Global solutions for higher-dimensional stretched small black holes

Chiang-Mei Chen,[1,*] Dmitri V. Gal'tsov,[2,†] Nobuyoshi Ohta,[3,‡] and Dmitry G. Orlov[4,§]

[1]*Department of Physics and Center for Mathematics and Theoretical Physics, National Central University, Chungli 320, Taiwan*
[2]*Department of Theoretical Physics, Moscow State University, 119899, Moscow, Russia*
[3]*Department of Physics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan*
[4]*Department of Physics, National Central University, Chungli 320, Taiwan*

(Received 28 October 2009; published 5 January 2010)

Small black holes in heterotic string theory have a vanishing horizon area at the supergravity level, but the horizon is stretched to the finite radius $AdS_2 \times S^{D-2}$ geometry once higher curvature corrections are turned on. This has been demonstrated to give good agreement with microscopic entropy counting. Previous considerations, however, were based on the classical local solutions valid only in the vicinity of the event horizon. Here we address the question of global existence of extremal black holes in the D-dimensional Einstein-Maxwell-Dilaton theory with the Gauss-Bonnet term introducing a variable dilaton coupling a as a parameter. We show that asymptotically flat black holes exist only in a bounded region of the dilaton couplings $0 < a < a_{cr}$,where a_{cr} depends on D. For $D \geq 5$ (but not for $D = 4$) the allowed range of a includes the heterotic string values. For $a > a_{cr}$ numerical solutions meet weak naked singularities at finite radii $r = r_{cusp}$ (spherical cusps), where the scalar curvature diverges as $|r - r_{cusp}|^{-1/2}$. For $D \geq 7$ cusps are met in pairs, so that solutions can be formally extended to asymptotically flat infinity choosing a suitable integration variable. We show, however, that radial geodesics cannot be continued through the cusp singularities, so such a continuation is unphysical.

DOI: 10.1103/PhysRevD.81.024002 PACS numbers: 04.20.Jb, 04.65.+e, 98.80.−k

I. INTRODUCTION

During recent years important progress has been achieved in understanding the entropy of the so-called small black holes (for reviews see [1–3]) which have a vanishing horizon area at the supergravity level [4–7]. The discrepancy with the microscopic counting which gives the finite entropy was resolved by the discovery that the area of the horizon is stretched to finite radius once curvature corrections are included. Indeed such corrections have long been known to exist in the low-energy effective theories of superstrings [8–12]. The classically computed entropy then differs from the Bekenstein-Hawking (BH) value [13–16], but agrees (at least up to a coefficient) with microscopic counting [17–29] including non-Bogomol'nyi-Prasad-Sommerfield (BPS) cases [30–46]. Within the models in which the supersymmetric versions of the curvature square terms are available, the correspondence was checked using the exact classical solutions [25,26,29]. It was also observed that good agreement is achieved if the curvature corrections are taken in the form of the Gauss-Bonnet (GB) term both in 4D and higher dimensions [37–40].

To compute the entropy of extremal black holes with the horizon $AdS_2 \times S^{D-2}$ from the classical side it is enough to construct local solutions in the vicinity of the horizon which is easily done analytically [47–49]. But this does not guarantee the existence of global asymptotically flat solutions. Construction of solutions with curvature corrections, apart from purely perturbative probes [50–52], requires numerical integration of the field equations. For nonextremal black holes this was done in [53–63]. The global existence of extremal black holes in the 4D model with the GB terms endowed with an arbitrary dilaton coupling a was proven in [64,65]. It turned out that global asymptotically flat black holes with the horizon $AdS_2 \times S^2$ existed for the dilaton coupling below the critical value of the order $a_{cr} \sim 1/2$ and less than $\frac{1}{2}$. This range does not include the heterotic string value $a = 1$ nor $\frac{1}{2}$. This result is modified in the presence of the magnetic charge [66], which extends the region of the allowed couplings and serve as the order parameter ensuring continuous transition to the theory without curvature corrections. It is worth noting that our model has neither continuous nor discrete S-duality, so properties of the purely electric solution essentially differ from that of dyons.

The purpose of the present paper is to investigate existence of global solutions for small stretched purely electric black holes in higher-dimensional Einstein-Maxwell-Dilaton theory with the Gauss-Bonnet term (EMDGB). We construct the local solutions in terms of series expansion around the degenerate event horizon for an arbitrary space-time dimension D and calculate the discrete sequence of black hole entropies using Sen's entropy function approach. The entropy is found to be monotonically increasing with D. Then we continue numerically these local solutions and show that in dimensions higher than four the heterotic string value of the dilaton coupling lies

*cmchen@phy.ncu.edu.tw
†galtsov@phys.msu.ru
‡ohtan@phys.kindai.ac.jp
§orlov_d@mail.ru

inside the range of the existence of global asymptotically flat static black holes.

We also investigate physical significance of the so-called turning points which were encountered in numerical solutions within the four-dimensional EMDGB theory [53–56,64,66]. They correspond to mild singularities at finite radii outside the horizon where the metric and its first derivatives are finite, but the second derivatives diverge. Numerical solutions can be extended through these singularities, which we call "cusps" in this paper, by suitable redefinition of the integration variable [56,67]. In four dimensions the solution extended this way then meets a stronger singularity at finite distance, so actually the cusp is just a precursor of the strong singularity. In higher dimensions ($D \geq 7$) we encounter an interesting new feature: the cusps come out in pairs of right and left turning points, so the extended solution finally may be even asymptotically flat. This could correspond to a novel type of black hole coated by cusp pairs. But somewhat disappointingly, our analysis shows that continuation of geodesics through the cusp singularities in the extended manifolds cannot be performed in a smooth way. Thus we are inclined to reject such extended manifolds as physical black hole solutions. Instead, we interpret the occurrence of cusp singularity as failure to produce asymptotically flat black holes. This gives an upper bound on the dilaton coupling. We find numerically the sequence of critical dilaton couplings for $4 \leq D \leq 10$ which turns out to be increasing with D.

Another novel feature of EMDGB black holes with a degenerate horizon in higher dimensions is that the role of the GB term in the near-critical solutions may still be significant. In four dimensions, as was shown in [64], the near-critical solutions saturate the BPS bounds of the corresponding theory without curvature corrections. This means that relative contribution of the GB term becomes negligible when the dilaton coupling approaches its upper boundary. We find that for $D \geq 7$ this is not so, and the BPS conditions are not satisfied in this limit.

This paper is organized as follows. In Sec. II, we define the action, present the field equations in various forms, and discuss symmetries of the system. In Sec. III, we review solutions for small black holes without GB corrections as well as the solutions with the GB term but without dilaton. Then we construct the local series solutions near the horizon and calculate the entropy of stretched black holes using Sen's entropy function. We obtain the discrete sequence of the entropies of curvature corrected black holes in various dimensions interpolating starting with twice the Hawking-Bekenstein value $A/2$ for $D = 4$ up to $41A/52$ for $D = 10$. In Sec. IV, we present asymptotic expansions of the desired solutions, introduce global charges and discuss the BPS conditions. Section V is devoted to the cusp problem. We explain why extension of solutions through the cusp singularity is physically unacceptable. Finally in Sec. VI we present numerical results for various dimensions and explore the fulfillment of the BPS conditions on the boundary of the allowed dilaton couplings.

II. SETUP

A low-energy bosonic effective action for the heterotic string theory with the curvature corrections is given by [11,12]

$$I = \frac{1}{16\pi G}\int d^D x\sqrt{-\tilde{g}}\Phi\Big(\tilde{R} + \Phi^{-2}\tilde{\partial}_\mu\Phi\tilde{\partial}^\mu\Phi - \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{\alpha'}{8}\tilde{\mathcal{L}}_{\rm GB}\Big), \tag{1}$$

where $\tilde{F}^{\mu\nu}$ is the Maxwell field (we use a truncation involving only one $U(1)$ field), $\tilde{\mathcal{L}}_{\rm GB}$ is the Euler density

$$\tilde{\mathcal{L}}_{\rm GB} = \tilde{R}^2 - 4\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + \tilde{R}_{\alpha\beta\mu\nu}\tilde{R}^{\alpha\beta\mu\nu}, \tag{2}$$

and α' is the Regge slope parameter. The tilde denotes the quantities related to the string frame metric $\tilde{g}_{\mu\nu}$. The action can be transformed to the Einstein frame with metric $g_{\mu\nu}$ by the conformal rescaling

$$g_{\mu\nu} = \Phi^{2/(D-2)}\tilde{g}_{\mu\nu}, \tag{3}$$

giving

$$I = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\Big(R - \frac{\Phi^{-2}}{D-2}\partial_\mu\Phi\partial^\mu\Phi - \Phi^{2/(D-2)}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha'}{8}\Phi^{2/(D-2)}\mathcal{L}_{\rm GB} + \mathcal{F}(\partial\Phi, R)\Big), \tag{4}$$

where $\mathcal{F}(\partial\Phi, R)$ denotes the cross terms of $\partial\Phi$ and curvature coming from the GB term under the frame transformation. For simplicity, we do not include these terms in our analysis. We expect that inclusion of these terms might affect the black hole properties only quantitatively but not qualitatively. Then redefining the dilaton field as

$$\Phi = e^{\sqrt{2(D-2)}\phi}, \tag{5}$$

we obtain the action

$$I = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\Big(R - 2\partial_\mu\phi\partial^\mu\phi - e^{2\sqrt{2/(D-2)}\phi}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha'}{8}e^{2\sqrt{2/(D-2)}\phi}\mathcal{L}_{\rm GB}\Big). \tag{6}$$

In this action we have the sequence of the dilaton couplings

$$a_{\rm str}^2 = \frac{2}{D-2}, \tag{7}$$

relevant for the string theory. If we do this in four dimensions, we have the dilaton coupling $a_{\rm str} = 1$, but if we do this in ten dimensions, we have $a_{\rm str} = 1/2$. It will be convenient, however, to consider the above action for two arbitrary dilaton couplings a and b:

$$I = \frac{1}{16\pi G}\int d^D x\sqrt{-g}(R - 2\partial_\mu\phi\partial^\mu\phi - e^{2a\phi}F_{\mu\nu}F^{\mu\nu} + \alpha e^{2b\phi}\mathcal{L}_{GB}), \tag{8}$$

where we also denoted the GB coupling $\alpha'/8 = \alpha$.

The space-time metric is parametrized by two functions $\omega(r)$ and $\rho(r)$:

$$ds^2 = -\omega(r)dt^2 + \frac{dr^2}{\omega(r)} + \rho^2(r)d\Omega^2_{D-2}. \tag{9}$$

For convenience, we list in the Appendix the relevant geometric quantities for more general static spherically symmetric metrics.

We will consider only purely electric static spherically symmetric configurations of the D-dimensional Maxwell field

$$A = -f(r)dt. \tag{10}$$

Then, integrating the Maxwell equations

$$(\rho^{D-2}f' e^{2a\phi})' = 0, \tag{11}$$

one obtains

$$f'(r) = q_e\rho^{2-D}e^{-2a\phi}, \tag{12}$$

where q_e is the electric charge, which is considered as a free parameter (note that the physical electric charge defined asymptotically differs from this quantity; see Sec. IVA).

A. Field equations

We present the Einstein equations in the form

$$G_{\mu\nu} = 8\pi G(T^{\rm mat}_{\mu\nu} + T^{\rm GB}_{\mu\nu}), \tag{13}$$

where $T^{\rm mat}_{\mu\nu}$ is the matter stress tensor

$$8\pi G T^{\rm mat}_{\mu\nu} = 2\Big[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi g_{\mu\nu} + e^{2a\phi}\Big(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}\Big)\Big], \tag{14}$$

and the $T^{\rm GB}_{\mu\nu}$ is the effective gravitational stresses due to the GB term

$$8\pi G T^{\rm GB}_{\mu\nu} = -\alpha e^{2b\phi}[H_{\mu\nu} + 8(2b^2\nabla^\alpha\phi\nabla^\beta\phi + b\nabla^\alpha\nabla^\beta\phi)P_{\mu\alpha\nu\beta}], \tag{15}$$

where

$$H_{\mu\nu} = 2(RR_{\mu\nu} - 2R_{\mu\alpha}R^\alpha{}_\nu - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} + R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma}) - \frac{1}{2}\mathcal{L}_{GB}g_{\mu\nu}, \tag{16}$$

$$P_{\mu\alpha\nu\beta} = R_{\mu\alpha\nu\beta} + 2g_{\mu[\beta}R_{\nu]\alpha} + 2g_{\alpha[\nu}R_{\beta]\mu} + Rg_{\mu[\nu}g_{\beta]\alpha}. \tag{17}$$

For the metric (9), the components of $G_{\mu\nu}$ are

$$\begin{aligned} G_{tt} &= -\frac{(D-2)\omega}{2\rho^2}[2\omega\rho\rho'' + \rho\omega'\rho' + (D-3)(\omega\rho'^2 - 1)],\\ G_{rr} &= \frac{D-2}{2\omega\rho^2}[\rho\omega'\rho' + (D-3)(\omega\rho'^2 - 1)],\\ G_{\theta\theta} &= \frac{1}{2}\rho^2\omega'' + \frac{D-3}{2}[2\omega\rho\rho'' + 2\rho\omega'\rho' + (D-4)(\omega\rho'^2 - 1)], \end{aligned} \tag{18}$$

while the energy-momentum due to matter fields is given by

$$\begin{aligned} 8\pi G T^{\rm mat}_{tt} &= \omega^2\phi'^2 + e^{2a\phi}\omega f'^2,\\ 8\pi G T^{\rm mat}_{rr} &= \phi'^2 - e^{2a\phi}\frac{f'^2}{\omega},\\ 8\pi G T^{\rm mat}_{\theta\theta} &= -\rho^2(\omega\phi'^2 - e^{2a\phi}f'^2). \end{aligned} \tag{19}$$

The energy-stress tensor due to the GB term is more complicated

$$\begin{aligned} \frac{8\pi G}{\alpha e^{2b\phi}}T^{\rm GB}_{tt} &= -\frac{D_4^2\omega}{\rho^3}(2\omega\rho'' + \omega'\rho')(\omega\rho'^2 - 1) - \frac{D_5^2\omega}{2\rho^4}(\omega\rho'^2 - 1)^2 - \frac{2bD_3^2\omega}{\rho^2}[(2\omega\phi'' + \omega'\phi')(\omega\rho'^2 - 1)\\ &\quad + 2\omega\rho'\phi'(2\omega\rho'' + \omega'\rho')] - \frac{4bD_4^2\omega^2\rho'\phi'}{\rho^3}(\omega\rho'^2 - 1) - \frac{8b^2D_3^2\omega^2\phi'^2}{\rho^2}(\omega\rho'^2 - 1),\\ \frac{8\pi G}{\alpha e^{2b\phi}}T^{\rm GB}_{rr} &= \frac{D_4^2\omega'\rho'}{\omega\rho^3}(\omega\rho'^2 - 1) + \frac{D_5^2}{2\omega\rho^4}(\omega\rho'^2 - 1)^2 + 2b\Big[\frac{D_3^2\omega'\phi'}{\omega\rho^2}(3\omega\rho'^2 - 1) + \frac{2D_4^2\rho'\phi'}{\rho^3}(\omega\rho'^2 - 1)\Big],\\ \frac{8\pi G}{\alpha e^{2b\phi}}T^{\rm GB}_{\theta\theta} &= D_4^3[\omega''(\omega\rho'^2 - 1) + 2\omega\omega'\rho'\rho'' + \omega'^2\rho'^2] + \frac{2D_5^3}{\rho}(\omega\rho')'(\omega\rho'^2 - 1) + \frac{D_6^3}{2\rho^2}(\omega\rho'^2 - 1)^2\\ &\quad + 4b\Big[D_3^3\rho(\omega\omega'\rho'\phi')' + D_4^3(\omega\phi')'(\omega\rho'^2 - 1) + 2D_4^3\omega\rho'\phi'(\omega\rho')' + \frac{D_5^3\omega\rho'\phi'}{\rho}(\omega\rho'^2 - 1)\Big]\\ &\quad + 8b^2[D_3^3\omega\omega'\rho\rho'\phi'^2 + D_4^3\omega\phi'^2(\omega\rho'^2 - 1)], \end{aligned} \tag{20}$$

where we have introduced the dimension-dependent coefficients

$$D_n^m = (D-m)_n = (D-m)(D-m-1)\cdots(D-n), \qquad n \geq m. \quad (21)$$

The dilaton equation reads

$$2(\omega\phi')' + 2D_2^2\omega\phi'\frac{\rho'}{\rho} + 2af'^2e^{2a\phi} + \alpha b D_3^2 e^{2b\phi}\Big\{2\frac{[\omega'(\omega\rho'^2-1)]'}{\rho^2} + D_5^4\frac{(\omega\rho'^2-1)^2}{\rho^4} + 4D_4^4(\omega\rho')'\frac{\omega\rho'^2-1}{\rho^3}\Big\} = 0. \quad (22)$$

From the Einstein equation, one can derive the following two second-order equations for the metric functions $\rho(r)$ and $\omega(r)$,[1] which are more convenient for numerical integration:

$$D_2^2\rho\rho'' + 2\rho^2\phi'^2 - 4\alpha b D_3^2[(\omega\rho'^2-1)\phi' e^{2b\phi}]' + 2\alpha D_3^2 e^{2b\phi}\Big(2b\omega'\rho'^2\phi' - D_4^4\frac{\omega\rho'^2-1}{\rho}\rho''\Big) = 0, \quad (23)$$

$$\rho\omega'' + 2D_3^3(\omega\rho')' + D_4^3\frac{\omega\rho'^2-1}{\rho} + 2\omega\rho\phi'^2 - 2\rho f'^2 e^{2a\phi} - 8\alpha b D_3^3(\omega\omega'\rho'\phi' e^{2b\phi})' - \alpha D_4^3 e^{2b\phi}\Big\{D_6^5\frac{(\omega\rho'^2-1)^2}{\rho^3} + 4D_5^5[(\omega\rho')' + 2b\omega\rho'\phi']\frac{\omega\rho'^2-1}{\rho^2} + 2[\omega'' + 4b(\omega\phi')' + 8b^2\omega\phi'^2]\frac{\omega\rho'^2-1}{\rho} + 2\frac{\rho'}{\rho}[2\omega\omega'\rho'' + \omega'^2\rho' + 8b\omega\phi'(\omega\rho')']\Big\} = 0. \quad (24)$$

B. Symmetries of the reduced action

One can check that equations of motion are invariant under a *three-parametric* group of global transformations which consist of the transformations of the field functions:

$$\omega \to \omega e^{\mu}, \qquad \rho \to \rho e^{\delta}, \qquad \phi \to \phi + \frac{\delta}{b}, \qquad f \to f e^{(\mu/2)-(a/b)\delta}, \quad (25)$$

accompanied by the shift and rescaling of the radial variable

$$r \to r e^{(\mu/2)+\delta} + \nu. \quad (26)$$

[1] Namely, the equation for ρ is $-\frac{\rho^2}{\omega^2}[(\text{Einstein equation})_{tt} + \omega^2(\text{Einstein equation})_{rr}]$ and the equation for ω is $\frac{2}{\rho}\times(\text{Einstein equation})_{\theta\theta}$.

Transformation of the electric potential is equivalent to rescaling of the electric charge

$$q_e \to q_e e^{(D-3+(a/b))\delta}. \quad (27)$$

Not all of these symmetries are the symmetries of the Lagrangian, however. Integrating the action (8) over the $(D-2)$-dimensional sphere and dropping integration over time integral, one obtains the one-dimensional reduced Lagrangian from the relation $I = \int L dr$. Up to the total derivative one has

$$L = D_2^2\rho'(\omega\rho^{D-3})' + D_3^2\rho^{D-4} - 2\rho^{D-2}(\omega\phi'^2 - f'^2e^{2a\phi}) - \frac{4}{3}\alpha D_4^2\rho'^3(\omega^2\rho^{D-5}e^{2b\phi})' + 4\alpha D_4^2\rho'(\omega\rho^{D-5}e^{2b\phi})' - \alpha e^{2b\phi}[4bD_3^2\rho^{D-4}\omega'\phi' - 2D_4^2\rho^{D-5}\omega'\rho' - D_5^2\rho^{D-6}(\omega\rho'^2-1)](\omega\rho'^2-1). \quad (28)$$

It is easy to check that the one-dimensional action remains invariant under the above transformations provided

$$\mu = -2(D-3)\delta, \quad (29)$$

namely, under the following *two-parametric* group of global transformations:

$$r \to re^{-(D-4)\delta} + \nu, \qquad \omega \to \omega e^{-2(D-3)\delta}, \qquad \rho \to \rho e^{\delta}, \qquad \phi \to \phi + \frac{\delta}{b}, \qquad f \to f e^{-(D-3+(a/b))\delta}. \quad (30)$$

They generate two conserved Noether currents,

$$J_g := \Big(\frac{\partial L}{\partial \Phi'^A}\Phi'^A - L\Big)\partial_g r|_{g=0} - \frac{\partial L}{\partial\Phi'^A}\partial_g\Phi^A|_{g=0}, \qquad \partial_r J_g = 0, \quad (31)$$

where Φ^A stands for ω, ρ, ϕ, f, and $g = \delta$, ν. The conserved quantity corresponding to ν is the Hamiltonian

$$H = D_2^2\rho'(\omega\rho^{D-3})' - D_3^2\rho^{D-4} - 2\omega\rho^{D-2}\phi'^2 + 2\rho^{D-2}f'^2e^{2a\phi} - \alpha e^{2b\phi}[4bD_3^2\rho^{D-4}\omega'\phi'(3\omega\rho'^2-1) - 2D_4^2\rho^{D-5}\omega'\rho'(3\omega\rho'^2-1) - D_5^2\rho^{D-6}(\omega\rho'^2-1)\times(3\omega\rho'^2+1)] - 4\alpha D_4^2\rho'^3(\omega^2\rho^{D-5}e^{2b\phi})' + 4\alpha D_4^2\rho'(\omega\rho^{D-5}e^{2b\phi})'. \quad (32)$$

This is known to vanish on shell for diffeomorphism invariant theories, $H = 0$. The Noether current corresponding to the parameter δ leads to the conservation equation $\partial_r J_\delta = 0$, where

$$J_\delta = -D_4^4 rH - D_2^2\omega'\rho^{D-2} + \frac{4}{b}\omega\rho^{D-2}\phi' + 4\Big(D-3+\frac{a}{b}\Big)q_e f + \alpha e^{2b\phi}[(\omega\rho'^2 - 1) \times (2D_2^2 D_3^2\omega'\rho^{D-4} - 8bD_3^2\omega\rho^{D-4}\phi') + 8bD_3^2\omega\omega'\rho^{D-3}\rho'\phi']. \tag{33}$$

Symmetry transformations will be used to rescale numerically obtained solutions to desired asymptotic form and obtain true physical parameters of the solution.

III. STRETCHING THE HORIZON OF A SMALL BLACK HOLE

A. Small dilatonic *D*-dimensional black hole without GB term

Let us first discuss the black hole solution without the GB term. It can be presented in the form [68]

$$ds^2 = -f_+ f_-^{-1+(4(D-3)/(D-2)\Delta)}dt^2 + f_+^{-1} f_-^{-1+(2/(D-3))-(4/(D-2)\Delta)}dr^2 + r^2 f_-^{(2/(D-3))-(4/(D-2)\Delta)}d\Omega_{D-2}^2, \tag{34}$$

$$e^{2a\phi} = e^{2a\phi_\infty} f_-^{-(2a^2/\Delta)}, \quad F_{\mathrm{tr}} = 4(D-3)\frac{(r_+r_-)^{(D-3)/2}}{\sqrt{\Delta}}e^{-a\phi_\infty}\frac{1}{r^{D-2}}, \tag{35}$$

where

$$f_\pm = 1 - \frac{r_\pm^{D-3}}{r^{D-3}}, \qquad \Delta = a^2 + \frac{2(D-3)}{D-2}. \tag{36}$$

The mass and the electric and dilaton charges are given by

$$\mathcal{M} = \frac{\Omega_{D-2}}{16\pi G}\Big[(D-2)(r_+^{D-3} - r_-^{D-3}) + \frac{4(D-3)}{\Delta}r_-^{D-3}\Big], \tag{37}$$

$$Q_e = \frac{(D-3)\Omega_{D-2}}{4\pi G}\sqrt{\frac{(r_+r_-)^{D-3}}{\Delta}}e^{a\phi_\infty}, \tag{38}$$

$$\mathcal{D} = -\frac{(D-3)a\Omega_{D-2}}{4\pi G\Delta}r_-^{D-3}. \tag{39}$$

For $a=0$, this solution reduces to the D-dimensional Reissner-Nordström solution. In the extremal limit $r_+ = r_- = r_0$, it contracts to

$$ds^2 = -f_0^2 dt^2 + f_0^{-1}dr^2 + r^2 d\Omega_{D-2}^2, \quad f_0 = 1 - \frac{r_0^{D-3}}{r^{D-3}}, \tag{40}$$

and has a degenerate event horizon $\mathrm{AdS}_2 \times S^{D-2}$. Note that for $a=0$, the GB term decouples from the system, so this solution remains valid in the full theory with $\alpha \neq 0$.

For $a \neq 0$, the extremal solution reads

$$ds^2 = -f_0^{(4(D-3)/(D-2)\Delta)}dt^2 + f_0^{-2(D-4)/(D-3)-(4/(D-2)\Delta)}dr^2 + r^2 f_0^{2/(D-3)-(4/(D-2)\Delta)}d\Omega_{D-2}^2. \tag{41}$$

This has a null singularity at the horizon. The Ricci scalar in the vicinity of this point diverges as

$$R \sim (r^{D-3} - r_0^{D-3})^{-(2/(D-3))+(4/(D-2)\Delta)}, \tag{42}$$

together with the dilaton function

$$e^{2a\phi} \sim (r^{D-3} - r_0^{D-3})^{-(2a^2/\Delta)}. \tag{43}$$

The divergence of the GB term near the horizon is[2]

$$e^{2a\phi}\mathcal{L}_{\mathrm{GB}}|_{r=r_+} \sim (r_+ - r_-)^{-(a^2(D^2-4)/(a^2(D-2)+2(D-3)))}, \tag{44}$$

so one can expect that the GB term will substantially modify the dilaton black hole solution in the extremal limit.

The mass, the dilaton charge, and the electric charge for this solution (defined as in Sec. IV below) are

$$\mathcal{M} = \frac{\Omega_{D-2}}{4\pi G}\frac{D_3^3}{\Delta}r_0^{D-3}, \quad \mathcal{Q}_e = Q_e e^{-a\phi_\infty} = \frac{\Omega_{D-2}}{4\pi G}\frac{D_3^3}{\sqrt{\Delta}}r_0^{D-3}, \quad \mathcal{D} = -\frac{\Omega_{D-2}}{4\pi G}\frac{aD_3^3}{\Delta}r_0^{D-3}. \tag{45}$$

They are determined by a single parameter r_0, so we have the following relations among the three quantities:

$$\mathcal{D} = a\mathcal{M}, \qquad \mathcal{Q}_e = \sqrt{\Delta}\mathcal{M}, \tag{46}$$

which imply the following BPS condition,

$$a^2\mathcal{M}^2 + \mathcal{D}^2 = \frac{2a^2}{\Delta}\mathcal{Q}_e^2. \tag{47}$$

B. Wiltshire black hole

Another limit in which our action admits an exact solution is that of vanishing dilaton. This is consistent with the field equations for $a = b = 0$. In this case an exact solution was found by Wiltshire [69]:

[2] We use this occasion to correct Eq. (33) of our previous paper for $D = 4$ [64].

$$\omega(r) = 1 + \frac{r^2}{2D_4^3\alpha}\left(1 \mp \sqrt{1 + \frac{64\pi D_4^3\alpha\mathcal{M}}{D_2^2\Omega_{D-2}r^{D-1}} - \frac{8D_4^4\alpha q_e^2}{D_2^2 r^{2(D-2)}}}\right),$$
$$\rho(r) = r. \tag{48}$$

The lower sign corresponds to an asymptotically AdS space-time for $\alpha > 0$ and to an asymptotically de Sitter solution for $\alpha < 0$. The upper sign leads to an asymptotically flat solution coinciding with the D-dimensional Reissner-Nordström solution. These solutions exist in dimensions $D \geq 5$ where the GB term is not the total derivative. The asymptotically flat solution has two horizons which coincide in the extremal limit for a special value of the electric charge. For the extremal solution, the mass and the charge can be expressed in terms of the single parameter, the radius of the horizon r_0:

$$\mathcal{M} = \frac{\Omega_{D-2}}{8\pi}(D-2)[r_0^2 + (D-4)^2\alpha]r_0^{D-5},$$
$$q_e^2 = \frac{D_3^2}{2}[r_0^2 + D_5^4\alpha]r_0^{2(D-4)}. \tag{49}$$

Conversely the radius can be expressed as

$$r_0^{D-3} = -\frac{4\pi D_5^5\mathcal{M}}{D_2^2\Omega_{D-2}} + \sqrt{\left(\frac{4\pi D_5^5\mathcal{M}}{D_2^2\Omega_{D-2}}\right)^2 + \frac{2D_4^4q_e^2}{D_3^2}}. \tag{50}$$

C. Local solution near the horizon

In what follows we set $b = a$ as relevant for the heterotic string theory case, but still keeping a arbitrary. Assuming that the full system with the GB term admits the $AdS_2 \times S^{D-2}$ horizon, $r = r_H$, we look for the series expansions of the metric function in powers of $x = r - r_H$:

$$\omega(r) = \sum_{i=2}^{\infty}\omega_i x^i, \qquad \rho(r) = \sum_{i=0}^{\infty}\rho_i x^i,$$
$$P(r) := e^{2a\phi(r)} = \sum_{i=0}^{\infty}P_i x^i. \tag{51}$$

The function ω starts with the quadratic term in view of the degeneracy of the horizon, while two other functions have the general Taylor's expansions. Denoting the physical radius of the horizon $\rho_0 = \rho(r_H)$, we obtain for the leading order coefficients:

$$\omega_2 = \frac{D_3^2}{2\rho_0^2}, \qquad P_0 = \frac{\rho_0^2}{4\alpha(2D-7)}, \tag{52}$$

and ρ_0 is related to the electric charge via

$$\rho_0^{D-2} = q_e\frac{4\sqrt{2\alpha}(2D-7)}{\sqrt{D_3^2(D^2-D-8)}}. \tag{53}$$

Note that the expression under the square root and the right-hand side as a whole are positive for $D \geq 4$. The horizon radius is fixed entirely by the electric charge, like in the extremal Reissner-Nordström case. In our units the GB parameter α has dimension L^2. When the GB term is switched off ($\alpha \to 0$), the horizon radius shrinks, as expected for small extremal black holes. Higher order expansion coefficients exhibit dependence on only one free parameter, namely, the P_1 in the dilaton expansion. Other coefficients are expressed in terms of the horizon radius ρ_0 and P_1, the first subleading coefficients being

$$\omega_3 = -\frac{2\alpha P_1}{3a^2\rho_0^4(D^3-9D^2+16D+8)}[D_3^2(3D^5-43D^4+213D^3-421D^2+236D+76)a^2 + D_3^3(2D-7)(3D^4-25D^3+70D^2-56D-40)],$$
$$\rho_1 = \frac{4\alpha P_1[(D^4-13D^3+54D^2-72D-2)a^2+(2D-7)(D^2-3D-2)]}{a^2\rho_0(D^3-9D^2+16D+8)}. \tag{54}$$

One can notice that the free parameter enters the expansion coefficients always in the combination P_1/a^2. This facilitates transition to the Wiltshire case.

In the limit of decoupled dilaton $a = 0$, the parameter $P_1 \to 0$, while the ratio P_1/a^2 remains finite. In this case we have nonvanishing coefficients P_0, ρ_1, and ω_i:

$$P(r) = P_0, \qquad \rho(r) = \rho_0 + \rho_1(r - r_0). \tag{55}$$

The asymptotic flatness requires $\rho_1 = 1$, so we have $\rho_0 = r_0$. For the Wiltshire solution $P_0 = 1$, and we obtain the following relation in the extremal case:

$$\rho_0^2 = r_0^2 = 4\alpha(2D-7), \tag{56}$$

Substituting (56) into (49), we find

$$\mathcal{M} = \frac{\Omega_{D-2}}{32\pi}\frac{D_2^2(D^2-12)}{2D-7}r_0^{D-3},$$
$$q_e^2 = \frac{1}{8}\frac{D_3^2(D^2-D-8)}{2D-7}r_0^{2(D-3)}, \tag{57}$$

and so our solution with the decoupled dilaton coincides with the extremal case of the Wiltshire solution.

We can consider subgroup of global symmetry transformation defined by two parameters δ and μ. We can eliminate the parameter ρ_0 from the expansion on the horizon if we apply the transformation with parameters $\mu = -2\ln\rho_0$ and $\delta = -\ln\rho_0$. The other transformation with parameters $\mu = 2\ln|P_1|$, $\delta = 0$ can take out P_1 from expansions. Choosing the absolute value $|P_1|$ in the second transformation allows us to get the remaining parameter $\xi = \frac{P_1}{|P_1|}$ in the expansion at the horizon which fixes the sign of ρ ($\rho_1 > 0$ to obtain a global solution) in the expansion. As a result, we have a map between parameters of expansion P_1, ρ_0 and parameters of global transformation μ, δ. Typically one can first investigate special solution with a simple choice of near horizon data, such as $P_1 = 1$ and $\rho_0 = 1$, which are the values we use for our numerical analysis below. Then general solutions with arbitrary values of free parameters can be simply obtained by the global transformation with $\mu = 2\ln\frac{|P_1|}{\rho_0}$, $\delta = -\ln\rho_0$. Also it is clear from the relation between ρ_0 and q_e that electrical charge plays the role of rescaling parameter. Finally the free parameter P_1 could be fixed in accordance with the boundary condition at infinity. We can also eliminate the GB coupling constant α from the system by introducing new dilaton function $F = \alpha P$ and rescaling charge q_e. In this way, we have only two parameters, the number of the dimension D and the dilaton coupling a, which affect the dynamics of solutions.

The values of the integrals of motion (32) and (33) in terms of the parameters of the local solution are

$$H = -\alpha\rho_0^{D-2}D_5^2P_0 - D_3^2\rho_0^{D-4} + \frac{2q_e^2}{P_0}\rho_0^{2-D}, \qquad J_\delta = 4(D-2)q_e f_0. \tag{58}$$

D. The entropy

Knowledge of the local solution near the horizon is enough to calculate the entropy of the black hole, assuming that the local solution can be extended to infinity. To compute the entropy, we apply Sen's entropy function approach [3] which is valid for the black holes with near horizon geometry of $AdS_2 \times S^{D-2}$. Using the notation of [3] we parametrize the near horizon geometry by two constants, v_1 and v_2 related to the radii of AdS_2 and S^{D-2}, as

$$ds^2 = v_1\left(-r^2d\tau^2 + \frac{dr^2}{r^2}\right) + v_2 d\Omega_{D-2}^2. \tag{59}$$

The scalar curvature and the GB term will read

$$R = -\frac{2}{v_1} + \frac{D_3^2}{v_2}, \qquad \mathcal{L}_{GB} = \frac{D_5^2}{v_2^2} - \frac{4D_3^2}{v_1 v_2}. \tag{60}$$

The dilaton field and gauge field strength are constant on the horizon

$$\phi = u, \qquad F_{\tau r} = p. \tag{61}$$

Sen's entropy function is defined to be the integrand of the action after integrating all angular coordinates of S^{D-2}. Using (8) we obtain

$$f = \frac{\Omega_{D-2}}{16\pi G}v_1 v_2^{(D-2)/2}\left[-\frac{2}{v_1} + \frac{D_3^2}{v_2} + e^{2au}\frac{2p^2}{v_1^2} + \alpha e^{2au}\left(\frac{D_5^2}{v_2^2} - \frac{4D_3^2}{v_1 v_2}\right)\right]. \tag{62}$$

The parameters v_1, v_2, u, e are related to the near horizon expansion coefficients by (note the rescaling of time coordinates $\tau = \omega_2 t$)

$$\omega_2 = \frac{1}{v_1}, \qquad \rho_0 = \sqrt{v_2}, \qquad P_0 = e^{2au}, \qquad q_e = p v_1^{-1} v_2^{(D-2)/2} e^{2au}. \tag{63}$$

According to the equations of motion, the value of parameters should minimize the entropy function:

$$\partial_{v_1} f = 0, \qquad \partial_{v_2} f = 0, \qquad \partial_u f = 0, \tag{64}$$

which lead to the following constraints,

$$v_2 = \frac{D_3^2}{2}v_1, \qquad v_1 = \frac{4(2D-7)p^2}{D^2-D-8}e^{2au}, \qquad p^2 = \frac{2(D^2-D-8)}{D_3^2}\alpha, \tag{65}$$

and furthermore imply $f = 0$. These three constraints are exactly identical with the relations (52) and (53) from the near horizon analysis. The physical electric charge, q (i.e. Q_e defined in Sec. IVA), can be obtained via $q = \partial_e f$

$$q = \frac{\Omega_{D-2}}{4\pi G}p v_1^{-1} v_2^{(D-2)/2} e^{2au} = \frac{\Omega_{D-2}}{4\pi G}q_e. \tag{66}$$

The entropy of black holes is related to the entropy function by a Legendre transformation

$$S = 2\pi(qp - f) = 2\pi qp = \frac{D^2-D-8}{8(2D-7)G}\Omega_{D-2}v_2^{(D-2)/2}. \tag{67}$$

The horizon area of $AdS_2 \times S^2$ is $A = \mathrm{vol}(\Omega_{D-2})\times v_2^{(D-2)/2}$, thus the entropy can be expressed in terms of area of horizon as

$$S = \frac{D^2-D-8}{8(2D-7)G}A = \frac{A}{4G} + \frac{D^2-5D+6}{8(2D-7)G}A = S_{BH} + S_{GB}, \tag{68}$$

and the deviation of the entropy from Bekenstein-Hawking relation by the GB term increases for higher and higher dimensions. For example, the ratio of S_{GB}/S_{BH} from $D = 4$ to 10 is

$$\frac{S_{GB}}{S_{BH}} = \left\{1, 1, \frac{6}{5}, \frac{10}{7}, \frac{5}{3}, \frac{21}{11}, \frac{28}{13}\right\}. \tag{69}$$

A general discussion on the entropy of theories with quadratic curvature correction and Lovelock theory is given in [49].

IV. ASYMPTOTICS

Now consider the asymptotic expansions of the metric function by substituting the following expansions into the equations of motion:

$$\omega(r) = 1 + \sum_{i=1} \frac{\bar{\omega}_i}{r^i}, \qquad \rho(r) = r + \sum_{i=1} \frac{\bar{\rho}_i}{r^i}, \qquad \phi(r) = \bar{\phi}_\infty + \sum_{i=1} \frac{\bar{\phi}_i}{r^i}. \tag{70}$$

According to the falloff of the Newton potential in different dimensions, one has the first nonzero term in the expansion for ω and that for dilaton starting from $i = D-3$, while ρ differs from r in $(2D-7)$-th terms:

$$\omega(r) = 1 + \frac{\bar{\omega}_{D-3}}{r^{D-3}} + \frac{2q_e^2 e^{-2a\bar{\phi}_\infty}}{D_3^2} \frac{1}{r^{2(D-3)}} + O\left(\frac{1}{r^{2D-4}}\right),$$
$$\rho(r) = r - \frac{(D-3)\bar{\phi}_{D-3}^2}{(D-2)(2D-7)} \frac{1}{r^{2D-7}} + O\left(\frac{1}{r^{2D-5}}\right),$$
$$\phi(r) = \bar{\phi}_\infty + \frac{\bar{\phi}_{D-3}}{r^{D-3}} - \frac{1}{2}\left[\frac{aq_e^2 e^{-2a\bar{\phi}_\infty}}{(D-3)^2} + \bar{\omega}_{D-3}\bar{\phi}_{D-3}\right] \times \frac{1}{r^{2(D-3)}} + O\left(\frac{1}{r^{2D-4}}\right). \tag{71}$$

One can notice, that these terms of expansion do not contain the GB coupling α. The contribution of the GB term is manifest in the third nonvanishing coefficient in ρ. If the GB term is switched off, $\alpha = 0$, the third nonvanishing coefficient of ρ is

$$\bar{\rho}_{3D-10} = \frac{4}{3D_3^2(3D-10)} \bar{\phi}_{D-3}[(D-3)^2\bar{\omega}_{D-3}\bar{\phi}_{D-3} + aq_e^2 e^{-2a\bar{\phi}_\infty}]. \tag{72}$$

In presence of the GB term it is

$$\bar{\rho}_{2D-5} = \frac{2(D-3)^2}{2D-5} \alpha a \bar{\omega}_{D-3}\bar{\phi}_{D-3} e^{2a\bar{\phi}_\infty}. \tag{73}$$

So in $D = 4$ the general relativity (GR) contribution dominates appearing as $\bar{\rho}_2$, and in $D = 5$ both GR and GB contributions appear in $\bar{\rho}_5$, but in higher dimensions, the GB contribution is leading.

For the asymptotically flat geometry the global physical quantities, such as mass and charges, can be read out from the asymptotic expansion. Since the first subleading coefficients are independent of the GB coupling, we can still use the formula of global charges for the theories without higher curvature corrections.

A. Global charges

The Arnowitt-Deser-Misner (ADM) mass is given in our notation by[3]

$$\mathcal{M} = \frac{\Omega_{D-2}}{8\pi G}(D-2)\left[r^{D-3}\left(\frac{1}{\sqrt{\omega}} - \frac{\rho}{r}\right) - r^{D-2}\left(\frac{\rho}{r}\right)'\right]_{r\to\infty}, \tag{74}$$

and reduces to

$$\mathcal{M} = -\frac{\Omega_{D-2}}{16\pi G}(D-2)[\bar{\omega}_{D-3} - 2(D-4)\bar{\rho}_{D-4}]. \tag{75}$$

For $D > 4$, in general, the ADM mass could depend not only on the first subleading coefficient $\bar{\omega}_{D-3}$ of ω, but also on the subleading coefficient $\bar{\rho}_{D-4}$ of ρ. But we have seen this coefficient is zero, so we have

$$\bar{\omega}_{D-3} = -\frac{16\pi G\mathcal{M}}{(D-2)\Omega_{D-2}}. \tag{76}$$

The definition of the dilaton charge $\mathcal{D}$ is

$$\mathcal{D} = \frac{1}{4\pi G}\int_{r\to\infty} d\Omega_{D-2} r^{D-2}\partial_r\phi, \tag{77}$$

which has a contribution from the expansion coefficient

$$\bar{\phi}_{D-3} = -\frac{4\pi G\mathcal{D}}{(D-3)\Omega_{D-2}}. \tag{78}$$

The physical electric charge can be computed by the flux

$$Q_e = \frac{1}{4\pi G}\int_{r\to\infty} d\Omega_{D-2} r^{D-2} e^{2a\phi_\infty} F_{\rm tr}, \tag{79}$$

so we have the following relation between this quantity and the charge introduced as an integration constant in the previous section:

$$q_e = \frac{4\pi G}{\Omega_{D-2}} Q_e. \tag{80}$$

The asymptotic values of two integrals of motion (32) and (33) are

$$H^\infty = [D_3^2(r\rho')_\infty^{D-4} - \alpha D_5^2 P_\infty (r\rho')_\infty^{D-6}](\omega_\infty \rho_\infty'^2 - 1), \tag{81}$$

$$J_\delta^\infty = [\alpha e^{2a\phi_\infty} D_5^2 D_4^4 (r\rho')_\infty^{D-5}(\omega_\infty\rho_\infty'^2 - 1) - D_4^2(r\rho')_\infty^{D-3}] \times \frac{\omega_\infty\rho_\infty'^2 - 1}{\rho'_\infty} + 4D_2^2 q_e f_\infty - 2D_2^2 D_3^2 \mathcal{M}\rho_\infty'^{D-2} - 4D_3^3\omega_\infty\rho_\infty'^{D-2}\frac{\mathcal{D}}{a}. \tag{82}$$

From $H = 0$, we can see that $\omega_\infty\rho_\infty'^2 \to 1$ for $r \to \infty$, which also regularizes the second integral of motion. For Minkowski space ($\omega_\infty = \rho'_\infty = 1$), we have

$$J_\delta^\infty = 4D_2^2 q_e f_\infty - 2D_2^2 D_3^2 \mathcal{M} - 4D_3^3 \frac{\mathcal{D}}{a}. \tag{83}$$

It is possible to apply global transformation to satisfy asymptotically flat condition which fixes one of the free parameters in expansion around horizon, P_1. Note that the values of the integral of motion are 4 times of what we have

[3] The volume of S^{D-2} is $\Omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma(\frac{D-1}{2})}$ and the gamma function is either $\Gamma(n+1) = n!$ or $\Gamma(\frac{n}{2}+1) = \sqrt{\pi}\frac{n!!}{2^{(n+1)/2}}$ for integer n. This gives $\Omega_{D-2} = \{4\pi, 2\pi^2, \frac{8}{3}\pi^2, \pi^3, \frac{16}{15}\pi^3, \frac{1}{3}\pi^4, \frac{32}{105}\pi^4\}$ for $D = 4, \cdots, 10$.

in [64]:

$$H = \frac{1}{2}(\omega_\infty \rho'^2_\infty - 1), \qquad J_\delta = 2q_e f_\infty - \mathcal{M} - \frac{\mathcal{D}}{a}.$$

B. BPS condition

The theory we are considering here does not necessarily have an underlying supersymmetry. However, it is instructive to investigate the fulfilment of the no-force condition which is usually associated with the supersymmetry. In particular, in the $D = 4$ case the supersummetric embedding into the heterotic string theory in the supergravity limit gives the BPS condition for the extremal small black holes ($\mathcal{Q}_e = Q_e e^{-a\phi_\infty}$)

$$\mathcal{M}^2 + \mathcal{D}^2 = \mathcal{Q}_e^2. \tag{84}$$

This corresponds to the vanishing of the sum of the gravitational and dilaton attractive forces and the electric repulsion. This does not hold if the GB term is turned on. However, it was demonstrated in [64] that on the boundary of the allowed domain of the dilaton coupling the role of the GB term is diminished, and the BPS condition is restored. Our aim here is to confirm this property.

In the higher-dimensional cases the gravitational, Coulomb and dilaton forces are

$$F_g \sim -\frac{8\pi G(D-3)}{(D-2)\Omega_{D-2}}\frac{\mathcal{M}^2}{r^{D-2}}, \qquad F_A \sim \frac{4\pi G}{\Omega_{D-2}}\frac{\mathcal{Q}_e^2}{r^{D-2}},$$
$$F_\phi \sim -\frac{4\pi G}{\Omega_{D-2}}\frac{\mathcal{D}^2}{r^{D-2}}, \tag{85}$$

so the no-force condition reads

$$2(D-3)\mathcal{M}^2 + (D-2)\mathcal{D}^2 = (D-2)\mathcal{Q}_e^2. \tag{86}$$

In the case that the GB term is decoupled, i.e. $\alpha = 0$, the no-force condition (86) at infinity is equivalent to the degenerated horizon obtained for the exact extremal dilatonic black hole solutions in the previous section,

$$\mathcal{D} = a\mathcal{M},$$
$$\mathcal{Q}_e = \sqrt{\Delta}\mathcal{M} \Rightarrow a^2\mathcal{M}^2 + \mathcal{D}^2 = \frac{2a^2}{\Delta}\mathcal{Q}_e^2. \tag{87}$$

Note that the relation (86) does not involve explicitly the dilaton coupling (though it appears in the definition of $\mathcal{Q}_e$). The special case of $D = 4$ ($\Delta = a^2 + 1$) was earlier discussed in [64] in which case $\bar{\omega}_1 = -2\mathcal{M}$, $\bar{\phi}_1 = -\mathcal{D}$ (with different sign convention).

V. CUSPS

It was discovered in [64] that the 4D EMDGB static spherically symmetric gravity typically develops cusps at some points $r = r_c$ in the vicinity of where the metric functions vanish. There they have Taylor expansions in terms of

$$y = |r - r_c|. \tag{88}$$

The metric and its first derivative are regular there, while the second derivatives diverge as $y^{-1/2}$. There are therefore the cusp hypersurfaces which are the spheres S^{D-2} of finite radius. These cusp spheres have curvature singularity which is rather mild (the Ricci scalar diverges only as $y^{-1/2}$, and the Kretchmann scalar as $1/y$). They are in fact the singular turning points of the radial variable $\rho(r)$.

The presence of turning points in the numerical solutions was encountered in the case $D = 4$ in [55,64,67]. The numerical solution can be extended through these points using the technique of [67] and then the solution evolves into a strong singularity. Here we find that the situation is similar in higher dimensions $D \leq 6$, but starting from $D = 7$ the solution can be extended to an asymptotically flat one.

A. Expansion near the turning points

The general property of the turning points is that the metric functions and the exponential of the dilaton field, $f = \{\omega(r), \rho(r), F(r) = \alpha e^{2a\Phi(r)}\}$, have finite first derivative and divergent second derivative, i.e.

$$f'(r_{\rm tp}) = \text{constant}, \qquad f''(r_{\rm tp}) \to \infty. \tag{89}$$

The metric functions and the dilaton can be expanded in terms of the fractional powers of the variable y

$$f(y) = f_0 + \sum_{i=2} f_i y^{i/2}, \tag{90}$$

where we have either $y = r_{\rm tp} - r$ (the right turning point) or $y = r - r_{\rm tp}$ (the left turning point). These two types of turning points have opposite signs of the odd-order derivatives,

$$f^{(2n+1)}(r - r_{\rm tp}) = -f^{(2n+1)}(r_{\rm tp} - r), \tag{91}$$

and the expansion coefficients, $\{f_i\}$, have the same "iterative" relations for both type turning points. The expansions read

$$\omega = \omega_0 + \omega_2 y + \omega_3 y^{3/2} + O(y^2), \tag{92}$$

$$\rho = \rho_0 + \rho_2 y + \rho_3 y^{3/2} + O(y^2), \tag{93}$$

$$F = F_0 + F_2 y + F_3 y^{3/2} + O(y^2). \tag{94}$$

They contain four free parameters, namely, ω_0, ρ_0, F_0 and ρ_2 (for the fixed charge parameter q_e), other coefficients depending on them. The coefficient ρ_3 is given by the square roots of a second order equation which can have two branches (positive and negative) corresponding to double valued solution near turning points. Similarly, ω_3 and F_3 also have two-branch solutions.

The exponents in the turning point expansions are independent of D. Therefore, the rate of divergence of geometric quantities is universal. More precisely, the scalar curvature is

$$R \sim -\frac{3}{4}\frac{\omega_3\rho_0 + 2(D-2)\omega_0\rho_3}{\rho_0} y^{-1/2}, \tag{95}$$

and the matter stress tensor is finite. Indeed, ω_3 is proportional to ρ_3 which has double values (with opposite sign) near the turning point. Therefore, the sign of divergent scalar curvature also changes. Moreover, one expects that the GB combination should have y^{-1} divergence, but actually it is weaker, namely, $y^{-1/2}$.

For numerical integration, we rewrite the equations of motion as a matrix equation of the dynamical system

$$\boldsymbol{A}\,\boldsymbol{x}' = \boldsymbol{b}, \tag{96}$$

where 6D vector $\boldsymbol{x}$ denotes $\boldsymbol{x}(r) = \{\omega(r), \omega'(r), \rho(r), \rho'(r), F(r), F'(r)\}$. The solution is ill defined at the points where $\det\boldsymbol{A} = 0$. The turning points are special cases of the general situation (see [67] for complete classification). We can extend solutions through the turning points introducing a suitable new parameter σ:

$$\dot{r} = \frac{dr}{d\sigma} = \lambda \det\boldsymbol{A}, \tag{97}$$

and generalizing the dynamical system to one dimension more, i.e. $\tilde{\boldsymbol{x}}(\sigma) = \{\omega(\sigma), \dot{\omega}(\sigma), \rho(\sigma), \dot{\rho}(\sigma), F(\sigma), \dot{F}(\sigma), r(\sigma)\}$. The matrix equation then becomes

$$\tilde{\boldsymbol{A}}\,\dot{\tilde{\boldsymbol{x}}} = \tilde{\boldsymbol{b}}, \qquad \tilde{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{A} & -\boldsymbol{b} \\ 0 & 1 \end{pmatrix}, \qquad \tilde{\boldsymbol{b}} = \begin{pmatrix} 0 \\ \lambda \det\boldsymbol{A} \end{pmatrix}. \tag{98}$$

The parameter λ can be fixed by normalization of $\dot{\tilde{\boldsymbol{x}}}$. We choose $\dot{\tilde{\boldsymbol{x}}}^2 = \dot{\boldsymbol{x}}^2 + \dot{r}^2 = 1$ and this ensures that $\dot{r}$ is finite for both small and large values of $\det\boldsymbol{A}$ which is useful for numerical calculation.

In terms of σ, we have the following result near turning point:

$$r(\sigma) = r_{\rm tp} + r_1(\sigma_{\rm tp} - \sigma)^2 + \cdots, \tag{99}$$

where $r_1 < 0$ for the right turning points and $r_1 > 0$ for the left ones. The metric can be rewritten as

$$ds^2 = -\omega(\sigma)dt^2 + \frac{d^2\sigma}{W(\sigma)} + \rho^2(\sigma)d\Omega^2_{D-2}, \tag{100}$$

where $W = \omega/\dot{r}^2$. Now, the functions ω, W, ρ are single valued functions of σ.

B. Geodesics near the turning points

As mentioned before, the curvature weakly diverges at turning point. One would expect the property near turning points is much better than near the singularity. So let us check the radial geodesic of t and σ as functions of the proper time λ ($d\theta/d\lambda = 0 = d\phi/d\lambda$). The relevant Christoffel symbols are

$$\Gamma^t{}_{t\sigma} = \frac{1}{2}\frac{\dot{\omega}}{\omega}, \qquad \Gamma^\sigma{}_{tt} = \frac{1}{2}W\dot{\omega}, \qquad \Gamma^\sigma{}_{\sigma\sigma} = -\frac{1}{2}\frac{\dot{W}}{W}, \tag{101}$$

The geodesic equation for t

$$\frac{d^2t}{d\lambda^2} + \frac{\dot{\omega}}{\omega}\frac{dt}{d\lambda}\frac{d\sigma}{d\lambda} = 0, \tag{102}$$

can be simplified as

$$\frac{d}{d\lambda}\left(\omega\frac{dt}{d\lambda}\right) = 0, \tag{103}$$

or

$$\frac{dt}{d\lambda} = \frac{C}{\omega}, \tag{104}$$

where the integration constant $C > 0$ means the "energy" per unit mass of test particle at infinity. The geodesic equation for σ coordinate is

$$\frac{d^2\sigma}{d\lambda^2} + \frac{1}{2}\left[W\dot{\omega}\left(\frac{dt}{d\lambda}\right)^2 - \frac{\dot{W}}{W}\left(\frac{d\sigma}{d\lambda}\right)^2\right] = 0. \tag{105}$$

After integration, it reduces to

$$k = \omega\left(\frac{dt}{d\lambda}\right)^2 - \frac{1}{W}\left(\frac{d\sigma}{d\lambda}\right)^2, \tag{106}$$

or

$$\left(\frac{d\sigma}{d\lambda}\right)^2 = W\left(\frac{C^2}{\omega} - k\right) = \frac{C^2 - k\omega}{\dot{r}^2}, \tag{107}$$

where $k = 1$ for timelike geodesic and $k = 0$ for null geodesic.

The geodesic solutions are

$$t = \frac{C}{\omega_0}\lambda, \qquad (\sigma - \sigma_{\rm tp})^2 = -\frac{k\omega_2}{4r_1^2}\lambda^2 \pm \frac{\sqrt{C^2 - k\omega_0}}{|r_1|}\lambda. \tag{108}$$

There are two possible solutions for σ, the minus branch is valid for $-\infty < \lambda \le 0$ and plus branch for $0 \le \lambda < \infty$ and both geodesics are terminated at the turning point ($\lambda = 0$). The only possible extension for the geodesic solution is "gluing" these two solutions, i.e.

$$(\sigma - \sigma_{\rm tp})^2 = -\frac{k\omega_2}{4r_1^2}\lambda^2 + {\rm sign}(\lambda)\frac{\sqrt{C^2 - k\omega_0}}{|r_1|}\lambda. \tag{109}$$

However, one can easy check that the result is not a smooth solution at $\lambda = 0$ and the second derivative of σ with respect to λ generates a delta function. Therefore, such extension, in general, is not a solution of (105). Similar situation happens for the timelike geodesic. Hence, the cusp turning points are not extendable for the geodesics. However there is an exception for the special values of $C^2 = k\omega_0$ and $k\omega_2 < 0$ (if $\omega_2 > 0$ the timelike geodesic cannot reach the turning point).

It is instructive to compare divergences in various cases. To study this, we use Krechman scalar $K = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$. The Schwarzschild black hole at $r = 0$ has $K \sim r^{-2(D-1)}$; the Reissner-Nordström black hole at $r = 0$ has $K \sim r^{-4(D-2)}$; the GB pure gravity black hole ($D > 5$) at $r = 0$ has (1) $K \sim r^{-2(D-1)}$ for a charged solution, and (2) $K \sim r^{-(D-1)}$ for a neutral one. The extremal dilatonic black hole with a GB term at a turning point $K \sim y^{-1}$ for any dimension has

$$R = \pm \frac{3}{4\rho_0} \frac{2w_0\rho_3(D-2) + \rho_0 w_3}{\sqrt{y}}, \tag{110}$$

where $y = r_{\rm tp} - r$ for an upper sign (a first type of turning point) and $y = r - r_{\rm tp}$ for a lower sign (a second type). If $\rho_3 = 0$, the scalar curvature at such a type of point is regular, but it imposes some constraint on the parameters because ρ_3 depends on the parameters.

VI. NUMERICAL RESULTS

Lets consider the special limit $a \to 0$ in which the dilaton field decouples. The analytical general solution is

$$\omega(r) = 1 + \frac{r^2}{2D_4^3\alpha}\left(1 \mp \sqrt{1 + \frac{64\pi D_4^3 \alpha \mathcal{M}}{(D-2)\mathrm{vol}(\Omega_{D-2})r^{D-1}} - \frac{8D_4^4\alpha q_e^2}{D_2^2 r^{2(D-2)}} + \frac{4D_4^3\alpha\Lambda}{D_2^1}}\right), \qquad \rho(r) = r. \tag{111}$$

For the case $\Lambda = 0$, the radius of degenerated (extremal) horizon is $r_0^2 = 4(2D-7)\alpha$ which can be obtained from (56). It is clear that the horizon shrinks to a point when we turn off the GB term. Although the dilaton field is decoupled, in the dimensions $D \geq 5$, the GB term still gives a nontrivial contribution to equations of motion which will break the BPS (nonforce) condition. In more detail, the ratio of the mass and the electrical charge for an extremal solution can be computed

$$\frac{\Delta\mathcal{M}}{\mathcal{Q}_e^2} = \frac{(D^2-12)^2}{4(D^2-D-8)(2D-7)} \geq 1, \tag{112}$$

and our numerical analysis give consistent results in the decoupling limit. Moreover, the numerical analysis also indicates that the dilaton charge is proportional to the dilation coupling times mass and

$$\frac{a^2\mathcal{M}^2}{\mathcal{D}^2} \geq 1, \tag{113}$$

but the exact form for the ratio is still unknown. In these two ratios, the equality holds for $D = 4$ which saturates the BPS condition (87).

In the numerical results, we are going to show the following quantities. From the symmetries (30), we know that if we regard the electrical charge as a scaling parameter, the following ratios will depend only on the dilaton coupling:

$$k_M(a) = \frac{\mathcal{M}^{(D-2)/(D-3)}}{\mathcal{Q}_e}, \qquad k_D(a) = \frac{\mathcal{D}^{(D-2)/(D-3)}}{\mathcal{Q}_e},$$
$$k_F(a) = \frac{\alpha e^{(D-2)a\phi_\infty}}{\mathcal{Q}_e}. \tag{114}$$

For verifying the BPS conditions, we will analyze the ratios

$$\frac{a^2\mathcal{M}^2}{\mathcal{D}^2}, \qquad \frac{\Delta\mathcal{M}^2}{\mathcal{Q}_e^2}, \qquad k_{\rm BPS} = \frac{\Delta(a^2\mathcal{M}^2 + \mathcal{D}^2)}{2a^2\mathcal{Q}_e^2}. \tag{115}$$

We now present our numerical results.

A. $D = 4$

For convenience, we first recall the results for $D = 4$ found in [64]. Starting with small a (for $a = 0$ one has the Reissner-Nordström solution) one finds that asymptotically flat black holes with degenerate event horizons exist up to $a = a_{\rm cr} = 0.488\,219\,703$. The critical value of dilaton coupling $a_{\rm cr}$ separates the regions where there exist regular asymptotically flat solutions for $a < a_{\rm cr}$ and the singular ones for $a > a_{\rm cr}$, which firstly have a throat ($\rho' = 0$), then a turning point where ρ'' changes sign and ρ'' diverges and finally a singular point. In the limit $a \to a_{\rm cr}$, the mass $\mathcal{M}$ diverges, and somewhat surprisingly, the BPS condition of the theory without curvature corrections holds for the ratios of parameters. This can be understood as dominance of the Einstein term over Gauss-Bonnet. Indeed, if we keep the mass fixed, the limit corresponds to gravitational constant G going to zero. Then the Einstein term becomes greater, unless the GB term is increasing similarly.

The critical value of the dilaton coupling is less than the heterotic string value $a = 1$ or $1/2$, so no asymptotically flat extremal EMDGB black holes exist in $D = 4$.

B. $D = 5$ and 6

For $D = 5$ and 6, the dynamics of the system is similar and numerical results are presented in Figs. 1–6: the asymptotically flat solutions exist up to the critical value of dilaton coupling $a_{\rm cr}$, and in the limit $a \to a_{\rm cr}$ the BPS conditions are satisfied, namely, $k_{\rm BPS} \to 1$ as shown in Figs. 1 and 4.

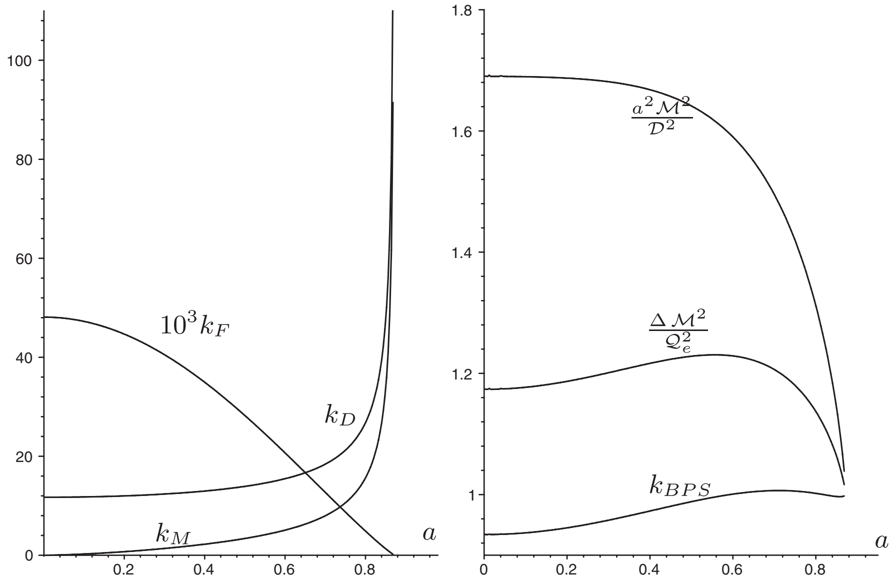

FIG. 1. Left panel: $k_M(a)$, $k_D(a)$ and $k_F(a)$ (multiplied by a factor 10^3) as functions of a in $D = 5$. Right panel: $k_{\rm BPS}$ and the ratios $\frac{\Delta\mathcal{M}^2}{\mathcal{Q}_e^2}$, $\frac{a^2\mathcal{M}^2}{\mathcal{D}^2}$.

The critical value of dilaton coupling for $D = 5$ is found to be $a_{\rm cr}^{D=5} = 0.872\,422\,988$. The corresponding string value (7) is smaller: $a_{\rm str}^{D=5} = 0.816\,496\,580$, so asymptotically flat stretched dilatonic black holes exist in five dimensions.

For $D = 6$ we obtain $a_{\rm cr}^{D=6} = 1.432\,881\,972$, the corresponding string value (7) being $a_{\rm str}^{D=6} = 0.707\,106\,781$. We observe that while the critical dilaton coupling is increasing with D, the string value (7) is decreasing, so we expect that for $D > 4$ we will have always $a_{\rm str} < a_{\rm cr}$.

The metric functions and the dilaton exponential F for asymptotically flat black holes are given in Fig. 2 for $a = 0.3$ and 0.8 (which are smaller than the critical value in $D = 5$), and in Fig. 5 for $a = 0.4$ and 1.4 in $D = 6$. Those for a larger than the critical value are displayed in Fig. 3 in $D = 5$ and in Fig. 6 in $D = 6$.

C. $D = 7$

The critical value for the dilaton coupling $a_{\rm cr}^{D=7} = 1.793\,909\,999$, the heterotic string value being $a_{\rm str}^{D=7} = 0.632\,455\,532$. As in lower dimensions, the critical value

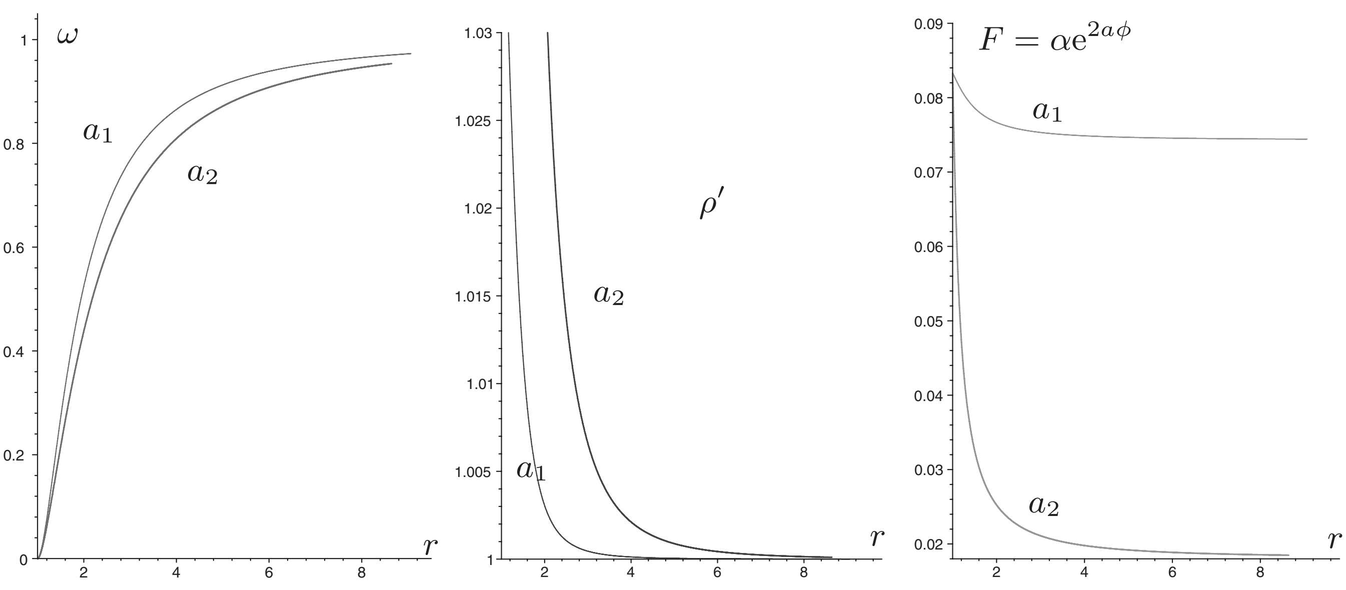

FIG. 2. Radial dependence of metric functions ω, ρ' and the dilaton exponential F for asymptotically flat black holes ($a < a_{\rm cr}^{D=5}$) with dilaton couplings $a_1 = 0.3$ and $a_2 = 0.8$ in $D = 5$.

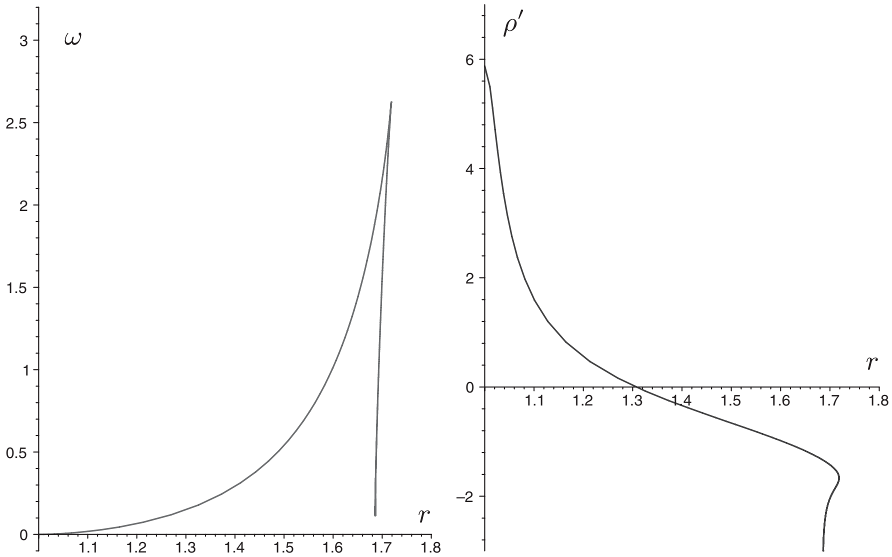

FIG. 3. Radial dependence of metric functions ω, ρ' for singular solutions ($a > a_{\rm cr}^{D=5}$) for dilaton couplings $a = 0.9$ in $D = 5$.

corresponds to the appearance of the first cusp (turning point) in the solution. The novel feature in $D = 7$ is that after the right turning point the left one appears, and the solution can be extended along the lines of [56]. Using the same procedure one can extend the solution to an asymptotically flat one, as shown in Fig. 7. With further increasing dilaton coupling the number of pairs of the turning points increases, so the asymptotically flat extended solutions look as shown in Fig. 8. The global parameters change in step-function-like manner each time when one new turning point is created; see Fig. 9. However, the extended solution cannot be considered as true black hole solution, since geodesics, as we have shown in Sec. V B, cannot be continued smoothly through the cusp singularities. Therefore we have to consider the critical value of the dilaton coupling in $D = 7$ as the true boundary of the range of a.

In the case $D = 4$ [64] it was observed that on both boundaries of the dilaton coupling $a \to 0$ and $a \to a_{\rm cr}$, the BPS condition of the EMD theory is saturated. This can be understood as an indication that the GB term is decoupled in these two limits. Indeed, for $a = 0$ it is obvious, while in the limit $a \to a_{\rm cr}$ the mass tends to infinity, in which case the Einstein term turns out to be dominant. In higher dimensions the situation is different. For $D \geq 5$ decoupling of the dilaton $a = 0$ does not switch off the GB term; instead we have to deal with Wiltshire solutions of the EMDGB theory. So the BPS saturation is not expected for

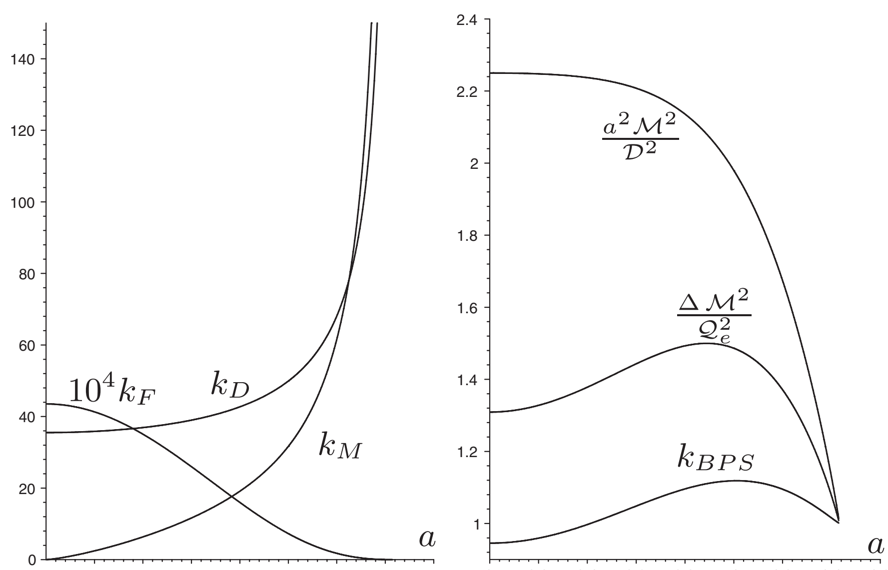

FIG. 4. Left panel: $k_M(a)$, $k_D(a)$, and $k_F(a)$ (multiplied by a factor 10^4) in $D = 6$. Right panel: the ratios of $k_{\rm BPS}$ and $\frac{\Delta \mathcal{M}^2}{\mathcal{Q}_e^2}$, $\frac{a^2 \mathcal{M}^2}{\mathcal{D}^2}$.

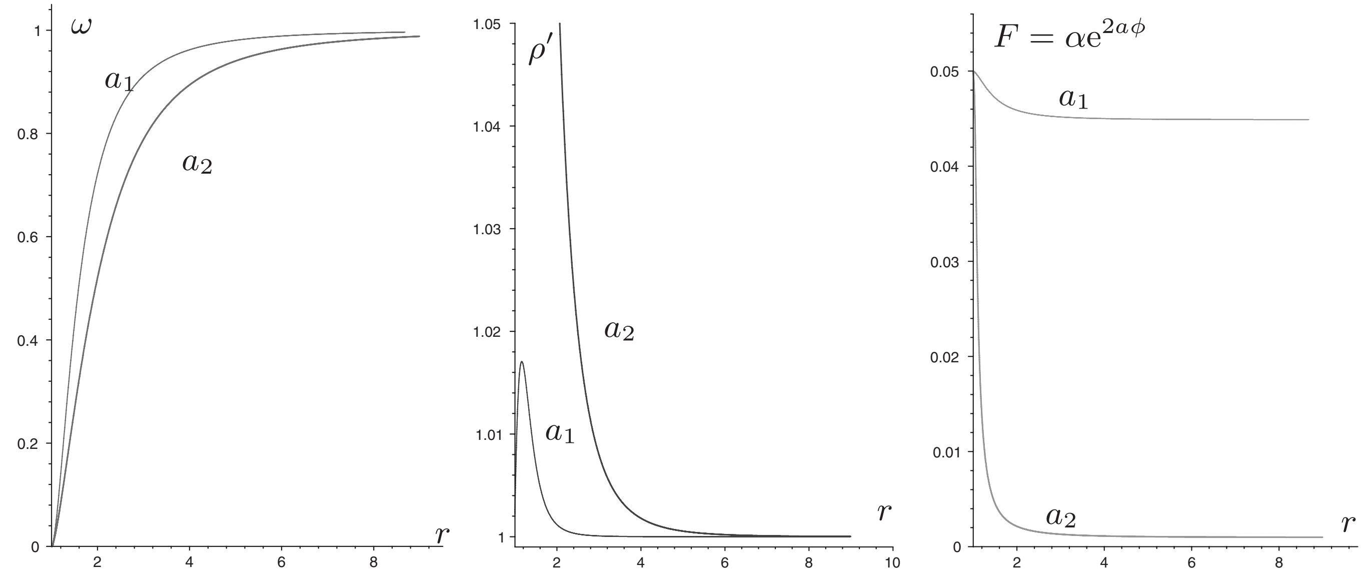

FIG. 5. Radial dependence of metric functions ω, ρ' and an exponential of dilaton field F for asymptotically flat black holes ($a < a_{cr}^{D=6}$) of dilaton couplings $a_1 = 0.4$, $a_2 = 1.4$ in $D = 6$.

$a = 0$, and this is confirmed by numerical calculations. For $D = 5, 6$ the BPS condition still holds on the right boundary of the dilaton coupling $a \rightarrow a_{cr}$ (see Figs. 1 and 4). However, in $D = 7$, the BPS condition does not hold anymore (see Fig. 10). This means that the GB term does not decouple in the limit $a \rightarrow a_{cr}$ as in lower dimensions. The results in terms of σ-coordinate are given in Figs. 11–13

D. $D = 8, 9, 10$

Properties similar to those in the case $D = 7$ were observed for higher-dimensional solutions $D = 8, 9, 10$. The critical values of dilaton coupling are $a_{cr}^{D=8} = 1.887\,653\,885$ ($a_{str}^{D=8} = 0.577\,350\,269$), $a_{cr}^{D=9} = 2.002\,906\,751$ ($a_{str}^{D=9} = 0.534\,522\,483$), and $a_{cr}^{D=10} = 2.121\,748\,877$ ($a_{str}^{D=10} = 0.5$). The numerical results are presented in Figs. 14–19. The BPS condition is not fulfilled on both boundaries of a. Supercritical solutions can be formally continued to infinity (as asymptotically flat) through the cusps which are met in pairs like in the case of $D = 7$. However we do not qualify them as physical black holes for the reasons explained in Sec. V.

VII. CONCLUSIONS

Here we summarize our findings. First, we have constructed an explicit local solution of the EMDGB static extremal black holes in the vicinity of the horizon and

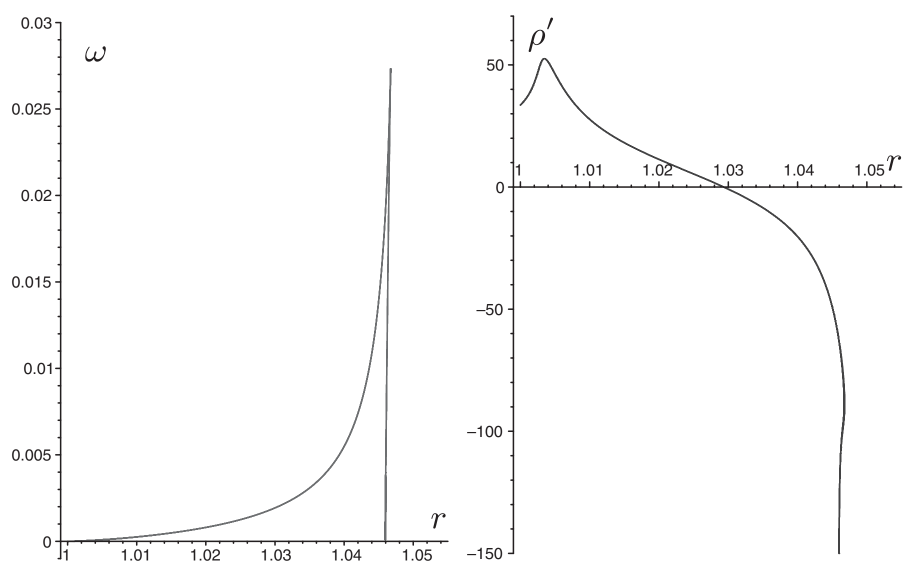

FIG. 6. Radial dependence of metric functions ω, ρ' for singular solutions ($a > a_{cr}^{D=6}$) of dilaton coupling $a = 1.5$ in $D = 6$.

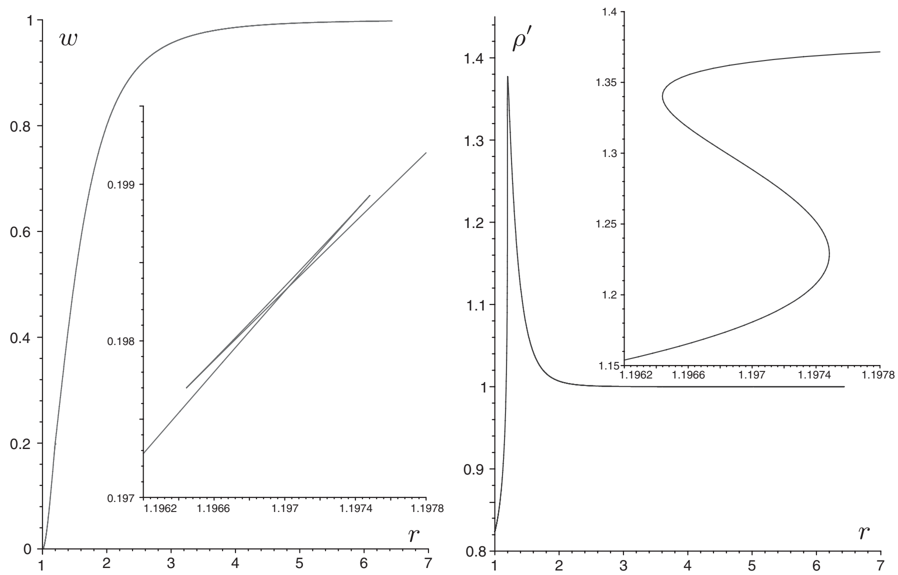

FIG. 7. Dependence of metric functions w and ρ' on the radial coordinate for $a = 1.8$ in $D = 7$ with two turning points. (Insets describe properties of solution nearby turning points.)

calculated the corresponding entropies. The ratios of entropy to the Hawking-Bekenstein one $A/4$ increases from 1 for $D = 4$, 5 to $41/13$ for $D = 10$. The entropy does not depend on the dilaton coupling. Contrary to this, the asymptotic behavior of the solutions crucially depend on dilaton coupling and asymptotically flat black holes exist only for $a < a_{\rm cr}$. The critical value of the dilaton coupling depend on D and increases with D. For $D = 4$, $a_{\rm cr}$ is smaller than the heterotic string value, therefore no stretched black holes exist in the effective heterotic theory. In contrast, for $D \geq 5$ the heterotic values of a lie inside the allowed region. Numerical solutions for asymptotically flat black holes are constructed for $4 \leq D \leq 10$. We investigated the ratios of the mass, dilaton charge and electric charges which show the degree of deviation form the BPS bounds in the absence of GB term as functions of the dilaton coupling. It is observed that for $D < 5$ the BPS bound is saturated near the threshold value $a_{\rm cr}$, thus demonstrating that the contribution of the GB term is effectively small there. For larger D such a behavior was not observed, indicating that the GB term remains important on the boundary.

The failure to reach the flat asymptotic in numerical integration manifests itself as emergence of turning points

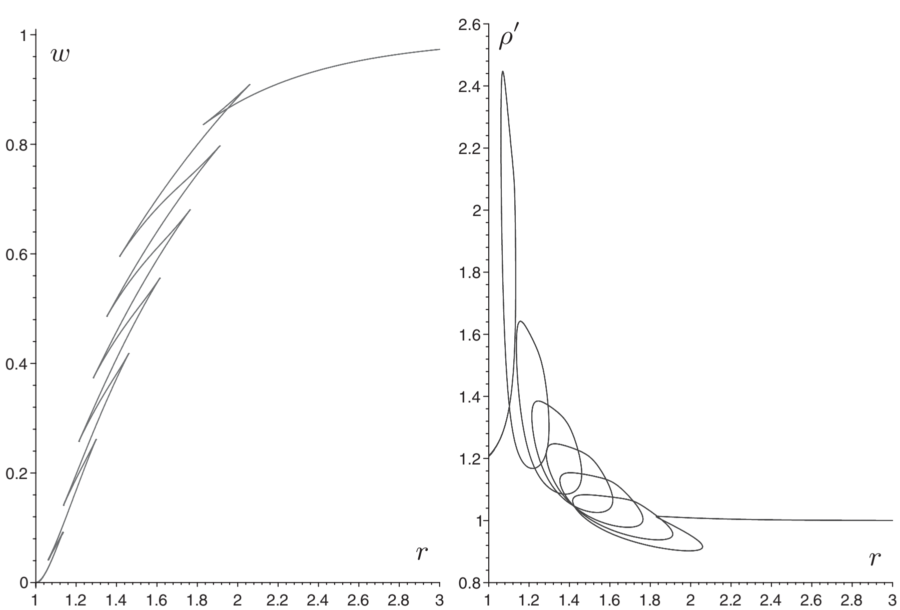

FIG. 8. Dependence of metric functions w and ρ' of the radial coordinate for $a = 1.81$ in $D = 7$. There are 14 turning points.

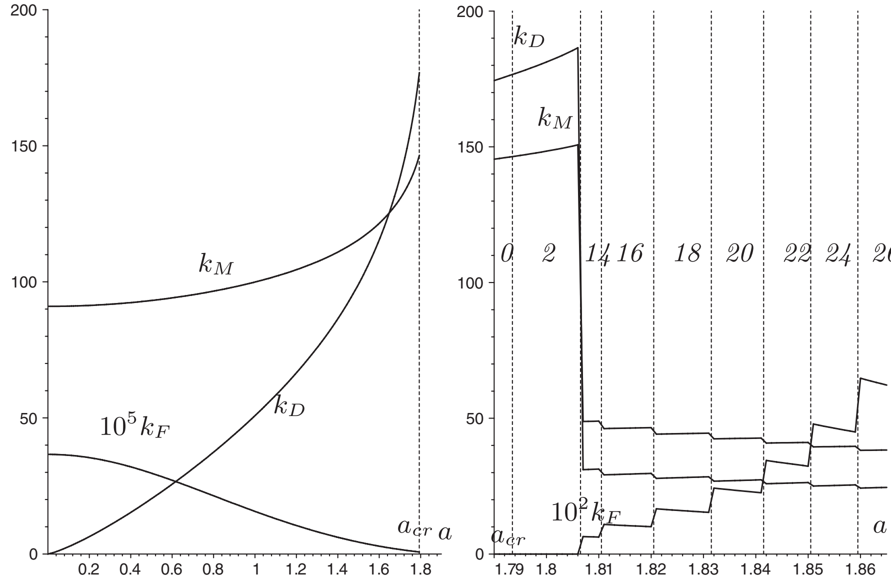

FIG. 9. Left panel: $k_M(a)$, $k_D(a)$, and $k_F(a)$ in the region before formation of turning points in $D = 7$. Right panel: same quantities after formation of turning points (the number of turning points is denoted by italic font).

of the radial variable in which the scalar curvature has very mild divergence. The solutions then exhibit typical cusp-shaped behavior. It was suggested before that these turning points should be passed by changing the integration variable in a suitable way so that the solution can be continued through these singularities. We have found that in dimensions $D \geq 7$ the turning points comes in pairs, and the solution can be formally extended to the flat asymptotic. However an inspection of radial geodesics reveals that they cannot be analytically continued through cusp singularities, so we do not believe that continuation of numerical solutions through the cusps is physically meaningful.

ACKNOWLEDGMENTS

C. M. C. is grateful to the AEI, Postdam for its hospitality in the early stages of this work. The work of C. M. C. and D. G. O. was supported by the National Science Council of the R. O. C. under the Grant No. NSC 96-2112-M-008-006-MY3 and in part by the National Center of Theoretical Sciences (NCTS). The work of

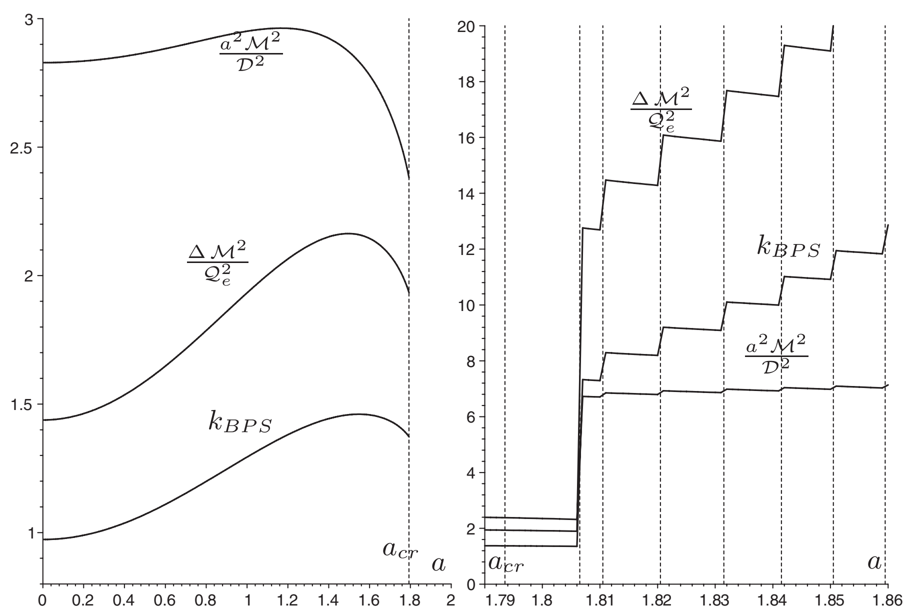

FIG. 10. Left panel: Ratios of $k_{\rm BPS}$ and $\frac{\Delta\mathcal{M}^2}{\mathcal{Q}_e^2}$, $\frac{a^2\mathcal{M}^2}{\mathcal{D}^2}$ in the region before formation of turning points in $D = 7$. Right panel: those after formation of turning points (the number of turning points is denoted by italic font).

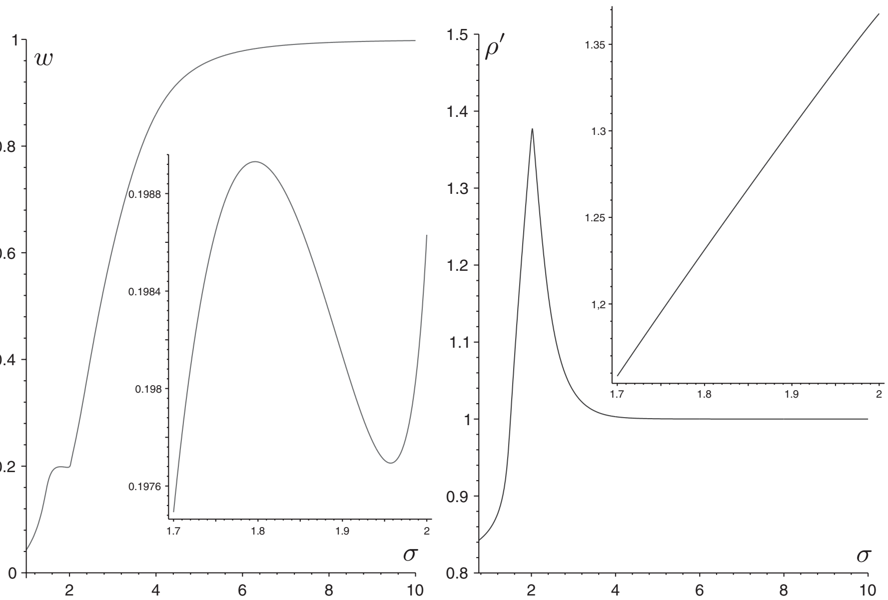

FIG. 11. Dependence of metric functions w and ρ' of parameter σ for $a = 1.8$ in $D = 7$ with two turning points. (Insets describe properties of solution nearby turning points.)

D. G. was supported by the RFBR under the project 08-02-01398-a. The work of N. O. was supported in part by the Grant-in-Aid for Scientific Research Fund of the JSPS No. 20540283, and also by the Japan-U.K. Research Cooperative Program.

APPENDIX A: GEOMETRIC QUANTITIES FOR SPHERICAL SYMMETRIC METRIC

This appendix gives detailed geometric quantities associated with the following spherical symmetric metric in dimensions D

$$ds^2 = -e^{2u(r)}dt^2 + e^{2v(r)}dr^2 + e^{2w(r)}d\Omega^2_{D-2,k}, \qquad \text{(A1)}$$

where k denotes the spatial curvature. The Riemann and Ricci tensors have the following components:

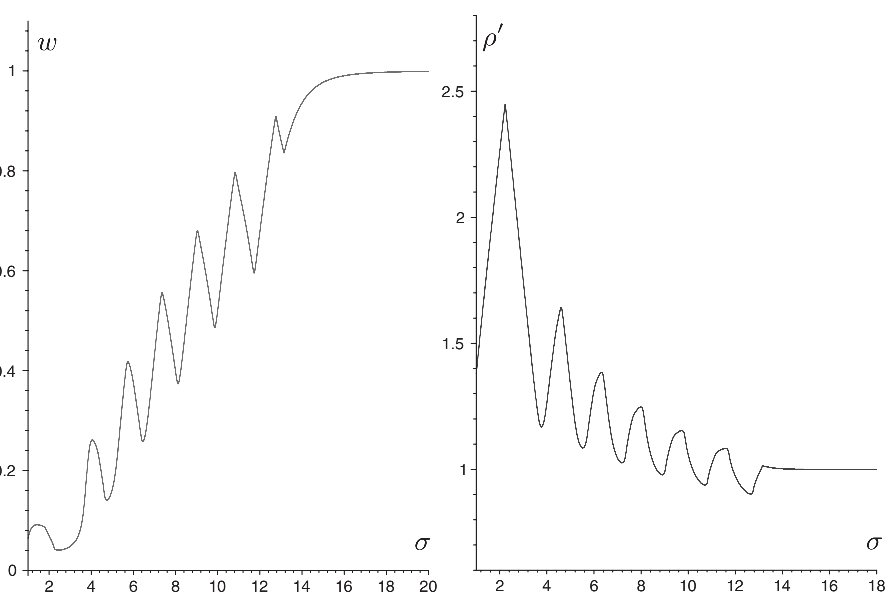

FIG. 12. Dependence of metric functions w and ρ' of parameter σ for $a = 1.81$ in $D = 7$ with 14 turning points.

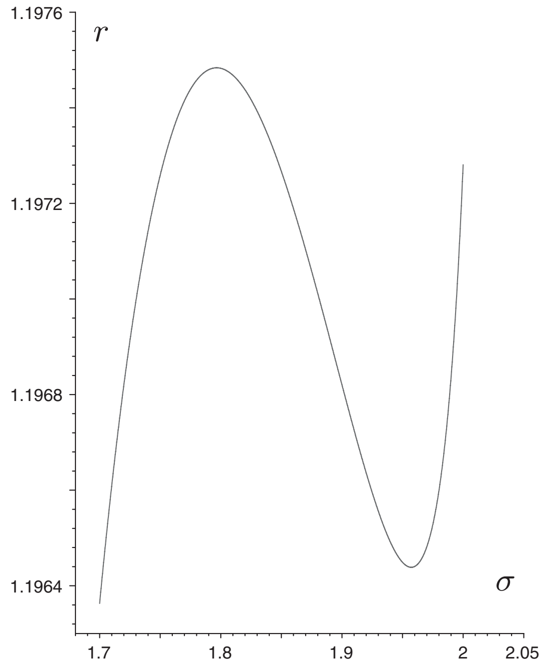

FIG. 13. Dependence of original radial coordinate r of parameter σ for $a = 1.8$ in $D = 7$ with two turning points.

$$\begin{aligned} R_{trtr} &= \mathrm{e}^{2u}(u'' + u'^2 - u'v'), \\ R_{tatb} &= \mathrm{e}^{2u-2v+2w}u'w'g_{ab}, \\ R_{rarb} &= -\mathrm{e}^{2w}(w'' + w'^2 - v'w')g_{ab}, \\ R_{acbd} &= (-\mathrm{e}^{4w-2v}w'^2 + k\mathrm{e}^{2w})(1 - \delta_{ac}\delta_{bd})g_{ab}g_{cd}, \\ R_{tt} &= \mathrm{e}^{2u-2v}(u'' + u'H'), \\ R_{rr} &= -(u'' + u'^2 - u'v') \\ &\quad - (D-2)(w'' + w'^2 - v'w'), \\ R_{ab} &= [-\mathrm{e}^{2w-2v}(w'' + w'H') + k(D-3)]g_{ab}, \end{aligned} \tag{A2}$$

and then the scalar curvature, Ricci square ($R^2_{\mu\nu} = R_{\mu\nu}R^{\mu\nu}$) and Riemann square ($R^2_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$) are

$$\begin{aligned} R &= -\mathrm{e}^{-2v}[2u'' + u'H' + u'^2 - u'v' + D_2^2(2w'' + w'H' + w'^2 - v'w')] + kD_3^2\mathrm{e}^{-2w}, \\ &= -\mathrm{e}^{-2v}[2(u'' + u'^2 - u'v') + 2D_2^2(w'' + u'w' - v'w' + w'^2) + D_3^2(w'^2 - k\mathrm{e}^{2v-2w})], \\ R^2_{\mu\nu} &= \mathrm{e}^{-4v}(u'' + u'H')^2 + \mathrm{e}^{-4v}[u'' + u'^2 - u'v' + D_2^2(w'' + w'^2 - v'w')]^2 + D_2^2[\mathrm{e}^{-2v}(w'' + w'H') - kD_3^3\mathrm{e}^{-2w}]^2, \\ R^2_{\alpha\beta\mu\nu} &= 4\mathrm{e}^{-4v}(u'' + u'^2 - u'v')^2 + 4D_2^2\mathrm{e}^{-4v}u'^2w'^2 + 4D_2^2\mathrm{e}^{-4v}(w'' + w'^2 - v'w')^2 + 2D_3^2(\mathrm{e}^{-2v}w'^2 - k\mathrm{e}^{-2w})^2, \end{aligned} \tag{A3}$$

where H is defined

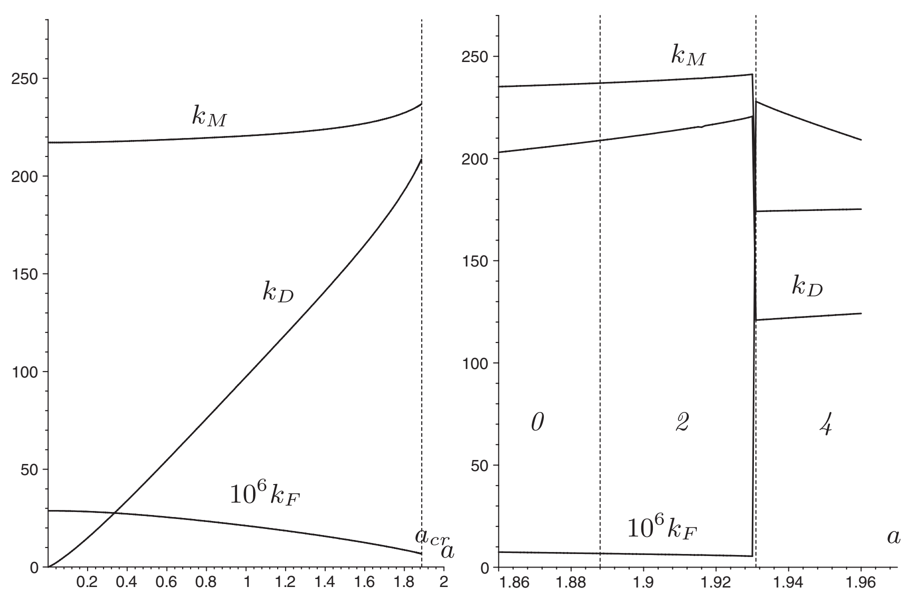

FIG. 14. Left panel: $k_M(a)$, $k_D(a)$, and $k_F(a)$ in $D = 8$ in the region before formation of turning points. Right panel: those after formation of turning points.

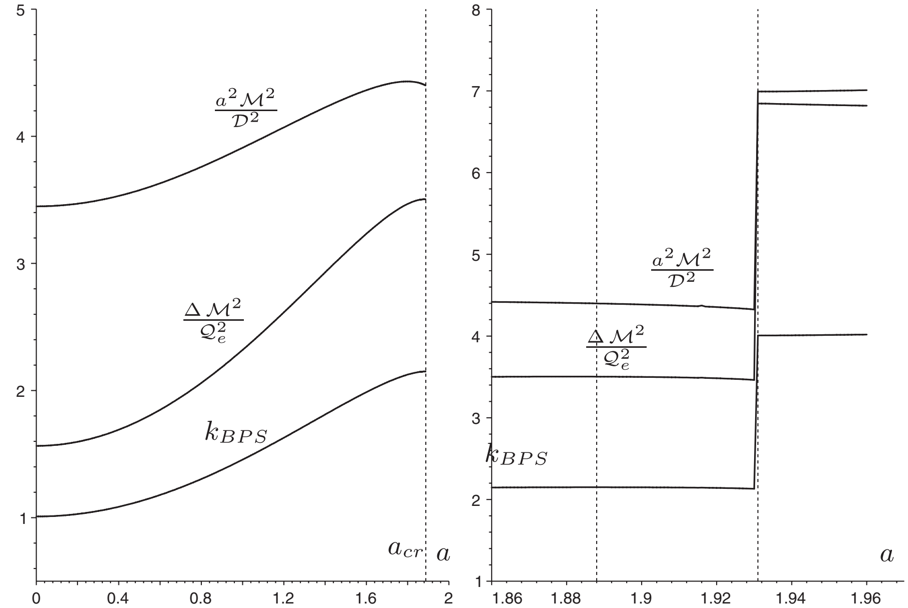

FIG. 15. Left panel: Ratios of $k_{\rm BPS}$ and $\frac{\Delta\mathcal{M}^2}{\mathcal{Q}_e^2}$, $\frac{a^2\mathcal{M}^2}{\mathcal{D}^2}$ in$D = 8$ in the region before formation of turning points. Right panel: those after formation of turning points.

$$H = u - v + (D - 2)w, \tag{A4}$$

and the following notation is used,

$$D^m_n = (D - m)_n = (D - m)(D - m - 1)\cdots(D - n), \qquad n \geq m. \tag{A5}$$

The GB combination is

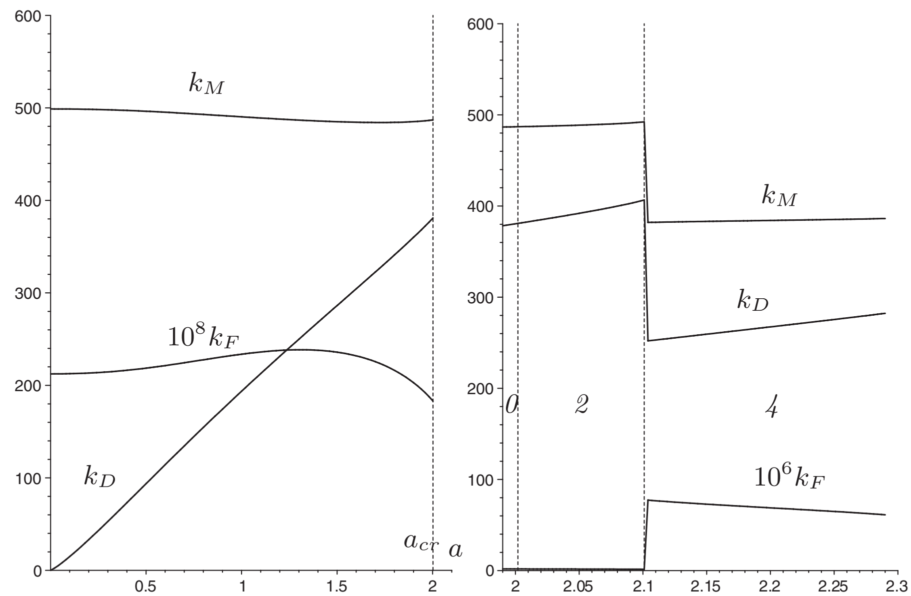

FIG. 16. Left panel: $k_M(a)$, $k_D(a)$, and $k_F(a)$ in $D = 9$ in the region before formation of turning points. Right panel: those after formation of turning points.

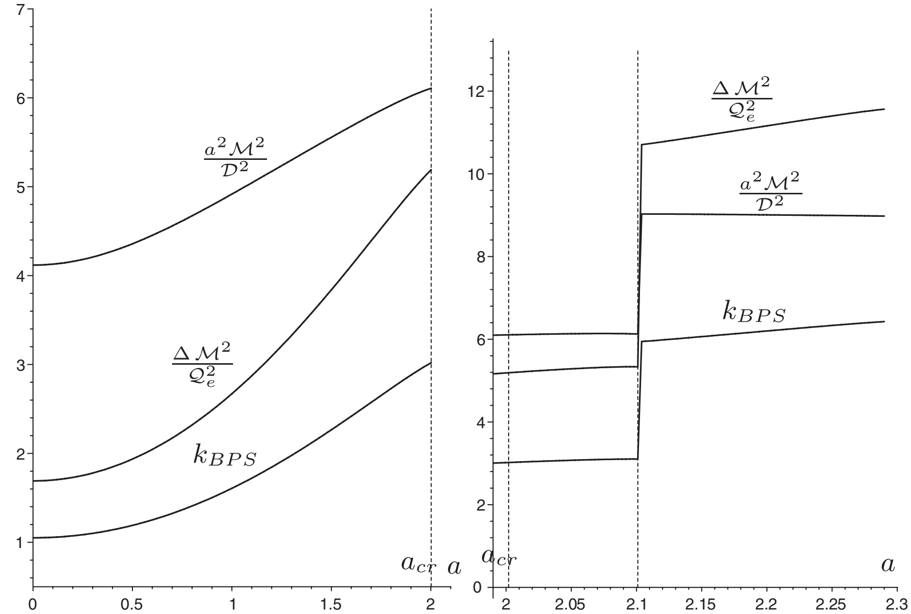

FIG. 17. Left panel: ratios of $k_{\rm BPS}$ and $\frac{\Delta\mathcal{M}^2}{\mathcal{Q}_e^2}$, $\frac{a^2\mathcal{M}^2}{\mathcal{D}^2}$ in $D=9$ in the region before formation of turning points. Right panel: those after formation of turning points.

$$\begin{aligned}\mathcal{L}_{\rm GB} &= D_3^2 e^{-4v}\{D_5^4(w'^2 - ke^{2v-2w})^2 + 8u'w'(w'' + w'^2 - w'v') + 4[u'' + u'^2 - u'v' + D_4^4(w'' + u'w' - v'w' + w'^2)] \\ &\quad \times (w'^2 - ke^{2v-2w})\} \\ &= 4D_3^2 e^{-u-v-2w}[e^{u-3v+2w}u'(w'^2 - ke^{2v-2w})]' + 4D_4^2 e^{-4v}(w'' + u'w' - v'w' + w'^2)(w'^2 - ke^{2v-2w}) \\ &\quad + D_5^2 e^{-4v}(w'^2 - ke^{2v-2w})^2. \end{aligned} \tag{A6}$$

One can easy check that, in four dimensions, the GB term of $\sqrt{-g}\,\mathcal{L}_{\rm GB}$ is a total derivative. For the gauge choice of coordinates

$$u = -v = \frac{1}{2}\ln\omega, \qquad w = \ln\rho, \tag{A7}$$

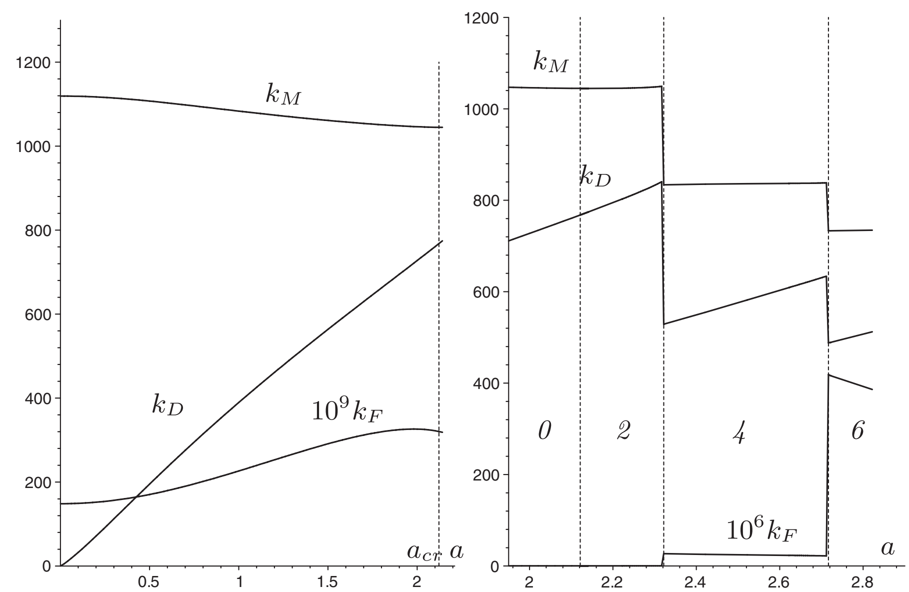

FIG. 18. Left panel: $k_M(a)$, $k_D(a)$, and $k_F(a)$ in $D=10$ in the region before formation of turning points. Right panel: those after formation of turning points.

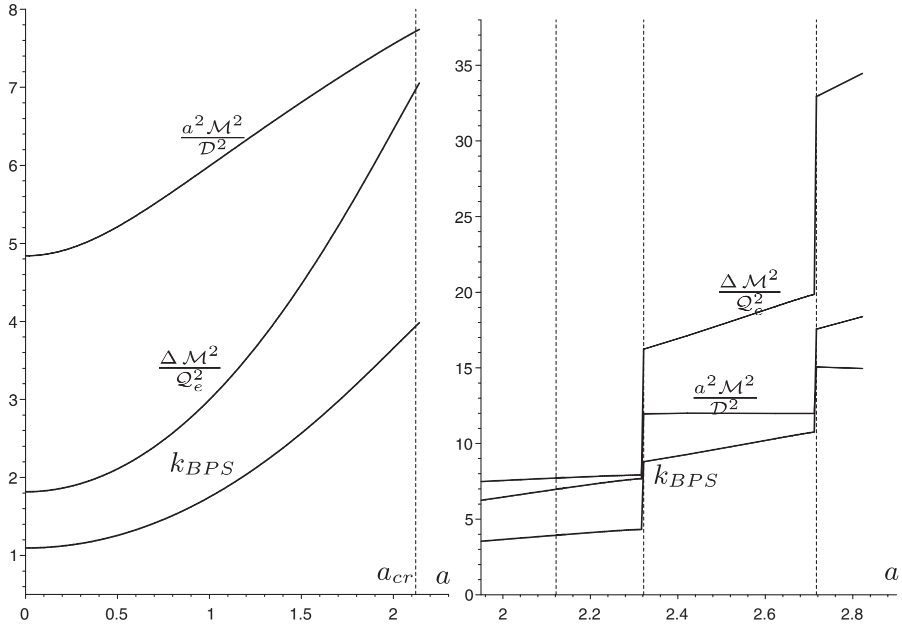

FIG. 19. Left panel: ratios of $k_{\rm BPS}$ and $\frac{\Delta\mathcal{M}^2}{\mathcal{Q}_e^2}$, $\frac{a^2\mathcal{M}^2}{\mathcal{D}^2}$ in $D = 10$ in the region before formation of turning points. Right panel: those after formation of turning points.

the relevant quantities become

$$\begin{aligned} R &= -\rho^{-2}[\omega''\rho^2 + 2D_2^2\rho(\omega\rho'' + \omega'\rho') + D_3^2(\omega\rho'^2 - k)], \\ &= \rho^{2-D}[-(\omega'\rho^{D-2} + 2D_2^2\omega\rho'\rho^{D-3})' \\ &\quad + D_2^2\rho'(\omega\rho^{D-3})' + D_3^2\rho^{D-4}k], \end{aligned} \tag{A8}$$

$$\begin{aligned} \mathcal{L}_{\rm GB} &= D_3^2\rho^{-4}\{2\omega'\rho'\rho^2(2\omega\rho'' + \omega'\rho') + 2\rho[\omega''\rho \\ &\quad + 2D_4^4(\omega\rho')'](\omega\rho'^2 - k) + D_5^4(\omega\rho'^2 - k)^2\}, \\ &= 2D_3^2\rho^{-2}[\omega'(\omega\rho'^2 - k)]' + 4D_4^2\rho^{-3}(\omega\rho'' \\ &\quad + \omega'\rho')(\omega\rho'^2 - k) + D_5^2\rho^{-4}(\omega\rho'^2 - k)^2. \end{aligned} \tag{A9}$$

[1] B. de Wit, Fortschr. Phys. **54**, 183 (2006).
[2] T. Mohaupt, arXiv:hep-th/0512048.
[3] A. Sen, Gen. Relativ. Gravit. **40**, 2249 (2008).
[4] G. W. Gibbons, Nucl. Phys. **B207**, 337 (1982).
[5] G. W. Gibbons and K. I. Maeda, Nucl. Phys. **B298**, 741 (1988).
[6] G. W. Gibbons and D. L. Wiltshire, Ann. Phys. (N.Y.) **167**, 201 (1986); **176**, 393(E) (1987).
[7] D. Garfinkle, G. T. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991); **45**, 3888(E) (1992).
[8] B. Zwiebach, Phys. Lett. **156B**, 315 (1985).
[9] C. G. Callan, I. R. Klebanov, and M. J. Perry, Nucl. Phys. **B278**, 78 (1986).
[10] D. J. Gross and E. Witten, Nucl. Phys. **B277**, 1 (1986).
[11] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. **B293**, 385 (1987).
[12] D. J. Gross and J. H. Sloan, Nucl. Phys. **B291**, 41 (1987).
[13] R. M. Wald, Phys. Rev. D **48**, R3427 (1993).
[14] T. Jacobson, G. Kang, and R. C. Myers, Phys. Rev. D **49**, 6587 (1994).
[15] V. Iyer and R. M. Wald, Phys. Rev. D **50**, 846 (1994).
[16] T. Jacobson, G. Kang, and R. C. Myers, arXiv:gr-qc/9502009.
[17] K. Behrndt, G. Lopes Cardoso, B. de Wit, D. Lust, T. Mohaupt, and W. A. Sabra, Phys. Lett. B **429**, 289 (1998).
[18] G. Lopes Cardoso, B. de Wit, and T. Mohaupt, Phys. Lett. B **451**, 309 (1999).
[19] G. Lopes Cardoso, B. de Wit, and T. Mohaupt, Fortschr. Phys. **48**, 49 (2000).
[20] G. Lopes Cardoso, B. de Wit, and T. Mohaupt, Nucl. Phys. **B567**, 87 (2000).
[21] G. Lopes Cardoso, B. de Wit, and T. Mohaupt, Classical Quantum Gravity **17**, 1007 (2000).
[22] T. Mohaupt, Fortschr. Phys. **49**, 3 (2001).
[23] G. Lopes Cardoso, B. de Wit, J. Kappeli, and T. Mohaupt, J. High Energy Phys. 12 (2000) 019.
[24] G. Lopes Cardoso, B. de Wit, J. Kappeli, and T. Mohaupt, Fortschr. Phys. **49**, 557 (2001).
[25] A. Dabholkar, Phys. Rev. Lett. **94**, 241301 (2005).
[26] A. Dabholkar, R. Kallosh, and A. Maloney, J. High Energy Phys. 12 (2004) 059.
[27] A. Sen, J. High Energy Phys. 05 (2005) 059.

[28] V. Hubeny, A. Maloney, and M. Rangamani, J. High Energy Phys. 05 (2005) 035.
[29] D. Bak, S. Kim, and S. J. Rey, arXiv:hep-th/0501014.
[30] K. Goldstein, N. Iizuka, R. P. Jena, and S. P. Trivedi, Phys. Rev. D **72**, 124021 (2005).
[31] R. Kallosh, J. High Energy Phys. 12 (2005) 022.
[32] P. K. Tripathy and S. P. Trivedi, J. High Energy Phys. 03 (2006) 022.
[33] A. Giryavets, J. High Energy Phys. 03 (2006) 020.
[34] K. Goldstein, R. P. Jena, G. Mandal, and S. P. Trivedi, J. High Energy Phys. 02 (2006) 053.
[35] R. Kallosh, N. Sivanandam, and M. Soroush, J. High Energy Phys. 03 (2006) 060.
[36] R. Kallosh, arXiv:hep-th/0603003.
[37] P. Prester, J. High Energy Phys. 02 (2006) 039.
[38] M. Cvitan, P. D. Prester, A. Ficnar, S. Pallua, and I. Smolic, Fortschr. Phys. **56**, 406 (2008).
[39] M. Cvitan, P. D. Prester, S. Pallua, and I. Smolic, J. High Energy Phys. 11 (2007) 043.
[40] P. D. Prester and T. Terzic, J. High Energy Phys. 12 (2008) 088.
[41] M. Alishahiha and H. Ebrahim, J. High Energy Phys. 03 (2006) 003.
[42] A. Sinha and N. V. Suryanarayana, Classical Quantum Gravity **23**, 3305 (2006).
[43] B. Chandrasekhar, S. Parvizi, A. Tavanfar, and H. Yavartanoo, J. High Energy Phys. 08 (2006) 004.
[44] S. Parvizi and A. Tavanfar, Mod. Phys. Lett. A **23**, 1869 (2008).
[45] B. Sahoo and A. Sen, J. High Energy Phys. 09 (2006) 029.
[46] D. Astefanesei, K. Goldstein, and S. Mahapatra, Gen. Relativ. Gravit. **40**, 2069 (2008).
[47] A. Sen, J. High Energy Phys. 09 (2005) 038.
[48] A. Sen, J. High Energy Phys. 03 (2006) 008.
[49] R. G. Cai, C. M. Chen, K. i. Maeda, N. Ohta, and D. W. Pang, Phys. Rev. D **77**, 064030 (2008).
[50] C. G. Callan, R. C. Myers, and M. J. Perry, Nucl. Phys. **B311**, 673 (1989).
[51] S. Mignemi and N. R. Stewart, Phys. Rev. D **47**, 5259 (1993).
[52] S. Mignemi, Phys. Rev. D **51**, 934 (1995).
[53] P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis, and E. Winstanley, Phys. Rev. D **54**, 5049 (1996).
[54] T. Torii, H. Yajima, and K. I. Maeda, Phys. Rev. D **55**, 739 (1997).
[55] S. O. Alexeev and M. V. Pomazanov, Phys. Rev. D **55**, 2110 (1997).
[56] S. O. Alexeev and M. V. Pomazanov, arXiv:gr-qc/9706066.
[57] M. Melis and S. Mignemi, Classical Quantum Gravity **22**, 3169 (2005).
[58] M. Melis and S. Mignemi, Phys. Rev. D **73**, 083010 (2006).
[59] Z. K. Guo, N. Ohta, and T. Torii, Prog. Theor. Phys. **120**, 581 (2008).
[60] Z. K. Guo, N. Ohta, and T. Torii, Prog. Theor. Phys. **121**, 253 (2009).
[61] N. Ohta and T. Torii, Prog. Theor. Phys. **121**, 959 (2009).
[62] N. Ohta and T. Torii, Prog. Theor. Phys. **122**, 1477 (2009).
[63] K. i. Maeda, N. Ohta, and Y. Sasagawa, Phys. Rev. D **80**, 104032 (2009).
[64] C. M. Chen, D. V. Gal'tsov, and D. G. Orlov, Phys. Rev. D **75**, 084030 (2007).
[65] C. M. Chen, Prog. Theor. Phys. Suppl. **172**, 161 (2008).
[66] C. M. Chen, D. V. Gal'tsov, and D. G. Orlov, Phys. Rev. D **78**, 104013 (2008).
[67] M. V. Pomazanov, arXiv:math-ph/0007008.
[68] D. Gal'tsov, S. Klevtsov, D. Orlov, and G. Clement, Int. J. Mod. Phys. A **21**, 3575 (2006).
[69] D. L. Wiltshire, Phys. Rev. D **38**, 2445 (1988).