarrow b_6 - \frac{35}{16} k_2 \frac{M^3}{r^3}, \tag{6.19c}$$

$$b_7 \rightarrow b_7 + k_1 \frac{M^2}{r^2}, \tag{6.19d}$$

$$b_8 \rightarrow b_8 + k_2 \frac{M^3}{r^3}, \tag{6.19e}$$

$$b_9 \rightarrow b_9 + k_3 \frac{M^4}{r^4}. \tag{6.19f}$$

Here the residual gauge freedom was reduced from a functional family characterized by an arbitrary function $\alpha(v)$ to a three-parameter family (with parameters k_1, k_2, and k_3). This loss of generality is a choice that is motivated by the observation that the odd-parity, $l = 2$ piece of the perturbed metric should be driven by a single function $\mathcal{B}(v)$, so that $\alpha(v)$ should involve only $\mathcal{B}(v)$ and its derivatives.

Below we shall seek choices for k_1, k_2, and k_3 that enforce the horizon-locking conditions $b_4 = b_5 = b_6 = 0$ at $r = 2M$. Since b_4 is gauge invariant, this will be achieved if and only if b_4 automatically vanishes at $r = 2M$. Using Eqs. (6.18) we find that the horizon-locking constraint of Eq. (6.9) implies

$$b_7(2M) = -M b_4'(2M), \tag{6.20a}$$

$$b_8(2M) = -\frac{3}{5} M b_5'(2M), \tag{6.20b}$$

$$b_9(2M) = -\frac{4}{9} M b_6'(2M), \tag{6.20c}$$

in which a prime indicates differentiation with respect to r. The last equation will be used to determine k_3.

Implementing the redefinition $\mathcal{B} \rightarrow \mathcal{B} - \frac{1}{2} q_1 M \dot{\mathcal{B}} + \frac{4}{21} q_2 M^2 \ddot{\mathcal{B}} + \cdots$ in Eqs. (6.18) has the effect of changing the identity of the radial functions $b_4 \cdots b_9$. They become

$$b_4 \rightarrow b_4, \tag{6.21a}$$

$$b_5 \rightarrow b_5 + q_1 \frac{M}{r} b_4, \tag{6.21b}$$

$$b_6 \rightarrow b_6 + \frac{21}{16} q_1 \frac{M}{r} b_5 + q_2 \frac{M^2}{r^2} b_4, \tag{6.21c}$$

$$b_7 \rightarrow b_7, \tag{6.21d}$$

$$b_8 \rightarrow b_8 + \frac{3}{5} q_1 \frac{M}{r} b_7, \tag{6.21e}$$

$$b_9 \rightarrow b_9 + \frac{35}{36} q_1 \frac{M}{r} b_8 + \frac{4}{9} q_2 \frac{M^2}{r^2} b_7. \tag{6.21f}$$

Below we shall seek choices for q_1 and q_2 that enforce $b_8 = b_9 = 0$ at $r = 2M$; this does not alter the conditions already imposed by the horizon-locking gauge. With all these choices implemented, only b_7 will be nonvanishing at $r = 2M$.

7. *Odd-parity sector,* $l = 3$

According to Eqs. (6.2) and Table IX, the odd-parity, $l = 3$ piece of the perturbation is described by

$$h_v = \frac{1}{12} r^4 b_4 \mathcal{B} - \frac{1}{18} r^5 b_5 \dot{\mathcal{B}} + \cdots, \tag{6.22a}$$

$$h_2 = \frac{1}{18} r^5 b_7 \mathcal{B} - \frac{1}{20} r^6 b_8 \dot{\mathcal{B}} + \cdots, \tag{6.22b}$$

in which $b_n := b_n^{\mathsf{o}}(r)$ and $\mathcal{B} := \mathcal{B}_m^{\mathsf{o}}(v)$. For ease of notation we have omitted the label $lm = 3m$ on the perturbation functions; the radial functions do not depend on m. Under a residual gauge transformation generated by the function $\alpha(v) = -\frac{1}{36} k_1 M^3 \mathcal{B} + \frac{1}{40} k_2 M^4 \dot{\mathcal{B}} + \cdots$, the radial functions change according to

$$b_4 \rightarrow b_4, \tag{6.23a}$$

$$b_5 \rightarrow b_5 - \frac{1}{2} k_1 \frac{M^3}{r^3}, \tag{6.23b}$$

$$b_7 \rightarrow b_7 + k_1 \frac{M^3}{r^3}, \tag{6.23c}$$

$$b_8 \rightarrow b_8 + k_2 \frac{M^4}{r^4}. \tag{6.23d}$$

Here the residual gauge freedom was reduced to a two-parameter family.

Below we shall seek choices for k_1 and k_2 that enforce the horizon-locking conditions $b_4 = b_5 = 0$ at $r = 2M$. Since b_4 is gauge invariant, this will be achieved if and only if b_4 automatically vanishes at $r = 2M$. Using Eqs. (6.22) we find that the horizon-locking constraint of Eq. (6.9) implies

$$b_7(2M) = -\frac{3}{5}Mb_4'(2M), \tag{6.24a}$$

$$b_8(2M) = -\frac{4}{9}Mb_5'(2M), \tag{6.24b}$$

in which a prime indicates differentiation with respect to r. The last equation will be used to determine k_2.

Implementing the redefinition $\mathcal{B} \to \mathcal{B} - \frac{2}{3}q_1 M\dot{\mathcal{B}} + \cdots$ in Eqs. (6.22) has the effect of changing the identity of the radial functions. They become

$$b_4 \to b_4, \tag{6.25a}$$

$$b_5 \to b_5 + q_1\frac{M}{r}b_4, \tag{6.25b}$$

$$b_7 \to b_7, \tag{6.25c}$$

$$b_8 \to b_8 + \frac{20}{27}q_1\frac{M}{r}b_7. \tag{6.25d}$$

Below we shall seek a choice for q_1 that enforces $b_8 = 0$ at $r = 2M$; this does not alter the conditions already imposed by the horizon-locking gauge. With all these choices implemented, only b_7 will be nonvanishing at $r = 2M$.

8. *Odd-parity sector,* $l = 4$

According to Eqs. (6.2) and Table IX, the odd-parity, $l = 4$ piece of the perturbation is described by

$$h_v = \frac{1}{60}r^5 b_4\mathcal{B} + \cdots, \tag{6.26a}$$

$$h_2 = \frac{1}{120}r^6 b_7\mathcal{B} + \cdots, \tag{6.26b}$$

in which $b_n := b_n^{\mathsf{h}}(r)$ and $\mathcal{B} := \mathcal{B}_m^{\mathsf{h}}(v)$. For ease of notation we have omitted the label $lm = 4m$ on the perturbation functions; the radial functions do not depend on m. Under a residual gauge transformation generated by the function $\alpha(v) = -\frac{1}{240}k_1 M^4\mathcal{B} + \cdots$, the radial functions change according to

$$b_4 \to b_4, \tag{6.27a}$$

$$b_7 \to b_7 + k_1\frac{M^4}{r^4}. \tag{6.27b}$$

Here the residual gauge freedom was reduced to a one-parameter family. The horizon-locking condition is $b_4 = 0$ at $r = 2M$, and this will be achieved if and only if b_4 automatically vanishes at $r = 2M$. Using Eqs. (6.26) we find that the horizon-locking constraint of Eq. (6.9) implies

$$b_7(2M) = -\frac{4}{9}Mb_4'(2M), \tag{6.28}$$

in which a prime indicates differentiation with respect to r. This equation will be used to determine k_1.

B. Linear perturbation: Field equations

1. *Even-parity sector*

We describe in detail the method by which the perturbation equations are integrated in the quadrupole case $l = 2$; the same strategy is employed for the other multipoles. We rely on the formalism of black-hole perturbation theory outlined in Martel and Poisson [33]. The method unfolds in a number of steps.

In the first step the forms $h_{vv} = -r^2 e_1(r)\mathcal{E}(\epsilon v) + \cdots$, $j_v = -\frac{1}{3}r^3 e_4(r)\mathcal{E}(\epsilon v) + \cdots$, and $G = -\frac{1}{3}r^2 e_7(r)\mathcal{E}(\epsilon v) + \cdots$ are inserted within the even-parity field equations. Here ϵ is a bookkeeping parameter that reminds us that derivatives with respect to v are considered to be small; in this first step the field equations are expanded to order ϵ^0, and all terms in $\dot{\mathcal{E}}$ are neglected. In the notation of Martel and Poisson, the equation $Q^{vv} = 0$ is automatically satisfied, and the equations $Q^{vr} = 0$, $Q^{rr} = 0$, and $Q^v = 0$ give rise to a system of three independent differential equations for the three unknown radial functions; the other field equations are related to these by the Bianchi identities. The general solution to the system of equations is easily obtained, and it depends on three integration constants. Imposing regularity at the event horizon determines one constant, and removes all terms proportional to $\log(r - 2M)$ from the radial functions. One of the two remaining integration constants is an overall multiplicative factor that is chosen so that when $r \gg 2M$, the functions e_1, e_4, and e_7 all approach unity. The remaining constant is equivalent to the parameter c_1 that appears in Eqs. (6.11); it characterizes the residual gauge freedom that is still contained within the light-cone class of gauges. This last constant can be selected by simultaneously enforcing $e_1 = e_4 = 0$ at $r = 2M$; this is the horizon-locking condition of Eq. (6.7). The functions e_1, e_4, and e_7 are now completely determined, and the constant c_1 has been chosen.

In the second step the forms

$$h_{vv} = -r^2 e_1(r)\mathcal{E}(\epsilon v) + \frac{1}{3}r^3 e_2(r)\dot{\mathcal{E}}(\epsilon v) + \cdots, \tag{6.29a}$$

$$j_v = -\frac{1}{3}r^3 e_4(r)\mathcal{E}(\epsilon v) + \frac{1}{6}r^4 e_5(r)\dot{\mathcal{E}}(\epsilon v) + \cdots, \tag{6.29b}$$

$$G = -\frac{1}{3}r^2 e_7(r)\mathcal{E}(\epsilon v) + \frac{5}{18}r^3 e_8(r)\dot{\mathcal{E}}(\epsilon v) + \cdots \tag{6.29c}$$

are inserted within the even-parity field equations. The functions e_1, e_4, and e_7 are already known from the first step, and the goal of the second step is to determine e_2, e_5, and e_8. The overdot still indicates differentiation with respect to v; the terms in $\dot{\mathcal{E}}$ are therefore of order ϵ, and higher derivatives are of order ϵ^2 and higher. In the second step the field equations are expanded to order ϵ, and all terms in $\ddot{\mathcal{E}}$ are neglected. As in the first step, the equations $Q^{vr} = 0$, $Q^{rr} = 0$, and $Q^v = 0$ give rise to a system of three independent differential equations for the three unknown radial functions. The general solution is easily obtained, and it depends on three integration constants. As before, imposing regularity at the event horizon determines one of these, and removes all terms proportional to $\log(r - 2M)$ from the radial functions. One of the two remaining constants is equivalent to the parameter c_2 that appears in Eqs. (6.11); it characterizes the residual gauge freedom that is still contained within the light-cone class of gauges. The other constant is equivalent to the parameter p_1 that appears in Eqs. (6.12); it corresponds to a redefinition of the quadrupole tidal moment $\mathcal{E}$. The constant c_2 is selected by simultaneously enforcing $e_2 = e_5 = 0$ at $r = 2M$; this is once more the horizon-locking condition of Eq. (6.7). The constant p_1 is selected by also enforcing $e_8 = 0$ at $r = 2M$. The functions e_2, e_5, and e_8 are now completely determined, and the constants c_2 and p_1 have been chosen.

In the third and last step Eqs. (6.10) are inserted within the field equations, which are expanded to order ϵ^2. The general solution to the system of differential equations for e_3, e_6, and e_9 is easily obtained, and as always it depends on three integration constants. The two that remain after imposing regularity at the event horizon are equivalent to the parameters c_3 and p_2 that appear in Eqs. (6.11) and (6.12). The constant c_3 is selected by simultaneously enforcing $e_3 = e_6 = 0$ at $r = 2M$. The constant p_2 is selected by also enforcing $e_9 = 0$ at $r = 2M$. The functions e_3, e_6, and e_9 are now completely determined, and the constants c_3 and p_2 have been chosen.

The task of solving the linearized field equations in the even-parity, $l = 2$ sector is now completed, and we proceed in the same fashion for the $l = 3$ and $l = 4$ sectors. The radial functions obtained here are listed in Table XIV.

2. Odd-parity sector

Essentially the same strategy is adopted for the odd-parity sector of the metric perturbation. We describe the details for the quadrupole ($l = 2$) sector.

In the first step we insert the expressions $h_v = \frac{1}{3} r^3 b_4(r)\mathcal{B}(\epsilon v) + \cdots$ and $h_2 = \frac{1}{3} r^4 b_7(r)\mathcal{B}(\epsilon v) + \cdots$ within the odd-parity field equations, which are expanded to order ϵ^0. In the notation of Martel and Poisson, the $P^v = 0$ and $P^r = 0$ equations give rise to a system of two differential equations for the two unknowns b_4 and b_7. The general solution to this system depends on three constants of integration. The first is determined by demanding regularity at $r = 2M$. The second is an overall multiplicative factor that is chosen so that b_4 and b_7 approach unity when $r \gg 2M$. The third and final constant of integration is equivalent to the parameter k_1 that appears in Eq. (6.19); it characterizes the residual gauge freedom that is still contained within the light-cone class of gauges. We find that b_4 automatically vanishes at $r = 2M$, and we set the value of k_1 by imposing the first horizon-locking constraint of Eqs. (6.20). The functions b_4 and b_7 are now completely determined, and the constant k_1 has been chosen.

In the second step we insert the expressions

$$h_v = \frac{1}{3} r^3 b_4(r)\mathcal{B}(\epsilon v) - \frac{1}{6} r^4 b_5(r)\dot{\mathcal{B}}(\epsilon v) + \cdots, \tag{6.30a}$$

$$h_2 = \frac{1}{3} r^4 b_7(r)\mathcal{B}(\epsilon v) - \frac{5}{18} r^5 b_8(r)\dot{\mathcal{B}}(\epsilon v) + \cdots \tag{6.30b}$$

within the odd-parity field equations, which are expanded to order ϵ. As in the first step, the general solution to the system of differential equations for b_5 and b_8 depends on three constants of integration. The first is determined by imposing regularity at $r = 2M$. The second constant is equivalent to the gauge parameter k_2 that appears in Eqs. (6.19), and the third is equivalent to the parameter q_1 that appears in Eqs. (6.21); this corresponds to a redefinition of the quadrupole tidal moment $\mathcal{B}$. We find that b_5 automatically vanishes at $r = 2M$, and we choose k_2 so that the second horizon-locking constraint of Eqs. (6.20) is satisfied. Finally, we choose q_1 so that b_8 also vanishes at $r = 2M$. The functions b_5 and b_8 are now completely determined, and the constants k_2 and q_1 have been chosen.

In the third and final step we insert Eqs. (6.18) within the odd-parity field equations, which are now expanded to order ϵ^2. As before the general solution to the system of differential equations for b_6 and b_9 depends on three constants of integration. The first is set by imposing regularity at $r = 2M$. The second is equivalent to the gauge parameter k_3 that appears in Eqs. (6.19), and the third is equivalent to the parameter q_2 that appears in Eqs. (6.21). We find that b_6 automatically vanishes at $r = 2M$. We choose k_3 so that the third of Eqs. (6.20) is satisfied, and q_2 so that b_9 also vanishes at $r = 2M$. The functions b_6 and b_9 are now completely determined, and the constants k_3 and q_2 have been chosen.

The task of solving the linearized field equations in the odd-parity, $l = 2$ sector is now completed, and we proceed in the same fashion for the $l = 3$ and $l = 4$ sectors. The radial functions obtained here are listed in Table XIV.

3. Light-cone gauge and horizon-locking condition

The perturbation obtained in this section is presented in the light-cone gauge introduced in Sec. VI A. The gauge is completely fixed: The residual gauge freedom that was initially left over was removed by imposing the horizon-locking conditions of Eqs. (6.7) and (6.9). In terms of the even-parity radial functions, this means that e_1^{q}, e_2^{q}, e_3^{q}, e_4^{q}, e_5^{q}, e_6^{q}, e_1^{o}, e_2^{o}, e_4^{o}, e_5^{o}, e_1^{h}, and e_4^{h} all vanish at $r = 2M$. In terms of the odd-parity radial functions, we have that b_1^{q},

b_2^{q}, b_3^{q}, b_4^{q}, b_5^{q}, b_6^{q}, b_1^{o}, b_2^{o}, b_4^{o}, b_5^{o}, b_1^{h}, and b_4^{h} all vanish at $r = 2M$. In addition, the freedom described in Sec. VI A to redefine the tidal moments was exploited to force e_8^{q}, e_9^{q}, e_8^{o}, b_8^{q}, b_9^{q}, and b_8^{o} also to vanish at $r = 2M$. With these conditions, the only radial functions that do not vanish at $r = 2M$ are e_7^{q}, e_7^{o}, e_7^{h}, b_7^{q}, b_7^{o}, and b_7^{h}; their values at $r = 2M$ are listed in the caption of Table XIV.

C. Bilinear perturbation: Preparation

1. Form of the metric perturbation and considerations of gauge

We next move on to the bilinear piece of the perturbed metric, which is generated by the linear quadrupole terms. The relevant pieces of the perturbed metric are

$$g_{vv} = -f + \text{linear} + \frac{1}{15} r^4 (p_1^{\mathsf{m}} \mathcal{P}^{\mathsf{m}} + q_1^{\mathsf{m}} \mathcal{Q}^{\mathsf{m}}) + \frac{2}{15} r^4 g_1^{\mathsf{d}} \mathcal{G}^{\mathsf{d}} + \frac{2}{7} r^4 (p_1^{\mathsf{q}} \mathcal{P}^{\mathsf{q}} + q_1^{\mathsf{q}} \mathcal{Q}^{\mathsf{q}}) + \frac{2}{3} r^4 g_1^{\mathsf{o}} \mathcal{G}^{\mathsf{o}} - \frac{1}{3} r^4 (p_1^{\mathsf{h}} \mathcal{P}^{\mathsf{h}} + q_1^{\mathsf{h}} \mathcal{Q}^{\mathsf{h}}) + \cdots, \tag{6.31a}$$

$$g_{vr} = 1, \tag{6.31b}$$

$$g_{vA} = \text{linear} - \frac{8}{75} r^5 g_2^{\mathsf{d}} \mathcal{G}_A^{\mathsf{d}} + \frac{8}{21} r^5 h_2^{\mathsf{q}} \mathcal{H}_A^{\mathsf{q}} + \frac{4}{105} r^5 (p_2^{\mathsf{q}} \mathcal{P}_A^{\mathsf{q}} + 11 q_2^{\mathsf{q}} \mathcal{Q}_A^{\mathsf{q}}) + \frac{2}{5} r^5 g_2^{\mathsf{o}} \mathcal{G}_A^{\mathsf{o}} + r^5 h_2^{\mathsf{h}} \mathcal{H}_A^{\mathsf{h}} - \frac{2}{15} r^5 (p_2^{\mathsf{h}} \mathcal{P}_A^{\mathsf{h}} + q_2^{\mathsf{h}} \mathcal{Q}_A^{\mathsf{h}}) + \cdots, \tag{6.31c}$$

$$g_{AB} = r^2 \Omega_{AB} + \text{linear} + \frac{8}{225} r^6 \Omega_{AB} (p_3^{\mathsf{m}} \mathcal{P}^{\mathsf{m}} + q_3^{\mathsf{m}} \mathcal{Q}^{\mathsf{m}}) + \frac{32}{225} r^6 \Omega_{AB} g_3^{\mathsf{d}} \mathcal{G}^{\mathsf{d}} - \frac{16}{105} r^6 \Omega_{AB} (p_3^{\mathsf{q}} \mathcal{P}^{\mathsf{q}} + q_3^{\mathsf{q}} \mathcal{Q}^{\mathsf{q}}) - \frac{3}{14} r^6 (p_4^{\mathsf{q}} \mathcal{P}_{AB}^{\mathsf{q}} - q_4^{\mathsf{q}} \mathcal{Q}_{AB}^{\mathsf{q}}) + \frac{3}{7} r^6 h_3^{\mathsf{q}} \mathcal{H}_{AB}^{\mathsf{q}} - \frac{8}{45} r^6 \Omega_{AB} g_3^{\mathsf{o}} \mathcal{G}^{\mathsf{o}} + r^6 g_4^{\mathsf{o}} \mathcal{G}_{AB}^{\mathsf{o}} + \frac{2}{45} r^6 \Omega_{AB} (p_3^{\mathsf{h}} \mathcal{P}^{\mathsf{h}} + q_3^{\mathsf{h}} \mathcal{Q}^{\mathsf{h}}) + r^6 (p_4^{\mathsf{h}} \mathcal{P}_{AB}^{\mathsf{h}} + q_4^{\mathsf{h}} \mathcal{Q}_{AB}^{\mathsf{h}}) + r^6 h_3^{\mathsf{h}} \mathcal{H}_{AB}^{\mathsf{h}} + \cdots. \tag{6.31d}$$

The bilinear metric perturbation is decomposed into multipole moments, and with the help of Tables X, XI, XII, and XIII, it can be expressed as in Eqs. (6.3) and (6.4), in terms of scalar, vector, and tensor harmonics. The metric of Eq. (6.31) includes all the tidal potentials that can contribute at bilinear order, including $\mathcal{P}^{\mathsf{q}}$, $\mathcal{H}_A^{\mathsf{h}}$, $\mathcal{G}_{AB}^{\mathsf{o}}$, $\mathcal{P}_{AB}^{\mathsf{h}}$, $\mathcal{Q}_{AB}^{\mathsf{h}}$, and $\mathcal{H}_{AB}^{\mathsf{h}}$ that were missing from Eqs. (3.4). All radial functions are required to approach unity as $r \to \infty$, except for p_1^{q}, g_4^{o}, p_4^{h}, q_4^{h}, h_2^{h}, and h_3^{h}, which are expected to approach zero as a power of $2M/r$.

The form of the metric perturbation can be altered by a residual gauge transformation that preserves the light-cone nature of the coordinate system. Because the gauge freedom was completely exhausted in the linear problem (by enforcing the horizon-locking conditions), the freedom that is left over in the bilinear problem is purely bilinear—the coordinate shifts are necessarily proportional to $\mathcal{E}\mathcal{E}$, $\mathcal{B}\mathcal{B}$, or $\mathcal{E}\mathcal{B}$. Under these circumstances the treatment of bilinear gauge transformations is identical to the linear treatment, and the residual gauge freedom is described by Preston and Poisson [34]. In this context the residual gauge freedom is wider than in the linear problem: We can no longer enforce the tracefree condition $\Omega^{AB} p_{AB} = 0$, which was maintained in the linear problem.

According to Preston and Poisson, the residual gauge freedom in the even-parity sector of the metric perturbation is described by

$$h_{vv} \to h_{vv} + 2r\ddot{a} + 2\left(1 - \frac{3M}{r}\right)\dot{a} - 2\dot{b} + \frac{2M}{r^2} b, \tag{6.32a}$$

$$j_v \to j_v + fa - b - r^2 \dot{c}, \tag{6.32b}$$

$$K \to K + 2\dot{a} + \frac{l(l+1)}{r} a - \frac{2}{r} b + l(l+1)c, \tag{6.32c}$$

$$G \to G - \frac{2}{r} a - 2c, \tag{6.32d}$$

where $a(v)$, $b(v)$, and $c(v)$ are the generators of the gauge transformation. The transformation is specific to each lm mode, and the complete gauge transformation is obtained by summing over the relevant multipoles ($l = \{0, 1, 2, 3, 4\}$).

The residual gauge freedom in the odd-parity sector of the metric perturbation is described by

$$h_v \to h_v - r^2 \dot{\alpha}, \tag{6.33a}$$

$$h_2 \to h_2 - 2r^2 \alpha, \tag{6.33b}$$

where $\alpha(v)$ is the generator of the gauge transformation. This transformation also is specific to each lm mode, and the complete gauge transformation is obtained by summing over the relevant multipoles ($l = \{2, 4\}$).

As we did in the linear problem, we shall exploit the residual gauge freedom to specialize the light-cone gauge to a *horizon-locking gauge* defined by the requirements

$$h_{vv}^{lm} = j_v^{lm} = 0 = h_v^{lm} \quad \text{at } r = 2M. \tag{6.34}$$

These conditions, together with the field equations, give rise to the same *horizon-locking constraint* as in the linear problem:

$$h_2^{lm}(v, r = 2M) = -\frac{8M^2}{(l-1)(l+2)} \frac{\partial h_v^{lm}}{\partial r}\bigg|_{r=2M}. \tag{6.35}$$

This comes about for the following reason: In the notation of Martel and Poisson [33], we find that while P_{lm}^r is no longer identically zero in the bilinear problem (because the linear, quadrupole piece of the metric perturbation produces an effective energy-momentum tensor for the bilinear metric perturbation), its value turns out to be zero at $r = 2M$; the derivation leading to Eq. (6.35) is therefore the same as the one leading to Eq. (6.9) in the linear problem.

In addition to this first set of horizon-locking constraints, another set arises as a consequence of the even-parity perturbation equations, in particular, the Q_{lm}^{rr} equation (again in the notation of Martel and Poisson). When evaluated at $r = 2M$, this equation implies

$$\partial_v K^{lm}(v, r = 2M) = 4M Q_{lm}^{rr}(v, r = 2M) \tag{6.36}$$

when we also impose the horizon-locking conditions $h_{vv}^{lm} = j_v^{lm} = 0$ at $r = 2M$. The right-hand side of this equation is not zero, and the bilinear Q_{lm}^{rr} can be computed from the linear pieces of the metric perturbation. Calculation reveals that the contributions to Q_{lm}^{rr} that originate from terms involving $\dot{\mathcal{E}}_{ab}$ and $\dot{\mathcal{B}}_{ab}$ in the linear perturbation vanish at $r = 2M$, so that $Q_{lm}^{rr}(r = 2M)$ comes entirely from terms that involve $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$. The end result is that $Q_{lm}^{rr}(r = 2M)$ is equal to a quantity quadratic in $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ that is differentiated with respect to v. Equation (6.36) can therefore be integrated with respect to v, and this gives rise to conditions on the value of K^{lm} at $r = 2M$. A detailed examination of Eq. (6.36) shows that these are equivalent to the *horizon-locking constraints*

$$p_3^{\mathsf{m}} = q_3^{\mathsf{m}} = p_3^{\mathsf{q}} = q_3^{\mathsf{q}} = p_3^{\mathsf{h}} = q_3^{\mathsf{h}} = \frac{5}{16} \tag{6.37}$$

and

$$g_3^{\mathsf{d}} = g_3^{\mathsf{o}} = -\frac{5}{16} \tag{6.38}$$

on the value of the radial functions at $r = 2M$.

2. Redefinitions

In addition to making use of the residual gauge freedom, the form of the metric perturbation can be adjusted by redefining the black-hole mass parameter M and the tidal moments $\mathcal{E}_{ab}$, $\mathcal{B}_{ab}$, $\mathcal{E}_{abc}$, and $\mathcal{B}_{abc}$. We consider the changes

$$M \to M + \frac{1}{15} m_1 M^5 \mathcal{E}_{pq}\mathcal{E}^{pq} + \frac{1}{15} m_2 M^5 \mathcal{B}_{pq}\mathcal{B}^{pq}, \tag{6.39a}$$

$$\mathcal{E}_{ab} \to \mathcal{E}_{ab} - \frac{2}{7} m_3 M^2 \mathcal{E}_{p\langle a}\mathcal{E}^p{}_{b\rangle} - \frac{2}{7} m_4 M^2 \mathcal{B}_{p\langle a}\mathcal{B}^p{}_{b\rangle}, \tag{6.39b}$$

$$\mathcal{B}_{ab} \to \mathcal{B}_{ab} + \frac{4}{7} m_5 M^2 \mathcal{E}_{p\langle a}\mathcal{B}^p{}_{b\rangle}, \tag{6.39c}$$

$$\mathcal{E}_{abc} \to \mathcal{E}_{abc} - 2m_6 M \epsilon_{pq\langle a}\mathcal{E}^p{}_b \mathcal{B}^q{}_{c\rangle}, \tag{6.39d}$$

$$\mathcal{B}_{abc} \to \mathcal{B}_{abc}, \tag{6.39e}$$

where the parameters m_n are dimensionless constants, and where the factors of $\frac{1}{15}$, $\frac{2}{7}$, $\frac{4}{7}$, and 2 were inserted for convenience. The redefinition of M formally introduces a time dependence in the black-hole mass parameter; this can be ignored at the level of accuracy maintained in this work, because according to the first of Eqs. (6.39), $\dot{M} = O(M^5/\mathcal{R}^5)$. The impact of the redefinitions on the radial functions will be examined below. We observe that there is no need to consider a change such as $\mathcal{E}_{abcd} \to \mathcal{E}_{abcd} + m_9 \mathcal{E}_{\langle ab}\mathcal{E}_{cd\rangle} + m_{10}\mathcal{B}_{\langle ab}\mathcal{B}_{cd\rangle}$, because this redefinition does not involve M; this ambiguity can be resolved at the level of the background metric.

The redefinitions of Eqs. (6.39) are compatible with the parity rules spelled out in Sec. II C. The rules forbid, for example, the presence of an $\mathcal{E}\mathcal{B}$ term in the shift in $\mathcal{E}_{ab}$, and the presence of $\mathcal{E}^2$ and $\mathcal{B}^2$ terms in the shift in $\mathcal{B}_{ab}$. Similarly, a shift in $\mathcal{B}_{abc}$ is ruled out, because the combinations of $\mathcal{E}$ and $\mathcal{B}$ that could be involved would violate the parity rules.

3. Even-parity sector, $l = 0$

It is easy to pick out the monopole piece of the bilinear metric perturbation from Eqs. (6.31); it involves the tidal moments $\mathcal{P}^{\mathsf{m}} := \mathcal{E}_{pq}\mathcal{E}^{pq}$ and $\mathcal{Q}^{\mathsf{m}} := \mathcal{B}_{pq}\mathcal{B}^{pq}$. Examining the gauge transformation of Eqs. (6.32), we find that the changes for $l = 0$ concern h_{vv} and K only, and that those are generated by two functions of time, $\dot{a}(v)$ and $b(v)$. Restricting the gauge freedom as we have done in the linear problem, we write $\dot{a} = \frac{1}{15} M^4 (c_1 \mathcal{P}^{\mathsf{m}} + d_1 \mathcal{Q}^{\mathsf{m}})$ and $b = \frac{1}{15} M^5 (c_2 \mathcal{P}^{\mathsf{m}} + d_2 \mathcal{Q}^{\mathsf{m}})$, with c_n, d_n denoting dimensionless constants. Neglecting time derivatives, substitution within Eqs. (6.32) and comparison with Eqs. (6.31) reveals that the gauge transformation produces the following changes in the radial functions:

$$p_1^{\mathsf{m}} \to p_1^{\mathsf{m}} + 2c_1\frac{M^4}{r^4} - 6c_1\frac{M^5}{r^5} + 2c_2\frac{M^6}{r^6}, \tag{6.40a}$$

$$p_3^{\mathsf{m}} \to p_3^{\mathsf{m}} + \frac{15}{4}c_1\frac{M^4}{r^4} - \frac{15}{4}c_2\frac{M^5}{r^5}, \tag{6.40b}$$

$$q_1^{\mathsf{m}} \to q_1^{\mathsf{m}} + 2d_1\frac{M^4}{r^4} - 6d_1\frac{M^5}{r^5} + 2d_2\frac{M^6}{r^6}, \tag{6.40c}$$

$$q_3^{\mathsf{m}} \to q_3^{\mathsf{m}} + \frac{15}{4}d_1\frac{M^4}{r^4} - \frac{15}{4}d_2\frac{M^5}{r^5}. \tag{6.40d}$$

The redefinition of M in Eqs. (6.39) produces an additional change in both p_1^{m} and q_1^{m}:

$$p_1^{\mathsf{m}} \to p_1^{\mathsf{m}} + 2m_1\frac{M^5}{r^5}, \tag{6.41a}$$

$$q_1^{\mathsf{m}} \to q_1^{\mathsf{m}} + 2m_2\frac{M^5}{r^5}. \tag{6.41b}$$

Below we shall use the horizon-locking condition $p_1^{\mathsf{m}} = 0$ at $r = 2M$ to determine c_1, leaving p_1^{m} and p_3^{m} dependent on c_2 and m_1. We shall next impose the horizon-locking constraint $p_3^{\mathsf{m}} = 5/16$ at $r = 2M$ to determine m_1. These conditions leave c_2 undetermined, and this is chosen so as to simplify the form of the radial functions. Similarly, we shall use the horizon-locking condition $q_1^{\mathsf{m}} = 0$ at $r = 2M$ to determine d_1, leaving q_1^{m} and q_3^{m} dependent on d_2 and m_2. We shall next impose the horizon-locking constraint $q_3^{\mathsf{m}} = 5/16$ at $r = 2M$ to determine m_2. These conditions leave d_2 undetermined, and this is chosen so as to simplify the form of the radial functions.

4. Even-parity sector, $l = 1$

The dipole piece of the bilinear perturbation involves the tidal potentials $\mathcal{G}^{\mathsf{d}}$ and $\mathcal{G}_A^{\mathsf{d}}$ defined in Table XII. The gauge transformation for $l = 1$ concerns h_{vv}, j_v, and K only. It is generated by three functions of time, which we write as $a = \frac{2}{15}c_1M^5\mathcal{G}^{\mathsf{d}}$, $b = \frac{2}{15}c_2M^5\mathcal{G}^{\mathsf{d}}$, and $c = \frac{2}{15}c_3M^4\mathcal{G}^{\mathsf{d}}$, with c_n denoting dimensionless constants. (There is an abuse of notation here. The functions a, b, and c are specific to each mode $l = 1$, $m = \{0, 1c, 1s\}$; the notation $\mathcal{G}^{\mathsf{d}}$ therefore refers to each harmonic component of the bilinear tidal moment, as listed in Table XII.) Neglecting time derivatives, substitution within Eqs. (6.32) and comparison with Eqs. (6.31) reveal that the gauge transformation produces the following changes in the radial functions:

$$g_1^{\mathsf{d}} \to g_1^{\mathsf{d}} + 2c_2\frac{M^6}{r^6}, \tag{6.42a}$$

$$g_2^{\mathsf{d}} \to g_2^{\mathsf{d}} - \frac{5}{4}(c_1 - c_2)\frac{M^5}{r^5} + \frac{5}{2}c_1\frac{M^6}{r^6}, \tag{6.42b}$$

$$g_3^{\mathsf{d}} \to g_3^{\mathsf{d}} + \frac{15}{8}c_3\frac{M^4}{r^4} + \frac{15}{8}(c_1 - c_2)\frac{M^5}{r^5}. \tag{6.42c}$$

Below we shall use the horizon-locking conditions $g_1^{\mathsf{d}} = g_2^{\mathsf{d}} = 0$ at $r = 2M$ to determine c_2, leaving g_n^{d} dependent on c_1 and c_3. We shall next impose the horizon-locking constraint $g_3^{\mathsf{d}} = -5/16$ at $r = 2M$ to determine c_1. These conditions leave c_3 undetermined, and this is chosen so as to simplify the form of the radial functions.

5. Even-parity sector, $l = 2$

The quadrupole piece of the bilinear perturbation involves the tidal potentials $\mathcal{P}^{\mathsf{q}}$ and $\mathcal{Q}^{\mathsf{q}}$, as well as their vectorial and tensorial counterparts; these are defined in Tables X and XI. The residual gauge transformation is generated by $a = \frac{2}{7}M^5(c_1\mathcal{P}^{\mathsf{q}} + d_1\mathcal{Q}^{\mathsf{q}})$, $b = \frac{2}{7}M^5(c_2\mathcal{P}^{\mathsf{q}} + d_2\mathcal{Q}^{\mathsf{q}})$, and $c = \frac{2}{7}M^4(c_3\mathcal{P}^{\mathsf{q}} + d_3\mathcal{Q}^{\mathsf{q}})$, in which $\mathcal{P}^{\mathsf{q}}$ and $\mathcal{Q}^{\mathsf{q}}$ stand for the harmonic components of the tidal moments listed in Tables X and XI. Neglecting time derivatives, substitution within Eqs. (6.32) and comparison with Eqs. (6.31) reveal that the gauge transformation produces the following changes in the radial functions:

$$p_1^{\mathsf{q}} \to p_1^{\mathsf{m}} + 2c_2\frac{M^6}{r^6}, \tag{6.43a}$$

$$p_2^{\mathsf{q}} \to p_2^{\mathsf{q}} + 15(c_1 - c_2)\frac{M^5}{r^5} - 30c_1\frac{M^6}{r^6}, \tag{6.43b}$$

$$p_3^{\mathsf{q}} \to p_3^{\mathsf{q}} - \frac{45}{4}c_3\frac{M^4}{r^4} - \frac{15}{4}(3c_1 - c_2)\frac{M^5}{r^5}, \tag{6.43c}$$

$$p_4^{\mathsf{q}} \to p_4^{\mathsf{q}} + \frac{8}{3}c_3\frac{M^4}{r^4} + \frac{8}{3}c_1\frac{M^5}{r^5}, \tag{6.43d}$$

$$q_1^{\mathsf{q}} \to q_1^{\mathsf{m}} + 2d_2\frac{M^6}{r^6}, \tag{6.43e}$$

$$q_2^{\mathsf{q}} \to q_2^{\mathsf{q}} + \frac{15}{11}(d_1 - d_2)\frac{M^5}{r^5} - \frac{30}{11}d_1\frac{M^6}{r^6}, \tag{6.43f}$$

$$q_3^{\mathsf{q}} \to q_3^{\mathsf{q}} - \frac{45}{4}d_3\frac{M^4}{r^4} - \frac{15}{4}(3d_1 - d_2)\frac{M^5}{r^5}, \tag{6.43g}$$

$$q_4^{\mathsf{q}} \to q_4^{\mathsf{q}} - \frac{8}{3}d_3\frac{M^4}{r^4} - \frac{8}{3}d_1\frac{M^5}{r^5}. \tag{6.43h}$$

The redefinitions of Eqs. (6.39) produce the additional changes

$$p_1^{\mathsf{q}} \to p_1^{\mathsf{q}} + m_3\frac{M^2}{r^2}f^2, \tag{6.44a}$$

$$p_2^{\mathsf{q}} \to p_2^{\mathsf{q}} + 5m_3\frac{M^2}{r^2}f, \tag{6.44b}$$

$$p_4^{\mathsf{q}} \to p_4^{\mathsf{q}} - \frac{4}{9}m_3\frac{M^2}{r^2}\left(1 - \frac{2M^2}{r^2}\right), \tag{6.44c}$$

$$q_1^{\mathsf{q}} \to q_1^{\mathsf{q}} + m_4\frac{M^2}{r^2}f^2, \tag{6.44d}$$

$$q_2^{\mathsf{q}} \to q_2^{\mathsf{q}} + \frac{5}{11}m_4\frac{M^2}{r^2}f, \tag{6.44e}$$

$$q_4^{\mathsf{q}} \to q_4^{\mathsf{q}} + \frac{4}{9}m_4\frac{M^2}{r^2}\left(1 - \frac{2M^2}{r^2}\right). \tag{6.44f}$$

Below we shall use the horizon-locking conditions $p_1^{\mathsf{q}} = p_2^{\mathsf{q}} = 0$ at $r = 2M$ to determine c_2, leaving p_n^{q} dependent on c_1, c_3, and m_3. We shall next impose the horizon-locking constraint $p_3^{\mathsf{q}} = 5/16$ at $r = 2M$ to determine c_1 and the additional condition $p_4^{\mathsf{q}} = 0$ at $r = 2M$ to determine m_3. These conditions leave c_3 undetermined, and this is chosen so as to simplify the form of the radial functions. Similarly, we shall use the horizon-locking conditions $q_1^{\mathsf{q}} = q_2^{\mathsf{q}} = 0$ at $r = 2M$ to determine d_2, leaving q_n^{q} dependent on d_1, d_3, and m_4. We shall next impose the horizon-locking constraint $q_3^{\mathsf{q}} = 5/16$ at $r = 2M$ to determine d_1 and the additional condition $q_4^{\mathsf{q}} = 0$ at $r = 2M$ to determine m_4. These conditions leave d_3 undetermined, and this is chosen so as to simplify the form of the radial functions.

6. *Even-parity sector,* $l = 3$

The octupole piece of the bilinear perturbation involves the tidal potentials $\mathcal{G}^{\mathsf{o}}$, $\mathcal{G}_A^{\mathsf{o}}$, and $\mathcal{G}_{AB}^{\mathsf{o}}$, which are defined in Table XII. The residual gauge transformation is generated by $a = \frac{2}{3}c_1 M^5 \mathcal{G}^{\mathsf{o}}$, $b = \frac{2}{3}c_2 M^5 \mathcal{G}^{\mathsf{o}}$, and $c = \frac{2}{3}c_3 M^4 \mathcal{G}^{\mathsf{o}}$, in which $\mathcal{G}^{\mathsf{o}}$ stands for the harmonic components of the tidal moments listed in Table XII. Neglecting time derivatives, substitution within Eqs. (6.32) and comparison with Eqs. (6.31) reveal that the gauge transformation produces the following changes in the radial functions:

$$g_1^{\mathsf{o}} \to g_1^{\mathsf{o}} + 2c_2 \frac{M^6}{r^6}, \tag{6.45a}$$

$$g_2^{\mathsf{o}} \to g_2^{\mathsf{o}} + 5(c_1 - c_2)\frac{M^5}{r^5} - 10c_1 \frac{M^6}{r^6}, \tag{6.45b}$$

$$g_3^{\mathsf{o}} \to g_3^{\mathsf{o}} - 45c_3 \frac{M^4}{r^4} - \frac{15}{2}(6c_1 - c_2)\frac{M^5}{r^5}, \tag{6.45c}$$

$$g_4^{\mathsf{o}} \to g_4^{\mathsf{o}} - 4c_3 \frac{M^4}{r^4} - 4c_1 \frac{M^5}{r^5}. \tag{6.45d}$$

The redefinitions of Eqs. (6.39) produce the additional changes

$$g_1^{\mathsf{o}} \to g_1^{\mathsf{o}} + m_6 \frac{M}{r}\left(1 - \frac{M}{r}\right)f^2, \tag{6.46a}$$

$$g_2^{\mathsf{o}} \to g_2^{\mathsf{o}} + \frac{5}{4}m_6 \frac{M}{r}\left(1 - \frac{4M}{3r}\right)f, \tag{6.46b}$$

$$g_4^{\mathsf{o}} \to g_4^{\mathsf{o}} + \frac{1}{3}m_6 \frac{M}{r}\left(f + \frac{4M^3}{5r^3}\right). \tag{6.46c}$$

Below we shall use the horizon-locking conditions $g_1^{\mathsf{o}} = g_2^{\mathsf{o}} = 0$ at $r = 2M$ to determine c_2, leaving g_n^{o} dependent on c_1, c_3, and m_6. We shall next impose the horizon-locking constraint $g_3^{\mathsf{o}} - 5/16$ at $r = 2M$ to determine c_1, and the additional condition $g_4^{\mathsf{o}} = 0$ at $r = 2M$ to determine m_6. These conditions leave c_3 undetermined, and this is chosen so as to simplify the form of the radial functions.

7. *Even-parity sector,* $l = 4$

The hexadecapole piece of the bilinear perturbation involves the tidal potentials $\mathcal{P}^{\mathsf{h}}$ and $\mathcal{Q}^{\mathsf{h}}$, as well as their vectorial and tensorial counterparts; these are defined in Tables X and XI. The residual gauge transformation is generated by $a = -\frac{1}{3}M^5(c_1\mathcal{P}^{\mathsf{h}} + d_1\mathcal{Q}^{\mathsf{h}})$, $b = -\frac{1}{3}M^5(c_2\mathcal{P}^{\mathsf{h}} + d_2\mathcal{Q}^{\mathsf{h}})$, and $c = -\frac{1}{3}M^4(c_3\mathcal{P}^{\mathsf{q}} + d_3\mathcal{Q}^{\mathsf{q}})$, in which $\mathcal{P}^{\mathsf{h}}$ and $\mathcal{Q}^{\mathsf{h}}$ denote the harmonic components of the tidal moments listed in Tables X and XI. Neglecting time derivatives, substitution within Eqs. (6.32) and comparison with Eqs. (6.31) reveal that the gauge transformation produces the following changes in the radial functions:

$$p_1^{\mathsf{h}} \to p_1^{\mathsf{h}} + 2c_2 \frac{M^6}{r^6}, \tag{6.47a}$$

$$p_2^{\mathsf{h}} \to p_2^{\mathsf{h}} + 10(c_1 - c_2)\frac{M^5}{r^5} - 20c_1 \frac{M^6}{r^6}, \tag{6.47b}$$

$$p_3^{\mathsf{h}} \to p_3^{\mathsf{h}} - 150c_3 \frac{M^4}{r^4} - 15(10c_1 - c_2)\frac{M^5}{r^5}, \tag{6.47c}$$

$$p_4^{\mathsf{h}} \to p_4^{\mathsf{h}} + 4c_3 \frac{M^4}{r^4} + 4c_1 \frac{M^5}{r^5}, \tag{6.47d}$$

$$q_1^{\mathsf{h}} \to q_1^{\mathsf{h}} + 2d_2 \frac{M^6}{r^6}, \tag{6.47e}$$

$$q_2^{\mathsf{h}} \to q_2^{\mathsf{h}} + 10(d_1 - d_2)\frac{M^5}{r^5} - 20d_1 \frac{M^6}{r^6}, \tag{6.47f}$$

$$q_3^{\mathsf{h}} \to q_3^{\mathsf{h}} - 150d_3 \frac{M^4}{r^4} - 15(10d_1 - d_2)\frac{M^5}{r^5}, \tag{6.47g}$$

$$q_4^{\mathsf{h}} \to q_4^{\mathsf{h}} + 4d_3 \frac{M^4}{r^4} + 4d_1 \frac{M^5}{r^5}. \tag{6.47h}$$

Below we shall use the horizon-locking conditions $p_1^{\mathsf{h}} = p_2^{\mathsf{h}} = 0$ at $r = 2M$ to determine c_2, leaving p_n^{h} dependent on c_1 and c_3. We shall next impose the horizon-locking constraint $p_3^{\mathsf{h}} = 5/16$ at $r = 2M$ to determine c_1. These conditions leave c_3 undetermined. We shall find that $p_4^{\mathsf{h}} \neq 0$ at $r = 2M$, but that its value is independent of c_3, which is chosen so as to simplify the form of the radial functions. Similarly, we shall use the horizon-locking conditions $q_1^{\mathsf{h}} = q_2^{\mathsf{h}} = 0$ at $r = 2M$ to determine d_2, leaving q_n^{h} dependent on d_1 and d_3. We shall next impose the horizon-locking constraint $q_3^{\mathsf{h}} = 5/16$ at $r = 2M$ to determine d_1. These conditions leave d_3 undetermined. We shall find that $q_4^{\mathsf{h}} \neq 0$ at $r = 2M$, but that its value is independent of d_3, which is chosen so as to simplify the form of the radial functions.

8. *Odd-parity sector,* $l = 2$

The quadrupole piece of the bilinear perturbation involves the tidal potentials $\mathcal{H}_A^{\mathsf{q}}$ and $\mathcal{H}_{AB}^{\mathsf{q}}$ defined in Table XIII. The residual gauge transformation is generated by $\alpha = \frac{3}{7}k_1 M^4 \mathcal{H}^{\mathsf{q}}$, in which $\mathcal{H}^{\mathsf{q}}$ stands for the harmonic components listed in Table XIII. Neglecting time derivatives, substitution within Eqs. (6.33) and comparison with Eqs. (6.31) reveal that the gauge transformation produces the following changes in the radial functions:

$$h_2^{\mathsf{q}} \rightarrow h_2^{\mathsf{q}}, \tag{6.48a}$$

$$h_3^{\mathsf{q}} \rightarrow h_3^{\mathsf{q}} - 2k_1\frac{M^4}{r^4}. \tag{6.48b}$$

The redefinitions of Eqs. (6.39) produce the additional changes

$$h_2^{\mathsf{q}} \rightarrow h_2^{\mathsf{q}} + m_5\frac{M^2}{r^2}f, \tag{6.49a}$$

$$h_3^{\mathsf{q}} \rightarrow h_3^{\mathsf{q}} + \frac{4}{9}m_5\frac{M^2}{r^2}\left(1 - \frac{6M^2}{r^2}\right). \tag{6.49b}$$

The horizon-locking constraint of Eq. (6.35) gives rise to

$$h_3^{\mathsf{q}}(2M) = -\frac{4}{9}M\frac{dh_2^{\mathsf{q}}}{dr}\bigg|_{r=2M}. \tag{6.50}$$

We shall verify that the horizon-locking condition $h_2^{\mathsf{q}}(2M) = 0$ is automatically satisfied, and use Eq. (6.50) to determine k_1. We shall then choose m_5 so that $h_3^{\mathsf{q}}(2M) = 0$.

9. Odd-parity sector, $l = 4$

The hexadecapole piece of the bilinear perturbation involves the tidal potentials $\mathcal{H}_A^{\mathsf{h}}$ and $\mathcal{H}_{AB}^{\mathsf{h}}$ defined in Table XIII. The residual gauge transformation is generated by $\alpha = \frac{1}{6}k_1M^4\mathcal{H}^{\mathsf{h}}$, in which $\mathcal{H}^{\mathsf{h}}$ denotes the harmonic components listed in Table XIII. Neglecting time derivatives, substitution within Eqs. (6.33) and comparison with Eqs. (6.31) reveal that the gauge transformation produces the following changes in the radial functions:

$$h_2^{\mathsf{h}} \rightarrow h_2^{\mathsf{h}}, \tag{6.51a}$$

$$h_3^{\mathsf{h}} \rightarrow h_3^{\mathsf{h}} - 2k_1\frac{M^4}{r^4}. \tag{6.51b}$$

The horizon-locking constraint of Eq. (6.35) gives rise to

$$h_3^{\mathsf{h}}(2M) = -\frac{1}{3}M\frac{dh_2^{\mathsf{h}}}{dr}\bigg|_{r=2M}. \tag{6.52}$$

We shall verify that $h_2^{\mathsf{h}}(2M) = 0$, and use Eq. (6.52) to determine k_1. We shall find that h_3^{h} does not vanish at $r = 2M$, but that its value is gauge invariant.

D. Bilinear perturbation: Field equations

1. General strategy

The solution to the bilinear problem requires a generalization of the perturbation formalism employed in the linear problem. The perturbed metric takes the schematic form $g = \hat{g} + \epsilon p_1 + \epsilon^2 p_2 + O(\epsilon^3)$, in which $\hat{g}$ stands for the Schwarzschild metric, ϵp_1 is the linear perturbation calculated previously, and $\epsilon^2 p_2$ the bilinear perturbation that we now wish to obtain; $\epsilon \sim \mathcal{R}^{-2}$ is a perturbative bookkeeping parameter. While ϵp_1 is linear in the tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$, $\epsilon^2 p_2$ involves terms of the schematic form $\mathcal{E}\mathcal{E}$, $\mathcal{E}\mathcal{B}$, and $\mathcal{B}\mathcal{B}$; those are decomposed into multipole moments and expressed in terms of the potentials $\mathcal{P}^{\mathsf{m}}$, $\mathcal{P}^{\mathsf{q}}$, $\mathcal{P}^{\mathsf{h}}$, $\mathcal{Q}^{\mathsf{m}}$, $\mathcal{Q}^{\mathsf{q}}$, $\mathcal{Q}^{\mathsf{h}}$, $\mathcal{G}^{\mathsf{d}}$, $\mathcal{G}^{\mathsf{o}}$, $\mathcal{H}^{\mathsf{q}}$, and $\mathcal{H}^{\mathsf{o}}$. Notice that the bilinear problem involves the quadrupole tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ only; at order $\epsilon^2 \sim \mathcal{R}^{-4}$ there is no need to involve the octupole and hexadecapole moments that also appear in the linear perturbation.

The Einstein tensor for the perturbed metric takes the schematic form $G[\hat{g}] + \epsilon G_1[p_1] + \epsilon^2 G_1[p_2] + \epsilon^2 G_2[p_1] + O(\epsilon^3)$, where $G[\hat{g}] = 0$ is the Einstein tensor of the Schwarzschild metric, and $\epsilon G_1[p_1] = 0$ is the first-order perturbation created by the linear perturbation ϵp_1. The remaining terms appear at second order. The first contribution, $\epsilon^2 G_1[p_2]$, is the Einstein tensor generated entirely from the second-order perturbation $\epsilon^2 p_2$; G_1 is the same linear differential operator that was encountered in the linear problem. The second contribution, $\epsilon^2 G_2[p_1]$, is generated by the first-order perturbation, and it originates in the nonlinearities of the vacuum field equations.

The bilinear perturbation problem consists of finding solutions to the field equations

$$G_1[p_2] = -G_2[p_1]. \tag{6.53}$$

These have the same formal structure as the linear field equations $G_1[p_1] = 0$, except for the fact that there is a source term on the right-hand side. The problem is tractable because the differential operator on the left-hand side is the same as in the linear problem, and because the source term on the right-hand side can be computed directly from the known solution ϵp_1. Notice than time derivatives of $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ can be ignored when computing $G_1[p_2]$ and $G_2[p_1]$; these contribute at order $\mathcal{R}^{-5}$ and higher, and they do not affect the field equations at order $\epsilon^2 \sim \mathcal{R}^{-4}$.

The strategy to solve the bilinear field equations is the same as in the linear problem. First p_2 is decomposed in scalar, vector, and tensor harmonics, a task that was already accomplished in Eqs. (6.31). Second, the linear differential operator $G_1[p_2]$ is allowed to act on the perturbation, and the result is again decomposed in spherical harmonics. Third, the effective source term $-G_2[p_1]$ is computed and decomposed in spherical harmonics. And fourth, the perturbation equations are integrated, one mode at a time. In practice we rely on the Martel-Poisson perturbation formalism [33] to implement this strategy.

2. Even-parity sector

The even-parity sector of the perturbation p_2 involves the terms in $\mathcal{P}^{\mathsf{m}}$, $\mathcal{P}^{\mathsf{q}}$, $\mathcal{P}^{\mathsf{h}}$, $\mathcal{Q}^{\mathsf{m}}$, $\mathcal{Q}^{\mathsf{q}}$, $\mathcal{Q}^{\mathsf{h}}$, $\mathcal{G}^{\mathsf{d}}$, and $\mathcal{G}^{\mathsf{o}}$; these are decomposed as in Eqs. (6.3). The source terms $-G_2[p_1]$ are computed from p_1 (which, we recall, involves the potentials $\mathcal{E}^{\mathsf{q}}$ and $\mathcal{B}^{\mathsf{q}}$ only), and these also are decomposed as in Eqs. (6.3). In the notation of Martel and Poisson, the reduced source functions are denoted Q^{vv},

Q^{vr}, Q^{rr}, Q^{v}, Q^{r}, $Q^{\flat}$, and $Q^{\sharp}$, in which we suppress the use of the lm labels. Examination of p_2 and calculation of $-G_2[p_1]$ reveal that the relevant multipoles are $l = \{0, 1, 2, 3, 4\}$.

The perturbation equations for $l = 0$ decouple into two sets, the first proportional to $\mathcal{P}^{\sf m}$ and involving the radial functions $p_1^{\sf m}$ and $p_3^{\sf m}$, the second proportional to $\mathcal{Q}^{\sf m}$ and involving the radial functions $q_1^{\sf m}$ and $q_3^{\sf m}$. For each set of equations the general solution depends on three integration constants; these were denoted $\{c_1, c_2, m_1\}$ and $\{d_1, d_2, m_2\}$ in Eqs. (6.40) and (6.41). The constants c_n and d_n are chosen so as to enforce the horizon-locking condition $p_n^{\sf m} = q_n^{\sf m} = 0$ at $r = 2M$. The constants m_1 and m_2 are chosen so as to simplify the form of the radial functions; we use this freedom to remove terms proportional to x^{-5} in $p_1^{\sf m}/f$ and $q_1^{\sf m}/f$, and terms proportional to x^{-4} in $p_3^{\sf m}/f$ and $q_3^{\sf m}/f$. The end results are listed in Table XV.

The perturbation equations for $l = 1$ involve the dipole potentials $\mathcal{G}^{\sf d}$ and the three radial functions $g_n^{\sf d}$. The general solution to the set of coupled differential equations for the radial functions depends on three integration constants; these were denoted $\{c_1, c_2, c_3\}$ in Eqs. (6.42). The constants c_1 and c_2 can be chosen so as to enforce the horizon-locking condition $g_n^{\sf d} = 0$ at $r = 2M$. This leaves c_3 undetermined, and it can be chosen so as to simplify the form of the radial functions; we use this freedom to remove a term proportional to x^{-5} in $g_2^{\sf d}/f$. The end results are displayed in Table XV.

The perturbation equations for $l = 2$ decouple into two sets, the first proportional to $\mathcal{P}^{\sf q}$ and involving the four radial functions $p_n^{\sf q}$, the second proportional to $\mathcal{Q}^{\sf q}$ and involving the four radial functions $q_n^{\sf q}$. For each set of differential equations the general solution depends on five integration constants; one can be eliminated by removing all terms proportional to $\log(r - 2M)$ from the radial functions, and the others correspond to $\{c_1, c_2, c_3, m_3\}$ and $\{d_1, d_2, d_3, m_4\}$ in Eqs. (6.43) and (6.44). The constants c_1, c_2, and m_3 are chosen so as to enforce the horizon-locking condition $p_n^{\sf q} = 0$ at $r = 2M$, while d_1, d_2, and m_4 are chosen to enforce $q_n^{\sf q} = 0$ at $r = 2M$. This leaves c_3 and d_3 undetermined, and those are chosen so as to simplify the form of the radial functions; we use this freedom to remove terms proportional to x^{-5} in both $p_2^{\sf q}/f$ and $q_2^{\sf q}/f$. The end results are listed in Table XV.

The perturbation equations for $l = 3$ involve the octupole potentials $\mathcal{G}^{\sf o}$ and the four radial functions $g_n^{\sf o}$. The general solution to the set of coupled differential equations for the radial functions depends on five integration constants; one can be eliminated by removing all terms proportional to $\log(r - 2M)$ from the radial functions, and the remaining constants were denoted $\{c_1, c_2, c_3, m_6\}$ in Eqs. (6.45) and (6.46). The constants c_1, c_2, and m_6 can be chosen so as to enforce the horizon-locking condition $g_n^{\sf o} = 0$ at $r = 2M$. This leaves c_3 undetermined, and it can be chosen so as to simplify the form of the radial functions; we use this freedom to remove a term proportional to x^{-5} in $g_2^{\sf o}/f$. The end results are displayed in Table XV.

The perturbation equations for $l = 4$ decouple into two sets, the first proportional to $\mathcal{P}^{\sf h}$ and involving the four radial functions $p_n^{\sf h}$, the second proportional to $\mathcal{Q}^{\sf h}$ and involving the four radial functions $q_n^{\sf h}$. For each set of differential equations the general solution depends on five integration constants; one can be eliminated by removing all terms proportional to $\log(r - 2M)$ from the radial functions, and another must be set so that the radial functions (except for $p_4^{\sf h}$ and $q_4^{\sf h}$) approach unity as $r \to \infty$. The remaining three are denoted $\{c_1, c_2, c_3\}$ and $\{d_1, d_2, d_3\}$ in Eqs. (6.47). The constants c_1 and c_2 are chosen so as to enforce the horizon-locking condition $p_1^{\sf h} = p_2^{\sf h} = p_3^{\sf h} = 0$ at $r = 2M$, and we find that the value of $p_4^{\sf h}$ at $r = 2M$ is independent of c_3 (and therefore gauge invariant); we choose c_3 so that a term proportional to x^{-5} is removed from $p_2^{\sf h}/f$. Similarly, d_1 and d_2 are chosen so as to enforce the horizon-locking condition $q_1^{\sf h} = q_2^{\sf h} = q_3^{\sf h} = 0$ at $r = 2M$, and we find that the value of $q_4^{\sf h}$ at $r = 2M$ is independent of d_3 (and therefore gauge invariant); we choose d_3 so that a term proportional to x^{-5} is removed from $q_2^{\sf h}/f$. The end results are listed in Table XV.

3. Odd-parity sector

The odd-parity sector of the perturbation p_2 involves the terms in $\mathcal{H}^{\sf q}$ and $\mathcal{H}^{\sf o}$; these are decomposed as in Eq. (6.4). The source terms $-G_2[p_1]$ are computed from p_1 and also decomposed as in Eq. (6.4). In the notation of Martel and Poisson, the reduced source functions are denoted P^v, P^r, and P, in which we suppress the use of the lm labels. Examination of p_2 and calculation of $-G_2[p_1]$ reveal that the relevant multipoles are $l = \{2, 4\}$.

The perturbation equations for $l = 2$ involve the potentials $\mathcal{H}^{\sf q}$ and the radial functions $h_2^{\sf q}$ and $h_3^{\sf q}$. The general solution to the set of coupled differential equations for the radial functions depends on three integration constants; one can be eliminated by removing all terms proportional to $\log(r - 2M)$ from the radial functions, and the others correspond to $\{k_1, m_5\}$ in Eqs. (6.48) and (6.49). We observe that $h_2^{\sf q} = 0$ at $r = 2M$, and we use Eq. (6.50) to set the value of k_1. Finally, we choose m_5 to enforce the additional condition that $h_3^{\sf q} = 0$ at $r = 2M$. The end results are listed in Table XV.

The perturbation equations for $l = 4$ involve the potentials $\mathcal{H}^{\sf h}$ and the radial functions $h_2^{\sf h}$ and $h_3^{\sf h}$. The general solution to the set of coupled differential equations for the radial functions depends on three integration constants. One of these can be eliminated by removing all terms proportional to $\log(r - 2M)$ from the radial functions, and another must be set so that both $h_2^{\sf h}$ and $h_3^{\sf h}$ approach zero as $r \to \infty$. The remaining constant is denoted k_1 in Eq. (6.51). We observe that $h_2^{\sf h} = 0$ at $r = 2M$, and we use Eq. (6.50) to set the value of k_1. The end results are listed in Table XV.

E. Conclusion

The linear piece of the metric perturbation was computed in Secs. VI A and VI B, and the bilinear piece was computed in Secs. VI C and VI D. The calculation is complete, and after collecting results we obtain the black-hole metric of Eqs. (3.7). The quasi-Cartesian representation of the metric can immediately be constructed from this, and the expressions are given in Eqs. (3.5).

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APPENDIX A: DECOMPOSITION OF TIDAL POTENTIALS IN SPHERICAL HARMONICS

We wish to express the angular version of the tidal potentials listed in Tables I, II, III, IV, V, and VI as expansions in scalar, vector, and tensor harmonics. Before we proceed we record the useful identities

$$\epsilon_{AB} = \epsilon_{abc}\Omega^a_A\Omega^b_B\Omega^c, \tag{A1a}$$

$$\epsilon_A{}^B\Omega^b_B = -\Omega^a_A\epsilon_{ap}{}^b\Omega^p, \tag{A1b}$$

$$D_AD_B\Omega^a = D_BD_A\Omega^a = -\Omega^a\Omega_{AB}. \tag{A1c}$$

These quantities were all introduced in Sec. II E of the main text.

The general structure of the even-parity tidal potentials is

$$\mathcal{A}^{(l)} = \mathcal{A}_{k_1k_2\cdots k_l}\Omega^{k_1}\Omega^{k_2}\cdots\Omega^{k_l}, \tag{A2a}$$

$$\mathcal{A}^{(l)}_a = \gamma_a{}^c\mathcal{A}_{ck_2\cdots k_l}\Omega^{k_2}\cdots\Omega^{k_l}, \tag{A2b}$$

$$\mathcal{A}^{(l)}_{ab} = 2\gamma_a{}^c\gamma_b{}^d\mathcal{A}_{cdk_3\cdots k_l}\Omega^{k_3}\cdots\Omega^{k_l} + \gamma_{ab}\mathcal{A}^{(l)}, \tag{A2c}$$

in which $\mathcal{A}_{k_1k_2\cdots k_l}$ is a constant STF tensor of rank l. It is not difficult to show that these satisfy the eigenvalue equations

$$r^2\gamma^{cd}D_cD_d\mathcal{A}^{(l)} + l(l+1)\mathcal{A}^{(l)} = 0, \tag{A3a}$$

$$r^2\gamma^{cd}D_cD_d\mathcal{A}^{(l)}_a + [l(l+1)-1]\mathcal{A}^{(l)}_a = 0, \tag{A3b}$$

$$r^2\gamma^{cd}D_cD_d\mathcal{A}^{(l)}_{ab} + [l(l+1)-4]\mathcal{A}^{(l)}_{ab} = 0, \tag{A3c}$$

where D_a is a projected differential operator that acts as follows on arbitrary tensor fields: $D_aT_{b_1b_2\cdots} := \gamma_a{}^p\gamma_{b_1}{}^{q_1}\gamma_{b_2}{}^{q_2}\cdots\partial_pT_{q_1q_2\cdots}$.

The transformed potentials are

$$\mathcal{A}^{(l)} = \mathcal{A}_{k_1k_2\cdots k_l}\Omega^{k_1}\Omega^{k_2}\cdots\Omega^{k_l}, \tag{A4a}$$

$$\mathcal{A}^{(l)}_A = \Omega^a_A\mathcal{A}_{ak_2\cdots k_l}\Omega^{k_2}\cdots\Omega^{k_l}, \tag{A4b}$$

$$\mathcal{A}^{(l)}_{AB} = 2\Omega^a_A\Omega^b_B\mathcal{A}_{abk_3\cdots k_l}\Omega^{k_3}\cdots\Omega^{k_l} + \Omega_{AB}\mathcal{A}^{(l)}, \tag{A4c}$$

and we wish to expand them in the even-parity harmonics of Eqs. (2.20). These satisfy the eigenvalue equations

$$\Omega^{CD}D_CD_DY^{lm} + l(l+1)Y^{lm} = 0, \tag{A5a}$$

$$\Omega^{CD}D_CD_DY^{lm}_A + [l(l+1)-1]Y^{lm}_A = 0, \tag{A5b}$$

$$\Omega^{CD}D_CD_DY^{lm}_{AB} + [l(l+1)-4]Y^{lm}_{AB} = 0, \tag{A5c}$$

which are in a close correspondence with Eqs. (A3).

We begin with $\mathcal{A}^{(l)}$, which we decompose as

$$\mathcal{A}^{(l)} = \sum_m \mathcal{A}^{(l)}_m Y^{lm}, \tag{A6}$$

in terms of harmonic components $\mathcal{A}^{(l)}_m$. There are $2l+1$ real terms in the sum, and the $2l+1$ independent components of $\mathcal{A}_{k_1k_2\cdots k_l}$ are in a one-to-one correspondence with the coefficients $\mathcal{A}^{(l)}_m$. Returning to the original representation of Eq. (A2), we find after differentiation that $D_A\mathcal{A}^{(l)} = l\Omega^a_AA_{ak_2\cdots k_l}\Omega^{k_2}\cdots\Omega^{k_l}$, and we conclude that

$$\mathcal{A}^{(l)}_A = \frac{1}{l}D_A\mathcal{A}^{(l)} = \frac{1}{l}\sum_m\mathcal{A}^{(l)}_mY^{lm}_A. \tag{A7}$$

This is the decomposition of $\mathcal{A}^{(l)}_A$ in vectorial, even-parity harmonics; this equation is valid for $l \neq 0$. An additional differentiation using the last of Eqs. (A1) reveals that $D_AD_B\mathcal{A}^{(l)} = -l\Omega_{AB}\mathcal{A}^{(l)} + l(l-1)\Omega^a_A\Omega^b_B\mathcal{A}_{abk_3\cdots k_l}\Omega^{k_3}\cdots\Omega^{k_l}$. From this we conclude that

$$\mathcal{A}^{(l)}_{AB} = \frac{2}{l(l-1)}\left[D_AD_B + \frac{1}{2}l(l+1)\Omega_{AB}\right]\mathcal{A}^{(l)} = \frac{2}{l(l-1)}\sum_m\mathcal{A}^{(l)}_mY^{lm}_{AB}. \tag{A8}$$

This is the decomposition of $\mathcal{A}^{(l)}_{AB}$ in tensorial, even-parity harmonics; this equation is valid for $l \neq \{0, 1\}$.

We next examine the odd-parity potentials. Their general structure is

$$\mathcal{B}^{(l)}_a = \epsilon_{apq}\Omega^p\mathcal{B}^q{}_{k_2\cdots k_l}\Omega^{k_2}\cdots\Omega^{k_l}, \tag{A9a}$$

$$\mathcal{B}^{(l)}_{ab} = (\epsilon_{apq}\Omega^p\mathcal{B}^q{}_{dk_3\cdots k_l}\gamma^d{}_b + \epsilon_{bpq}\Omega^p\mathcal{B}^q{}_{ck_3\cdots k_l}\gamma^c{}_a) \times\Omega^{k_3}\cdots\Omega^{k_l}, \tag{A9b}$$

and they satisfy the eigenvalue equations

$$r^2\gamma^{cd}D_cD_d\mathcal{B}^{(l)}_a + [l(l+1)-1]\mathcal{B}^{(l)}_a = 0, \tag{A10a}$$

$$r^2\gamma^{cd}D_cD_d\mathcal{B}^{(l)}_{ab} + [l(l+1)-4]\mathcal{B}^{(l)}_{ab} = 0, \tag{A10b}$$

the same as in the even-parity case. The potentials become

$$\mathcal{B}^{(l)}_A = \Omega^a_A \epsilon_{apq}\Omega^p \mathcal{B}^q{}_{k_2\cdots k_l}\Omega^{k_2}\cdots\Omega^{k_l}, \tag{A11a}$$

$$\mathcal{B}^{(l)}_{AB} = (\Omega^a_A \epsilon_{apq}\Omega^p \mathcal{B}^q{}_{bk_3\cdots k_l}\Omega^b_B + \Omega^b_B \epsilon_{bpq}\Omega^p \mathcal{B}^q{}_{ak_3\cdots k_l}\Omega^a_A)\Omega^{k_3}\cdots\Omega^{k_l} \tag{A11b}$$

after transformation to angular coordinates. We wish to decompose these in the odd-parity harmonics of Eqs. (2.21), which satisfy the eigenvalue equations

$$\Omega^{CD}D_C D_D X^{lm}_A + [l(l+1)-1]X^{lm}_A = 0, \tag{A12a}$$

$$\Omega^{CD}D_C D_D X^{lm}_{AB} + [l(l+1)-4]X^{lm}_{AB} = 0, \tag{A12b}$$

the same as in the even-parity case.

We once more begin with $\mathcal{B}^{(l)} := \mathcal{B}_{k_1\cdots k_l}\Omega^{k_1}\cdots\Omega^{k_l}$ and its decomposition

$$\mathcal{B}^{(l)} = \sum_m \mathcal{B}^{(l)}_m Y^{lm}. \tag{A13}$$

Differentiating the first expression, multiplying this by the Levi-Civita tensor, and involving the second of Eqs. (A1) returns $\epsilon_A{}^B D_B \mathcal{B}^{(l)} = -l\Omega^a_A\epsilon_{apq}\Omega^p\mathcal{B}^q{}_{k_2\cdots k_l}\Omega^{k_2}\cdots\Omega^{k_l}$. From this we conclude that

$$\mathcal{B}^{(l)}_A = \frac{1}{l}(-\epsilon_A{}^B D_B)\mathcal{B}^{(l)} = \frac{1}{l}\sum_m \mathcal{B}^{(l)}_m X^{lm}_A. \tag{A14}$$

This is the decomposition of $\mathcal{B}^{(l)}_A$ in vectorial, odd-parity harmonics; this equation is valid for $l \neq 0$. After a second differentiation we get $-\epsilon_A{}^C D_B D_C \mathcal{B}^{(l)} = l\epsilon_{AB}\mathcal{B}^{(l)} + l(l-1)\Omega^a_A\epsilon_{apq}\Omega^p\mathcal{B}^q{}_{bk_3\cdots k_l}\Omega^b_B\Omega^{k_3}\cdots\Omega^{k_l}$, and after symmetrization of the indices we obtain

$$\mathcal{B}^{(l)}_{AB} = -\frac{1}{l(l-1)}(\epsilon_A{}^C D_B + \epsilon_B{}^C D_A)D_C\mathcal{B}^{(l)} = \frac{2}{l(l-1)}\sum_m \mathcal{B}^{(l)}_m X^{lm}_{AB}. \tag{A15}$$

This is the decomposition of $\mathcal{B}^{(l)}_{AB}$ in tensorial, odd-parity harmonics; this equation is valid for $l \neq \{0, 1\}$.

APPENDIX B: DETERMINANT OF THE HORIZON METRIC

The horizon metric of Eq. (4.3) can be expressed as $\gamma_{AB} = 4M^2\Omega_{AB} + p_{AB}$. The metric determinant is given by $\sqrt{\gamma} = 4M^2\sin\theta(1 + \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 - \frac{1}{4}\varepsilon^A{}_B\varepsilon^B{}_A + \cdots)$, where $\varepsilon^A{}_B := \frac{1}{4}M^{-2}\Omega^{AC}p_{CB}$ and $\varepsilon = \varepsilon^A{}_A$. Evaluating this gives

$$\sqrt{\gamma} = 4M^2\sin\theta\Big[1 + \frac{8}{45}M^4(\mathcal{P}^{\mathsf{m}} + \mathcal{Q}^{\mathsf{m}}) - \frac{32}{45}M^4\mathcal{G}^{\mathsf{d}} - \frac{16}{21}M^4(\mathcal{P}^{\mathsf{q}} + \mathcal{Q}^{\mathsf{q}}) + \frac{8}{9}M^4\mathcal{G}^{\mathsf{o}} + \frac{2}{9}M^4(\mathcal{P}^{\mathsf{h}} + \mathcal{Q}^{\mathsf{h}}) - \frac{1}{9}M^4(\mathcal{E}^{\mathsf{q}}_{AB} + \mathcal{B}^{\mathsf{q}}_{AB})(\mathcal{E}^{\mathsf{q}AB} + \mathcal{B}^{\mathsf{q}AB}) + O(5)\Big], \tag{B1}$$

where the indices on $\mathcal{E}^{\mathsf{q}}_{AB}$ and $\mathcal{B}^{\mathsf{q}}_{AB}$ are raised with Ω^{AB}. This convention will be used consistently below: All indices on tidal potentials will be lowered with Ω_{AB} and raised with Ω^{AB}.

The identities

$$\mathcal{E}^{\mathsf{q}}_{ab}\mathcal{E}^{\mathsf{q}ab} = \mathcal{E}^{\mathsf{q}}_{AB}\mathcal{E}^{\mathsf{q}AB} = \frac{8}{5}\mathcal{P}^{\mathsf{m}} - \frac{48}{7}\mathcal{P}^{\mathsf{q}} + 2\mathcal{P}^{\mathsf{h}}, \tag{B2a}$$

$$\mathcal{E}^{\mathsf{q}}_{ab}\mathcal{B}^{\mathsf{q}ab} = \mathcal{E}^{\mathsf{q}}_{AB}\mathcal{B}^{\mathsf{q}AB} = -\frac{16}{5}\mathcal{G}^{\mathsf{d}} + 4\mathcal{G}^{\mathsf{o}}, \tag{B2b}$$

$$\mathcal{B}^{\mathsf{q}}_{ab}\mathcal{B}^{\mathsf{q}ab} = \mathcal{B}^{\mathsf{q}}_{AB}\mathcal{B}^{\mathsf{q}AB} = \frac{8}{5}\mathcal{Q}^{\mathsf{m}} - \frac{48}{7}\mathcal{Q}^{\mathsf{q}} + 2\mathcal{Q}^{\mathsf{h}} \tag{B2c}$$

can be established by direct computation, by making use of the definitions of the tidal potentials provided in Tables I, II, III, IV, V, and VI. Inserting them into our expression for $\sqrt{\gamma}$ returns the simple expression displayed in Eq. (4.4).

APPENDIX C: CALCULATION OF TIDAL HEATING

In this appendix we provide calculational details regarding the heating of the black hole by the tidal interaction. The results derived here are used in various places in Sec. IV D.

To a degree of accuracy that is sufficient for our purposes, the inverse to the horizon metric of Eq. (4.3) is

$$\gamma^{AB} = \frac{1}{4}M^{-2}\Omega^{AB} + \frac{1}{6}(\mathcal{E}^{\mathsf{q}AB} + \mathcal{B}^{\mathsf{q}AB}) + M^{-2}O(3). \tag{C1}$$

It is understood that on the right-hand side of this equation, upper-case Latin indices are raised with Ω^{AB}.

We use γ^{AB} to raise indices on the shear tensor of Eq. (4.8). This yields

$$\sigma^{AB} = -\frac{1}{12}(\dot{\mathcal{E}}^{\mathsf{q}AB} + \dot{\mathcal{B}}^{\mathsf{q}AB}) - \frac{1}{60}M(\dot{\mathcal{E}}^{\mathsf{o}AB} + \dot{\mathcal{B}}^{\mathsf{o}AB}) - \frac{1}{420}M^2(\dot{\mathcal{E}}^{\mathsf{h}AB} + \dot{\mathcal{B}}^{\mathsf{h}AB}) - \frac{1}{45}M^2\Omega^{AB}(\dot{\mathcal{P}}^{\mathsf{m}} + \dot{\mathcal{Q}}^{\mathsf{m}}) + \frac{4}{45}M^2\Omega^{AB}\dot{\mathcal{G}}^{\mathsf{d}} + \frac{2}{21}M^2\Omega^{AB}(\dot{\mathcal{P}}^{\mathsf{q}} + \dot{\mathcal{Q}}^{\mathsf{q}}) - \frac{1}{9}M^2\Omega^{AB}\dot{\mathcal{G}}^{\mathsf{o}} - \frac{1}{36}M^2\Omega^{AB}(\dot{\mathcal{P}}^{\mathsf{h}} + \dot{\mathcal{Q}}^{\mathsf{h}}) + \frac{1}{42}M^2\dot{\mathcal{P}}^{\mathsf{h}AB} - \frac{5}{63}M^2\dot{\mathcal{Q}}^{\mathsf{h}AB} + \frac{13}{126}M^2\dot{\mathcal{H}}^{\mathsf{h}AB} + M^{-3}O(6). \tag{C2}$$

To arrive at this result we made use of the identity $\mathcal{A}_{AC}\mathcal{A}^C{}_B = \frac{1}{2}\Omega_{AB}\mathcal{A}_{CD}\mathcal{A}^{CD}$ satisfied by any symmetric-tracefree tensor $\mathcal{A}_{AB}$ on the unit two-sphere, as well as Eqs. (B2) from Appendix B. We emphasize that while indices on σ_{AB} are raised with the physical horizon metric γ^{AB}, indices on all tidal potentials are raised with Ω^{AB}.

To evaluate the right-hand side of Eq. (4.12) we use Eqs. (4.8) and (C2) to construct $\sigma_{AB}\sigma^{AB}$, which we integrate over the horizon with the help of Eq. (4.4). The shear tensor is expressed as a multipole expansion, and its square consists of products of multipole moments. Some of these products involve moments of the same order. For example, $\sigma_{AB}\sigma^{AB}$ contains the term $\frac{1}{9}M^4(\dot{\mathcal{E}}^{\mathsf{q}}_{AB} + \dot{\mathcal{B}}^{\mathsf{q}}_{AB}) \times (\dot{\mathcal{E}}^{\mathsf{q}AB} + \dot{\mathcal{B}}^{\mathsf{q}AB})$, which is a product of quadrupole moments; such terms survive an angular integration and contribute to the right-hand side of Eq. (4.12). Other products involve moments of different orders, and those integrate to zero.

To evaluate an angular integral such as $\int \dot{\mathcal{E}}^{\mathsf{q}}_{AB}\dot{\mathcal{E}}^{\mathsf{q}AB} d\Omega$, where $d\Omega := \sin\theta d\theta d\phi$, we recall the definition $\dot{\mathcal{E}}^{\mathsf{q}}_{AB} = \dot{\mathcal{E}}^{\mathsf{q}}_{ab}\Omega^a_A\Omega^b_B$ and deduce the identity $\dot{\mathcal{E}}^{\mathsf{q}}_{AB}\dot{\mathcal{E}}^{\mathsf{q}AB} = \dot{\mathcal{E}}^{\mathsf{q}}_{ab}\dot{\mathcal{E}}^{\mathsf{q}ab}$. We next substitute the expression for $\dot{\mathcal{E}}^{\mathsf{q}}_{ab}$ found in Table I and carry out the integration. These steps are repeated for all other relevant products of multipole moments, and we obtain

$$\frac{1}{4\pi}\int \dot{\mathcal{E}}^{\mathsf{q}}_{AB}\dot{\mathcal{E}}^{\mathsf{q}AB} d\Omega = \frac{8}{5}\dot{\mathcal{E}}_{ab}\dot{\mathcal{E}}^{ab}, \tag{C3a}$$

$$\frac{1}{4\pi}\int \dot{\mathcal{B}}^{\mathsf{q}}_{AB}\dot{\mathcal{B}}^{\mathsf{q}AB} d\Omega = \frac{8}{5}\dot{\mathcal{B}}_{ab}\dot{\mathcal{B}}^{ab}, \tag{C3b}$$

$$\frac{1}{4\pi}\int \dot{\mathcal{E}}^{\mathsf{o}}_{AB}\dot{\mathcal{E}}^{\mathsf{o}AB} d\Omega = \frac{8}{21}\dot{\mathcal{E}}_{abc}\dot{\mathcal{E}}^{abc}, \tag{C3c}$$

$$\frac{1}{4\pi}\int \dot{\mathcal{B}}^{\mathsf{o}}_{AB}\dot{\mathcal{B}}^{\mathsf{o}AB} d\Omega = \frac{128}{189}\dot{\mathcal{B}}_{abc}\dot{\mathcal{B}}^{abc}. \tag{C3d}$$

All other integrations vanish, or contribute to the right-hand-side of Eq. (4.12) at order $(M/\mathcal{R})^9$ and beyond.

The general solution to Eq. (4.13) is

$$\dot{\mathcal{A}}(v) = e^{\kappa_0 v}\dot{\mathcal{A}}(0) - 8\pi\int_0^v \mathcal{F}(v')e^{\kappa_0(v-v')}dv'. \tag{C4}$$

After three integration by parts this becomes

$$\frac{\kappa_0}{8\pi}\dot{\mathcal{A}}(v) = e^{\kappa_0 v}\bigg[\frac{\kappa_0}{8\pi}\dot{\mathcal{A}}(0) - \mathcal{F}(0) - \frac{1}{\kappa_0}\dot{\mathcal{F}}(0) - \frac{1}{\kappa_0^2}\ddot{\mathcal{F}}(0)\bigg] + \mathcal{F}(v) + \frac{1}{\kappa_0}\dot{\mathcal{F}}(v) + \frac{1}{\kappa_0^2}\ddot{\mathcal{F}}(v) + \frac{1}{\kappa_0^2}\int_0^v \frac{d^3\mathcal{F}}{dv'^3}e^{\kappa_0(v-v')}dv'. \tag{C5}$$

The last term can be neglected, because $\kappa_0^{-3}d^3\mathcal{F}/dv^3$ is of order $(M/\mathcal{R})^9$ and beyond the accuracy maintained in the computation of the flux function. The first collection of terms, those that depend on the initial conditions at $v = 0$, grow exponentially over a short time scale of order $\kappa_0^{-1} = 4M$. We do not expect the black-hole area to behave in this way; we expect instead that the tidal interaction will produce a slow growth over a much longer time scale. To eliminate the run-away solution we demand that the solution satisfy the initial condition

$$\frac{\kappa_0}{8\pi}\dot{\mathcal{A}}(0) = \mathcal{F}(0) + \frac{1}{\kappa_0}\dot{\mathcal{F}}(0) + \frac{1}{\kappa_0^2}\ddot{\mathcal{F}}(0) + O(9). \tag{C6}$$

Under this restriction, Eq. (C5) reduces to

$$\frac{\kappa_0}{8\pi}\dot{\mathcal{A}}(v) = \mathcal{F}(v) + \frac{1}{\kappa_0}\dot{\mathcal{F}}(v) + \frac{1}{\kappa_0^2}\ddot{\mathcal{F}}(v) + O(9), \tag{C7}$$

and this is just Eq. (4.15). Notice that Eq. (C7) is compatible with the initial conditions of Eq. (C6).

It is unusual, when dealing with horizons, to impose *initial* conditions on solutions to differential equations, as we have done in Eq. (C6). The reason, of course, is that event horizons are always identified by *final* conditions. We chose to proceed in this way because our horizon is not necessarily an event horizon, as we explained in Sec. IV A. We also explained that the horizon *becomes* an event horizon when the tidal interaction switches off in the remote future. Under this condition we may impose the *final* condition that $\dot{\mathcal{A}} = 0$ when $v = \infty$. The exact solution to Eq. (4.13) is then

$$\dot{A}(v) = 8\pi\int_v^\infty \mathcal{F}(v')e^{-\kappa_0(v'-v)}dv', \tag{C8}$$

and this leads once more to Eq. (C7) after three integrations by parts.

APPENDIX D: ALTERNATIVE DERIVATION OF THE BACKGROUND METRIC

In this appendix we give a brief sketch of an alternative derivation of the background metric of Eqs. (3.3). The derivation described in Sec. V relied on Zhang's observation [18] that the metric of a vacuum region of spacetime around a timelike geodesic γ is a functional of two sets of tidal moments $\mathcal{E}_{a_1a_2\cdots a_l}$ and $\mathcal{B}_{a_1a_2\cdots a_l}$; these are STF tensors that depend on proper time on the world line, and they are related to components of the Weyl tensor (and its derivatives) evaluated on the world line. Here we provide a precise definition of the light-cone coordinates and construct the metric systematically through order $(r/\mathcal{R})^4$; we recover the metric of Eqs. (3.3) and therefore confirm the validity of Zhang's observation. Our development follows the general methods developed in Ref. [2].

1. Formal definition of light-cone coordinates

As in Sec. II A we consider a timelike geodesic γ described by the parametric relations $z^\alpha(\tau)$, with τ denoting proper time. On γ we install an orthonormal tetrad $(u^{\alpha'}, e^{\alpha'}_a)$ of parallel-transported vectors. (We use primed indices to refer to points on the world line; unprimed indices will refer to points off the world line.) For the time being we do not assume that γ is situated in a vacuum region of spacetime; this assumption will be incorporated at a later stage. To assign light-cone coordinates (v, x^a) to a point x off the world line we locate the unique future-directed null geodesic segment β that begins at x and ends at a point x' on the world line. (The construction requires that x be in the normal convex neighborhood of x'; the coordinates are defined in this neighborhood only.) The advanced-time coordinate v is the value of the proper-time parameter τ at this point: $x' = z(\tau = v)$. And the quasi-Cartesian coordinates x^a are defined by

$$x^a := -e^a_{\alpha'}\sigma^{\alpha'}(x, x'), \tag{D1}$$

where $e^a_{\alpha'} := \delta^{ab} g_{\alpha'\beta'} e^{\beta'}_b$ and $\sigma^{\alpha'} := \nabla^{\alpha'}\sigma$ is Synge's world function $\sigma(x, x')$ [30,71] differentiated with respect to its second argument. The points x and x' are related by the condition $\sigma(x, x') = 0$, which indicates that the points are linked by a null geodesic segment.

We define

$$r := -\sigma_{\alpha'} u^{\alpha'} \tag{D2}$$

and state without proof that r is an affine parameter on β; it decreases as the null geodesic approaches the world line. (The proof of this statement is contained in Ref. [2].) The completeness relation $g^{\alpha'\beta'} = -u^{\alpha'}u^{\beta'} + \delta^{ab} e^{\alpha'}_a e^{\beta'}_b$ and the identity $\sigma^{\alpha'}\sigma_{\alpha'} = 2\sigma = 0$ imply that $r^2 = \delta_{ab} x^a x^b$. It is useful to introduce

$$\Omega^a := x^a / r, \tag{D3}$$

and we use the completeness relation to write

$$\sigma^{\alpha'} = r(u^{\alpha'} - \Omega^a e^{\alpha'}_a). \tag{D4}$$

This is a decomposition of the displacement vector $\sigma^{\alpha'}(x, x')$ in the tetrad $(u^{\alpha'}, e^{\alpha'}_a)$. The vector

$$\ell_\alpha := \sigma_\alpha / r \tag{D5}$$

is future-directed and tangent to null geodesic segment β. Here $\sigma_\alpha := \nabla_\alpha \sigma$ is the world function $\sigma(x, x')$ differentiated with respect to its first argument.

Suppose now that the point x is moved to a neighboring point $x + \delta x$. The coordinates of the new point will be $v + \delta v$ and $x^a + \delta x^a$, and to calculate the coordinate displacements we must locate the new point $x' + \delta x'$ on the world line, which is linked to $x + \delta x$ by a new geodesic segment $\beta + \delta\beta$. Using $\delta x^{\alpha'} = u^{\alpha'}\delta v$ and expanding $\sigma(x + \delta x, x' + \delta x') = 0$ to first order in the displacements, it is easy to show that $\delta v = -\ell_\alpha \delta x^\alpha$, so that

$$\partial_\alpha v = -\ell_\alpha. \tag{D6}$$

The definition of the coordinates x^a in terms of the world function then implies that $\delta x^a = -(e^a_{\alpha'}\sigma^{\alpha'}{}_{\beta'} u^{\beta'})\delta v - e^a_{\alpha'}\sigma^{\alpha'}{}_\beta \delta x^\beta$, so that

$$\partial_\alpha x^a = (e^a_{\alpha'}\sigma^{\alpha'}{}_{\beta'} u^{\beta'})\ell_\alpha - e^a_{\beta'}\sigma^{\beta'}{}_\alpha. \tag{D7}$$

Here $\sigma^{\alpha'}{}_{\beta'} := \nabla^{\alpha'}\nabla_{\beta'}\sigma$ is the second covariant derivative of the world function with respect to x', while $\sigma^{\beta'}{}_\alpha := \nabla^{\beta'}\nabla_{\alpha'}\sigma$ denotes a mixed derivative with respect to each argument.

2. Metric

We begin with a computation of the inverse metric, with components

$$g^{vv} = g^{\alpha\beta}\partial_\alpha v \partial_\beta v, \tag{D8a}$$

$$g^{va} = g^{\alpha\beta}\partial_\alpha x^a \partial_\beta v, \tag{D8b}$$

$$g^{ab} = g^{\alpha\beta}\partial_\alpha x^a \partial_\beta x^b. \tag{D8c}$$

Using Eq. (D6) and the fact that ℓ_α is a null vector we immediately find that $g^{vv} = 0$. With Eq. (D7) we get $g^{va} = e^a_{\alpha'}\sigma^{\alpha'}{}_\beta \ell^\beta$, and we simplify this with the help of Eq. (D5) and the identity $\sigma^{\alpha'}{}_\beta \sigma^\beta = \frac{1}{2}\nabla^{\alpha'}(\sigma_\beta\sigma^\beta) = \nabla^{\alpha'}\sigma$. With Eqs. (D1) and (D3), this is $g^{va} = \Omega^a$. To obtain the components

$$g^{ab} = e^a_{\alpha'} e^b_{\beta'} g^{\alpha\beta}\sigma^{\alpha'}{}_\alpha \sigma^{\beta'}{}_\beta - (e^a_{\alpha'}\sigma^{\alpha'}{}_{\beta'} u^{\beta'})\Omega^b - (e^b_{\alpha'}\sigma^{\alpha'}{}_{\beta'} u^{\beta'})\Omega^a \tag{D9}$$

requires a much longer computation, and the result will be expressed as an expansion in powers of r.

We rely on the known expansions (see, for example, Ref. [72])

$$\sigma_{\alpha'\beta'} = g_{\alpha'\beta'} - \frac{1}{3} R_{\alpha'\gamma'\beta'\delta'}\sigma^{\gamma'}\sigma^{\delta'} + \frac{1}{12} R_{\alpha'\gamma'\beta'\delta';\epsilon'}\sigma^{\gamma'}\sigma^{\delta'}\sigma^{\epsilon'} - \left(\frac{1}{60} R_{\alpha'\gamma'\beta'\delta';\epsilon'\iota'} + \frac{1}{45} R_{\alpha'\gamma'\delta'\mu'} R^{\mu'}{}_{\epsilon'\iota'\beta'}\right)\sigma^{\gamma'}\sigma^{\delta'}\sigma^{\epsilon'}\sigma^{\iota'} + \cdots, \tag{D10}$$

$$\sigma_{\alpha'\beta} = g^{\beta'}{}_\beta\Big[-g_{\alpha'\beta'} - \frac{1}{6} R_{\alpha'\gamma'\beta'\delta'}\sigma^{\gamma'}\sigma^{\delta'} + \frac{1}{12} R_{\alpha'\gamma'\beta'\delta';\epsilon'}\sigma^{\gamma'}\sigma^{\delta'}\sigma^{\epsilon'} - \left(\frac{1}{40} R_{\alpha'\gamma'\beta'\delta';\epsilon'\iota'} + \frac{7}{360} R_{\alpha'\gamma'\delta'\mu'} R^{\mu'}{}_{\epsilon'\iota'\beta'}\right)\sigma^{\gamma'}\sigma^{\delta'}\sigma^{\epsilon'}\sigma^{\iota'} + \cdots\Big] \tag{D11}$$

for the derivatives of the world function, in which $g^{\beta'}{}_\beta(x, x')$ is the parallel propagator [30,71]. We make the substitutions in Eq. (D9) and express $\sigma^{\alpha'}$ as in Eq. (D4).

The end result of a lengthy computation is

$$g^{vv} = 0, \tag{D12a}$$

$$g^{va} = \Omega^a, \tag{D12b}$$

$$\begin{aligned} g^{ab} = {}& \delta^{ab} + \frac{1}{3}r^2(P^{ab} + P^a\Omega^b + P^b\Omega^a) - \frac{1}{12}r^3(2\dot{P}^{ab} + \dot{P}^a\Omega^b + \dot{P}^b\Omega^a) - \frac{1}{12}r^3(2Q^{ab} + Q^a\Omega^b + Q^b\Omega^a) \\ & + \frac{1}{60}r^4(3\ddot{P}^{ab} + \ddot{P}^a\Omega^b + \ddot{P}^b\Omega^a) + \frac{1}{30}r^4(3\dot{Q}^{ab} + \dot{Q}^a\Omega^b + \dot{Q}^b\Omega^a) + \frac{1}{60}r^4(3S^{ab} + S^a\Omega^b + S^b\Omega^a) \\ & + \frac{1}{60}r^4(3U^{ab} + U^a\Omega^b + U^b\Omega^a) + \frac{1}{45}r^4(3V^{ab} + V^a\Omega^b + V^b\Omega^a) + O(r^5). \end{aligned} \tag{D12c}$$

The inverse metric is expressed in terms of the potentials

$$P_{ab} = R_{a0b0} - (R_{acb0} + R_{bca0})\Omega^c + R_{acbd}\Omega^c\Omega^d = P_{ba}, \tag{D13a}$$

$$P_a = R_{a0c0}\Omega^c - R_{acd0}\Omega^c\Omega^d = P_{ab}\Omega^b, \tag{D13b}$$

$$P = R_{c0d0}\Omega^c\Omega^d = P_a\Omega^a, \tag{D13c}$$

$$Q_{ab} = -R_{a0b0|c}\Omega^c + (R_{acb0|d} + R_{bca0|d})\Omega^c\Omega^d - R_{acbd|e}\Omega^c\Omega^d\Omega^e = Q_{ba}, \tag{D13d}$$

$$Q_a = -R_{a0c0|d}\Omega^c\Omega^d + R_{acd0|e}\Omega^c\Omega^d\Omega^e = Q_{ab}\Omega^b, \tag{D13e}$$

$$Q = -R_{c0d0|e}\Omega^c\Omega^d\Omega^e = Q_a\Omega^a, \tag{D13f}$$

$$S_{ab} = R_{a0b0|cd}\Omega^c\Omega^d - (R_{acb0|de} + R_{bca0|de})\Omega^c\Omega^d\Omega^e + R_{acbd|ef}\Omega^c\Omega^d\Omega^e\Omega^f = S_{ba}, \tag{D13g}$$

$$S_a = R_{a0c0|de}\Omega^c\Omega^d\Omega^e - R_{acd0|ef}\Omega^c\Omega^d\Omega^e\Omega^f = S_{ab}\Omega^b, \tag{D13h}$$

$$S = R_{c0d0|ef}\Omega^c\Omega^d\Omega^e\Omega^f = S_a\Omega^a, \tag{D13i}$$

$$\begin{aligned} U_{ab} = {}& (R_{a0m0}R^m{}_{bc0} + R_{b0m0}R^m{}_{ac0} + R_{amb0}R^m{}_{0c0} + R_{bma0}R^m{}_{0c0})\Omega^c + (2R_{a0c0}R_{b0d0} + R_{amc0}R^m{}_{db0} \\ & + R_{bmc0}R^m{}_{da0} - 2R_{a0b0}R_{0c0d} - R_{amb0}R^m{}_{cd0} - R_{bma0}R^m{}_{cd0} - R_{acm0}R^m{}_{bd0} - R_{bcm0}R^m{}_{ad0} - R_{acbm}R^m{}_{0d0} \\ & - R_{bcam}R^m{}_{0d0})\Omega^c\Omega^d + (-R_{a0c0}R_{bde0} - R_{b0c0}R_{ade0} + R_{acmd}R^m{}_{be0} + R_{bcmd}R^m{}_{ae0} + R_{acb0}R_{d0e0} \\ & + R_{bca0}R_{d0e0} + R_{acbm}R^m{}_{de0} + R_{bcam}R^m{}_{de0})\Omega^c\Omega^d\Omega^e = U_{ba}, \end{aligned} \tag{D13j}$$

$$U_a = (R_{a0m0}R^m{}_{cd0} + R_{cma0}R^m{}_{0d0})\Omega^c\Omega^d - (R_{acm0}R^m{}_{de0} + R_{acdm}R^m{}_{0e0})\Omega^c\Omega^d\Omega^e = U_{ab}\Omega^b, \tag{D13k}$$

$$V_{ab} = P_{ac}P^c{}_b - P_aP_b = V_{ba}, \tag{D13l}$$

$$V_a = P_{ac}P^c - P_aP, \tag{D13m}$$

$$V = P_cP^c - P^2, \tag{D13n}$$

which are defined in terms of the frame components of the Riemann tensor (and its derivatives) evaluated on the world line. We adopt the notation introduced in Sec. II A, and an overdot indicates differentiation with respect to proper time. For example, $\dot{P}_{ab} := \dot{R}_{a0b0} - (\dot{R}_{acb0} + \dot{R}_{bca0})\Omega^c + \dot{R}_{acbd}\Omega^c\Omega^d$, where, for example, $\dot{R}_{a0b0} := R_{\alpha'\mu'\beta'\nu';\lambda'}e^{\alpha'}_a u^{\mu'} e^{\beta'}_b u^{\nu'} u^{\lambda'}$. Notice that the derivative operator acts on the Riemann tensor only. The fact that the tetrad is parallel-transported on the world line implies that the right-hand side can also be written as $(R_{\alpha'\mu'\beta'\nu'}e^{\alpha'}_a u^{\mu'} e^{\beta'}_b u^{\nu'})_{;\lambda'}u^{\lambda'}$, and we find that $\dot{R}_{a0b0} = dR_{a0b0}/d\tau$. To avoid ambiguities with second derivatives of the Riemann tensor, we always differentiate with respect to proper time in the last step; for example, $\dot{R}_{a0b0|c} := R_{\alpha'\mu'\beta'\nu';\gamma'\lambda'}e^{\alpha'}_a u^{\mu'} e^{\beta'}_b u^{\nu'} e^{\gamma'}_c u^{\lambda'}$.

The inverse of Eqs. (D12) is calculated using the techniques introduced in Sec. V A. We obtain

$$g_{vv} = -1 - r^2P + \frac{1}{3}r^3\dot{P} + \frac{1}{3}r^3Q - \frac{1}{12}r^4\ddot{P} - \frac{1}{6}r^4\dot{Q} - \frac{1}{12}r^4S + \frac{1}{3}r^4V + O(r^5), \tag{D14a}$$

$$g_{va} = \Omega_a + \gamma^c_a\bigg[-\frac{2}{3}r^2P_c + \frac{1}{4}r^3\dot{P}_c + \frac{1}{4}r^3Q_c - \frac{1}{15}r^4\ddot{P}_c - \frac{2}{15}r^4\dot{Q}_c - \frac{1}{15}r^4S_c - \frac{1}{15}r^4U_c + \frac{2}{15}r^4V_c + O(r^5)\bigg], \tag{D14b}$$

$$\begin{aligned} g_{ab} = {}& \gamma_{ab} + \gamma_a{}^c\gamma_b{}^d\bigg[-\frac{1}{3}r^2P_{cd} + \frac{1}{6}r^3\dot{P}_{cd} + \frac{1}{6}r^3Q_{cd} - \frac{1}{20}r^4\ddot{P}_{cd} - \frac{1}{10}r^4\dot{Q}_{cd} - \frac{1}{20}r^4S_{cd} - \frac{1}{20}r^4U_{cd} \\ & + \frac{2}{45}r^4V_{cd} + O(r^5)\bigg], \end{aligned} \tag{D14c}$$

where $\gamma_{ab} := \delta_{ab} - \Omega_a\Omega_b$. We recognize the structure of Eqs. (5.7), with a clear decomposition of the metric into longitudinal and transverse pieces.

3. Decomposition of the Weyl tensor

At this stage we demand that the Ricci tensor, its first derivatives, and its second derivatives, all vanish on the world line γ. This implies that the Riemann tensor $R_{\alpha'\beta'\gamma'\delta'}$ and its derivatives are equal to the Weyl tensor $C_{\alpha'\beta'\gamma'\delta'}$ and its derivatives. The symmetries of the Weyl tensor, the Bianchi identities, and the Ricci identities then imply that the Weyl tensor (and its derivatives) can be expressed in terms of the tidal moments $\mathcal{E}_{ab}$, $\mathcal{E}_{abc}$, $\mathcal{E}_{abcd}$, $\mathcal{B}_{ab}$, $\mathcal{B}_{abc}$, and $\mathcal{B}_{abcd}$; these were defined in Sec. II A.

The frame components of the Weyl tensor on the world line are given by

$$C_{a0b0} = \mathcal{E}_{ab}, \tag{D15a}$$
$$C_{abc0} = \epsilon_{abp}\mathcal{B}^p{}_c, \tag{D15b}$$
$$C_{abcd} = -\epsilon_{abp}\epsilon_{cdq}\mathcal{E}^{pq}. \tag{D15c}$$

The last equation can also be written as

$$C_{abcd} = \delta_{ac}\mathcal{E}_{bd} - \delta_{ad}\mathcal{E}_{bc} - \delta_{bc}\mathcal{E}_{ad} + \delta_{bd}\mathcal{E}_{ac}, \tag{D16}$$

by making use of the general identity $\epsilon_{abp}\epsilon_{cdq} = \delta_{ac}(\delta_{bd}\delta_{pq} - \delta_{bq}\delta_{dp}) - \delta_{ad}(\delta_{bc}\delta_{pq} - \delta_{bq}\delta_{cp}) + \delta_{aq}(\delta_{bc}\delta_{pd} - \delta_{bd}\delta_{cp})$.

The frame components of the first spatial derivatives of the Weyl tensor are

$$C_{a0b0|c} = \mathcal{E}_{ab|c}, \tag{D17a}$$
$$C_{abc0|d} = \epsilon_{abp}\mathcal{B}^p{}_{c|d}, \tag{D17b}$$
$$C_{abcd|e} = -\epsilon_{abp}\epsilon_{cdq}\mathcal{E}^{pq}{}_{|e}, \tag{D17c}$$

where $\mathcal{E}_{ab|c} := C_{a0b0|c}$ and $\mathcal{B}_{ab|c} := \frac{1}{2}\epsilon_{apq}C^{pq}{}_{b0|c}$. These are related to the tidal moments by

$$\mathcal{E}_{ab|c} = \mathcal{E}_{abc} + \frac{1}{3}(\epsilon_{acp}\dot{\mathcal{B}}^p{}_b + \epsilon_{bcp}\dot{\mathcal{B}}^p{}_a), \tag{D18a}$$
$$\mathcal{B}_{ab|c} = \frac{4}{3}\mathcal{B}_{abc} - \frac{1}{3}(\epsilon_{acp}\dot{\mathcal{E}}^p{}_b + \epsilon_{bcp}\dot{\mathcal{E}}^p{}_a). \tag{D18b}$$

The frame components of the (symmetrized) second spatial derivatives of the Weyl tensor are

$$C_{a0b0|(cd)} = \mathcal{E}_{ab|(cd)}, \tag{D19a}$$
$$C_{abc0|(de)} = \epsilon_{abp}\mathcal{B}^p_{c|(de)}, \tag{D19b}$$
$$C_{abcd|(ef)} = -\epsilon_{abp}\epsilon_{cdq}\mathcal{E}^{pq}{}_{|(ef)}, \tag{D19c}$$

where $\mathcal{E}_{ab|(cd)} := C_{a0b0|(cd)}$ and $\mathcal{B}_{ab|(cd)} := \frac{1}{2}\epsilon_{apq}C^{pq}{}_{b0|(cd)}$. These are related to the tidal moments by

$$\begin{aligned}\mathcal{E}_{ab|(cd)} = {}& 2\mathcal{E}_{abcd} + \frac{1}{3}(\epsilon_{acp}\dot{\mathcal{B}}^p{}_{bd} + \epsilon_{adp}\dot{\mathcal{B}}^p{}_{bc} + \epsilon_{bcp}\dot{\mathcal{B}}^p{}_{ad} + \epsilon_{bdp}\dot{\mathcal{B}}^p{}_{ac}) + \frac{4}{21}\delta_{ab}\ddot{\mathcal{E}}_{cd} \\ & - \frac{1}{7}(\delta_{ac}\ddot{\mathcal{E}}_{bd} + \delta_{ad}\ddot{\mathcal{E}}_{bc} + \delta_{bc}\ddot{\mathcal{E}}_{ad} + \delta_{bd}\ddot{\mathcal{E}}_{ac}) + \frac{11}{21}\delta_{cd}\ddot{\mathcal{E}}_{ab} - \frac{4}{3}(\mathcal{E}_{ab}\mathcal{E}_{cd} - \mathcal{B}_{ab}\mathcal{B}_{cd}) + \frac{2}{3}(\mathcal{E}_{ac}\mathcal{E}_{bd} - \mathcal{B}_{ac}\mathcal{B}_{bd}) \\ & + \frac{2}{3}(\mathcal{E}_{ad}\mathcal{E}_{bc} - \mathcal{B}_{ad}\mathcal{B}_{bc}) - \frac{22}{21}\delta_{ab}G_{cd} + \frac{19}{42}(\delta_{ac}G_{bd} + \delta_{ad}G_{bc} + \delta_{bc}G_{ad} + \delta_{bd}G_{ac}) + \frac{20}{21}\delta_{cd}G_{ab} \\ & - \frac{4}{21}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})G,\end{aligned} \tag{D20a}$$

$$\begin{aligned}\mathcal{B}_{ab|(cd)} = {}& \frac{10}{3}\mathcal{B}_{abcd} - \frac{1}{4}(\epsilon_{acp}\dot{\mathcal{E}}^p{}_{bd} + \epsilon_{adp}\dot{\mathcal{E}}^p{}_{bc} + \epsilon_{bcp}\dot{\mathcal{E}}^p{}_{ad} + \epsilon_{bdp}\dot{\mathcal{E}}^p{}_{ac}) + \frac{4}{21}\delta_{ab}\ddot{\mathcal{B}}_{cd} \\ & - \frac{1}{7}(\delta_{ac}\ddot{\mathcal{B}}_{bd} + \delta_{ad}\ddot{\mathcal{B}}_{bc} + \delta_{bc}\ddot{\mathcal{B}}_{ad} + \delta_{bd}\ddot{\mathcal{B}}_{ac}) + \frac{11}{21}\delta_{cd}\ddot{\mathcal{B}}_{ab} - \frac{4}{3}(\mathcal{E}_{ab}\mathcal{B}_{cd} + \mathcal{B}_{ab}\mathcal{E}_{cd}) + \frac{2}{3}(\mathcal{E}_{ac}\mathcal{B}_{bd} + \mathcal{B}_{ac}\mathcal{E}_{bd}) \\ & + \frac{2}{3}(\mathcal{E}_{ad}\mathcal{B}_{bc} + \mathcal{B}_{ad}\mathcal{E}_{bc}) - \frac{22}{21}\delta_{ab}H_{cd} + \frac{19}{42}(\delta_{ac}H_{bd} + \delta_{ad}H_{bc} + \delta_{bc}H_{ad} + \delta_{bd}H_{ac}) + \frac{20}{21}\delta_{cd}H_{ab} \\ & - \frac{4}{21}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})H.\end{aligned} \tag{D20b}$$

We introduced the notation

$$G_{ab} := \mathcal{E}_{ap}\mathcal{E}^p{}_b - \mathcal{B}_{ap}\mathcal{B}^p{}_b, \tag{D21a}$$
$$G := \mathcal{E}^{pq}\mathcal{E}_{pq} - \mathcal{B}^{pq}\mathcal{B}_{pq}, \tag{D21b}$$
$$H_{ab} := \mathcal{E}_{ap}\mathcal{B}^p{}_b + \mathcal{B}_{ap}\mathcal{E}^p{}_b, \tag{D21c}$$
$$H := 2\mathcal{E}^{pq}\mathcal{B}_{pq}. \tag{D21d}$$

4. Metric in irreducible form

The decomposition of the Weyl tensor and its derivatives in terms of the tidal moments is substituted within P_{ab}, P_a, P, Q_{ab}, Q_a, Q, S_{ab}, S_a, S, U_{ab}, U_a, V_{ab}, V_a, and V. All of this is next substituted within our previous expression for the metric tensor. The manipulations required to simplify the expressions are extremely lengthy (on the order of 35 pages of small handwriting). We find that the terms that are linear in the Weyl tensor and its derivatives group themselves automatically into the irreducible tidal potentials introduced in Tables I and II.

The organization of the terms that are quadratic in the Weyl tensor requires more work. For example, our initial expression for the quadratic terms in g_{vv} is

$$\begin{aligned} g_{vv}^{\text{quadratic}} &= \frac{1}{21} r^4 (\mathcal{E}_{pq}\mathcal{E}^{pq} - \mathcal{B}_{pq}\mathcal{B}^{pq}) \\ &\quad + \frac{2}{21} r^4 (2\mathcal{E}_{ap}\mathcal{E}^p{}_b + 5\mathcal{B}_{ap}\mathcal{B}^p{}_b)\Omega^a\Omega^b \\ &\quad - \frac{1}{3} r^4 (\mathcal{E}_{ab}\mathcal{E}_{cd} + \mathcal{B}_{ab}\mathcal{B}_{cd})\Omega^a\Omega^b\Omega^c\Omega^d \\ &\quad + \frac{2}{3} r^4 \epsilon_{cpq}\mathcal{E}^p{}_a\mathcal{B}^q{}_b\Omega^a\Omega^b\Omega^c. \end{aligned} \tag{D22}$$

To write this in terms of irreducible tidal potentials we write a product of unit radial vectors such as $\Omega^a\Omega^b\Omega^c\Omega^d$ as the STF decomposition

$$\begin{aligned} \Omega^a\Omega^b\Omega^c\Omega^d &= \Omega^{\langle abcd\rangle} + \frac{1}{7}(\delta^{ab}\Omega^{\langle cd\rangle} + \delta^{ac}\Omega^{\langle bd\rangle} \\ &\quad + \delta^{ad}\Omega^{\langle bc\rangle} + \delta^{bc}\Omega^{\langle ad\rangle} + \delta^{bd}\Omega^{\langle ac\rangle} \\ &\quad + \delta^{cd}\Omega^{\langle ab\rangle}) + \frac{1}{15}(\delta^{ab}\delta^c d + \delta^{ac}\delta^{bd} \\ &\quad + \delta^{ad}\delta^{bc}), \end{aligned} \tag{D23}$$

and we make the substitution within the metric function. Looking at the term $\mathcal{E}_{ab}\mathcal{E}_{cd}\Omega^a\Omega^b\Omega^c\Omega^d$, for example, we obtain $\mathcal{E}_{ab}\mathcal{E}_{cd}\Omega^{\langle abcd\rangle} + \frac{4}{7}\mathcal{E}_{pa}\mathcal{E}^p{}_b\Omega^{\langle ab\rangle} + \frac{2}{15}\mathcal{E}_{pq}\mathcal{E}^{pq}$, which can be expressed in the equivalent form $\mathcal{E}_{\langle ab}\mathcal{E}_{cd\rangle}\Omega^a\Omega^b\Omega^c\Omega^d + \frac{4}{7}\mathcal{E}_{p\langle a}\mathcal{E}^p{}_{b\rangle}\Omega^a\Omega^b + \frac{2}{15}\mathcal{E}_{pq}\mathcal{E}^{pq}$. According to the definitions listed in Table III, this is $\mathcal{P}^{\mathsf{h}} + \frac{4}{7}\mathcal{P}^{\mathsf{q}} + \frac{2}{15}\mathcal{P}^{\mathsf{m}}$, a superposition of hexadecapole, quadrupole, and monopole tidal potentials.

Proceeding in a similar way with g_{va} and g_{ab}, we eventually arrive at Eqs. (3.3).

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