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Matter instabilities in general Gauss-Bonnet gravity

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We study the evolution of cosmological perturbations in $f(\mathcal{G})$ gravity, where the Lagrangian is the sum of a Ricci scalar R and an arbitrary function f in terms of a Gauss-Bonnet term $\mathcal{G}$. We derive the equations for perturbations assuming matter to be described by a perfect fluid with a constant equation of state w. We show that density perturbations in perfect fluids exhibit negative instabilities during both the radiation and the matter domination, irrespective of the form of $f(\mathcal{G})$. This growth of perturbations gets stronger on smaller scales, which is difficult to be compatible with the observed galaxy spectrum unless the deviation from general relativity is very small. Thus $f(\mathcal{G})$ cosmological models are effectively ruled out from this ultraviolet instability, even though they can be compatible with the late-time cosmic acceleration and local gravity constraints.

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I. INTRODUCTION

Independent observational evidence for dark energy has stimulated the idea that general relativity (GR) may be modified on large distances to give rise to a late-time cosmic acceleration [1]. A simple dark energy scenario constructed in this vein is so-called $f(R)$ gravity in which f is a function of the Ricci scalar R [2]. Although there are some restrictions to the functional form of $f(R)$ to satisfy both cosmological and local gravity constraints, it is possible to design viable models [3] that can be distinguished from GR at least in the metric formalism of $f(R)$ gravity.

The $f(R)$ gravity in the metric formalism corresponds to the so-called Brans-Dicke theory with a parameter $\omega_{\rm BD} = 0$ in the presence of a potential of gravitational origin [4]. One can generalize this to scalar-tensor theories with an arbitrary Brans-Dicke parameter $\omega_{\rm BD}$. In fact, it is possible to construct scalar-field potentials that can be responsible for the cosmic acceleration, while at the same time satisfying local gravity constraints [5]. These models, including $f(R)$ gravity, exhibit several interesting observational signatures such as the phantom equation of state [6], the modified matter power spectrum [7], and the modified weak lensing spectrum [8].

The Ricci scalar R is not the only scalar quantity which is used to change gravity, as we can easily construct other scalar quantities such as $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ from the Ricci tensor $R_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu\rho\sigma}$ [9]. However, for the Gauss-Bonnet (GB) curvature invariant

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \qquad (1)$$

one can avoid the appearance of spurious spin-2 ghosts [10,11]. If a scalar-field ϕ with an exponential potential $V(\phi) = V_0 e^{-\lambda\phi}$ couples to the GB term [12], a scaling matter era can be followed by a late-time de Sitter (dS) solution for the exponential GB coupling $F(\phi) \propto e^{\mu\phi}$ with $\mu > \lambda$ [13,14]. While this GB coupling is well motivated by low-energy effective string theory [15], the joint likelihood analysis using observational data of big bang nucleosynthesis, large-scale structure, and baryon acoustic oscillations disfavors such a model, provided that it aims to account for dark energy [16]. In addition the energy contribution coming from the GB term needs to be strongly suppressed for consistency with solar-system experiments [17,18]. This cannot be compatible with the requirement of cosmic acceleration today (see also Ref. [19]), at least in the presence of a kinetic term for the scalar field and some forms of the potential. The instability of tensor perturbations is also present in those models during the epoch of cosmic acceleration [14,20,21].

However, it is possible to explain the late-time cosmic acceleration for the modified gravity scenario in which the Lagrangian density is given by $\mathcal{L} = R + f(\mathcal{G})$, where $f(\mathcal{G})$ is an arbitrary function in terms of $\mathcal{G}$ [22], provided the function f satisfies some conditions [23]. This is equivalent to a theory with a scalar field coupled to the GB term in the absence of a kinetic term [22,24,25]. A number of $f(\mathcal{G})$ models that have a matter era followed by a dS attractor have been proposed in Ref. [23] (see also Refs. [26–28]). These models can be also consistent with local gravity constraints for a wide range of parameter space [29].

In order to test the cosmological viability of $f(\mathcal{G})$ dark energy models, it is important to study cosmological perturbations responsible for structure formation. In Ref. [27] the evolution of density perturbations has been discussed for nonrelativistic matter with an equation of state $w = 0$, under the approximation that the background cosmological evolution mimics that of the ΛCDM model. In this paper we derive the equation for density perturbations with a general constant equation of state w. Therefore our analysis includes the perturbations in radiation ($w = 1/3$) as well as those in nonrelativistic matter. Moreover we use concrete $f(\mathcal{G})$ models that satisfy both local gravity constraints and cosmological constraints at the background level. This is particularly important when we discuss the evolution of perturbations at late times, because the deviation from the ΛCDM model can be significant.

We will show that, in the Universe dominated by a single perfect fluid with an equation of state parameter w, the perturbations in the fluid exhibit violent instabilities for $w > -1/2$ in the small-scale limit. This is associated with a negative speed squared c_s^2 of one eigenvector mode. The perturbations of nonrelativistic matter as well as radiation, at some scale, will start to grow exponentially during the matter/radiation domination, unless the deviation from GR is very small. In GR, this same mode does not exist, so that there is no smooth limit from one theory to the other.

This paper is organized as follows. In Sec. II we review $f(\mathcal{G})$ models that satisfy cosmological constraints at the background level as well as local gravity constraints. Section III is devoted to the analysis of cosmological perturbations in $f(\mathcal{G})$ gravity in the presence of a perfect fluid with the equation of state w. In this section we discuss the presence of an instability at small scales due to a negative speed squared for the propagating mode. In Sec. IV we analyze more in detail for perturbations in nonrelativistic matter in order to describe the growth of a large-scale structure. We shall numerically integrate the perturbation equations for concrete $f(\mathcal{G})$ models and estimate how much deviation from GR can be allowed by the observations of galaxy clustering in the linear regime. We conclude in Sec. V.

II. DARK ENERGY MODELS BASED ON $f(\mathcal{G})$ GRAVITY

Let us start with the following action:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g_M}[R + f(\mathcal{G})] + S_m, \qquad (2)$$

where G is a bare gravitational constant and g_M is the determinant for the space-time metric $g_{\mu\nu}$. For the matter action S_m we shall consider a perfect fluid whose equation of state $w = p_m/\rho_m$ is strictly constant, where p_m and ρ_m are the pressure and the energy density, respectively. Taking the variation of the action (2) with respect to $g_{\mu\nu}$, we obtain the field equation

$$\begin{aligned} &G_{\mu\nu} + 8[R_{\mu\rho\nu\sigma} + R_{\rho\nu}g_{\sigma\mu} - R_{\rho\sigma}g_{\nu\mu} - R_{\mu\nu}g_{\sigma\rho} \\ &\quad + R_{\mu\sigma}g_{\nu\rho} + (R/2)(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho})]\nabla^\rho\nabla^\sigma f_{,\mathcal{G}} \\ &\quad + (\mathcal{G}f_{,\mathcal{G}} - f)g_{\mu\nu} = 8\pi G T_{\mu\nu}, \end{aligned} \qquad (3)$$

where $f_{,\mathcal{G}} = \partial f/\partial\mathcal{G}$, $G_{\mu\nu} = R_{\mu\nu} - (1/2)Rg_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor of matter.

For the flat Friedmann-Lemaître-Robertson-Walker (FLRW) background with a scale factor a, we obtain the following dynamical equation:

$$3H^2 = \mathcal{G}f_{,\mathcal{G}} - f - 24H^3\dot{f}_{,\mathcal{G}} + 8\pi G\rho_m, \qquad (4)$$

where $H \equiv \dot{a}/a$, a dot represents a time derivative in terms of cosmic time t, and the GB term is given by

$$\mathcal{G} = 24H^2(H^2 + \dot{H}) = -12H^4(1 + 3w_{\rm eff}). \qquad (5)$$

Here $w_{\rm eff}$ is an effective equation of state defined by

$$w_{\rm eff} \equiv -1 - \frac{2\dot{H}}{3H^2}. \qquad (6)$$

The matter energy density ρ_m satisfies the standard continuity equation,

$$\dot{\rho}_m + 3H(1 + w)\rho_m = 0, \qquad (7)$$

which has the solution $\rho_m \propto a^{-3(1+w)}$ for constant w.

It is possible to realize a late-time cosmic acceleration by the existence of a dS point that satisfies the condition $3H_1^2 = \mathcal{G}_1 f_{\mathcal{G}}(\mathcal{G}_1) - f(\mathcal{G}_1)$, where H_1 and $\mathcal{G}_1$ are the Hubble parameter and the GB term at the dS point, respectively. The condition,

$$0 < H_1^6 f_{,\mathcal{G}\mathcal{G}}(H_1) < 1/384, \qquad (8)$$

is required from the stability of the dS point [23]. We have $\mathcal{G} < 0$ and $\dot{\mathcal{G}} > 0$ during both radiation and matter domination. However the GB term changes its sign from negative to positive during the transition from the matter era ($\mathcal{G} = -12H^4$) to the dS epoch ($\mathcal{G} = 24H^4$). For the existence of standard radiation and matter eras we require that $f_{,\mathcal{G}\mathcal{G}} \equiv \partial^2 f/\partial\mathcal{G}^2 > 0$ for $\mathcal{G} \le \mathcal{G}_1$ [23]. Since the term $24H^3\dot{f}_{,\mathcal{G}}$ in Eq. (4) is of the order of $H^8 f_{,\mathcal{G}\mathcal{G}}$, this is suppressed relative to $3H^2$ for $H^6 f_{,\mathcal{G}\mathcal{G}} \ll 1$ during the radiation and matter domination. In order for this condition to hold, we require that $f_{,\mathcal{G}\mathcal{G}}$ approaches $+0$ in the limit $|\mathcal{G}| \to \infty$. Recall that even around the de Sitter point the condition $H^6 f_{,\mathcal{G}\mathcal{G}} \ll 1$ is satisfied from Eq. (8).

A couple of representative models that can satisfy these conditions are [23]

$$\text{(A)}\ f(\mathcal{G}) = \lambda\frac{\mathcal{G}}{\sqrt{\mathcal{G}_*}}\arctan\left(\frac{\mathcal{G}}{\mathcal{G}_*}\right) - \frac{1}{2}\lambda\sqrt{\mathcal{G}_*}\ln\left(1 + \frac{\mathcal{G}^2}{\mathcal{G}_*^2}\right) - \alpha\lambda\sqrt{\mathcal{G}_*}, \qquad (9)$$

$$\text{(B)}\ f(\mathcal{G}) = \lambda\frac{\mathcal{G}}{\sqrt{\mathcal{G}_*}}\arctan\left(\frac{\mathcal{G}}{\mathcal{G}_*}\right) - \alpha\lambda\sqrt{\mathcal{G}_*}, \qquad (10)$$

where α, λ, and $\mathcal{G}_*$ are positive constants. The second derivatives of f in terms of $\mathcal{G}$ for the models (A) and (B) are $f_{,\mathcal{G}\mathcal{G}} = \lambda/[\mathcal{G}_*^{3/2}(1 + \mathcal{G}^2/\mathcal{G}_*^2)]$ and $f_{,\mathcal{G}\mathcal{G}} = 2\lambda/[\mathcal{G}_*^{3/2}(1 + \mathcal{G}^2/\mathcal{G}_*^2)^2]$, respectively (both of which are positive for all $\mathcal{G}$).

The quantity defined by

$$\xi \equiv f_{,\mathcal{G}} \qquad (11)$$

is constant for the ΛCDM model, $f(\mathcal{G}) = -2\Lambda + c\mathcal{G}$ (here we have included the linear term $c\mathcal{G}$ because this also gives rise to the equations of motion the same as those in the ΛCDM model). In order to discuss cosmological perturbations in the next section, it is convenient to introduce the following quantity:

$$\mu \equiv H\dot{\xi} = H\dot{\mathcal{G}}f_{,\mathcal{G}\mathcal{G}} = 72H^6 f_{,\mathcal{G}\mathcal{G}}[(1+w_{\rm eff})(1+3w_{\rm eff}) - w'_{\rm eff}/2], \quad (12)$$

where a prime represents a derivative with respect to $N = \ln a$. This quantity characterizes the deviation from the ΛCDM model. During the radiation and matter domination one has $\mu = 192H^6 f_{,\mathcal{G}\mathcal{G}}$ and $\mu = 72H^6 f_{,\mathcal{G}\mathcal{G}}$, respectively, whereas at the de Sitter attractor $\mu = 0$.

In Fig. 1 we plot the evolution of μ and $w_{\rm eff}$ in model (A) for $\alpha = 100$ and $\lambda = 3\times 10^{-4}$. In this case the quantity μ is much smaller than unity in the deep matter era ($w_{\rm eff} \simeq 0$) and it grows to the order of 10^{-4} prior to the accelerated epoch. This is followed by the decrease of μ toward 0 with small oscillations, as the solution approaches the de Sitter attractor with $w_{\rm eff} = -1$. For smaller α and larger λ, it is also possible to realize larger maximum values of μ such as $\mu_{\rm max} \gtrsim 0.1$. The qualitative behavior shown in Fig. 1 is generic for viable $f(\mathcal{G})$ models at the background level.

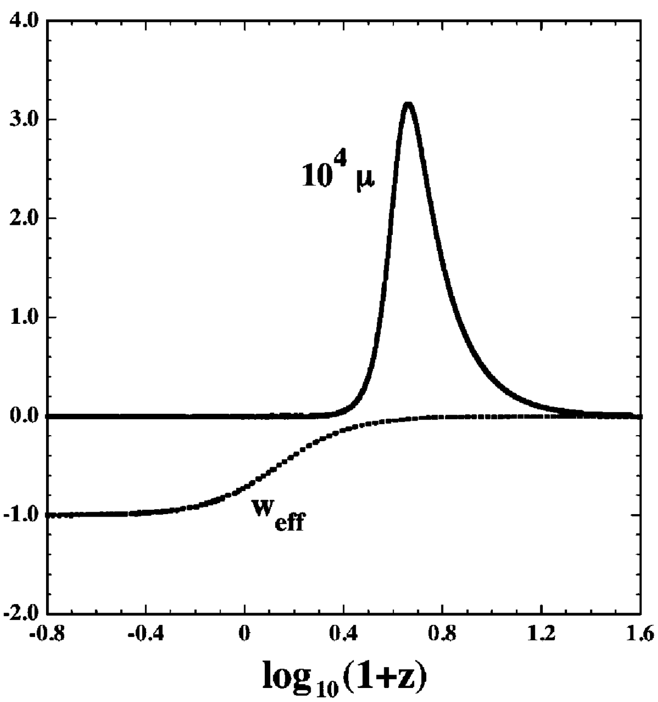

FIG. 1. Evolution of μ (multiplied by 10^4) and $w_{\rm eff}$ versus the redshift $z = a_0/a - 1$ for the model (9) with parameters $\alpha = 100$ and $\lambda = 3\times 10^{-4}$. The initial conditions are chosen to be $x = -1.499\,985$, $y = 20$, and $\Omega_m = 0.999\,99$ (see Appendix B for the definition of x and y).

III. COSMOLOGICAL PERTURBATIONS

In order to study cosmological perturbations in $f(\mathcal{G})$ gravity we introduce a perturbed metric with 4 scalar perturbations α, β, ϕ, and γ about a spatially flat FLRW cosmological background [30],

$$ds^2 = -(1+2\alpha)dt^2 - 2a\beta_{i}dtdx^i + a(t)^2[(1+2\phi)\delta_{ij} + 2\partial_i\partial_j\gamma]dx^i dx^j. \quad (13)$$

Let us decompose the energy-momentum tensor T^μ_ν into background and perturbed parts, i.e. $T^0_0 = -(\rho_m + \delta\rho_m)$ and $T^0_\alpha = -\rho_m v_{,\alpha}$, where v is a velocity potential. We define the gauge-invariant matter density perturbation δ_m, as

$$\delta_m \equiv \frac{\delta\rho_m}{\rho_m} + \frac{\dot{\rho}_m}{\rho_m}v. \quad (14)$$

We also introduce two gauge-invariant combinations

$$\Phi_1 \equiv \phi + Hv, \quad (15)$$

$$\Phi_2 \equiv \delta\xi + \dot{\xi}v, \quad (16)$$

where $\delta\xi$ is the perturbation of the quantity $\xi = f_{,\mathcal{G}}$.

A. Perturbation equations

Following a similar procedure to the one developed recently in Ref. [31], one can show that the dynamics of cosmological perturbations in $f(\mathcal{G})$ gravity in the presence of a perfect fluid with an equation of state w reduces to that of two propagating fields Φ_1 and Φ_2 defined by the following perturbed action:

$$\delta S = \int d^4x[A_1\dot{\Phi}_1^2 + 2A_2\dot{\Phi}_1\dot{\Phi}_2 + A_3\dot{\Phi}_2^2 - g_1(\vec{\nabla}\Phi_1)^2 - 2g_2\vec{\nabla}\Phi_1\cdot\vec{\nabla}\Phi_2 - g_3(\vec{\nabla}\Phi_2)^2 + B(\dot{\Phi}_2\Phi_1 - \dot{\Phi}_1\Phi_2) - m_3\Phi_2^2 - 2m_2\Phi_1\Phi_2], \quad (17)$$

where A_i, g_i, B, and m_i are time-dependent coefficients whose explicit forms are given in Appendix A. We also have the following relation:

$$\alpha + \dot{v} = \frac{1+4\mu}{H(1+6\mu)}\dot{\Phi}_1 + \frac{2H}{1+6\mu}\dot{\Phi}_2 - \frac{2H^2}{1+6\mu}\Phi_2 = -\frac{w}{1+w}\delta_m, \quad (18)$$

where μ is defined in Eq. (12).

From the action (17) we obtain the perturbation equations in Fourier space

$$\frac{d}{dt}(A_1\dot{\Phi}_1 + A_2\dot{\Phi}_2) + g_1k^2\Phi_1 + g_2k^2\Phi_2 + 2m_2\Phi_2 - B\dot{\Phi}_2 = 0, \quad (19)$$

$$\frac{d}{dt}(A_2\dot{\Phi}_1 + A_3\dot{\Phi}_2) + g_2k^2\Phi_1 + g_3k^2\Phi_2 + m_3\Phi_2 + B\dot{\Phi}_1 = 0, \quad (20)$$

where k is a comoving wave number.

B. Instability at small scales

Let us show the presence of a small-scale instability in $f(\mathcal{G})$ gravity associated with a negative speed squared of one propagating mode. This instability appears at large redshifts for the models that look like GR ($\mu \ll 1$) at early times. For sufficiently small scales it is easy to show that the highest derivative terms prevail over any other term in the differential equations. Then one can approximately find a harmonic oscillatorlike dispersion relation in the follow way.

For large k, one can look for an approximate solution in the following way. The dominant contribution to the equation of motion will be in the form

$$A\ddot{\vec{\Phi}} - g\nabla^2\vec{\Phi} \approx 0, \quad (21)$$

where A is the 2×2 symmetric matrix whose diagonal elements are A_1 and A_3 and nondiagonal element is A_2. Along the same lines one defines the matrix g with the elements g_1, g_2, and g_3. Introducing the time-dependence $\Phi_j \propto \exp(i\omega t)$ with $j = 1, 2$ in Fourier space, it follows that

$$(-\omega^2 A + k^2 g)\vec{\Phi} \approx 0, \quad (22)$$

which has solutions for some values of ω^2. This approximation tends to be more accurate for larger k. These expressions give the dispersion relation for the propagating modes under consideration.

Nonzero solutions for $\vec{\Phi}$ exist provided that the following relation holds:

$$\det(\omega^2 A - k^2 g) = 0. \quad (23)$$

After finding the eigenvalues ω^2, one can proceed to look for the eigenvectors which diagonalize the kinetic operator. We find that one eigenvector mode propagates with a speed squared $c_1^2 = w$, as expected, and the other one with a speed squared

$$c_2^2 = 1 + \frac{2\dot{H}}{H^2} + \frac{1+w}{1+4\mu}\frac{8\pi G\rho_m}{3H^2}. \quad (24)$$

This coincides with the result of the vacuum case found in Ref. [32] by taking the limit $\rho_m \to 0$.

In the Universe dominated by a single fluid one has $3H^2 \simeq 8\pi G\rho_m$ and $\dot{H}/H^2 \simeq -3(1+w)/2$. Under the condition that $\mu \ll 1$, the speed squared (24) reduces to

$$c_2^2 \simeq -1 - 2w. \quad (25)$$

This shows the existence of a negative instability for $w > -1/2$. Hence the perturbations in radiation and nonrelativistic matter are subject to this instability during the radiation and matter domination, respectively. In the matter-dominated epoch ($\Omega_m \simeq 1$) the result (25) agrees with the value $\tilde{c}_2^2 = 1 + 4\dot{H}/(3H^2)$ that appears as a coefficient of the term k^2/a^2 in Eq. (47) of Ref. [27]. During the transition from the matter era to the accelerated epoch c_2^2 can be quite different from $\tilde{c}_2^2$, because Ω_m is smaller than 1 and the quantity μ is not necessarily negligible relative to 1. Therefore the stability at late times must be checked against the quantity c_2^2.

The reason why the values c_2^2 and $\tilde{c}_2^2$ are different can be understood as follows. Looking at Eqs. (47) and (48) in Ref. [27], the modes are not diagonalized, as the k^2/a^2 term still appears for two different fields. Even if the coefficient of the k^2/a^2 term for one of two equations is negative, this is not enough to state that there is an instability in the system. In other words, even if g_1 is negative, the eigenvalues of $A^{-1}g$ can still both be positive. Another difference among these two studies is that for the evolution of perturbations, the authors in Ref. [27] chose a background solution that mimics the evolution in the ΛCDM model.[1] Strictly speaking, this is not a solution of the Einstein equations. This can affect the evolution of the perturbations especially at late times. In our work we use concrete $f(\mathcal{G})$ models to find cosmological evolution of both the background and the perturbations.

The Laplacian instability mentioned above appears at any time in the past at some sufficiently small scales (apart from the epoch of inflation during which $c_2^2 \approx 1$). This can place tight constraints on $f(\mathcal{G})$ models. Let us discuss this more precisely, without any approximation, for the growth of large-scale structure during the matter domination.

IV. GROWTH OF MATTER PERTURBATIONS

Let us focus on nonrelativistic matter with the equation of state $w = 0$. In order to derive the equation for matter perturbations, we first combine Eqs. (18)–(20) and finally take the limit $w \to 0$. From Eq. (18) one can express $\dot{\Phi}_1$ in terms of $\dot{\Phi}_2$, Φ_2, and δ_m. Multiplying Eq. (19) by g_2 and Eq. (20) by g_1 and subtracting one from the other, we find an equation which depends on Φ_1 only through its first time derivative. Combining these two equations and taking the limit $w \to 0$, one reaches the following equation of motion:

[1]Observations show that the current equation of state of dark energy is close to -1 to a pretty high redshift, so that the deviation from the GR background is not large. Reference [27] used this fact to neglect the $k^2\eta$ contribution relative to the term already on the left-hand side of Eq. (A6) for the realistic cosmological models one considers, for which $\mu \ll 1$ or equivalently $|H^6 f_{,\mathcal{G}\mathcal{G}}| \ll 1$. In this work we make a more general approach of computing the diagonalized modes accurately by including the contribution coming from the $k^2\eta$ term.

$$\ddot{\Phi}_2 - d_4\dot{\Phi}_2 + \left(d_3 + c_2^2\frac{k^2}{a^2}\right)\Phi_2 - d_1\dot{\delta}_m - d_2\delta_m = 0, \tag{26}$$

where

$$c_2^2 = \frac{1 + \Omega_m + 2x + 4\mu(1 + 2x)}{1 + 4\mu}, \tag{27}$$

$$d_1 = \frac{\mu[1 + \Omega_m + 2x + 4\mu(1 + 2x)]}{H(1 + 4\mu)}, \tag{28}$$

$$d_2 = \frac{\Omega_m[1 + 3\Omega_m + 4x + 4\mu(1 + 4x)]}{4(1 + 4\mu)}, \tag{29}$$

$$d_3 = H^2\{4x^2 + 4x + x' - 2\mu[4(1 - 3x^2 - 2x - x') + 3\Omega_m(1 + x) + 8\mu(2 - 2x^2 - x')]\}/[2\mu(1 + 4\mu)], \tag{30}$$

$$d_4 = -\frac{3H[1 + \Omega_m + 2x + 4\mu(1 + 2x)]}{1 + 4\mu}, \tag{31}$$

and $x \equiv \dot{H}/H^2$, $x' \equiv \dot{x}/H$, and $\Omega_m \equiv 8\pi G\rho_m/(3H^2)$.

To find the second dynamical equation we multiply Eq. (19) by g_3 and Eq. (20) by g_2 and then subtract the two equations. We then divide it by $g_1g_3 - g_2^2$ and differentiate it with respect to time. This gives rise to the equation which involves $\dot{\Phi}_1$, so that we can replace it with δ_m. Furthermore, the same equation will contain second and third time derivatives of Φ_2, which can be substituted by using Eq. (26). By doing so and taking the limit $w \to 0$, one finds the following dynamical equation:

$$\ddot{\delta}_m - d_5\dot{\delta}_m - d_6\delta_m - d_8\dot{\Phi}_2 + \left(d_9 + d_7\frac{k^2}{a^2}\right)\Phi_2 = 0, \tag{32}$$

where

$$d_5 = -\frac{2H(1 - 2x\mu + 2\mu)}{1 + 4\mu}, \qquad d_6 = \frac{3H^2\Omega_m(1 + x)}{1 + 4\mu}, \tag{33}$$

$$d_7 = \frac{4H^2(1 + x)}{1 + 4\mu}, \qquad d_8 = -\frac{12H^3(1 + x)}{1 + 4\mu}, \tag{34}$$

$$d_9 = \frac{3H^4[4x^2 + 4x + x' - 4\mu(1 - 3x^2 - 2x - x')]}{\mu(1 + 4\mu)}. \tag{35}$$

Note that in GR $\mu = 0$ and $\Phi_2 = 0$. From Eqs. (30) and (35) both d_3 and d_9 diverge in the limit $\mu \to 0$. Therefore we can solve Eq. (26) for Φ_2 and substitute it into Eq. (32). This results in the following equation:

$$\ddot{\delta}_m + C_1\dot{\delta}_m + C_2\delta_m = r\ddot{\Phi}_2 + (d_8 - rd_4)\dot{\Phi}_2, \tag{36}$$

where $C_1 \equiv rd_1 - d_5$, $C_2 \equiv rd_2 - d_6$, and

$$r \equiv \frac{M_B^2}{M_A^2}, \qquad M_A^2 \equiv d_3 + c_2^2\frac{k^2}{a^2}, \qquad M_B^2 \equiv d_9 + d_7\frac{k^2}{a^2}. \tag{37}$$

In order to derive analytic solutions let us consider the case $\mu(k/aH)^2 \ll 1$. Since we are interested in subhorizon modes ($k \gg aH$), the condition $\mu \ll 1$ also follows. During the matter domination characterized by $H \simeq 2/(3t)$ and $\Omega_m \simeq 1$, we have

$$C_1 \simeq 2H(1 - 2\mu), \qquad C_2 \simeq -\frac{3}{2}H^2\left[1 + \frac{8}{9}\mu\left(\frac{k}{aH}\right)^2\right]. \tag{38}$$

In the GR limit $\mu \to 0$ and $\Phi_2 \to 0$, Eq. (36) reduces to $\ddot{\delta}_m + 2H\dot{\delta}_m - (3/2)H^2\delta_m = 0$, which has the growing mode solution $\delta_m \propto t^{2/3}$ in the matter era.

In the regime $\mu(k/aH)^2 \ll 1$ the growth rate of δ_m gets larger than that in GR because of the presence of the $(8/9)\mu(k/aH)^2$ term in Eq. (38). For subhorizon modes ($k \gg aH$) this effect is more important than the reduction of the friction term C_1 induced by 2μ.

The quantity $\mu(k/aH)^2$ can grow to the order of 1 by the present epoch, depending on the wave number k. In this case the growth of matter perturbations is significantly different from that in GR. During the matter-dominated epoch one has $d_3 \simeq 3H^2/(2\mu) > 0$ and $c_2^2 \simeq -1$, so that the mass term $M_A^2 \simeq 3H^2/(2\mu) - k^2/a^2$ changes its sign from positive to negative at $\mu(k/aH)^2 = 3/2$. This leads to a negative instability for the perturbation Φ_2 through Eq. (26). The evolution of the mass term M_B^2 during the matter era is given by $M_B^2 \simeq H^2(9H^2/\mu - 2k^2/a^2)$, which changes from positive to negative at $\mu(k/aH)^2 = 9/2$. Thus the onset of the negative instability can be characterized by the condition

$$\mu \approx (aH/k)^2. \tag{39}$$

In the regime $\mu(k/aH)^2 \gg 1$ one can approximate $M_A^2 \simeq c_2^2k^2/a^2$ and $M_B^2 \simeq d_7k^2/a^2$, which results in the positive mass ratio $r \simeq 2H^2/(1 + 8\mu)$. Then Eq. (36) reduces to

$$\ddot{\delta}_m + 2H\dot{\delta}_m + \frac{H^2}{2(1 + 8\mu)}\delta_m = \frac{2H^2}{1 + 8\mu}\ddot{\Phi}_2. \tag{40}$$

Here we have not used the approximation $\mu \ll 1$. Notice that the coefficient in front of δ_m is positive and hence this term does not lead to the growth of δ_m. However, the rapid growth of Φ_2 induced by the negative c_2^2 works as a source term for the amplification of δ_m in Eq. (40). We have $c_2^2 = 1$ and $d_7 = 4H^2$ at the late-time dS point, which means that both c_2^2 and d_7 change signs from negative to positive during the transition from the matter era to the accelerated epoch. Hence we can expect that the growth of matter perturbations ends before reaching the dS attractor.

In Fig. 2 the evolution of matter perturbations is plotted for the model (9) with parameters $\alpha = 100$ and $\lambda = 3 \times 10^{-4}$ (see Appendix B for the details of numerical integration). In this case the quantity μ reaches the maximum value $\mu_{\rm max} = 3 \times 10^{-4}$ around the redshift $z = 3.6$ (see Fig. 1). Using the criterion (39), the perturbation with $k \approx 60aH$ is about to enter the negative instability region. The quantity μ decreases rapidly after it reaches the maximum, whereas aH at $z = 3.6$ is not much different from a_0H_0 today ($z = 0$). Hence one can estimate that the modes with $k \lesssim 60a_0H_0$ are hardly affected by the negative instability. In the numerical simulation of Fig. 2 this can be confirmed for the mode $k = 30a_0H_0$. Meanwhile Fig. 2 shows that the modes with $k \gtrsim 100a_0H_0$ exhibit violent negative instabilities. Note that the apparent discontinuous behavior seen in Fig. 2 for the mode $k = 150a_0H_0$ comes from the fact that δ_m temporally becomes negative.

The wave numbers relevant to the observed galaxy power spectrum in the linear regime correspond to $30a_0H_0 \lesssim k \lesssim 600a_0H_0$ (i.e. $0.01h\ {\rm Mpc}^{-1} \lesssim k \lesssim 0.2h\ {\rm Mpc}^{-1}$). For the model parameters used in Fig. 2, the resulting matter power spectrum is certainly ruled out from the observations of large-scale structure. In Fig. 2 we have chosen the initial conditions $\Phi = \dot{\Phi} = 0$ as a minimal case, but nonzero initial values of Φ and $\dot{\Phi}$ lead to even larger amplitude of δ_m. We also note that, irrespective

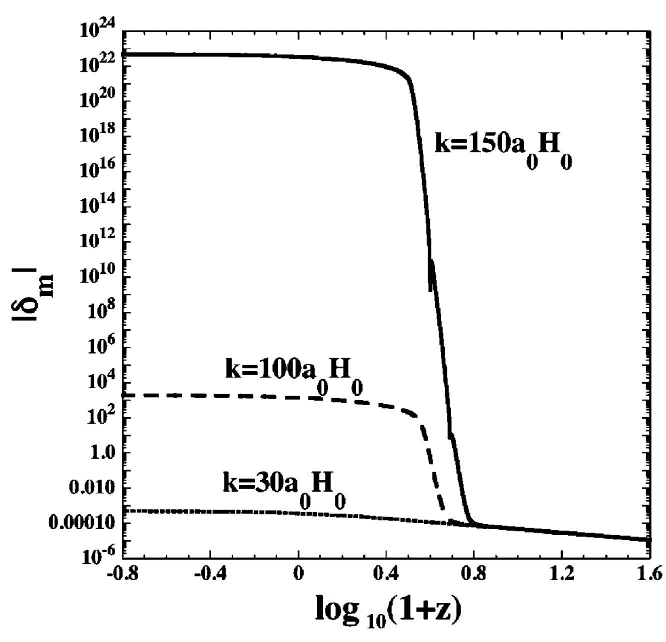

FIG. 2. Evolution of δ_m versus the redshift $z = a_0/a - 1$ for the model (9) with the same model parameters as given in Fig. 1. We choose three different wave numbers: (i) $k = 150a_0H_0$, (ii) $k = 100a_0H_0$, and (iii) $k = 30a_0H_0$. The initial conditions are $x = -1.499\,985$, $y = 20$, $\Omega_m = 0.999\,99$, and $\delta_m = \dot{\delta}_m/H = 10^{-5}$, $\Phi = \dot{\Phi} = 0$.

of the forms of $f(\mathcal{G})$ models discussed in Sec. II, the behavior of perturbations is similar to that discussed above.

The only way to avoid this negative instability is to make the parameter μ as small as possible by changing model parameters, so that the modes relevant to the matter power spectrum never reach the regime $\mu(k/aH)^2 = \mathcal{O}(1)$. If we take the smallest scale $k \approx 600a_0H_0$ of the linear matter power spectrum, the condition under which the negative instability can be avoided translates into

$$\mu_{\rm max} \lesssim 10^{-6}, \qquad (41)$$

where we have used the approximation $aH \approx a_0H_0$ at $\mu = \mu_{\rm max}$. Hence the deviation from the ΛCDM model is constrained to be very small. Nonetheless, even by introducing by hand this effective cutoff for the wavelength due to the experimental apparatus we use to observe data, the theory does possess a UV instability, no matter how small but nonzero μ is. In this case perturbation theory at some small scale will break down, during anytime in the past up to the dark energy domination. Therefore, these theories cannot be studied by using perturbation theory, and in general, one should expect strong dynamical deviations from GR, as the background is not trustable any longer.

Furthermore we wish to stress that the negative instability cannot be avoided as we go to smaller scales. In order to avoid violent growth in the nonlinear regime of the matter power spectrum ($k \gtrsim 600a_0H_0$), the constraint on $\mu_{\rm max}$ becomes even severer than the one given in Eq. (41). Moreover the growth rate of matter perturbations gets enormously large for increasing k. The point is that we can always find the wave number k satisfying $\mu(k/aH)^2 \approx 1$ even for very small values of μ. This property also persists for the perturbations in radiation. Since the quantity μ during the radiation era is suppressed relative to that during the matter era, the scales of instabilities of radiation perturbations are much smaller than those of matter perturbations. The only way to consistently remove this UV instability is to set μ identically equal to zero; that is, the gravitational theory exactly reduces to GR.

V. CONCLUSIONS

In this paper we have studied cosmological perturbations in $f(\mathcal{G})$ gravity, in the presence of a perfect fluid with a constant equation of state w. In the Universe dominated by a single fluid with $w > -1/2$, we have shown the presence of an instability associated with a negative speed squared of one eigenvector mode. Hence the perturbations in radiation and nonrelativistic matter are affected by this instability during the radiation and matter domination, respectively. Our results are more general than those given in Ref. [27] in the sense that we have considered a general equation of state w and that we have not assumed the ΛCDM-like background evolution.

A useful quantity that characterizes the deviation from the ΛCDM model is $\mu = H\dot{f}_{,\mathcal{G}}$. In the limit that $\mu \to 0$

(i.e. the ΛCDM model) one can avoid the appearance of the negative instability. If $\mu \neq 0$, the instability of perturbations appears for $\mu \gtrsim (aH/k)^2$. Even for tiny values of μ much smaller than 1, there are small-scale modes that satisfy this condition. We have studied the evolution of nonrelativistic matter perturbations numerically and confirmed that the perturbations are strongly amplified once they enter the regime $\mu \gtrsim (aH/k)^2$. From the requirement that the matter power spectrum in the linear regime is not affected by this violent instability, we have found that the maximum value of the deviation parameter is constrained to be $\mu_{\rm max} \lesssim 10^{-6}$.

Nonetheless the UV limit of this theory remains unsatisfactory, as perturbation theory would break down eventually at some scale, and the background is not under control any longer at these scales. When this happens, there is not even an easy way for the cosmological background of this theory to be checked against observations. It is mostly this UV unpredictable behavior which sets the strongest bound. This feature will always remain unless one sets μ identically to 0 at any time, as in this case the theory reduces to GR and the unstable eigenvector mode automatically disappears. In this sense we believe that this theory is ruled out by our analysis, which explicitly shows the existence of eigenmodes with negative squared speed in the past.

While we have focused on linear perturbations, nonlinear effects become important once δ_m grows to the order of 1. It may be of interest to see whether such nonlinear effects strengthen or weaken the growth of perturbations.

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APPENDIX A: COEFFICIENTS OF THE ACTION (17)

Here we present the coefficients appearing in the action (17):

$$A_1 = \frac{(12H^5\dot{\xi}^3 w + 3H^4 w\dot{\xi}^2 + 16H^2\dot{\xi}^2\pi G\rho_m w + 16H^2\dot{\xi}^2\pi G\rho_m + 8\pi G\rho_m H\dot{\xi} + 8\pi G\rho_m wH\dot{\xi} + w\rho_m\pi G + \pi G\rho_m)a^3}{2\pi Gw(1+6\dot{\xi}H)^2H^2}, \tag{A1}$$

$$A_2 = \frac{(-12H^4 w\dot{\xi}^2 - 3w\dot{\xi}H^3 + 8\pi G\rho_m H\dot{\xi} + 8\pi G\rho_m wH\dot{\xi} + 2\pi G\rho_m + 2w\rho_m\pi G)a^3}{2wG\pi(1+6\dot{\xi}H)^2}, \tag{A2}$$

$$A_3 = \frac{(12w\dot{\xi}H^3 + 3wH^2 + 4w\rho_m\pi G + 4\pi G\rho_m)a^3H^2}{2wG\pi(1+6\dot{\xi}H)^2}, \tag{A3}$$

$$g_1 = \frac{1}{2\pi G(1+6\dot{\xi}H)^2H^2}(12H^5\dot{\xi}^3 + 3\dot{\xi}^2H^4 + 24H^3\dot{\xi}^3\dot{H} + 24H^2\dot{\xi}^2\pi G\rho_m + 6H^2\dot{H}\dot{\xi}^2 + 24H^2\dot{\xi}^2\pi G\rho_m w + 8\pi G\rho_m H\dot{\xi} + 8\pi G\rho_m wH\dot{\xi} + w\rho_m\pi G + \pi G\rho_m)a, \tag{A4}$$

$$g_2 = \frac{(-12\dot{\xi}^2H^4 - 3\dot{\xi}H^3 - 24H^2\dot{H}\dot{\xi}^2 - 6\dot{\xi}H\dot{H} + 2\pi G\rho_m + 2w\rho_m\pi G)a}{2\pi G(1+6\dot{\xi}H)^2}, \tag{A5}$$

$$g_3 = \frac{3aH^2(4\dot{\xi}H^3 + H^2 + 8\dot{\xi}H\dot{H} + 2\dot{H} + 4w\rho_m\pi G + 4\pi G\rho_m)}{2\pi G(1+6\dot{\xi}H)^2}, \tag{A6}$$

$$B = \frac{(1+4\dot{\xi}H)(-6w\dot{\xi}H^3 + 18H\dot{H}w\dot{\xi} + 3\dot{H}w + 4w\rho_m\pi G + 4\pi G\rho_m)Ha^3}{4wG\pi(1+6\dot{\xi}H)^2}, \tag{A7}$$

$$m_2 = -\dot{B}/2, \tag{A8}$$

$$m_3 = -\frac{1}{4(1+6\dot{\xi}H)^3\dot{\xi}wG\pi}[a^3H(-12w\dot{H}H^2 - 3wH\ddot{H} + 24H^5w\dot{\xi} + 432H^7w\dot{\xi}^3 + 192H^6w\dot{\xi}^2 + 192H\dot{\xi}w^2\rho_m^2\pi^2G^2 + 72\pi G\rho_m w^2\dot{\xi}H^3 + 240H^2\dot{\xi}^2\dot{H}\pi G\rho_m + 72H\dot{\xi}\dot{H}\pi G\rho_m + 384H\dot{\xi}w\rho_m^2\pi^2G^2 + 144\pi G\rho_m w^2\dot{\xi}^2H^4 + 96\pi G\rho_m w\dot{\xi}^2H^4 + 80\pi G\rho_m w\dot{\xi}H^3 - 432H^3\dot{\xi}^3\dot{H}^2w - 252H^3\dot{\xi}^2\ddot{H}w - 288H^2\dot{\xi}^2\dot{H}^2w - 48H^2\dot{\xi}\ddot{H}w - 72H\dot{\xi}\dot{H}^2w - 864H^5\dot{H}w\dot{\xi}^3 - 132H^3\dot{H}w\dot{\xi} - 552H^4\dot{H}w\dot{\xi}^2 - 432H^4\dot{\xi}^3\ddot{H}w + 192H\dot{\xi}\pi^2G^2\rho_m^2 - 48\pi G\rho_m\dot{\xi}^2H^4 + 8\pi G\rho_m\dot{\xi}H^3 + 672H^2\dot{\xi}^2\dot{H}w\rho_m\pi G + 432H^2\dot{\xi}^2\dot{H}w^2\rho_m\pi G + 72H\dot{\xi}\dot{H}w^2\rho_m\pi G + 144H\dot{\xi}\dot{H}w\rho_m\pi G - 6w\dot{H}^2)]. \tag{A9}$$

APPENDIX B: NUMERICAL INTEGRATION OF DYNAMICAL EQUATIONS

In addition to the dimensionless variables $x = \dot{H}/H^2$ and $\Omega_m = 8\pi G\rho_m/(3H^2)$, we introduce another variable $y \equiv H/H_*$, where H_* is a constant related to $\mathcal{G}_*$ (a typical scale of the GB term for dark energy) via $H_* = \mathcal{G}_*^{1/4}$. Then the background equations can be expressed as

$$x' = -4x^2 - 4x + \frac{1}{24^2H^6f_{,\mathcal{G}\mathcal{G}}}\left[\frac{\mathcal{G}f_{,\mathcal{G}} - f}{H^2} - 3(1-\Omega_m)\right], \tag{B1}$$

$$y' = xy, \tag{B2}$$

$$\Omega_m' = -(3+2x)\Omega_m. \tag{B3}$$

The quantities $H^6f_{,\mathcal{G}\mathcal{G}}$ and $(\mathcal{G}f_{,\mathcal{G}} - f)/H^2$ can be expressed by x and y once the model is specified. Introducing the following quantity:

$$\Phi \equiv H^2\Phi_2, \tag{B4}$$

the perturbation equations (26) and (32) can be written as

$$\Phi'' = \left(\frac{d_4}{H} + 3x\right)\Phi' - \left[\frac{d_3}{H^2} + 2\frac{d_4}{H}x + 2x^2 - 2x' + c_2^2\left(\frac{k}{aH}\right)^2\right]\Phi + d_1H\delta_m' + d_2\delta_m, \tag{B5}$$

$$\delta_m'' = \left(\frac{d_5}{H} - x\right)\delta_m' + \frac{d_6}{H^2}\delta_m + \frac{d_8}{H^3}\Phi' - \left[\frac{d_9}{H^4} + \frac{d_7}{H^2}\left(\frac{k}{aH}\right)^2 + 2\frac{d_8}{H^3}x\right]\Phi. \tag{B6}$$

Numerically we solve these equations together with the background equations (B1)–(B3).

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